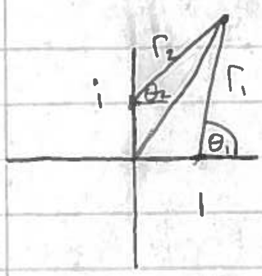
Nathan Yee
HW 3
Complex Variable
9/23/16

(24)

$$F(z) = (z-1)^{\frac{1}{2}} (z-i)^{\frac{1}{3}}$$

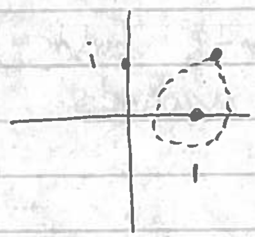


$$F(z) = \sqrt{r_1}\, e^{i\theta_1 \frac{1}{2}} \cdot \sqrt[3]{r_2}\, e^{i\theta_2 \frac{1}{3}}$$

(i) Branch points when $F(z) = 0$

$z_1 = 1$ degree 1
$z_2 = i$

To find degree, circle branch points individually



$\theta_1 + 2\pi$
$\theta_2 + 0$

$$\Rightarrow f_{new} = (r_1)^{\frac{1}{2}} e^{i\frac{\theta_1 + 2\pi}{2}} (r_2)^{\frac{1}{3}} e^{i\theta_2 \frac{1}{3}}$$

$$f_{new} = f_{old}\, e^{i\pi}$$

Circling 1 gives us increase of angle $e^{i\pi}$.
So, 1 has degree 1

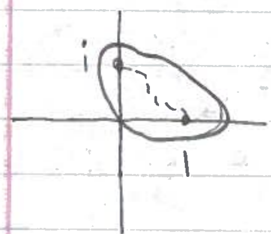
Circling i gives us increase of angle $e^{i\frac{2\pi}{3}}$
So, z has degree 2

$\theta_1 + 0$
$\theta_2 + 2\pi$

$$\Rightarrow f_{new} = (r_1)^{\frac{1}{2}} e^{i\frac{\theta_1}{2}} (r_2)^{\frac{1}{3}} e^{i\frac{\theta_2 + 2\pi}{3}}$$

$$f_{new} = f_{old}\, e^{i\frac{2\pi}{3}}$$

(ii)



If we make a closed loop around both branch points, we get an angle increase of

$$f_{new} = f_{old}\, e^{i\pi}\, e^{i\frac{2\pi}{3}}$$

In (35) our angle increase summed to 2π.

(iii) Going around both branch points adds angle $e^{i\frac{5\pi}{3}}$, so $f(z)$ has 6 values for a typical point

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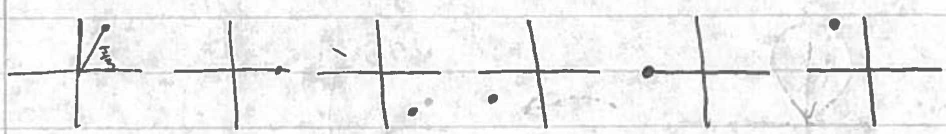
iii

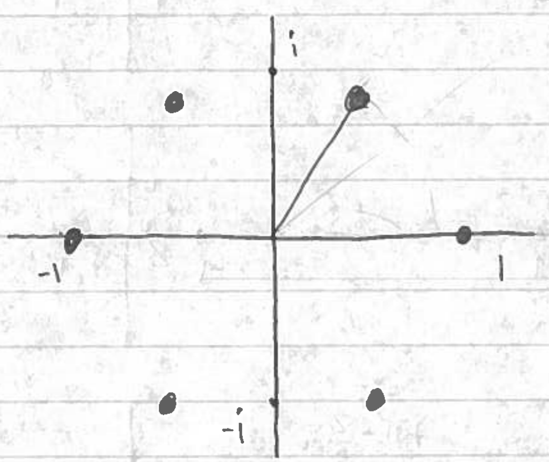
$$f(0) = \sqrt{0-1}\;\sqrt[3]{0-i}$$
$$= (-1)^{\frac{1}{2}}(-i)^{\frac{1}{3}}$$
$$= (-1)^{\frac{1}{2}}(-1)^{\frac{1}{6}}$$
$$= (-1)^{\frac{1}{3}}$$
$$= e^{i\frac{\pi}{3}}$$

