Random walks in different dimensions

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December 2025

1 Introduction

Random walks are one of the simplest models for random motion and they appear in many areas of science and engineering. They describe a particle that takes a long sequence of small random steps and they link ideas from probability such as expectation variance and the central limit theorem to concrete physical pictures of diffusion. In low dimensions they also show striking qualitative effects such as repeated returns to the origin and very long wandering times near the starting point.

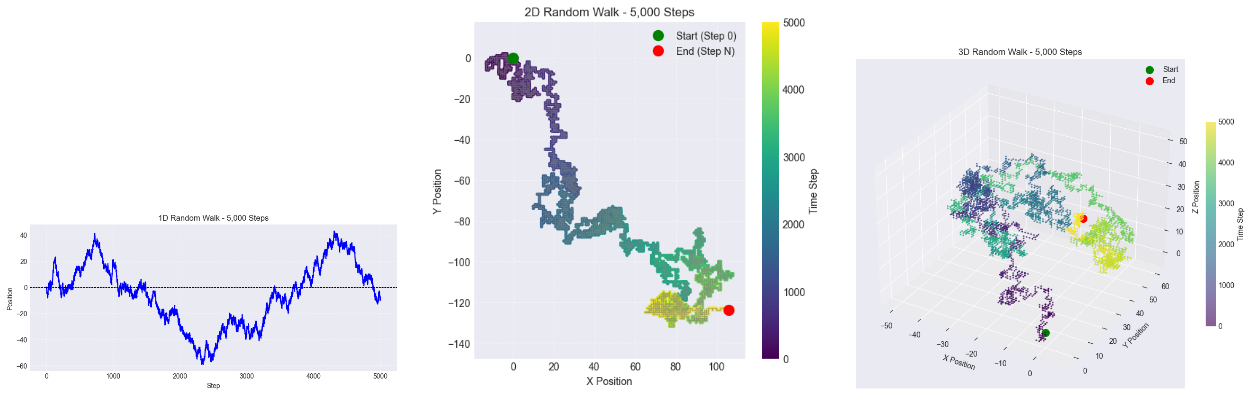
In this report we study simple symmetric lattice random walks in different dimensions using a combination of theory and large scale simulations. In all of our simulations the walker moves by one unit along a randomly chosen coordinate direction with equal probability of moving in the positive or the negative direction so the process matches the simple random walk model from class. For these walks we measure how the mean squared displacement grows in time how often and how reliably the walker returns to the origin how long it takes to reach a fixed distance from the origin for the first time and how the distribution of displacements compares to the prediction of the central limit theorem. By comparing the same experiments across different dimensions we see both the universal features that do not change such as linear growth of mean squared displacement in time and the sharp differences that do change such as the transition from recurrent behavior in one and two dimensions to transient behavior in three and higher dimensions.

2 Example Random Walk Trajectories

Before looking at ensembles of walks and numerical summaries it is useful to see what a single random walk looks like in practice. We first generated one simple symmetric random walk of length $n = 5000$ steps in the first three directions. Each walk starts at the origin and at every step the walker moves by one unit along a randomly chosen coordinate axis with equal probability of moving in the positive or the negative direction. This matches the simple random walk model from class and gives a direct visual picture of how the walk behaves in different dimensions.

In one dimension the trajectory is a jagged path on the integer line that repeatedly crosses the origin and wanders away from it over time. In two dimensions the path forms a tangled curve in the plane which tends to linger near the origin for a while before spreading outward and visiting points in a rough disk. In three dimensions the walk produces a cloud like spatial path that is harder to draw on paper but still shows the same idea of a particle taking a long sequence of small unbiased steps away from and back toward the origin.

We can understand the typical size of these example walks using a simple variance calculation. In one dimension we model the position after n steps by a partial sum $S_n = \sum_{k=1}^{n} X_k$ where each step X_k takes values 1 or -1 with equal probability and the steps are independent. Then $E(X_k) = 0$ and $\mathrm{Var}(X_k) = 1$ so additivity of variance gives $\mathrm{Var}(S_n) = n$ and since $E(S_n) = 0$ we



