Final

Spring 2017
(May 2, 2017)

Name: ______________________________ PUID: ______________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Signature)

Problem	Score
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Total	

Problem 1. (10 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is *always* true. Otherwise it is false.) Circle your answer. No partial credit will be given.

(a) Two events A and B are independent if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B].$$

TRUE or FALSE.

(b) Let X be a random variable with mean $\mathbb{E}[X] = \mu$. Then, the mean of $\frac{1}{X}$ is

$$\mathbb{E}\left[\frac{1}{X}\right] = \frac{1}{\mu}.$$

TRUE or FALSE.

(c) Let A, B, C be three events. Then,

$$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A \mid B \cap C]\mathbb{P}[B \mid C]\mathbb{P}[C]$$

TRUE or FALSE.

(d) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

TRUE or FALSE.

(e) The probability $\mathbb{P}[X = b]$ is determined by

$$\mathbb{P}[X = b] = \lim_{h \to 0} F_X(b+h) - F_X(b).$$

TRUE or FALSE.

(f) Let X be a random variable with a PDF

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}, & x > 0, \\ 1, & \tau \leq x \leq 0, \end{cases}$$

Then, $\tau = -1$.

TRUE or FALSE.

(g) Let X and Y be two random variables with correlation coefficient $\rho = 1$. Then, $X = Y$.

TRUE or FALSE.

Not Required (h) Let $X_1, \ldots, X_N$ be a sequence of i.i.d. random variables with mean $\mathbb{E}[X_n] = \mu$. Weak law of large number says for any $\varepsilon > 0$,

$$\mathbb{P}\left[\lim_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} X_n - \mu\right| \geq \varepsilon\right] = 0$$

TRUE or FALSE.

(i) Let X, Y, Z be three random variables. It is given that X and Y are uncorrelated. Then,

$$\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Z).$$

TRUE or FALSE.

Not Required (j) Let X be a random variable and $s > 0$ be a constant. If $\varepsilon > 0$, then

$$\mathbb{P}[e^{sX} \geq e^{s\varepsilon}] \leq \frac{\mathbb{E}[e^{sX}]}{e^{s\varepsilon}}.$$

TRUE or FALSE.

Problem 2. (15 POINTS)
Circle one and only one answer. If I cannot tell which answer you are circling, I will give you a zero. There is no partial credit for this problem.

1. Two dice are tossed. Let X be the absolute difference in the number of dots facing up. For example, if the first dice is 5, the second dice is 6, then $X = |5-6| = 1$. The probability that $X \leq 2$ is

   (a) $\frac{6}{36}$
   (b) $\frac{8}{36}$
   (c) $\frac{10}{36}$
   (d) $\frac{12}{36}$
   (e) $\frac{18}{36}$
   (f) $\frac{20}{36}$
   (g) $\frac{24}{36}$
   (h) $\frac{32}{36}$
   (i) None of the above
   (j) Problem undefined

2. Let X be a random variable with PMF

$$\mathbb{P}[X = k] = \left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \ldots$$

   Let A and B be the events that

   - $A = \{X \text{ is even}\}$
   - $B = \{X \text{ is a multiple of } 5\}$.

   Find $\mathbb{P}[A \cap B]$.

   (a) $\frac{1}{2}$
   (b) $\frac{1}{3}$
   (c) $\frac{1}{4}$
   (d) $\frac{1}{31}$
   (e) $\frac{1}{127}$
   (f) $\frac{1}{511}$
   (g) $\frac{1}{1023}$
   (h) $\frac{1}{2055}$
   (i) None of the above.
   (j) Problem undefined.

3. Let X and Y be two independent random variables with PDF $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X - Y$. Then, $F_Z(z) =$

   (a) $\int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dx dy$
   (b) $\int_{-\infty}^{\infty} \int_{-\infty}^{y-z} f_X(x) f_Y(y) dx dy$
   (c) $\int_{-\infty}^{\infty} \int_{-\infty}^{y+z} f_X(x) f_Y(y) dx dy$
   (d) $\int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy dx$
   (e) $\int_{-\infty}^{\infty} \int_{-\infty}^{x-z} f_X(x) f_Y(y) dy dx$
   (f) $\int_{-\infty}^{\infty} \int_{-\infty}^{x+z} f_X(x) f_Y(y) dy dx$
   (g) (a) and (d)
   (h) (b) and (f)
   (i) (c) and (e)
   (j) None of the above

4. Let X be a random variable with CDF $F_X(x)$. Let $Z = 2X^2 + 1$. Then, $F_Z(z) =$?

   (a) $F_X(\sqrt{(z-1)/2})$
   (b) $F_X(\sqrt{(z+1)/2})$
   (c) $F_X(\sqrt{(z+1)/2}) + F_X(\sqrt{(z-1)/2})$
   (d) $F_X(\sqrt{(z+1)/2}) - F_X(\sqrt{(z-1)/2})$
   (e) $F_X(\sqrt{-(z+1)/2})$
   (f) $F_X(\sqrt{-(z-1)/2})$
   (g) $F_X(\sqrt{(z-1)/2}) + F_X(-\sqrt{(z-1)/2})$
   (h) $F_X(\sqrt{(z-1)/2}) - F_X(-\sqrt{(z-1)/2})$
   (i) Problem undefined.
   (j) None of the above.

