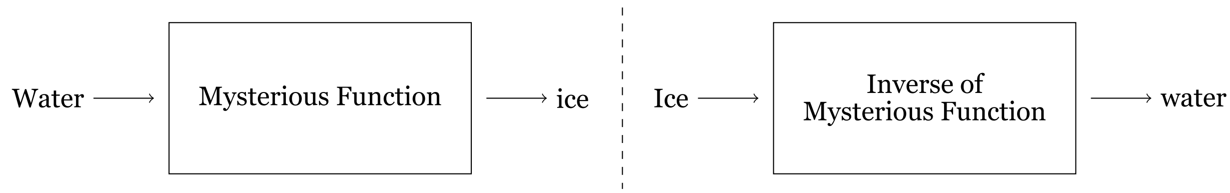
Inverse Functions - Note

An inverse function can be thought of as the exact opposite of the original function.

For example, consider a mysterious function where if we feed water into the function, the output will be ice. Then if we feed ice into the inverse of that mysterious function, then water should come out.



Let's look at a more concrete example. Consider the function $y = 3x - 2$.

What does this function do to x to produce y?

Well, the function **multiplies** x by 3 and **subtracts** 2 from it. The two operations in this function are multiplication and subtraction.

Then what does the inverse of this function need to do to get back the input?

The inverse of the function needs to do exactly the opposite of what the original function does. So it needs to **add** 2 to x and **divide** the resulting number by 3. Note how the order of these operations is very important. The notes will later elaborate on how to do this.

Notation

- If $f(x)$ represents a function, then the inverse of $f(x)$ is represented as $f^{-1}(x)$.
- $f^{-1}(x)$ is read as "f inverse of x".
- $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$. Express the latter in fraction form to avoid using exponents.

Determining if a Function has an Inverse.

Consider the function $f(x) = x^2$. If we set $x = 2$, then the function will spit out a 4. If we set $x = -2$, the function will also spit out a 4.

Now imagine that we have the inverse of the function $f^{-1}(x)$. What will $f^{-1}(4)$ output? Will it be 2 or -2?

The answer is that we can't know. This is why math has a rule set in place stating that only **one-to-one functions** have inverses.

> **One-to-one Function:** A function where each input has a unique output. This means that the function will never have 2 x-values that produce the same y-value.

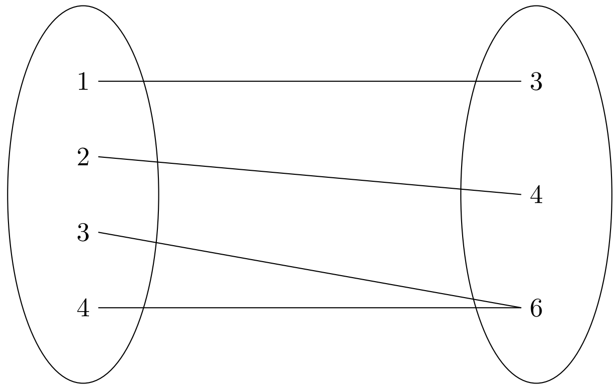
There are multiple ways to determine if a function has an inverse. The two main ways is using ordered pairs and the horizonatal line test.

Example: Determining if a Function is One-to-one using Ordered Pairs

Question: Determine if the function $f(x) = \{(1,3),(2,4),(3,6),(4,6),(1,3)\}$ is one-to-one.

Step-by-step Solution:

Here is a visual representation of the function f. You do not need to draw this if you can determine if a function is one-to-one by observing the ordered pairs.



Step 1) Identify the y-value of the ordered pair.

Let's start with the first ordered pair: $(1,3)$. The y-value of this pair is 3.

Step 2) Check every other y-value to see if the y-value you found in Step 1 repeats.

In this case, 3 also appears in the last pair.

Step 3) If it does repeat, then check if it has the same corresponding x-value as the ordered pair in Step 1, if not, this is not a one-to-one functions.

In this case, the last pair's x-value is also 1, meaning that the last pair is just a duplicate of our first pair. Therefore, there's still no problem with 3 being a unique y-value.

Step 4) Repeat Steps 1-3 until you checked every ordered pair. If you checked every ordered pair and haven't found 2 x-values that produce the same y-value in step 2-3, then the function is one-to-one.

If we repeat steps 1-3, eventually we get to $(3,6)$. The y-value of this coordinate is 6 and it repeats in the 4th pair: $(4,6)$. The x-value of the 4th pair is 4, which is different from our $(3,6)$ x-value. Therefore 6 is not a unique y-value which means that $f(x)$ is not a one-to-one function. This also means $f(x)$ does not have an inverse.

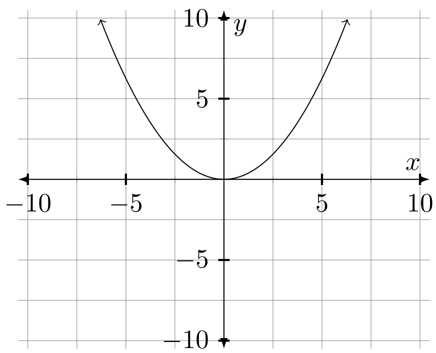
Note: You don't need to follow these steps exactly if you can just observe the ordered pairs and find multiple instances of y-values.