Looping both points gives $f_{new} = f_{old}\, e^{i\frac{5\pi}{3}}$

$$e^{i\frac{\pi}{3}},\ e^{i\frac{6\pi}{3}},\ e^{i\frac{11\pi}{3}},\ e^{i\frac{16\pi}{3}},\ e^{i\frac{21\pi}{3}},\ e^{i\frac{26\pi}{3}}$$

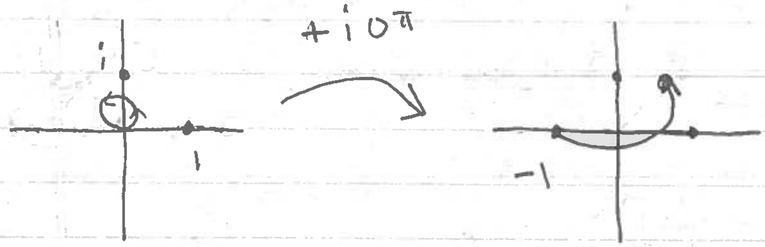
$$e^{i\frac{\pi}{3}},\ e^{i0},\ e^{i\frac{5\pi}{3}},\ e^{i\frac{4\pi}{3}},\ e^{i\frac{3\pi}{3}},\ e^{i\frac{2\pi}{3}}$$



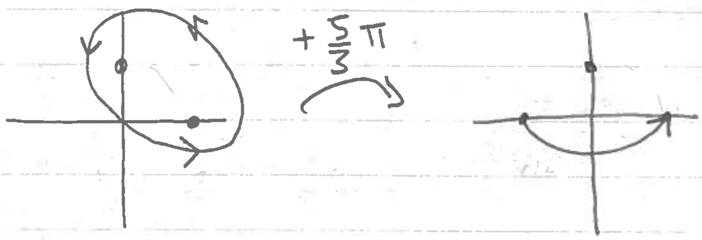


(iv)

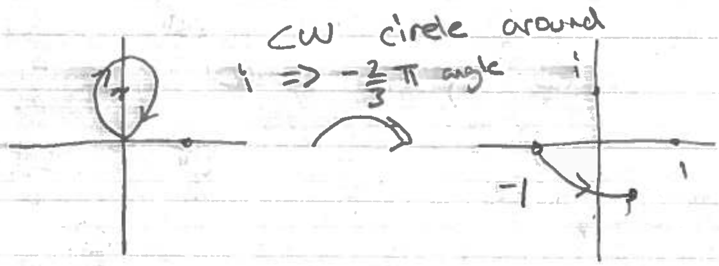
$e^{i\frac{\pi}{3}}$



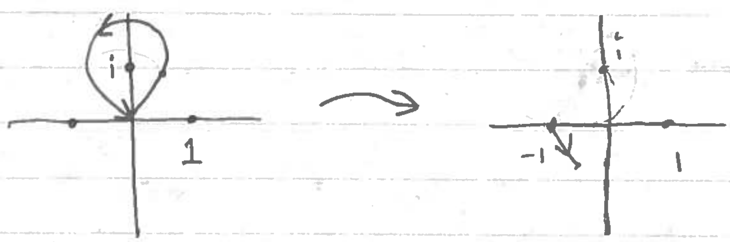
e^{i0}



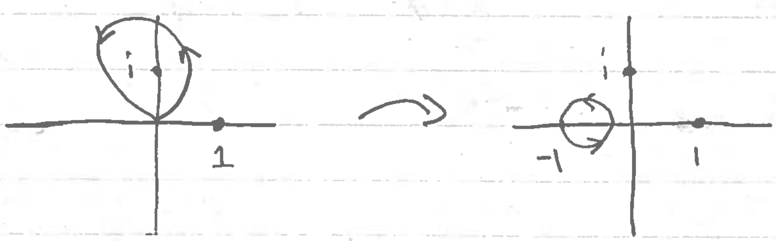
$e^{i\frac{5}{3}\pi} = e^{i(-\frac{\pi}{3})}$