Not Required

5. Let θ be an unknown parameter. Let $X_n = \mu + W_n$ where $W_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \theta)$ for $n = 1, \ldots, N$, i.e., W_n is i.i.d. Gaussian of mean zero and variance θ. Suppose that we have observed $x_1, \ldots, x_N$. The maximum likelihood estimate of θ is

   (a) $\widehat{\theta}_{ML} = \mu$
   (b) $\widehat{\theta}_{ML} = \mu^2$
   (c) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n$
   (d) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n^2$
   (e) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} (x_n - \mu)$
   (f) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} (x_n - \mu)^2$
   (g) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} (x_n^2 + \mu^2)$
   (h) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} \log(x_n)$
   (i) $\widehat{\theta}_{ML} = \frac{1}{N}\sum_{n=1}^{N} \sqrt{x_n}$
   (j) None of the above.

Problem 3. (10 POINTS)
Let X be a random variable with

$$\mathbb{P}[X = 1] = \mathbb{P}[X = -1] = \frac{1}{2}.$$

Let W be a Gaussian random variable with $W \sim \mathcal{N}(0, 1)$. Suppose that X and W are independent, and let $Y = X + W$.

Find $\mathbb{E}[X \mid Y = 1]$.

$\mathbb{E}[X \mid Y = 1] =$

Problem 4. (20 POINTS)
Let $X_1, \ldots, X_N$ be a sequence of i.i.d. random variables with PDF

$$f_{X_n}(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

Let $Z_N = X_1 + \ldots + X_N$, where N is a fixed constant.

(a) (10 points) Find the PDF of Z_N. (Hint: Use characteristic function.)

$$f_{Z_N}(z) = \begin{cases} \qquad\qquad\qquad\qquad , & z \geq 0, \\ \\ \qquad\qquad\qquad\qquad , & z < 0. \end{cases}$$

(b) (10 points) Now, suppose that N is a Poisson random variable with mean β, i.e.,

$$\mathbb{P}(N = n) = \frac{\beta^n}{n!} e^{-\beta}.$$

Find $\mathbb{E}[Z_N]$. Show your steps clearly or otherwise no point.

$\mathbb{E}[Z_N] =$

Problem 5. (25 POINTS)
Let $X(t)$ be a random process defined as

$$X(t) = \cos(\omega_0 t + \Theta),$$

where the random variable $\Theta \sim \text{Uniform}[0, 2\pi]$.

(a) (10 points) Find $\mathbb{E}[X(t)]$ and $\mathbb{E}[X(t+\tau)X(t)]$. Show your steps clearly or otherwise no point.

$\mathbb{E}[X(t)] =$	$\mathbb{E}[X(t+\tau)X(t)] =$

(b) (5 points) Is $X(t)$ W.S.S.? Explain your answer or otherwise no point.

YES / NO

(c) (10 points) Suppose that $X(t)$ is sent through a linear time invariant system with input/output relationship

$$2\frac{d^2}{dt^2}Y(t) + 2\frac{d}{dt}Y(t) + 4Y(t) = 3\frac{d^2}{dt^2}X(t) - 3\frac{d}{dt}X(t) + 6X(t).$$

Find $R_Y(\tau)$.

$R_Y(\tau) =$

Problem 6. (20 POINTS)
Consider a transmission system which sends a binary signal $\{+1, -1\}$ with equal probability $\mathbb{P}[X = +1] = \mathbb{P}[X = -1] = \frac{1}{2}$. Let Z be a noise random variable with PDF

$$f_Z(z) = \begin{cases} \alpha^2 z e^{-\alpha z}, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

Let $Y = X + Z$ be the received signal. The receiver implements a simple detection rule as follows:

- Say $X = +1$ if $Y \geq \ell$, where ℓ is a parameter that you need to determine;
- Say $X = -1$ if $Y < \ell$.

Suppose that $\alpha = \ln \sqrt{3}$.

(a) (10 points) Show that the probability of detection error is

$$P_e = C\left(1 - \int_A^B \alpha^2 x e^{-\alpha x} dx\right).$$

What are A, B, C?

(Hint: P_e is defined as $P_e \stackrel{\text{def}}{=} \mathbb{P}[X = +1 \text{ and decide } -1] + \mathbb{P}[X = -1 \text{ and decide } +1]$.)

$A =$	$B =$	$C =$

(b) (10 points) Determine ℓ that minimizes the probability of error. Use $\alpha = \ln \sqrt{3}$.
Hint: Use the Fundamental Theorem of Calculus.

$\ell =$

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1 - p + pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s-1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw}$, $a > 0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw}$, $a > 0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2}$, $a > 0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2}$, $a > 0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2}$, $a > 0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|}$, $a > 0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2}$, $a > 0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}}$, $a > 0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t - t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w - w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$