Final

Fall 2016
(Dec 16, 2016)

Name: ______________________________ PUID: ____________________

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

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(Please copy and write the above statement.)

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(Signature)

Problem	Score
Q1	
Q2	
Q3	
Q4	
Q5	
Total	

Problem 1. (20 POINTS)
Determine whether the following statements are TRUE or FALSE. (A statement is true if it is *always* true. Otherwise it is false.) Circle your answer. No partial credit will be given.

(a) Let A, B, X be three events, and assume that $A \cup B = \Omega$. Then,

$$\mathbb{P}[X] = \mathbb{P}[X \mid A \backslash B] + \mathbb{P}[X \mid A \cap B] + \mathbb{P}[X \mid B \backslash A].$$

TRUE or FALSE.

(b) For any random variable X, the variance has the property that $\text{Var}[aX + b] = a\text{Var}[X] + b$.

TRUE or FALSE.

(c) Let $X \geq 0$ and $Y \geq 0$ be two independent random variables. Then,

$$\mathbb{P}[X + Y \leq 1] \leq \mathbb{P}[X \leq 1 \ \cap \ Y \leq 1].$$

TRUE or FALSE.

(d) Let $X \sim \text{Uniform}[-1, 1]$. Then

$$F_X(x) = \frac{x+1}{2}.$$

TRUE or FALSE.

(e) Let X be a random variable with a PDF

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}, & x < 0, \\ c\lambda \exp\{-\lambda x\}, & x \geq 0. \end{cases}$$

Then, $c = 1$.

TRUE or FALSE.

(f) The probability $\mathbb{P}[X = b]$ is determined by

$$\mathbb{P}[X = b] = \lim_{h \to 0} F_X(b + h) - F_X(b).$$

TRUE or FALSE.

(g) Let X be a random variable with CDF $F_X(x)$. Let $Y = 2X + 3$. Then,

$$F_Y(y) = F_X\left(\frac{y-3}{2}\right).$$

TRUE or FALSE.

Not Required (h) Let $M_N = \frac{1}{N}\sum_{n=1}^{N} X_n$ be the sample mean of a sequence of random variables $X_1, \ldots, X_N$. If $M_N \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{N}\right)$, then

$$\sqrt{N}\left(\frac{M_N - \mu}{\sigma}\right) \sim \mathcal{N}(0, 1).$$

TRUE or FALSE.

(i) Let X, Y, Z be three random variables. It is given that X and Y are uncorrelated. Let a be a constant. Then,

$$\text{Cov}(X, Y + aZ) = a\text{Cov}(X, Z).$$

TRUE or FALSE.

(j) Let X, Y be two independent random variables such that $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Exponential}(\alpha)$. Let $U = 3XY$. Then, the moment generating function of U is

$$M_U(s) = 1 - p + p\left(\frac{3\alpha}{\alpha - s}\right).$$

TRUE or FALSE.

Problem 2. (20 POINTS)
Circle one and only one answer. If I cannot tell which answer you are circling, I will give you a zero. There is no partial credit for this problem.

1. Consider a pair of random variables X and Y with joint PMF given by the following table.

	Y= 1	2	3	4
X = 1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{0}{20}$
2	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
3	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
4	$\frac{0}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

The conditional expectation $\mathbb{E}[X \mid \{Y = 1 \cup Y = 2\}]$ is

(a) 1
(b) 2
(c) $\frac{7}{3}$
(d) $\frac{21}{20}$
(e) $\frac{9}{2}$
(f) $\frac{5}{2}$
(g) $\frac{7}{2}$
(h) $\frac{4}{3}$
(i) None of the above
(j) Problem undefined

