Backward Stochastic Differential Equations

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Introduction

A bit of history : backward stochastic differential equations (BSDEs) - [Bismut 1973] + [Pardoux & Peng 1990]

- Existence and uniqueness of solutions
- Applications in finance, control theory, mathematical physics...
- Today, BSDEs continue to be an active area of research

Plan

Agenda :

Mathematical framework and definitions

For the whole seminar we fix a $T > 0$, $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ a filtered probability space, and B a d-dimensional Brownian motion. We shall always assume $\mathbb{F} = \mathbb{F}^B$.

Theorem (Martingale representation theorem)

$\forall \xi \in \mathbb{L}^2(\mathcal{F}_T^B)$, $\exists ! \sigma \in \mathbb{L}^2(\mathbb{F}^B, \mathbb{R}^d)$ *such that :*

$$\xi = \mathbb{E}[\xi] + \int_0^T \sigma_t . dB_t$$

Consequently, for any $\mathbb{F}^B$*-martingale* M *such that* $\mathbb{E}[|M_T|^2] < \infty$, $\exists ! \sigma \in \mathbb{L}^2(\mathbb{F}^B, \mathbb{R}^d)$ *such that :*

$$M_t = M_0 + \int_0^t \sigma_s . dB_s$$

Mathematical framework and definitions

$$\begin{cases} dY_t = 0 \\ Y_T = \xi \end{cases} \tag{1}$$

$Y_t = \xi \ \forall t$? But then Y_t is not adapted to $\mathcal{F}_t^B \rightarrow$ We need a certain Z_t such that

$$\begin{cases} dY_t = Z_t dB_t \\ Y_T = \xi \end{cases} \tag{2}$$

and thus Y_t becomes adapted to $\mathcal{F}_t^B$. Idea: Martingale representation theorem!

Mathematical framework and definitions

Given $\xi \in \mathbb{L}^2(\mathcal{F}_T^B)$, it induces naturally a martingale $Y_t := \mathbb{E}[\xi|\mathcal{F}_t]$. And by the previous theorem $\exists! Z \in \mathbb{L}^2(\mathbb{F})$ such that :

$$dY_t = Z_t dB_t, \text{ or equivalently, } Y_t = \xi - \int_t^T Z_s dB_s.$$

This is a linear SDE with terminal condition $Y_T = \xi$ and Y is adapted thanks to $Z \rightarrow$ BSDE !

Mathematical framework and definitions

More generally,

Definition (Adapted solution of a BSDE)

An adapted solution of a BSDE is a *pair* of $\mathbb{F}$-measurable processes (Y,Z) with the terminal condition $Y_T = \xi$ such that for $t \in [0,T]$

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dB_t \ , \ \mathbb{P}\text{-a.s.}$$

or equivalently

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$$

where $Y \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2})$, $Z \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2 \times d})$

f is often called the "driver" or the "generator"

Mathematical framework and definitions

General assumptions

- $\mathbb{F} = \mathbb{F}^B$ holds;
- f maps $[0, T] \times \Omega \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_2 x d}$ onto $\mathbb{R}^{d_2}$ and is $\mathbb{F}$-measurable in all variables;
- f is uniformly Lipschitz continuous in (y,z) with a Lipschitz constant L;
- $\xi \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{R}^{d_2})$ and $f^0 := f(.,0,0) \in \mathbb{L}^{1,2}(\mathbb{F}, \mathbb{R}^{d_2})$

Under these assumptions, the data (f,ξ) are said to be standard data for the BSDE. For notational simplicity we shall assume $d_2 = d = 1$

The linear case

We consider the general linear BSDE with $d_2 = 1$

$$Y_t = \xi + \int_t^T [\alpha_s Y_s + \beta_s Z_s + f^0(s)] ds - \int_t^T Z_s dB_s \text{ (X)}$$

The well-posedness of this BSDE will follow from the general theory. Here we provide a representation formula for its solution.

