Nathan Yee
MTH 3160
Homework 1
9/9/16

(4)

$M = a^2 + b^2$

$N = c^2 + d^2$

$MN = p^2 + q^2$

Consider $|(a+ib)(c+id)|^2$ (1)

Useful to know $|z|^2 = z\bar{z}$ (2), $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (3)

$A = (a+ib) \Rightarrow \bar{A} = (a-ib)$

$$MN = |(a+ib)(c+id)|^2$$
$$= (a+ib)(c+id)\overline{(a+ib)(c+id)}$$
$$= (a+ib)\overline{(a+ib)}(c+id)\overline{(c+id)}$$
$$= (a+ib)(a-ib)(c+id)(c-id)$$
$$MN = (a^2+b^2)(c^2+d^2)$$

$$MN = |(a+ib)(c+id)|^2$$
$$= |ac + iad + ibc - bd|^2$$
$$= |ac - bd + i(ad+bc)|^2$$

Let $p = ac - bd$

$q = ad + bc$

$|p + iq|^2$

$(p+q)(p-q)$

$MN = p^2 + q^2$

So $MN = p^2 + q^2$

$p = ac - bd$

$q = ad + bc$

where a, b, c, d are integers

subsequently p, q are integers

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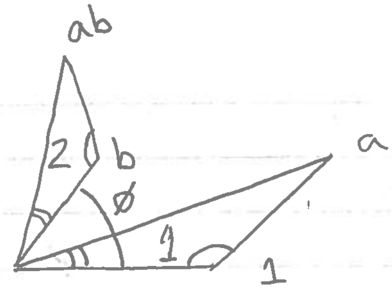

Because triangle 1 and 2 share angles ∠ and ∠, they are similar.

To scale triangle 1 to triangle 2, multiply side of triangle 1 by b.
Ex. $a \to a \cdot b$

The similar triangles also preserve the sum of angles. see how point ab is the sum of ø and ∠ where ∠ is the angle of a and ø the angle of b

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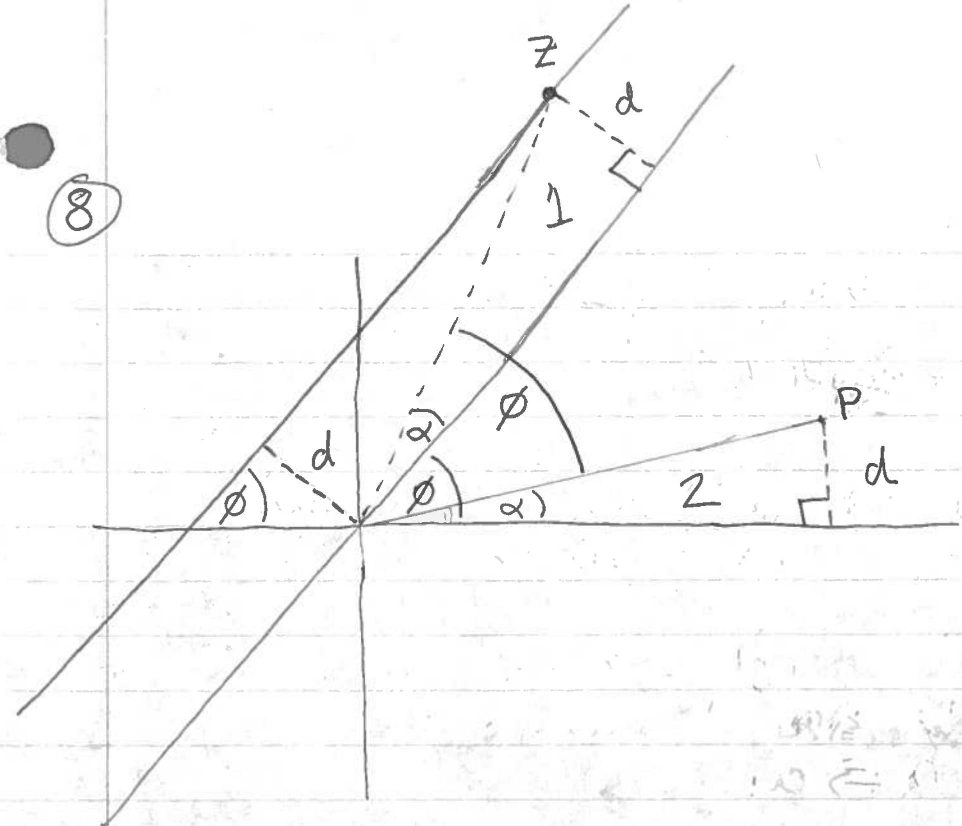