(a) Example one dimensional walk 5000 steps

(b) Example two dimensional walk 5000 steps

(c) Example three dimensional walk 5000 steps

Figure 1: Single realizations of simple symmetric random walks in one two and three dimensions each with $n = 5000$ steps starting at the origin

have $E(S_n^2) = n$. The root mean square distance from the origin is $\sqrt{E(S_n^2)} = \sqrt{n}$ so a typical position after n steps has size on the order of $\sqrt{n}$. In d dimensions the position after n steps is a vector $\mathbf{S}_n$ and the squared distance from the origin is $R_n^2 = \|\mathbf{S}_n\|^2$. In the lattice model used here each step has length one so each coordinate has variance of order n/d and summing over coordinates gives $E(R_n^2) = n$ which means that the root mean square radius is again $\sqrt{n}$. For our example trajectories with $n = 5000$ steps this theory predicts a natural scale of about $\sqrt{5000}$ and the measured final distances in one two and three dimensions all fall in this range which is consistent with the simple random walk model.

3 Visualizing ensemble behavior

Single trajectories give some intuition for how a random walk behaves but they do not show how a whole collection of walks fills space. To see the collective picture we simulated ensembles of independent walks and plotted all of their paths on the same axes. In one dimension we generated 300 walks of length 1000 steps. In two dimensions we generated 100 walks of length 1000 steps. In three dimensions we generated 50 walks of length 1000 steps. All walks used the same simple lattice step rule as before.

In one dimension the ensemble forms a dense fan that starts at the origin and spreads outward over time. Near the origin many paths cross each other and the central region looks almost solid while the outer ends show fewer paths. The overall picture matches the idea that typical positions grow like the square root of time while individual walks fluctuate up and down around zero.

In two and three dimensions the ensemble plots show how the walks begin to fill out regions of space rather than just a line. In two dimensions the paths form a cloud of tangled curves that roughly fills a disk around the origin with higher density near the center and gradually thinning toward the boundary. In three dimensions the projected paths look like overlapping threads in space and the overall cloud is more diffuse but still concentrated near the origin with fewer endpoints far away.

These ensemble views give a visual sense of how random walks with the same law can produce a wide variety of individual paths while still sharing the same overall spreading pattern in different dimensions.

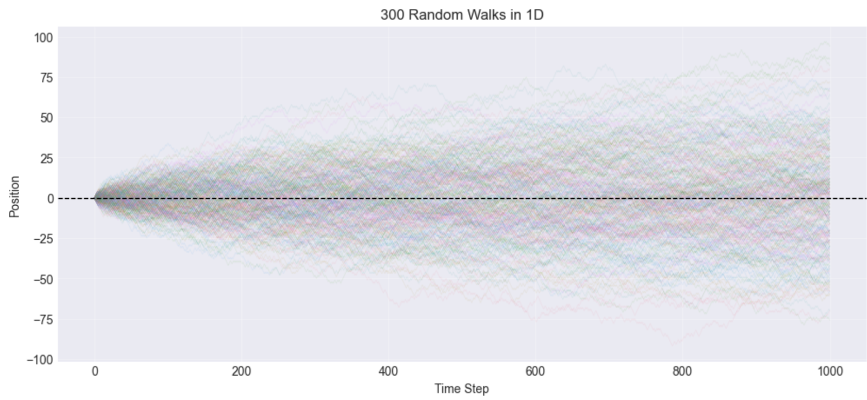
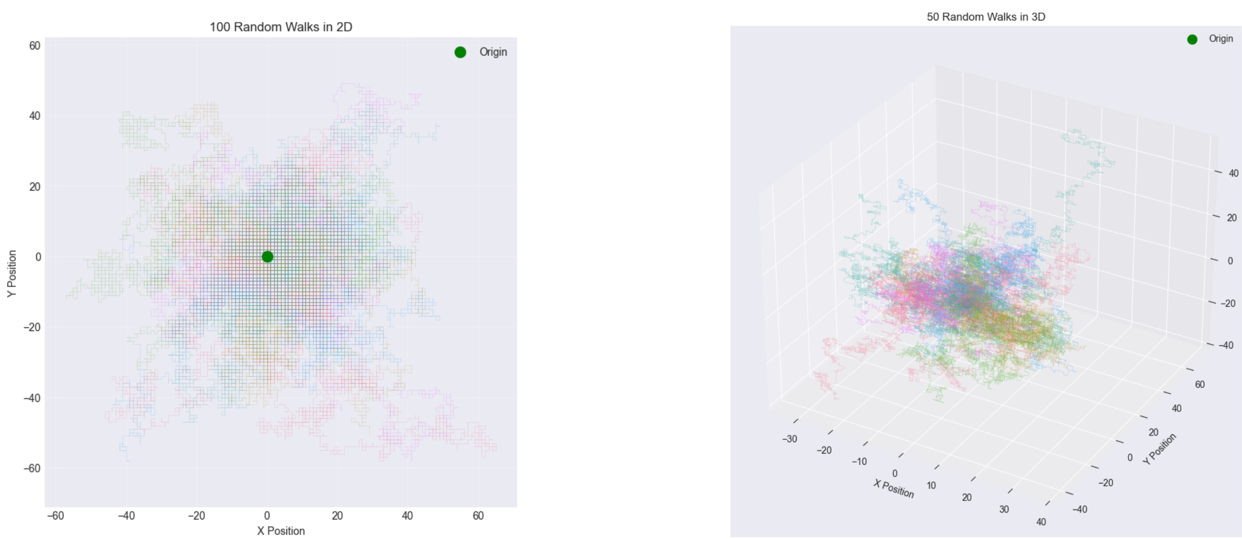