Example: Determining if a Function is One-to-one using The Horizontal Line Test

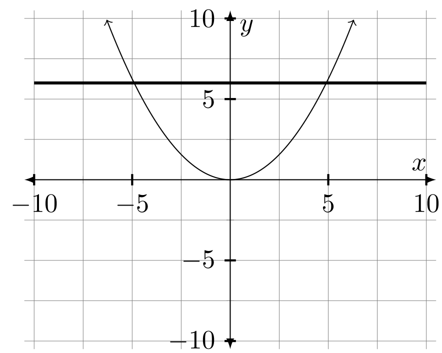
Question: Determine if the function $f(x) = \frac{1}{4}x^2$ is one-to-one.

Step-by-step Solution:

Step 1) Graph the function. You don't need to be really precise about graphing it. Sometimes even graphing the parent function alone is enough to determine if the function is one-to-one.



Step 2) Visualize a horizontal line going through the graph (or use a straight-edge) and move it around. If the horizontal line intersects at two or more different points of the graph, then function is not one-to-one. If it doesn't, then the function is one-to-one.

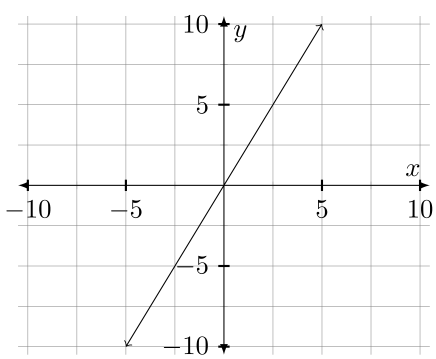


In this case, the horizontal line does intercept $f(x) = \frac{1}{4}x^2$ in two or more different points on the graph if the horizontal line is above the x-axis. Therefore, $f(x)$ is not one-to-one.

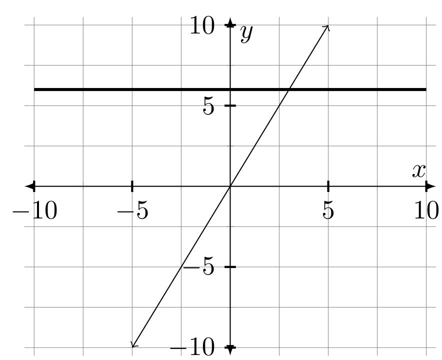
Question: Determine if the function $f(x) = 2x$ is one-to-one.

Step-by-step Solution:

Step 1) Graph the function.



Step 2) Visualize a horizontal line going through the graph (or use a straight-edge) and move it around to check if it intersects the graph more than once.



In this case, the horizontal line does not intercept $f(x) = 2x$ more than once on the graph no matter where the horizontal line is. Therefore, $f(x)$ is one-to-one.

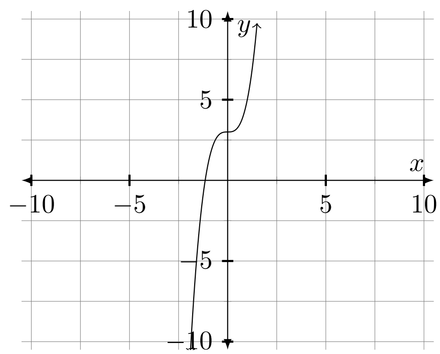
Now that we know how to determine if a function has an inverse or not, we can figure out the inverse of the function.

Example: Determining the Inverse of a Function

Question: Determine the inverse of $f(x) = 2x^3 + 3$.

Step-by-step solution:

Step 1) Determine whether or not the function has an inverse.



We can use the horizontal line test to see if the function has an inverse.

First, graph the function and see if a horizontal line will cross 2 or more points of the function.

In this case, any horizontal line will only intersect with the function at 1 point. Therefore, this function is a one-to-one, which means that $f(x)$ has an inverse.

Step 2) Replace $f(x)$ with y in the equation.

$$y = 2x^3 + 3$$

Step 3) Swap all x's and y's in the equation.

$$x = 2y^3 + 3$$

Step 4) Solve for y algebraically.

$$x = 2y^3 + 3$$

$$x - 3 = 2y^3 + 3 - 3$$

Move 3 over by subtracting 3 from both sides.

$$x - 3 = 2y^3$$

$$\frac{x-3}{2} = \frac{2y^3}{2}$$

Divide both sides by 2.

$$\frac{x-3}{2} = y^3$$

$$\sqrt[3]{\frac{x-3}{2}} = \sqrt[3]{y^3}$$

Cube root both sides to isolate y.

