HW 2
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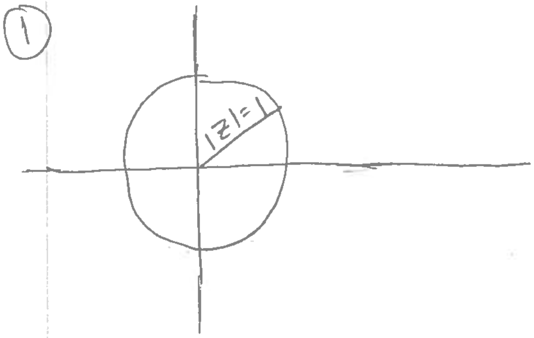
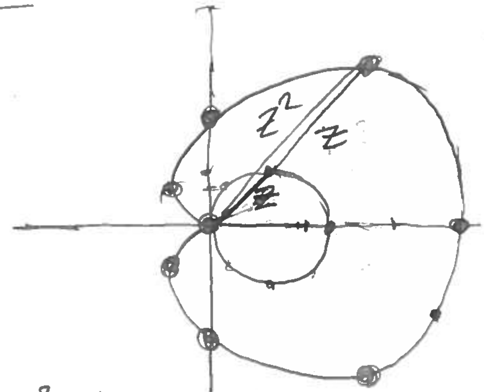
Complex Variable

$|z| = 1$

(1)





$||z-1|| = 1$

$\|z-1\|$		$z \mapsto z^2$	
2	1	4	4
$1+i$	1	$1^2 + 2i - 1$	$2i$
0	1	0	0
$1-i$	1	$-2i$	

$1 + e^{\pi/4}$

$z \mapsto z^2$

(10) $$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \frac{\sin((j+1)\phi)}{\sqrt{1+k^2}^{\,j+1}} X^j$$

$X = x-k$

$\phi = \arg(i-k)$

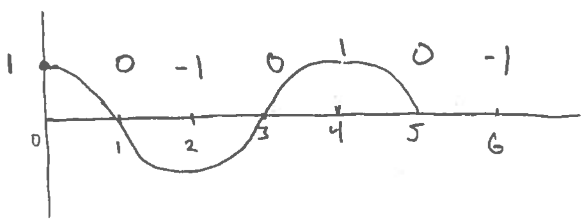
Recover $H(x) = \sum_{j=0}^{\infty} (-1)^j x^{2j}$

$k=0, \quad \phi = \frac{\pi}{2}, \quad X = x$

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \frac{\sin(\frac{\pi}{2}j + \frac{\pi}{2})}{1} x^j, \quad \sin(x+\tfrac{\pi}{2}) = \cos(x)$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \cos(\tfrac{\pi}{2}j)\, x^j$$

j	$\cos(\frac{\pi}{2}j)$
0	1
1	0
2	-1
3	0
4	1
5	0
6	-1



evens

$$= \sum_{j=0}^{\infty} \cos(\tfrac{\pi}{2}(2j)) x^{2j} + \sum_{j=0}^{\infty} \cos(\tfrac{\pi}{2}(2j+1)) x^{2j+1}$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \cos(\pi j)\, x^{2j} + 0$$

(10)

(ii) $$\sum_{j=0}^{\infty} \frac{\sin((j+1)\phi)}{\sqrt{1+k^2}^{\,j+1}} X^j \qquad X = x-k \qquad \phi = \arg(i-k)$$

$$\sum_{j=0}^{\infty} \frac{\sin((j+1)\arg(i-k))}{\sqrt{1+k}^{\,j+1}} (X-k)$$

$$\tan(\arg(z)) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$$

$$\sin((j+1)\arg(i-k)) = 0$$

$$\sin(\underbrace{j\arg(i-k)}_{\text{frequency modifier}} + \underbrace{\arg(i-k)}_{\text{offset}}) = 0$$

$$X^2 \rightarrow \sin(3\arg(i-k)) = 0$$

$$3\arg(i-k) = 0, \pi, 2\pi \ldots$$

$$\arg(i-k) = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$$

$$\frac{i}{-k} = \tan(0), \tan(\tfrac{\pi}{3}), \tan(\tfrac{2\pi}{3})$$

$$\frac{i}{-k} = 0, \sqrt{3}, -\sqrt{3}, 0, \sqrt{3}, -\sqrt{3} \ldots$$

$$k = \frac{-i}{\sqrt{3}} \qquad k = \frac{i}{\sqrt{3}}$$

k is real

$$k = \pm \frac{1}{\sqrt{3}}$$

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$P(z)$ $Q(z)$

R of $P(z)Q(z)$ $>$ R of $P(z)$ and R of $Q(z)$

think in terms of $\dfrac{z^2}{(5-z)^3}$

R is distance to closest singularity

$$\frac{(1-z)}{(5-z)} \quad \frac{(5-z)}{(1-z)(6-z)} = \frac{1}{(6-z)}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$R=5$ $\qquad$ $R=1$ $\qquad$ $R=6$