Figure 2: Ensemble of 300 simple symmetric random walks in one dimension each of length 1000 steps



(a) Ensemble of 100 walks in two dimensions

(b) Ensemble of 50 walks in three dimensions

Figure 3: Ensemble trajectory plots for simple symmetric random walks in two and three dimensions

4 Mean squared displacement across dimensions

A natural way to quantify how far a random walk has spread after n steps is the mean squared displacement. For a walk with position vector $\mathbf{S}_n$ after n steps we define $\mathrm{MSD}(n) = E(\|\mathbf{S}_n\|^2)$. In the simple symmetric lattice walk considered here each step has length one and the increments have zero mean and finite variance. Diffusion theory gives the relation $\mathrm{MSD}(t) = 2dDt$ where t is the step count n, d is the dimension, and D is the diffusion coefficient. For the step rule used in our simulations this theory predicts $D = \frac{1}{2d}$ so that $\mathrm{MSD}(n) = n$ in every dimension. This prediction says that the overall rate of spreading as measured by the mean squared displacement is universal even though the diffusion coefficient itself becomes smaller as the dimension increases.

To test this prediction we simulated ensembles of simple symmetric random walks in dimensions $d = 1, 2, 3, 4, 5$. For each dimension we generated 2000 independent walks of length $n = 5000$ steps and at each step we recorded the squared distance from the origin for every walker. The mean squared displacement at step n was computed by averaging these squared distances over all walks.

For each dimension we then fitted a line of the form $\mathrm{MSD}(n) = \kappa_d n$ to the empirical curve and used the fitted slope κ_d to estimate an effective diffusion coefficient $D_d = \frac{\kappa_d}{2d}$.

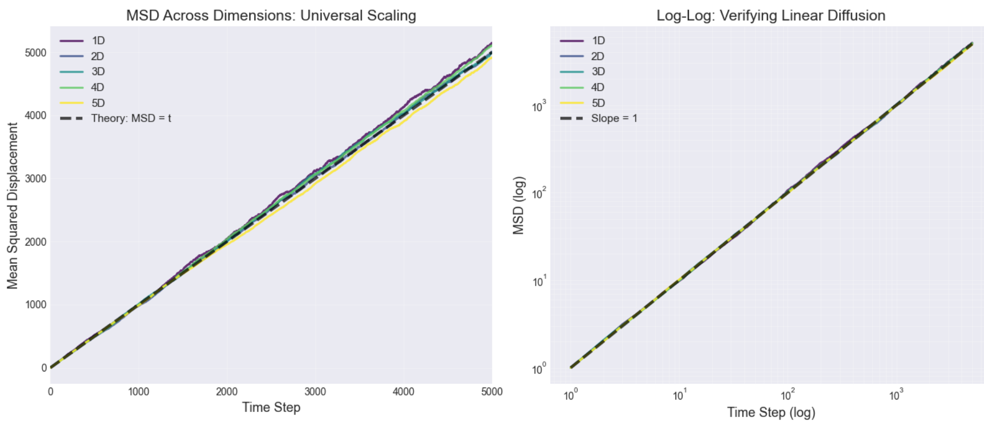


Figure 4: Mean squared displacement as a function of step count for simple symmetric random walks in dimensions one through five. Each curve is obtained by averaging the squared distance from the origin over 2000 independent walks of length 5000 steps. The black reference line has slope one and corresponds to the theoretical prediction $\mathrm{MSD}(n) = n$ for all dimensions.