$$d = \left| \operatorname{Im}\left[e^{-i\phi} z \right] \right|$$

$p = e^{-i\phi} z$

$|p| = |z|$

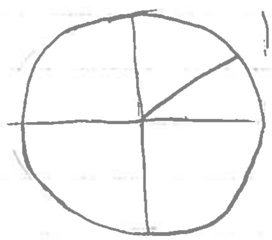
$\cancel{\phi} + \alpha = \cancel{\phi} + \alpha$

Triangles share angles α, 90°, and lengths $|z|$ so they are similar

Therefore triangle 2 has height d

(10)

$$|z| = 1 \longrightarrow \operatorname{Im}\left[\frac{z}{(z+1)^2}\right] = 0$$



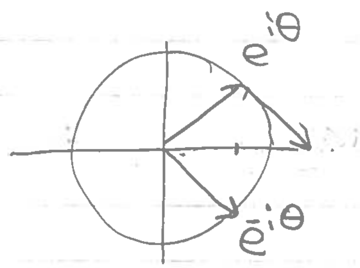
$$z = e^{i\theta}$$

$$\operatorname{Im}\left[\frac{e^{i\theta}}{(e^{i\theta}+1)^2}\right] = 0$$

$$\frac{e^{i\theta}}{e^{2i\theta} + 2e^{i\theta} + \frac{e^{i\theta}}{e^{i\theta}}} = 0$$

$$\frac{e^{i\theta}}{e^{i\theta}\left(e^{i\theta} + 2 + \frac{1}{e^{i\theta}}\right)} = 0$$

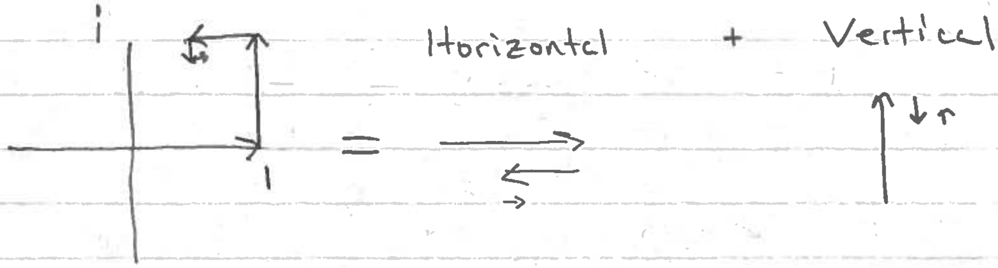
$$\frac{1}{e^{i\theta} + e^{-i\theta} + 2} = 0$$



$$e^{i\theta} + e^{-i\theta} = \text{Real}$$

$$\operatorname{Im}\left[\frac{1}{\text{Real} + 2}\right] = 0$$

(16)



Horizontal

$\longrightarrow + 1/0!$

$\longleftarrow - 1/2!$

$\rightarrow + 1/4!$

$\leftarrow - 1/6!$

Vertical

$\uparrow \quad \downarrow \quad \uparrow \quad \downarrow$

$$+\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$$

Horizontal $\quad S_n = \frac{(-1)^0}{(2\cdot 0)!} + \frac{(-1)^1}{(2\cdot 1)!} + \cdots + \frac{(-1)^n}{(2\cdot n)!}$

Vertical $\quad S_{in} = \frac{(-1)^0}{(2\cdot 0+1)!} + \frac{(-1)^1}{(2\cdot 1+1)!} + \cdots + \frac{(-1)^n}{(2\cdot n+1)!}$

Taylor series

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$, $\quad \cos(-1) = 1 - \frac{(-1)^2}{2!} + \frac{(-1)^4}{4!} + \cdots$