2. Let $X \sim \text{Exponential}(\lambda)$. Let Y be a random variable such that

$$Y = \begin{cases} X - c, & X \geq c, \\ 0, & X < c. \end{cases}$$

Then, the PDF of Y is

(a) $f_Y(0) = 0$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$.
(b) $f_Y(0) = 1$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$.
(c) $f_Y(0) = e^{-\lambda c}$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$.
(d) $f_Y(0) = 1 - e^{-\lambda c}$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$.
(e) $f_Y(0) = e^{-\lambda c}$ and $f_Y(y) = \lambda e^{-\lambda(y-c)}$ for $y > 0$.
(f) $f_Y(0) = e^{-\lambda c}$ and $f_Y(y) = \lambda e^{-\lambda(y+c)}$ for $y > 0$.
(g) $f_Y(0) = 1 - e^{-\lambda c}$ and $f_Y(y) = \lambda e^{-\lambda(y-c)}$ for $y > 0$.
(h) $f_Y(0) = 1 - e^{-\lambda c}$ and $f_Y(y) = \lambda e^{-\lambda(y+c)}$ for $y > 0$.
(i) None of the above.
(j) Problem undefined.

3. Let X and Y be two independent random variables with PDF $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Then, $F_Z(z) =$

   (a) $\int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dx dy$
   (b) $\int_{-\infty}^{\infty} \int_{-\infty}^{y-z} f_X(x) f_Y(y) dx dy$
   (c) $\int_{-\infty}^{\infty} \int_{-\infty}^{y+z} f_X(x) f_Y(y) dx dy$
   (d) $\int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy dx$
   (e) $\int_{-\infty}^{\infty} \int_{-\infty}^{x-z} f_X(x) f_Y(y) dy dx$
   (f) $\int_{-\infty}^{\infty} \int_{-\infty}^{x+z} f_X(x) f_Y(y) dy dx$
   (g) (a) and (d)
   (h) (b) and (e)
   (i) (c) and (f)
   (j) None of the above

Not Required

4. Let Θ be a random parameter with prior distribution $\Theta \sim \mathcal{N}(0, 1)$. Let X be a Gaussian random variable with mean Θ and variance σ^2. Suppose that we have observed two realizations x_1 and x_2. The maximum-a-posteriori (MAP) estimate of Θ is

   (a) $\widehat{\theta}_{\text{MAP}} = \frac{x_1+x_2}{2}$
   (b) $\widehat{\theta}_{\text{MAP}} = \frac{x_1+x_2}{2\sigma^2}$
   (c) $\widehat{\theta}_{\text{MAP}} = \frac{x_1}{\sigma^2} + x_2$
   (d) $\widehat{\theta}_{\text{MAP}} = x_1 + \frac{x_2}{\sigma^2}$
   (e) $\widehat{\theta}_{\text{MAP}} = \frac{x_1\sigma^2+x_2}{2}$
   (f) $\widehat{\theta}_{\text{MAP}} = \frac{x_1+x_2\sigma^2}{2}$
   (g) $\widehat{\theta}_{\text{MAP}} = \frac{\sigma^2(x_1+x_2)}{2}$
   (h) $\widehat{\theta}_{\text{MAP}} = \frac{x_1+x_2}{2+\sigma^2}$
   (i) $\widehat{\theta}_{\text{MAP}} = \frac{\sigma^2 x_1+x_2}{2+\sigma^2}$
   (j) None of the above.

5. Let $X(t)$ be a WSS process with power spectral density $S_X(\omega)$. Let $Y(t) = X(t) - X(t-d)$ for some positive constant d. The power spectral density $S_Y(\omega)$ is

   (a) $2S_X(\omega)(1 + \cos(\omega d))$
   (b) $2S_X(\omega)(1 - \cos(\omega d))$
   (c) $2S_X(\omega)\cos(\omega d)$
   (d) $2S_X(\omega)(1 + \sin(\omega d))$
   (e) $2S_X(\omega)(1 - \sin(\omega d))$
   (f) $S_X(\omega)(1 - e^{-j\omega d} - e^{j\omega d})$
   (g) $S_X(\omega)(1 + e^{-j\omega d})$
   (h) $S_X(\omega)(1 - e^{j\omega d})$
   (i) $S_X(\omega)(1 + e^{-|\omega d|})$
   (j) None of the above.

Problem 3. (20 POINTS)
Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = c(x+y),$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

(a) (3 points) Find c.