The curves in Figure 4 are almost perfectly linear in the step count and lie very close to the reference line with slope one in every dimension. This visual evidence already suggests that the walks are diffusive and that the rate of growth of the mean squared displacement is essentially the same for all dimensions.

Dim	D_d (measured)	D (theory)	Ratio	R^2
1	0.5141	0.5000	1.0282	0.999279
2	0.2490	0.2500	0.9961	0.999607
3	0.1736	0.1667	1.0419	0.999764
4	0.1275	0.1250	1.0203	0.999847
5	0.0966	0.1000	0.9657	0.999305

Table 1: Diffusion coefficient estimates from linear fits of the mean squared displacement curves for dimensions one through five. D_d is the measured diffusion coefficient, D is the theoretical value $\frac{1}{2d}$, the ratio column compares measured to theoretical values, and R^2 measures the quality of the linear fit.

These numerical values show that the measured coefficients D_d are very close to the theoretical values $\frac{1}{2d}$ in every dimension and that the fitted slopes are all near one. The coefficient of determination R^2 is above 0.999 in all cases which indicates an excellent linear fit. Taken together the figure and the table confirm that all simulated random walks are diffusive and that the basic scaling of the mean squared displacement proportional to the number of steps holds uniformly across dimensions.

5 Return to the origin and recurrence

Another natural question for a random walk is whether the walker ever returns to its starting point. For the simple symmetric random walk on the integer lattice there is a classical result called the

Polya recurrence theorem. It says that in one and two dimensions the walk is recurrent which means that with probability one it will visit the origin again at some time. In three or more dimensions the walk is transient which means that there is a positive probability that it never comes back. Intuitively in low dimensions the walker has only a few directions to go and keeps crossing the origin while in higher dimensions there are many more ways to escape and never return.

One way to summarize this theory is to look at the eventual return probability $p(d)$ for each dimension d. This is the probability that a walk which starts at the origin will ever return to the origin at some time in the future. The Polya recurrence theorem tells us that $p(1) = 1$ and $p(2) = 1$ while $p(d) < 1$ for all $d \geq 3$. More detailed calculations show that $p(3)$ is about 0.3405 and $p(4)$ is about 0.1932. These values mean that in three dimensions only about one third of all walks ever come back to the starting point and in four dimensions the fraction is even smaller.

To compare this theory with simulations we looked at the event that a walk has hit the origin at least once after leaving it. For each dimension $d = 1, 2, 3, 4$ we generated 5000 independent walks of length $n = 10000$ steps using the same lattice rule as in the previous sections. For every walk we recorded whether it ever returned to the origin during these 10000 steps. The fraction of walks that returned gives an estimate of the return probability in that dimension over this time window. In recurrent dimensions this fraction should move closer to 1 as we make the walks longer. In transient dimensions it should settle near a constant less than 1.

Dimension	Fraction returned	Theory	Classification
1	99.3%	100%	recurrent
2	73.7%	100%	recurrent
3	33.6%	34.05%	transient
4	20.2%	19.32%	transient

Table 2: Estimated fraction of walks that visited the origin at least once by step 10000 in each dimension based on 5000 simulated walks together with the corresponding theoretical eventual return probabilities and the expected classification

In one dimension the return fraction 99.3% is already very close to the theoretical value 100% and would move even closer if we increased the number of steps. In two dimensions the observed fraction 73.7% is lower because we have only run the walks for a finite time but it is clearly increasing toward 100% and would keep growing if we simulated longer trajectories. This behavior matches the idea that one and two dimensional walks are recurrent.

In three dimensions the measured fraction 33.6% is very close to the theoretical value $p(3) \approx$ 34.05%. Extending the simulation would not drive this number toward 100% but instead keep it near this level with a substantial group of walks that never return. In four dimensions the fraction 20.2% is similarly close to the theoretical value $p(4) \approx 19.32\%$ which again reflects transient behavior. Taken together these numbers show the same pattern predicted by the Polya recurrence theorem. Low dimensional walks almost surely come back to the origin at some time while higher dimensional walks have a significant chance to wander off forever.