$\sin(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$, $\quad -\sin(-1) = 1 + \frac{(-1)^3}{3!} - \frac{(-1)^5}{5!} \cdots$

$S_n + iS_{in} = \cos(-1) - i\sin(-1)$

$= .54 - i.84$

Converge to $.54 - i.84$

(24)

$$1 + z + z^2 + \dots + z^{n-1} = \frac{z^n - 1}{z - 1}$$

(i) $|z| > 1 \rightarrow |z|^n$ dominates $|z|$ so diverges

$|z| = 1 \rightarrow (1-1)/(1-1)$ so singularity

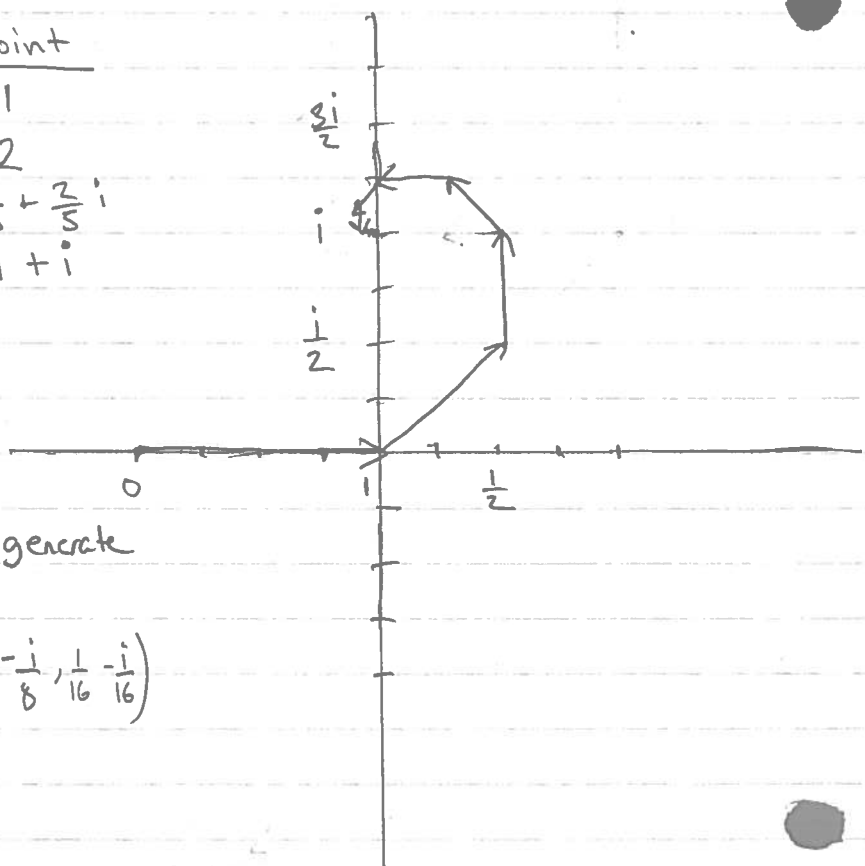
$-1 < z < 1 \rightarrow \lim_{n\to\infty} z^n = 0$ so $-\frac{1}{z-1}$

$|z| = -1 \rightarrow |z|^n = 2$ or 0 so singularity

$|z| < -1 \rightarrow |z|^n$ dominates $|z|$ so diverges

(ii) $-1 < z < 1$ $\lim_{n\to\infty} z^n = 0$ so $-\frac{1}{z-1}$

z	Point	Point
$0+i0$	$\frac{1}{1-(0+i0)}$	1
$\frac{1}{2}+i0$	$\frac{1}{1-1/2}$	2
$0+i\frac{1}{2}$	$\frac{1}{1-i/2}$	$\frac{4}{5}+\frac{2}{5}i$
$\frac{1}{2}+\frac{1}{2}i$	$\frac{1}{1-\frac{1}{2}-\frac{1}{2}i}$	$1+i$