(b) (4 points) Find $f_X(x)$ and $f_Y(y)$.

(c) (3 points) Find $\mathbb{E}[\frac{1}{X+\frac{1}{2}}]$.

(d) (3 points) Find $f_{Y|X}(y|x)$.

(e) (3 points) Find $\mathbb{E}[Y^2|X = x]$.

(f) (4 points) Find $\mathbb{P}[Y > X \mid X > \frac{1}{2}]$.

Problem 4. (20 POINTS)
Let θ be an unknown scalar. Let $W_n \sim \mathcal{N}(0, \sigma^2)$ be an i.i.d. Gaussian noise. Let $Y_n = \theta + W_n$, for $n = 1, \ldots, N$. Suppose that we have observed realizations $y_1, \ldots, y_N$ for $Y_1, \ldots, Y_N$.

(a) (5 points) Find the joint PDF of $Y_1, \ldots, Y_N$ given the parameter θ, i.e., $f_{Y_1,\ldots,Y_N}(y_1, \ldots, y_N \mid \theta)$.

Not Required (b) (5 points) Recall that the maximum likelihood estimate $\widehat{\theta}_{\text{ML}}$ is defined as

$$\widehat{\theta}_{\text{ML}} = \underset{\theta}{\text{argmax}} \quad f_{Y_1,\ldots,Y_N}(y_1, \ldots, y_N \mid \theta)$$

Find $\widehat{\theta}_{\text{ML}}$. Express your answer in terms of $y_1, \ldots, y_N$.

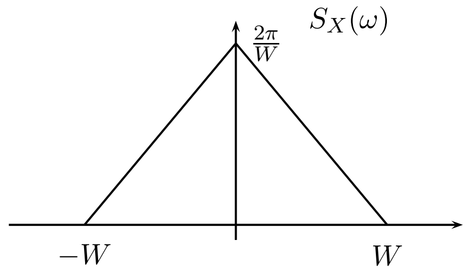
(c) (5 points) Let $M_N = \frac{1}{N}\sum_{n=1}^{N} Y_n$. Find the moment generating function of M_N.

Not Required (d) (5 points) Use central limit theorem, find the probability

$$\mathbb{P}[|M_N - \theta| \leq \sigma].$$

Express your answer in terms of the $\Phi(\cdot)$ function.

Problem 5. (20 POINTS)
Let $X(t)$ be a WSS process with autocorrelation function $R_X(\tau)$ and power spectral density $S_X(\omega)$ as shown below.



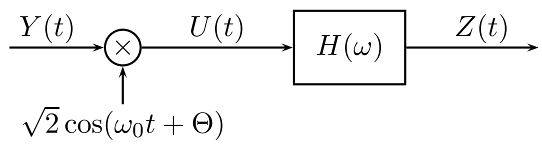
Let $\Theta \sim \text{Uniform}[0, 2\pi]$ be a uniformly distributed random variable. Define

$$Y(t) = \sqrt{2}X(t)\cos(\omega_0 t + \Theta).$$

(a) (5 points) Find $\mathbb{E}[Y(t+\tau)Y(t)]$, i.e., $R_Y(\tau)$.

(b) (5 points) Find $S_Y(\omega)$. Express your answer in terms of $S_X(\omega)$. (Hint: $f(t)e^{j\omega_0 t} \longleftrightarrow F(\omega - \omega_0)$.)

Now, assume that $Y(t)$ is passed through an LTI system shown in the following block diagram.



(c) (5 points) Find and sketch $S_U(\omega)$. Express your answer in terms of $S_X(\omega)$. Mark and label your figure clearly.

(d) (5 points) Assume that $H(\omega)$ is given by

$$H(\omega) = \begin{cases} 1, & -W \le \omega \le W, \\ 0, & \text{otherwise.} \end{cases}$$

Find $R_Z(\tau)$.

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1-p+pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s-1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb}-e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a>0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a>0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2},\ a>0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2},\ a>0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2},\ a>0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|},\ a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a>0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w-w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1-2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A\sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$