6 Displacement distributions and the central limit theorem

So far we have focused on averages such as the mean squared displacement but the full distribution of positions after many steps also carries important information. For the simple symmetric random walk there is a classical prediction from the central limit theorem. In one dimension we can write the position after n steps as a sum $S_n = \sum_{k=1}^{n} X_k$ where the steps X_k are independent and take values

1 or -1 with equal probability. The central limit theorem says that for large n the distribution of S_n is close to a normal distribution with mean 0 and variance n. In higher dimensions each coordinate of the position vector is a sum of many small steps and should also be approximately normal with mean 0 and variance proportional to n.

To test these predictions we first looked at the one dimensional case. We simulated 5000 independent walks of length $n = 2000$ steps and recorded the final position S_n of each walk. We then drew a histogram of the observed positions and overlaid the density of a normal distribution with mean 0 and variance n. We also created a Q Q plot that compares the ordered sample values to the quantiles of the reference normal distribution. If the points in the Q Q plot lie close to a straight line this indicates that the normal model is a good fit.

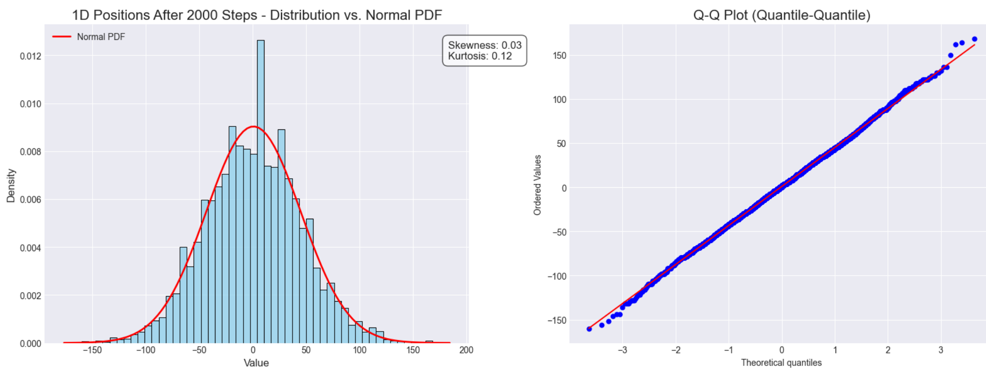


Figure 5: One dimensional displacement analysis after $n = 2000$ steps based on 5000 walks. The left panel shows a histogram of the final positions with a normal density curve using mean 0 and variance n. The right panel shows a Q Q plot of the sample against the corresponding normal distribution

In the one dimensional case the histogram is symmetric around zero and has a smooth bell shaped profile. The empirical mean is very close to zero and the empirical variance is close to $n = 2000$ which matches the prediction that the variance should grow linearly with the number of steps. The normal curve drawn on the histogram follows the data closely and the Q Q plot shows points that cluster near a straight line with only small deviations at the extremes. Together these features support the idea that the distribution of S_n is well approximated by a normal distribution when n is large.

We then repeated this analysis in three dimensions. We simulated another 5000 walks of length $n = 2000$ in $d = 3$ and recorded the final position vector for each walk. For each coordinate $X_n^{(1)}$, $X_n^{(2)}$, and $X_n^{(3)}$ we drew a histogram of the observed values and a Q Q plot against a normal distribution. Since each step only changes one coordinate at a time and picks that coordinate uniformly at random each coordinate is updated on about one third of the steps. The central limit theorem therefore suggests that each coordinate should be close to normal with mean 0 and variance about $n/3$.

The histograms for the three coordinates all show centered bell shaped curves with spreads that match the expected scale $\sqrt{n/3}$. The corresponding Q Q plots line up well along straight lines with only small random fluctuations. This indicates that each coordinate behaves like a one dimensional random walk that has been slowed down by the factor $1/3$. These results support the idea that the three dimensional walk can be modeled by a vector whose coordinates are approximately independent normal variables.

We also examined the distribution of the radial distance from the origin in three dimensions.