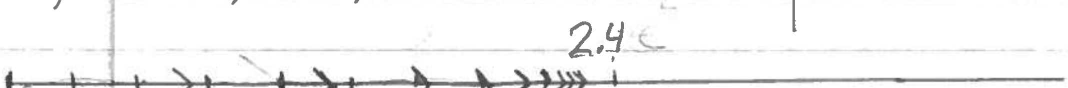


Used mathematica to generate list of points

$\left(\frac{1}{2}+\frac{i}{2}, \frac{i}{2}, -\frac{1}{4}+\frac{i}{4}, -\frac{1}{4}, -\frac{1}{8}-\frac{i}{8}, -\frac{i}{8}, \frac{1}{16}-\frac{i}{16}\right)$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\right)$

$(.71, .5, .35, .25, .176)$



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$1, 4, 6, 4, -4, -16, -24$

$S_1 = 1 \quad S_2 = 4$

(i) $S_{n+2} = 2(S_{n+1} - S_n)$

$z^2 - 2z + 2 = 0 \rightarrow S_n = z^n$

$z^{n+2} = 2(z^{n+1} - z^n)$ Divide by z^n

$z^2 = 2(z-1)$

$z^2 = 2z - 2$

$z^2 - 2z + 2 = 0$

(iii) $z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$

$z = \frac{2 \pm \sqrt{-4}}{2}$

$z = 1 \pm i$

(30) continued

(ii)

$$S_n = A(1+i)^n + B(1-i)^n$$
$$S_{n+1} = A(1+i)^{n+1} + B(1-i)^{n+1}$$
$$S_{n+2} = A(1+i)^{n+2} + B(1-i)^{n+2}$$

$$S_{n+2} = 2(S_{n+1} - S_n)$$

$$A(1+i)^{n+2} + B(1-i)^{n+2} = 2\left(A(1+i)^{n+1} + B(1+i)^{n+1} - (A(1+i)^n + B(1+i)^n)\right)$$

$$\frac{1}{(1-i)^n}\left[A(1+i)^{n+2} + B(1-i)^{n+2} = 2A(1+i)^{n+1} + 2B(1-i)^{n+1} - 2A(1+i)^n - 2B(1-i)^n\right]$$

$$\frac{1+i}{1-i} = i, \quad \frac{(1+i)^{n+2}}{(1-i)^n} = i^n(1+i)^2$$

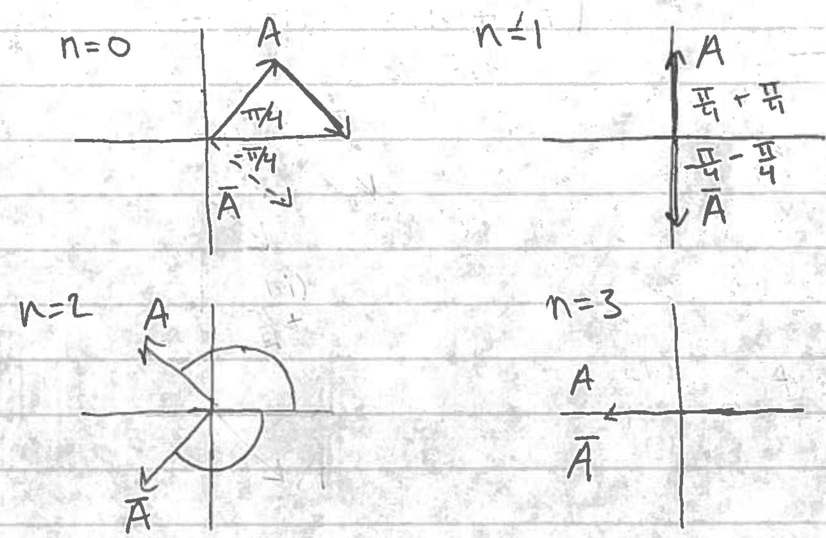
$$Ai^n(1+i)^2 + B(1-i)^2 = 2Ai^n(1+i) + 2B(1-i) - 2Ai^n - 2B$$
$$Ai^n(1+i)^2 - 2Ai^n(1+i) + 2Ai^n + B(1-i)^2 - 2B(1-i) + 2B = 0$$
$$Ai^n\left((1+i)^2 - 2(1+i) + 2\right) + B\left((1-i)^2 - 2(1-i) + 2\right) = 0$$
$$Ai^n(1+2i-1-2-2i+2) + B(1-2i-1-2+2i+2) = 0$$
$$Ai^n(0) + B(0) = 0$$
$$Ai^n(0) + B(0) = 0$$