The linear case

Theorem (Representation formula for LBSDE)

Let $\xi \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{R})$ *and* $f^0 \in \mathbb{L}^{1,2}(\mathbb{F}, \mathbb{R}^d)$, $\alpha \in \mathbb{L}^\infty(\mathbb{F}, \mathbb{R})$, $\beta \in \mathbb{L}^\infty(\mathbb{F}, \mathbb{R}^d)$ *and if the pair* $(Y, Z) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{1\times d})$ *satisfies the linear BSDE (X), then*

$$Y_t = \Gamma_t^{-1}\mathbb{E}[\Gamma_T \xi + \int_t^T \Gamma_s f^0(s) ds | \mathcal{F}_t],$$

where Γ_t *is the adjoint process defined by the forward LSDE,*

$$d\Gamma_t = \Gamma_t[\alpha_t dt + \beta_t . dB_t] \ ; \Gamma_0 = 1$$

The linear case

For the proof we need the following result :

Lemma

Let $\xi \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{R}^{d_2})$ *and* $f^0 \in \mathbb{L}^{1,2}(\mathbb{F}, \mathbb{R}^{d_2})$ *Then the following linear BSDE has a unique solution* $(Y, Z) \in \mathbb{S}^2(\mathbb{F}, \mathbb{R}^{d_2}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2 \times d})$

$$Y_t = \xi + \int_t^T f^0(s)ds - \int_t^T Z_s dB_s$$

Proof (lemma) : $Y_t = \mathbb{E}[\xi + \int_t^T f^0(s)ds|\mathcal{F}_t]$. And the uniqueness is deduced by the martingale reprensetation theorem

The linear case

Proof (Theorem) : Applying Itô formula we have

$$d(\Gamma_t Y_t) = -\Gamma_t f^0(t) + \Gamma_t[Y_t\beta_t^T + Z_t]dB_t$$

With the change of variables :

$$\hat{Y}_t := \Gamma_t Y_t \ ; \ \hat{Z}_t := \Gamma_t[Y_t\beta_t^T + Z_t] \ ; \ \hat{f^0}(t) := \Gamma_t f^0(t)$$

Then, one may rewrite it as

$$\hat{Y}_t = \hat{\xi} + \int_t^T \hat{f^0}(s)ds - \int_t^T \hat{Z}_s dB_s$$

Then, with the previous lemma

$$\hat{Y}_t = \mathbb{E}[\hat{\xi} + \int_t^T \hat{f^0}(s)ds|\mathcal{F}_t]$$

A Priori Estimates for BSDEs

We now investigate the nonlinear BSDE, and we have the following result:

Theorem

Suppose the data (f, ξ) to be standard data for the BSDE, and $(Y, Z) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2 \times d})$ be a solution to the BSDE. Then $Y \in \mathbb{S}^2(\mathbb{F}, \mathbb{R}^{d_2})$ and there exists a constant C, depending only on T, L and d, d_2 such that

$$||(Y, Z)||^2 := \mathbb{E}[|Y_T^*|^2 + \int_0^T |Z_t|^2 dt] \leq C I_0^2$$

Where $I_0^2 := \mathbb{E}[|\xi|^2 + (\int_0^T |f^0(t)| dt)^2]$

Proof : Omitted

A Priori Estimates for BSDEs

We deduce this theorem

Theorem

For $i = 1,2$ Suppose the data (f^i, ξ^i) to be standard data for the BSDE, and $(Y^i, Z^i) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2 \times d})$ be a solution to the BSDE with coefficients (f^i, ξ^i). Then

$$||(\Delta Y, \Delta Z)||^2 := \mathbb{E}[|\Delta\xi|^2 + (\int_0^T |\Delta f(t, Y_t^1, Z_t^1)|^2 dt)^2]$$

Where

$$\Delta Y := Y^1 - Y^2, \Delta Z := Z^1 - Z^2, \Delta\xi := \xi^1 - \xi^2, \Delta f := f^1 - f^2.$$

Proof : Omitted

Well-Posedness of BSDEs

We now establish the general well-posedness of BSDEs

Theorem

Suppose the data (f, ξ) to be standard, the BSDE has a unique solution $(Y, Z) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{d_2 \times d})$.

Proof : The uniqueness follows from the last Theorem on A Priori Estimates and the existence from the Picard iteration.

Remark : The well-posedness result can be extended to $\mathbb{L}^p(\mathbb{F})$ for $p \geq 2$.