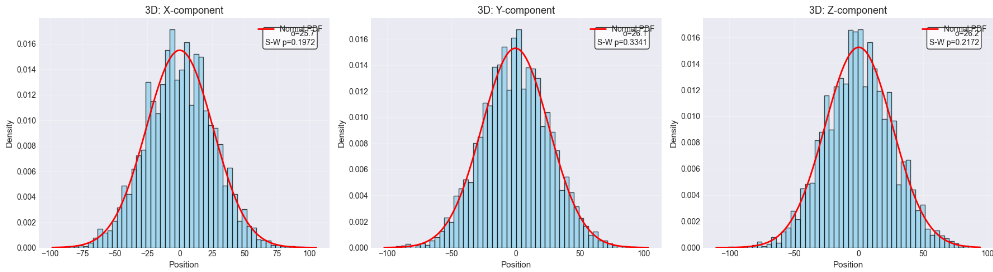


Figure 6: Coordinate wise displacement analysis for three dimensional walks after $n = 2000$ steps based on 5000 paths. The figure shows histograms and Q Q plots for each coordinate compared to a normal distribution with mean 0 and variance close to $n/3$

If each coordinate of the position vector is approximately normal with mean 0 and variance $n/3$, then the radial distance

$$R = \sqrt{X^2 + Y^2 + Z^2}$$

should follow what is known as the Maxwell distribution. A Maxwell distribution is simply the distribution of the length of a three–dimensional vector whose coordinates are independent normal variables with the same variance. Its shape comes from two effects: lengths are always nonnegative, and the number of spatial points at radius r grows like r^2, which produces a peak away from zero and a right–skewed tail. Because our random walk coordinates behave like independent normals, the radius of the walker's final position should therefore follow this Maxwell pattern.

To check this we computed the radius of the final position for each simulated walk and compared the empirical histogram to the corresponding Maxwell curve.

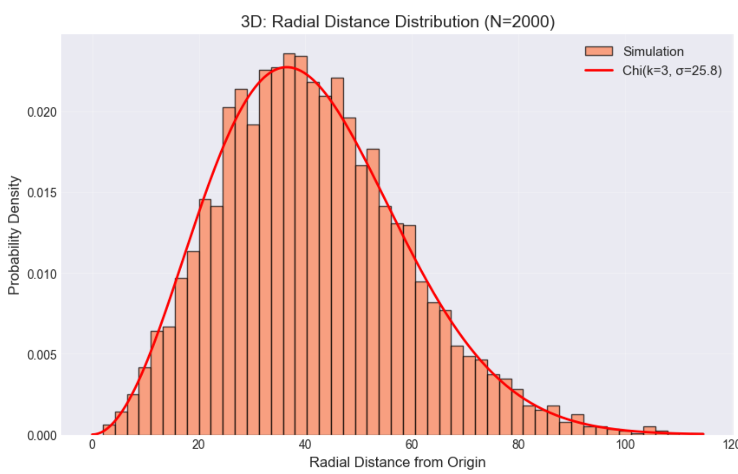


Figure 7: Distribution of the radial distance from the origin in three dimensions after $n = 2000$ steps based on 5000 simulated walks. The histogram shows the empirical radii and the smooth curve shows the Maxwell distribution for a three dimensional normal vector with matching variance

The histogram in Figure 7 has the same skewed shape with a peak away from zero that the Maxwell model predicts and the theoretical curve follows the data closely. Most walkers end at a distance near the peak of this curve with fewer walkers found very close to the origin or very far away. Taken together the one dimensional and three dimensional plots show that after many steps the position of the walker is well approximated by a multivariate normal distribution and that the continuous diffusion picture from the central limit theorem is a good description of the discrete lattice walks in our simulations.

7 First passage times and dimensional scaling

Another way to measure how a random walk spreads is to ask how long it takes to reach a given distance from the origin for the first time. For a radius L we define the first passage time T_{fp} as the number of steps until the distance from the origin first becomes at least L. This random variable measures the time needed to escape a ball of radius L around the starting point. From diffusion theory we expect a typical escape time that grows like L^2. The mean squared displacement after t steps is of order t so the typical distance reached after time t is of order $\sqrt{t}$. Inverting this relation suggests that the time needed to reach distance L should be of order L^2 in any dimension even though the full distribution of T_{fp} can look very different.

We first fixed a target radius $L = 30$ and compared first passage times across dimensions. For $d = 1, 2, 3$ we generated 5000 independent walks of length $n = 10000$ steps and for each walk we recorded the first step at which the distance from the origin reached or exceeded L. If a walk never reached L within 10000 steps it was counted as a failure although for this choice of L the success rate was essentially 100% in every dimension. The code also computed the empirical mean first passage time, the median, and the success rate in each case.