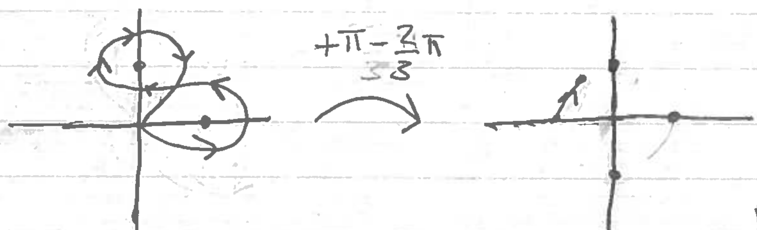
$e^{i\frac{4}{3}\pi}$



$e^{i\frac{3}{3}\pi} = e^{i\pi}$



$e^{i\frac{2}{3}\pi} =$



Not exactly sure what happens if we circle branch points in opposite directions. I think this works though.

(25)

$$f(z) = \frac{1}{\sqrt{1-z^4}} = \frac{1}{((1-z^2)(1+z^2))^{\frac{1}{2}}}$$

$$= \frac{1}{\left((1-z)(1+z)(i-z)(i+z)\right)^{\frac{1}{2}}}$$

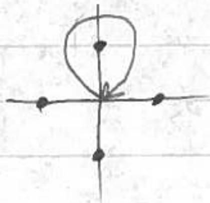
Branch $z = 1, i, -1, -i$

$$\frac{1}{1-z} = r_1 e^{i\theta_1} \qquad \frac{1}{i-z} = r_2 e^{i\theta_2}$$

$$\frac{1}{1+z} = r_3 e^{i\theta_3} \qquad \frac{1}{i+z} = r_4 e^{i\theta_4}$$

$$f(z) = \sqrt{r_1 r_2 r_3 r_4}\, e^{i\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)}$$

Degree is same for all branch points,



$$f_{new} = \sqrt{r_1 r_2 r_3 r_4}\, e^{i\left(\frac{\theta_1+\theta_2+2\pi+\theta_3+\theta_4}{2}\right)}$$

$$f_{new} = f_{old}\, e^{i\pi}$$

$$f_{new}(z) = -f_{old}(z)$$

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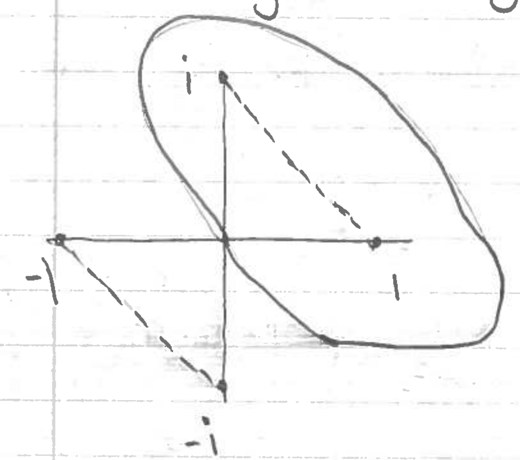
Cant go through 1 branch
Can go through 2 branch
cant go through 3 branch
Can go through 4 branch



(28)

