(2)

$$F(z) = u(x,y) + iv(x,y)$$

$a = u_x = v_y$

$b = u_y = -v_x$

$a_x = u_{xx} = v_{yx} = v_{xy} = -u_{yy} = -b_y$

$b_x = u_{yx} = -v_{xx} = u_{xy} = v_{yy} = a_y$

$a_y = u_{xy} = v_{yy}$

$b_y = u_{yy} = -v_{xy}$

$u_{xx} = -u_{yy}$ $\qquad$ $v_{xx} = -v_{yy}$

$\Delta u = u_{xx} + u_{yy} = 0$ $\qquad$ $\Delta v = v_{xx} + v_{yy} = 0$

3

i $u = e^x \cos y$

$$e^z = e^{x+iy} = e^x e^{iy}$$
$$e^z = e^x(\cos y + i\sin y)$$
$$f(z) = e^z, \quad u = e^x \cos y$$
$$v = e^x i\sin y$$

Since $f(z)$ is analytic, u must necessarily be harmonic

ii $u = e^{x^2-y^2} \cos 2xy$

$$z^2 = (x+iy)^2 = x^2 + i2xy - y^2$$
$$z^2 = x^2 - y^2 + i2xy$$
$$e^{z^2} = e^{x^2-y^2+i2xy}$$
$$h(z) = e^{z^2} = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$$
$$f(z) = z^2, \quad g(z) = e^z \Rightarrow h(z) = g(f(z)) = g \circ f$$

Since $f(z)$ and $g(z)$ are analytic, $h(z)$ is analytic. Therefore u is harmonic

iii $\ln|f(z)|$, where $f(z)$ is analytic

Lets try some $f(z)$'s

$f(z) = z$

$\ln|z|$, log of length is real, so u is harmonic

$f(z) = z^2$

$$\ln|z| = \ln|r^2 e^{2i\theta}|$$
$$= \ln|r^2|$$

= log of length is real, so u is harmonic

$f(z) = \log(z)$

$$\ln|\log z| = \ln|\ln(re^{i\theta})|$$
$$= \ln|\ln(r) + i\theta|$$
$$= \ln(\ln(r))$$

= log of length is real, so u is harmonic

Because $\ln|f(z)|$ is a log of a length for analytic f, u is harmonic

(4)

$f = u + iv$

$\boxed{z = x + iy}$

$$u = ax^2 + bxy + cy^2$$

$x = Re[z]$

$y = Im[z]$

$$u_x = 2ax + by + 0 = v_y$$

$$u_y = 0 + bx + 2cy = -v_x$$

Want something like $V = \int u_x\,dy - \int u_y\,dx$

$$V = \underbrace{2axy} + \frac{1}{2}by^2 - \frac{1}{2}bx^2 - \underbrace{2cyx} + \text{const}$$

Need v such that $v_x = 2ax + by$ and $v_y = bx + 2cy$. So $2axy$ and $-2cyx$ must be the same term in V. So $a = -c$

$$i\,V = \left(2axy + \frac{1}{2}by^2 - \frac{1}{2}bx^2\right)i + \text{const}$$

$$u = ax^2 + bxy - ay^2$$

$$f = u + iv$$

$$f = ax^2 + bxy - ay^2 + i2axy + i\frac{1}{2}by^2 - i\frac{1}{2}bx^2 + \text{const}$$

$$a(x^2 - y^2) + \frac{ib}{2}(-1)(x^2 - y^2) + bxy + i2axy$$

$$+ (i2a + b)\,xy$$

$$a(x+iy)(x+iy) - \frac{ib}{2}(x+iy)(x+iy)$$

$$\frac{ib}{2}$$

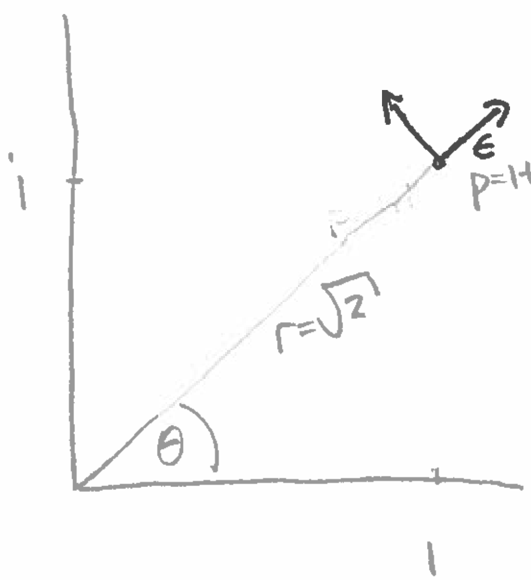
$$f = az^2 - \frac{ib}{2}z^2 + Re[z]\,Im[z]\,(i2a + b) + \text{const}$$