Dimension	Target L	Mean FPT (steps)	Median FPT (steps)	Success rate
1	30	904.5	700.0	100.0%
2	30	927.1	753.5	100.0%
3	30	918.8	750.0	100.0%

Table 3: Mean first passage time, median first passage time, and success rate for reaching radius $L = 30$ in one, two, and three dimensions based on ensembles of 5000 walks of length 10000 steps

To see the full distributions we also plotted histograms of T_{fp} for each dimension and overlaid vertical lines at the empirical mean and median.

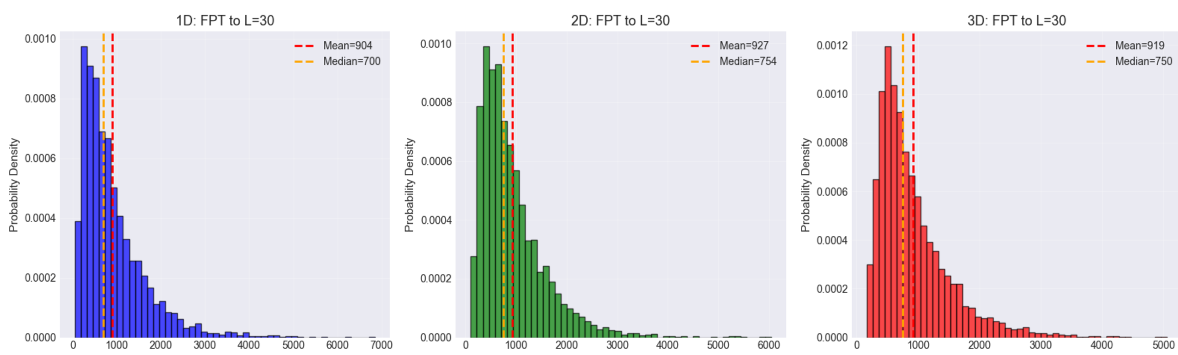


Figure 8: First passage time distributions T_{fp} to radius $L = 30$ in one, two, and three dimensions each based on 5000 simulated walks. Dashed lines mark the empirical mean and median in each panel

The mean first passage times in Table 3 are all close to $L^2 = 900$ steps which agrees with the simple diffusive scaling prediction. The fact that the means in one, two, and three dimensions stay in the same range supports the idea that the leading L^2 dependence does not change with dimension. At the same time the distributions in Figure 8 are broad and right skewed. In each dimension the median is noticeably smaller than the mean which shows that a significant fraction of walks escape faster than the average while a smaller group takes much longer and pulls the mean to the right. The one dimensional histogram is especially wide which reflects long periods where the walk wanders near the origin before finally crossing the target radius. In higher dimensions the peak is more concentrated and extremely long escape times are less common because the walker has more ways to move outward.

We then used a second experiment to test the predicted L^2 scaling more directly in one dimension. The code looped over target radii L in the set $\{15, 20, 25, 30, 35, 40\}$. For each L it generated 3000 one dimensional walks of length $n = 8000$ steps, computed the first passage time for each walk, and then recorded the mean first passage time $\langle T_{\text{fp}}(L)\rangle$. The same code also computed the power law slope by fitting a straight line to the log log plot of $\langle T_{\text{fp}}(L)\rangle$ versus L.

Target L	Mean FPT (steps)	L^2	Mean FPT divided by L^2
15	227.4	225	1.01
20	409.0	400	1.02
25	618.2	625	0.99
30	902.1	900	1.00
35	1226.2	1225	1.00
40	1564.7	1600	0.98

Table 4: Mean first passage times in one dimension for several target radii together with the corresponding values of L^2 and the ratio of mean FPT to L^2 based on ensembles of 3000 walks