Value of z^i at $z=-1$

Start at $z=1$ and $1\frac{1}{2}$ rotations

from (30) page 101:

$$z^k = e^{k\log(z)}$$

$$z^i = e^{i\log(z)}$$

$$f(z) = e^{i\log(z)}, \quad z = r_1 e^{i\theta_1}$$

$$f(z) = e^{i\log(r_1 e^{i\theta_1})}$$

$\theta_1 + 2\pi$

$$f_{new}(z) = e^{i\log(r_1 e^{i\theta_1 + 2\pi})}$$

$$= e^{i\log(r_1 e^{i\theta_1} e^{2\pi})}$$

$$= e^{i(\log(r_1 e^{i\theta_1}) + 2\pi)}$$

$$= e^{i\log(r_1 e^{i\theta_1})} e^{2\pi}$$

What about $1\frac{1}{2}$ rotation, $+3\pi$

$$f_{new}(z) = e^{i(\log(r_1 e^{i\theta_1}) + i3\pi)}$$

$$f_{new}(z) = f_{old}(z)\, e^{3\pi}$$

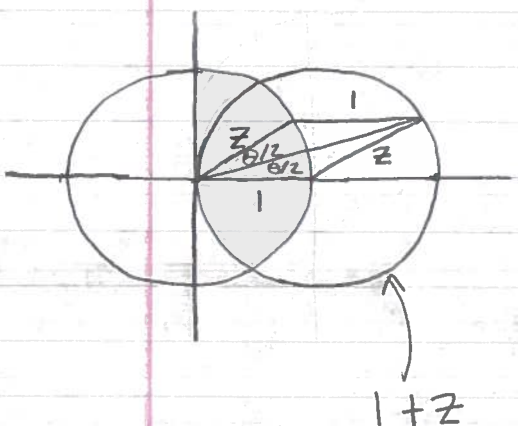
$$1\tfrac{1}{2} \text{ rotations} = e^{3\pi}$$

(37)

(i) $z = e^{i\theta}, \quad \operatorname{Im}[\operatorname{Log}(1+z)] = \frac{\Theta}{2}$

where Θ is principal values of θ

$-\pi < \Theta \leq \pi$



Same shape Yuzhong had last class. Angle is half. But what about Im[Log] ?

$$\operatorname{Im}[\operatorname{Log}(z)] = \operatorname{Im}\left[\ln|z| + i\operatorname{Arg}(z)\right]$$

$$= \operatorname{Im}\left[\ln|e^{i\theta}|\right] + \operatorname{Arg}(z)$$

$$= 0 + \Theta$$

$$\boxed{\operatorname{Im}[\operatorname{Log}(z+1)] = \operatorname{Arg}(z+1) = \frac{\Theta}{2}}$$

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ii

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \cdots$$

$$\text{Log}(1+e^{i\theta}) = e^{i\theta} - \frac{e^{i2\theta}}{2} + \frac{e^{i3\theta}}{3} - \frac{e^{i4\theta}}{4} + - - -$$

$$\text{Im}(e^{i\theta}) = \text{Im}(\cos\theta + i\sin\theta) = \sin\theta$$

$$F(\theta) = \text{Im}\left[\text{Log}(1+e^{i\theta})\right] = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4}$$

iii

$$F(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos n\theta + b_n \sin n\theta\right]$$

$$a_n = \frac{1}{\pi}\int_{\cancel{0}\ -\pi}^{\cancel{2\pi}\ \pi} F(\theta)\cos n\theta \, d\theta \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{\cancel{0}\ -\pi}^{\cancel{2\pi}\ \pi} F(\theta)\sin n\theta \, d\theta$$

Angles wrap around for Log $2\pi, 0 \to \pi, -\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(n\theta)\left[\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3}\right]d\theta = 0,$$ symmetry even function

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(n\theta)\left(\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \cdots\right)$$

$$\int u\,dv = uv - \int v\,du$$

I know b_n must equal $\frac{(-1)^{n+1}}{n}$.