3i

$$\ln|1+x| = \int_0^x \frac{1}{1+x}\,dx$$

$$= \ln|1+x|\Big|_0^x$$

$$= \ln|1+x| - \ln|1|$$

$$\ln(1+x) = \ln|1+x|$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{j=0}^{\infty} x^j = 1+(-x)+(-x)^2+(-x)^3+\cdots$$

$$= \int 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\begin{tikzcd}
\overset{\frac{1}{1+x}}{f} \arrow[r, "\text{int } \int"] \arrow[d, "\text{power series}"'] & \overset{\ln(x+1)}{g} \arrow[d, "\text{power series}"] \\
\Sigma \arrow[r, "\text{int} \int"] & \Sigma
\end{tikzcd}$$

$$(-1)^n \longrightarrow \frac{(-1)^k}{n}$$

(32)

$$L(z) = \mathrm{Log}(1+z)$$

Since $L(0) = 0$

$$L(z) = az + bz^2 + cz^3 + dz^4 + \dots$$

$$1 + z = e^L = 1 + L + \frac{1}{2!}L^2 + \frac{1}{3!}L^3 + \dots$$

$$e^L = 1 + z = 1 + L + \frac{1}{2!}L^2 + \frac{1}{3!}L^3 + \dots$$

$$1 + z = 1 + (az + bz^2 + cz^3 + dz^4) + \frac{1}{2!}(az + bz^2 + cz^3 + dz^4)^2$$

$$e^L = \cancel{1} + z = \cancel{1} + az + bz^2 + \boxed{cz^3} + dz^4 + \frac{1}{2}(\boxed{a^2z^2} + abz^3 + acz^4 + adz^5 +$$
$$abz^3 + b^2z^4 + bcz^5 + bdz^6 +$$
$$acz^4 + bcz^5 + c^2z^6 + cdz^7 +$$
$$adz^5 + bdz^6 + dcz^7 + d^2z^8)$$

$$1 = a + bz + cz^2 + \dots$$
$$a = 1$$

$$0 = bz^1 + c\cancel{z^2} + d\cancel{z^3} + \boxed{\frac{1}{2}a^2z^1} + abz^2$$
$$0 = b + \frac{1}{2}$$
$$b = -\frac{1}{2}$$

Mathematica

$$c = \frac{1}{3}$$

$$d = -\frac{1}{4}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(21) $z = re^{i\theta}$

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$e^{re^{i\theta}} = \sum_{n=0}^{\infty} \frac{r^n e^{in\theta}}{n!}$$

$$e^{r\cos\theta} e^{ri\sin\theta} = \sum_{n=0}^{\infty} \frac{r^n}{n!} \cos(n\theta) + i\sin(n\theta)$$

$$e^{r\cos\theta}\left[\cos(r\sin\theta) + i\sin(r\sin\theta)\right] = \sum_{n=0}^{\infty} \frac{r^n}{n!}\cos(n\theta) + i\sin(n\theta)$$

(1) $$e^{r\cos\theta}\cos(r\sin\theta) = \sum_{n=0}^{\infty} \frac{r^n}{n!}\cos(n\theta)$$

(2) $$e^{r\cos\theta}\sin(r\sin\theta) = \sum_{n=0}^{\infty} \frac{r^n}{n!}\sin(n\theta)$$

(i) $$\cos(\sin(\theta))e^{\cos\theta} = \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!}$$

When $r=1$ in (1)

$$\sin(\sin(\theta))e^{\cos\theta} = \sum_{n=0}^{\infty} \frac{\sin(n\theta)}{n!}$$

When $r=1$ in (2)

(ii) $\displaystyle\int_0^{2\pi} e^{\cos\theta}\,[\cos(\sin\theta)]\,\cos(m\theta)\,d\theta = \frac{\pi}{m!}$, $m = +\text{int}$

$$e^{\cancel{r}\cos\theta}\cos(r\sin\theta)\,\cos(m\theta) = \cos(m\theta)\sum_{n=0}^{\infty}\frac{\cancel{r^n}}{n!}\cos(n\theta)$$

$$\int_0^{2\pi}\cos(m\theta)\sum_{n=0}^{\infty}\frac{1}{n!}\cos(n\theta)\,d\theta = \frac{\pi}{m!}$$

$$\sum_{n=0}^{\infty}\int_0^{2\pi}\frac{1}{n!}\cos(m\theta)\cos(n\theta)\,d\theta = \frac{\pi}{m!}$$

$r = \sqrt{2}\, x$

(iii) $x = \frac{r}{\sqrt{2}}$ $\qquad f(x) = e^x \sin(x) =$

$$f(x) = e^x \sin(x) \approx e^{r\cos(\theta)} \sin(r\cos(\theta))$$

$$\cos(\theta) = 1/\sqrt{2}$$
$$\theta = \pi/4$$

$$f(x) = e^{r\cos\theta} \sin(r\cos(\theta)) = \sum_{n=0}^{\infty} \frac{r^n}{n!} \sin(r\cos\theta)$$

$$e^{r\cos(\frac{\pi}{4})} \sin(r\cos(\tfrac{\pi}{4})) = \sum_{n=0}^{\infty} \frac{r^n}{n!} \sin(r\cos(\tfrac{\pi}{4}))$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{2}x)^n}{n!} \sin(n\tfrac{\pi}{4})$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{2}^n x^n}{n!} \sin(n\tfrac{\pi}{4})$$

0 1 2

(iv) $f(x) = 0 + \sqrt{2}\, x \sin(\frac{\pi}{4}) + \frac{2x^2 \sin(2\frac{\pi}{4})}{2!}$

$$e^x \sin(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$$

This sound like a good thing to do if I had more time :)

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$a=1$
$b=1$
$t=x$

(v)

$$\frac{d^n}{dt^n}\left[e^{at}\sin bt\right] = (a^2+b^2)^{\frac{n}{2}} e^{at} \sin\left[bt + n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$\frac{d^n}{dt^n}\left[e^{x}\sin x\right] = (2)^{\frac{n}{2}} e^{x} \sin\left[x + n\frac{\pi}{4}\right]$$

$$f(x) = e^x \sin x$$
$$f^n(0) = 2^{\frac{n}{2}} e^0 \sin\left[0 + n\frac{\pi}{4}\right]$$
$$f^n(0) = \sqrt{2}^n \sin\left[n\frac{\pi}{4}\right]$$

Taylor Series $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n, \; a=0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}^n \sin\left[n\frac{\pi}{4}\right]}{n!} x^n$$

same as (iii)