8

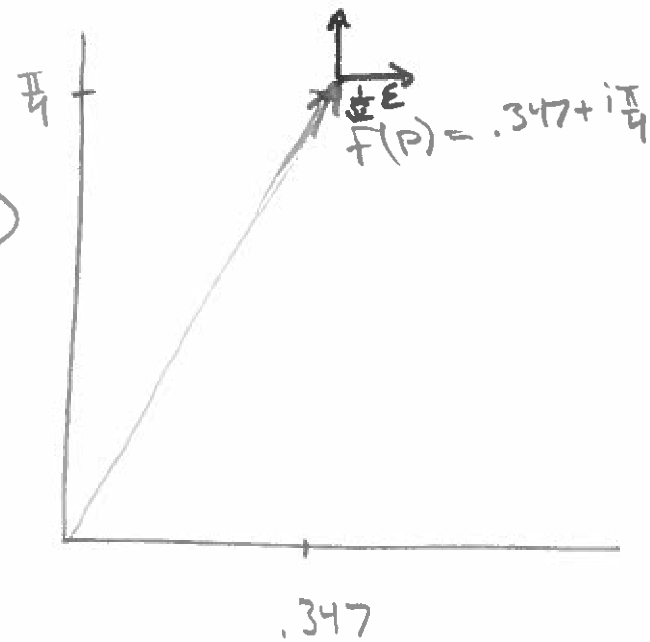
$$\ln(z) = \ln(re^{i\theta}) = \ln|r| + i\theta$$

$$\ln(1+i) = \ln(\sqrt{2}\,e^{i\frac{\pi}{4}}) = \ln|\sqrt{2}| + i\frac{\pi}{4}$$

$$= .347 + i\frac{\pi}{4}$$



f = log(z)



$$\text{Amp} = \frac{1}{r} = \frac{1}{\sqrt{2}}$$

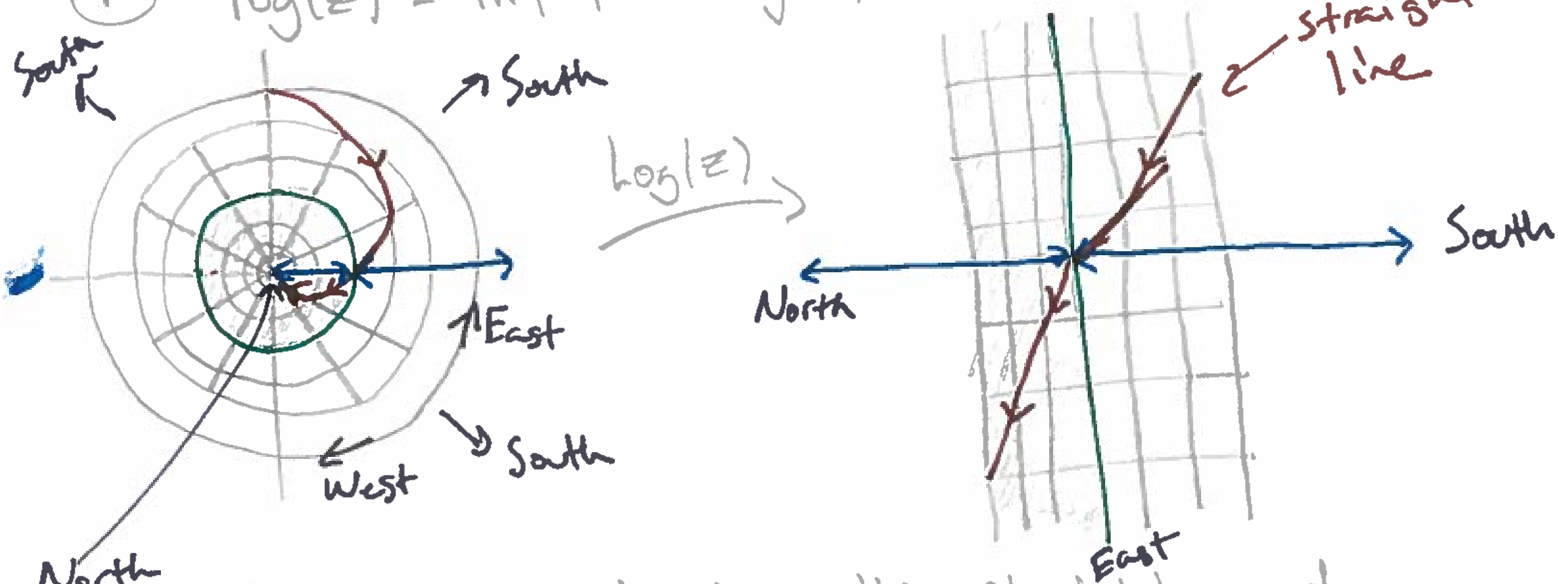
$$\text{Twist} = -\theta = -\theta$$

$$\text{Amplitwist} = \frac{1}{\sqrt{2}}e^{-i\theta} = \frac{1}{z}$$

(14) Mercator Projection

- cylindrical map projection
- conformal
- Distorts size of objects - infinite at poles

(i) $\log(z) = \ln|z| + i\,\mathrm{Arg}(z)$



(ii) You can travel along the straight red line in the image in the same direction.

The red line always has same compass reading.

The fact that red line never reaches north and makes same angle with squares means compass never changes.