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i $\Theta = \mathrm{Arg}(z)$

$$\frac{1}{2}\mathrm{Log}\left[\frac{z+1}{z-1}\right] = \frac{1}{2}\left[\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots\right) - \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \dots\right)\right]$$

$$\frac{1}{2}\mathrm{Log}\left[\frac{z+1}{z-1}\right] = z + \frac{z^3}{3} + \frac{z^5}{5} + \dots$$

iii $G(\theta) = \sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + \frac{\sin(7\theta)}{7} + \dots$

$z = e^{i\theta}$

$$G(\theta) = \mathrm{Im}\left[\frac{1}{2}\mathrm{Log}\left[\frac{1+e^{i\theta}}{1-e^{i\theta}}\right]\right] = \mathrm{Im}\left[e^{i\theta} + \frac{e^{i3\theta}}{3} + \frac{e^{i5\theta}}{5} + \dots\right]$$

$$G(\theta) = \sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5}$$

Integration:

I was having a bit of trouble integrating these for 37 and 38. I'll ask in class

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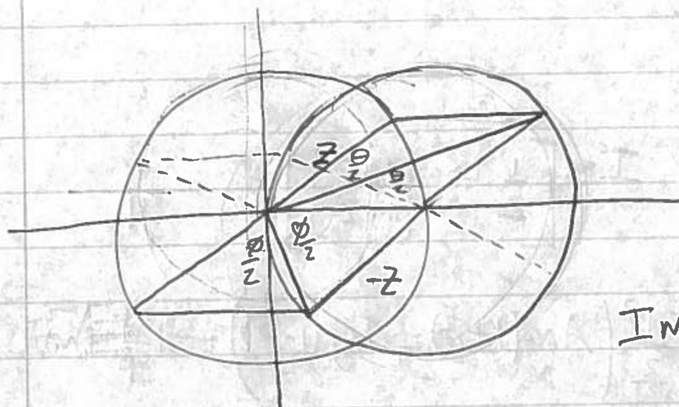
(ii) $\operatorname{Im}\left[\frac{1}{2}\operatorname{Log}\left[\frac{1+z}{1-z}\right]\right] = \text{sign of } \theta \ \frac{\pi}{4}$

When $0 < \theta \leq \pi$



$$2\frac{\theta}{2} + 2\frac{\phi}{2} = \pi$$

$$\theta + \phi = \pi$$

$$\frac{\theta+\phi}{2} = \frac{\pi}{2}$$

$$\operatorname{Im}\left[\frac{1}{2}\operatorname{Log}\left[\frac{1+z}{1-z}\right]\right] = +\frac{\pi}{4}$$

$$\frac{1}{2}\left[\operatorname{Arg}(1+z) - \operatorname{Arg}(1-z)\right] = +\frac{\pi}{4}$$

$$\frac{1}{2}\left[\frac{\theta}{2} - -\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right] = +\frac{\pi}{4}$$

$$+\frac{\pi}{4} = +\frac{\pi}{4}$$

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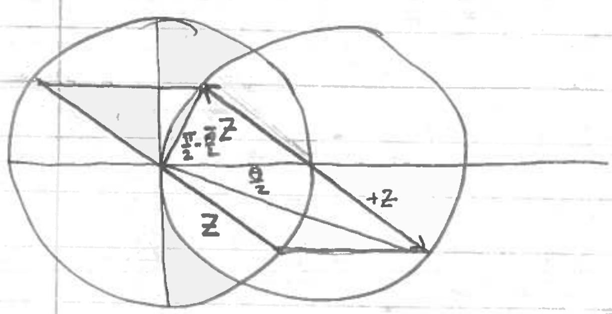
ii When $-\pi < \theta < 0$



$$\operatorname{Im}\left[\frac{1}{2}\operatorname{Log}\left[\frac{1+z}{1-z}\right]\right] = -\frac{\pi}{4}$$

$$\frac{1}{2}\left(\operatorname{Arg}(1+z) - \operatorname{Arg}(1-z)\right) = -\frac{\pi}{4}$$

$$\frac{1}{2}\left(\left(-\frac{\theta}{2}\right) - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) = -\frac{\pi}{4}$$

$$-\frac{\pi}{4} = -\frac{\pi}{4}$$

Not sure what happens when $\theta = 0$ geometrically

$$\operatorname{Im}\left[\frac{1}{2}\operatorname{Log}\left[\frac{1+z}{1-z}\right]\right] = -\frac{\pi}{4}$$

$$\frac{1}{2}\left(0 - \left(\frac{\pi}{2} - 0\right)\right)$$