Since the equation always equals 0 with arbitrary complex numbers A and B, $A(1+i)^n + B(1-i)^n$ is a solution to the recurrence relation

(30)

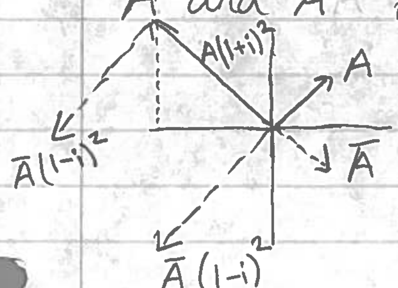
(iii) for only real solutions: $B = \bar{A}$
deduce that $S_n = 2\,\mathrm{Re}\left[A(1+i)^n\right]$

$$S_n = A(1+i)^n + \bar{A}(1-i)^n$$



We see that for $n = 1, 2, 3, 4, \ldots$
$A(1+i)^n$ and $\bar{A}(1-i)^n$ will always have same modulus and opposite angle, so we only get real numbers

To calculate S_n, simply add two vectors A and $\bar{A}$ with appropriate scaling, $|(1 \pm i)|^n$



$$A(1+i)^2 + \bar{A}(1-i)^2 = 2\,\mathrm{Re}\left[A(1+i)^2\right]$$

So $S_n = 2\,\mathrm{Re}\left[A(1+i)^n\right]$

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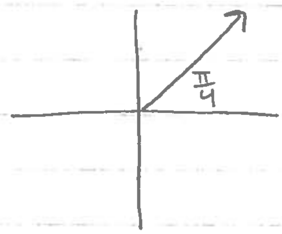
iv) $A = -\frac{1}{2} - i \rightarrow S_n = 2^{\frac{n}{2}} \sqrt{5} \cos\left[\frac{(n+4)\pi}{4} + \tan^{-1} 2\right]$

$S_n = 2 \operatorname{Re}\left[\left(-\frac{1}{2}-i\right)(1+i)^n\right]$

<u>Angle</u> of $\left(-\frac{1}{2}-i\right)(1+i)^n$



$-\pi + \tan^{-1}\left(\frac{-1}{-\frac{1}{2}}\right)$

$-\pi + \tan^{-1}(2)$

add 2π $\frac{4}{4}\pi + \tan^{-1}(2)$

$\frac{\pi}{4} \cdot n$

$\frac{4}{4}\pi + \tan^{-1}(2) + \frac{\pi}{4} \cdot n$

$\frac{(n+4)\pi}{4} + \tan^{-1}(2)$

<u>Cos</u> because we want real component

Scaling of $\left(-\frac{1}{2}-i\right)(1+i)^n = \left|-\frac{1}{2}-i\right|\left|(1+i)^n\right|$

$= \frac{\sqrt{5}}{2} \cdot \sqrt{2}^n$

$= \frac{\sqrt{5}}{2} \cdot 2^{\frac{n}{2}}$

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(iv) $S_n = 2\,\mathrm{Re}\left[\left(-\frac{1}{2} - i\right)(1+i)^n\right]$

$= 2 \cdot \text{scaling} \cos(\text{angle})$

$= 2 \frac{\sqrt{5}}{2} 2^{\frac{n}{2}} \cos\left(\frac{(n+4)\pi}{4} + \tan^{-1}(2)\right)$

$S_n = 2^{\frac{n}{2}} \sqrt{5} \cos\left[\frac{(n+4)\pi}{4} + \tan^{-1}(2)\right]$