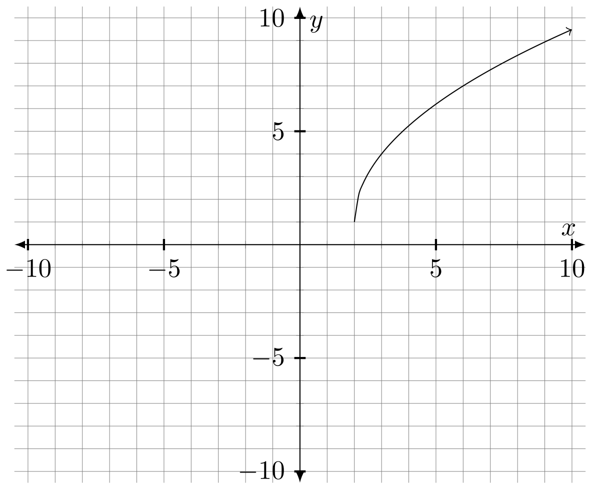
$$\sqrt[3]{\frac{x-3}{2}} = y$$

Step 5) Replace y with $f^{-1}(x)$. Swap the equation if necessary.

$$f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$$

Example: Graphing the Inverse of a Function from a Graph

Question: Graph the inverse of the following relation.



Step 1) Determine if the function has an inverse.

Using the horizontal line test, we can see that no matter where we place a horizontal line, it will always intersect the function at 1 or less points. Therefore, the function is one-to-one, which means that the function has an inverse.

Step 2) Determine an equation that describes the graph.

First, recognize the parent function. In this case, the graph looks like a square root. Therefore the function has to be in a similar form to $y = \sqrt{x}$. More specifically, $f(x) = a\sqrt{x+k} + b$ where a, k, and b are constants that we need to figure out.

Since the "base" of the function is easily identifiable, we can see that the graph shifted 2 units to the right and 1 unit up. This means that $k = -2$ and $b = 1$. Substituting these numbers into our general formula results in $f(x) = a\sqrt{x-2} + 1$.

To find the stretch, we can solve for a using algebra. Choose an easy coordinate on the graph that we can use to substitute into the equation. Let's use $(3, 4)$ for our example.

$4 = a\sqrt{3-2} + 1$	Substitute in the known values of x and y.
$4 = a\sqrt{1} + 1$	Simplify the inside of the square root.
$4 = a(1) + 1$	Simplify the square root.
$4 - 1 = a + 1 - 1$	Add 5 to both sides to isolate the a
$3 = a$	Congratulations, we found what a should be.

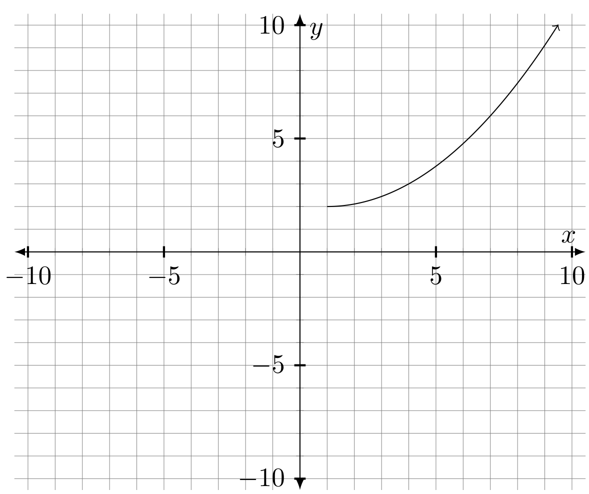
Substitute our value of a into the general formula along with k and b and we get the function for our graph:

$$f(x) = 3\sqrt{x-2} + 1$$

Step 3) Find the inverse of the function.

$y = 3\sqrt{x-2} + 1$	Substitute $f(x)$ for y in the equation.
$x = 3\sqrt{y-2} + 1$	Swap all x's with y's.
$x - 1 = 3\sqrt{y-2}$	Subtract both sides by 1 to cancel out the 1 from the right side.
$\frac{x-1}{3} = \sqrt{y-2}$	Divide both sides by 3 to cancel out the 3 from the right side.
$(\frac{x-1}{3})^2 = y - 2$	Square both sides to cancel out the square root on the right side.
$\frac{(x-1)^2}{3^2} = y - 2$	Simplify the left side.
$\frac{1}{9}(x-1)^2 = y - 2$	
$\frac{1}{9}(x-1)^2 + 2 = y$	Add 2 to both sides to isolate y.
$f^{-1}(x) = \frac{1}{9}(x-1)^2 + 2$	Replace the y with $f^{-1}(x)$ and flip the equation around to make it look pretty.

Step 4) Graph the inverse function.



Simply plot a couple of points using the inverse function like $(4, 3)$ and $(7, 6)$ and connect the dots.

Note how we only drew one side of the parabola. This is because our original function had a domain of $x \geq 2$ and a range of $y \geq 1$.

When we swapped x and y in step 3 to find the inverse, we also swapped the domain and range. Therefore, our new domain is $x \geq 1$ and our new range is $y \geq 2$. The left side of the parabola is where $x < 1$, which is outside of our new range so we cannot draw it.

Congratulations! You now know how to graph the inverse of a function from a graph!