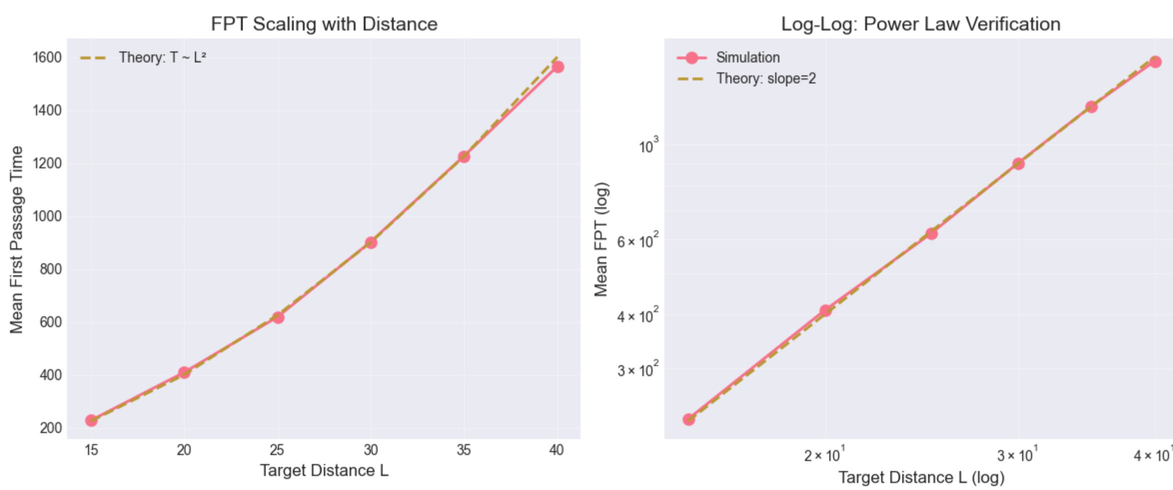


Figure 9: Scaling of the mean first passage time in one dimension. Left panel mean first passage time versus target radius L with the reference curve L^2. Right panel log log plot of the same data along with a reference line of slope 2

Table 4 shows that the mean first passage time is very close to L^2 for every target radius in this range. The ratio of mean FPT to L^2 stays within a few percent of 1 which is consistent with the theoretical scaling law. The log log fit reported by the code gives a power law exponent of about 1.970 which is very close to the ideal value 2.0 and the small difference can be explained by sampling noise and the finite simulation horizon. Together the cross dimensional comparison at fixed L and the one dimensional scaling test support the picture that first passage times have broad heavy tailed distributions but that their typical size grows like the square of the distance that the walk has to travel.

8 Conclusion and connections

In this project we used simulations and basic probability theory to study simple symmetric random walks in dimensions from one to five. For each experiment we compared numerical results to classical predictions. We saw that the mean squared displacement grows linearly in time in every dimension that we tested that low dimensional walks are recurrent while higher dimensional walks are transient that first passage times to a fixed distance have broad heavy tailed distributions with a typical scale that grows like L^2 and that the distribution of displacements is well approximated by a multivariate normal vector in line with the central limit theorem. The agreement between simulation and theory across all of these viewpoints gives a coherent picture of how random walks behave as the dimension changes.

Random walks of the type we studied appear in many areas of physics and applied science. In Brownian motion the random walk model is used as a microscopic description of the erratic motion of particles suspended in a fluid. In quantum physics path integral formulations sum over random paths that resemble random walks in space time. In polymer physics and protein folding models treat long chains as random walks or self avoiding random walks in three dimensions. Spin diffusion and exciton transport in condensed matter systems can be modeled by random walks on lattices or networks. Financial models often replace the simple symmetric walk with a geometric random walk but the basic idea of cumulative random steps in log price is the same. Random walk ideas also appear in cosmology in models of density fluctuations and in ecology and biology where they describe foraging movement chemotaxis and the dispersal of organisms.

From a computational point of view the project combined simple algorithms with careful implementation to reach large scales. To generate an ensemble of M walks of length N steps in d dimensions we used vectorized NumPy operations so that the total work scales like $O(NM)$ but with a small constant. Mean squared displacement curves were computed both by direct ensemble averaging of r^2 at each time and by a time averaged method based on fast Fourier transforms which reduces the cost of computing many lagged correlations from $O(N^2)$ to $O(N \log N)$. Return probabilities were measured by counting how many trajectories visited the origin at least once. First passage times were computed by scanning along each trajectory until the distance from the origin first crossed the target radius. Displacement distributions were studied by building histograms and Q Q plots for coordinates and for radial distance.

Overall the simulations involved on the order of fifty thousand independent random walks and roughly a few hundred million total steps across all experiments. Despite the simplicity of the model the numerical results showed strong agreement with exact and asymptotic theoretical formulas. This combination of analytic insight and computational experiments illustrates how random walks link abstract probability theory to concrete diffusive behavior in physical and applied systems and how modern numerical tools make it easy to explore these links across different dimensions.

All of our code is here: https://github.com/Nathanana/RandomWalk

Biblography

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