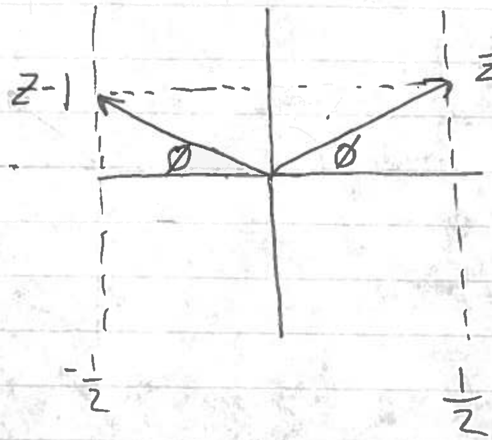
(v) $S_{33} = 65536$

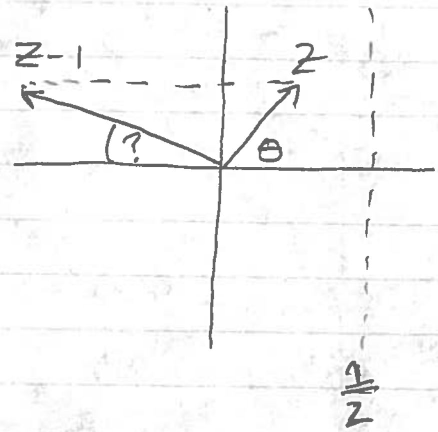
$S_{34} = 262144$

$S_{35} = 393216$

(33) $(z-1)^{10} = z^{10} \Rightarrow \left(\frac{z-1}{z}\right)^{10} = 1$

(1)





In order for the ratio $\left(\frac{z-1}{z}\right)^{10} = 1$,
$|z-1| = |z|$

Since the 1 in $z-1$ only implies horizontal translation of vector z, $z-1$ must be reflected across the y-axis. It is only for the points where $Re(z) = \frac{1}{2}$ does $z-1$ reflect across the y axis

A different argument is that for $Re(z) = \frac{1}{2}$, we can create similar triangles that share the same base length, and therefore same hypotenuse

Only 9 roots because of real axis symetry or because a polynomial of degree n has exactly n-roots

$$-z^{10} + z^{10} + z^9 + z^8 + \text{JUNK} = 0$$
$$z^9 + z^8 + \text{Junk} = 0$$

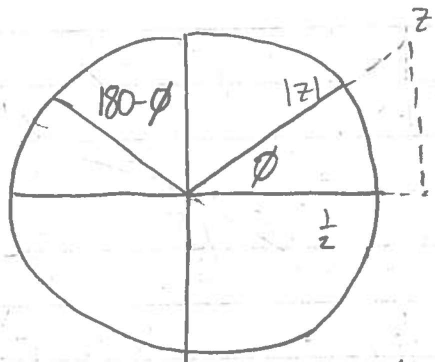
$\tan\theta = \frac{b}{a}$

$\cos\phi = \frac{\frac{1}{2}}{|z|}$

$|z| = \frac{1}{2\cos\phi}$

(33) $(z-1)^{10} = z^{10}$

(ii) $w^{10} = 1, \quad w = \frac{z-1}{z}$



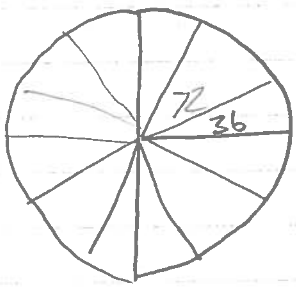
To preserve ratio

$\frac{(z-1)^{10}}{(z)^{10}} = 1$

(iii)

$10(180 - \phi) \bmod 2\pi = 10\phi$

$\phi = -72, -54, -36, -18, 0, 18, 36, 54, 72$



since $Re(z) = \frac{1}{2}$

$|z| = \frac{\cos(\phi)}{2}$

$\phi = -\frac{4\pi}{10}, -\frac{3\pi}{10}, \frac{-2\pi}{10}, \frac{-1\pi}{10}, 0, \frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{4\pi}{10}$

$|z| = 1.62, .85, .62, .53, .5, .53, .62, .85, 1.62$

$z = 1.62e^{-i\frac{4\pi}{10}}, .85e^{-i\frac{3\pi}{10}}, .62e^{-i\frac{2\pi}{10}}, .53e^{-i\frac{1\pi}{10}}, .5e^{0}, .53e^{i\frac{\pi}{10}}, .62e^{i\frac{2\pi}{10}}, .85e^{i\frac{3\pi}{10}}, 1.62e^{i\frac{4\pi}{2}}$

Exponent rather than $x+iy$