Basic Properties of BSDEs

We have a comparison result

Theorem (Comparison Theorem)

Let $d_2 = 1$. Assume for $i = 1,2$, (f^i, ξ^i) to be standard data, and $(Y^i, Z^i) \in \mathbb{S}^2(\mathbb{F}, \mathbb{R}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{1\times d})$ be the unique solution to the following BSDE :

$$Y^i_t = \xi^i + \int_t^T f^i(s, Y^i_s, Z^i_s)ds - \int_t^T Z^i_s dB_s$$

Assume further that
$\xi^1 \leq \xi^2, \mathbb{P} - a.s., \text{and}\, f^1(., y, z) \leq f^2(., y, z), dt \times d\mathbb{P} - a.s. \forall (y, z).$
Then,

$$Y^1_t \leq Y^2_t, 0 \leq t \leq T, \mathbb{P} - a.s.$$

Basic Properties of BSDEs

Theorem (Stability)

Let (ξ, f) and (ξ^n, f^n), $n = 1,2,\dots$, be standard data with the same Lipschitz constant L, and $(Y, Z), (Y^n, Z^n) \in \mathbb{S}^2(\mathbb{F}, \mathbb{R}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{R}^{1\times d})$ be the solution to the corresponding BSDE. Denote

$$\Delta Y^n := Y^n - Y, \Delta Z^n := Z^n - Z, \Delta\xi^n := \xi^n - \xi, \Delta f^n := f^n - f.$$

Assume further that

$$\lim_{n\to\infty} \mathbb{E}[|\Delta\xi^n|^2 + (\int_0^T |\Delta f^n(t, 0, 0)|dt)^2] = 0,$$

and that $\Delta f^n(y, z) \to 0$ in measure $dt \times d\mathbb{P}$, for all (y,z). Then,

$$\lim_{n\to\infty} ||(\Delta Y^n, \Delta Z^n)|| = 0,$$

Applications : PDE and Feynman-Kac formulae

Let us consider the linear parabolic partial differential equation (PDE):

$$\begin{cases} \dfrac{\partial v}{\partial t} + Lv + f = 0, & \text{on } [0,T) \times \mathbb{R}^d, \\ v(T,.) = h, & \text{on } \mathbb{R}^d, \end{cases} \tag{3}$$

where L is the second order Dynkin operator:

$$Lv = b(x) \cdot \nabla v + \frac{1}{2} \operatorname{tr}(\sigma\sigma^\top(x)\nabla^2 v). \tag{4}$$

Under suitable conditions on the functions b, σ, f and h defined on $\mathbb{R}^d$, there exists a unique solution v to (3), which may be represented by the Feynman-Kac formula:

$$v(t,x) = \mathbb{E}_h \left[\int_t^T f(X_{s,x})\, ds + h(X_{T,x}) \right], \quad (t,x) \in [0,T] \times \mathbb{R}^d \tag{5}$$

Applications : PDE and Feynman-Kac formulae

The solution to the (forward) diffusion process, $dX_s = b(X_s)ds + \sigma(X_s)dB_s$, $s \geq t$, starting from x at time t is denoted by $X_{t,x}$. With Ito's formula when v is smooth, we can define the pair of processes (Y, Z):

$$Y_t := v(t, X_t), \ Z_t \qquad := \sigma(X_t)^\top \frac{\partial v}{\partial x}(t, X_t), \quad 0 \leq t \leq T,$$

and applying Ito's formula to $v(s, X_s)$ between t and T, with v satisfying the PDE, we get:

$$Y_t = h(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \qquad (6)$$

where $h(X_T)$ is a terminal condition.

Applications in Stochastic Control : Stochastic maximum principle

We consider a stochastic control problem on a finite horizon : let X be a controlled diffusion on $\mathbb{R}^n$ governed by

$$dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dB_s, \quad (6.23)$$

where $\alpha \in A$, the control process, is a progressively measurable valued in A. The gain functional to maximize is

$$J(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)dt + g(X_T)\right],$$

where $f : [0, T] \times \mathbb{R}^n \times A \to \mathbb{R}$ is continuous in (t, x) for all a in A, $g : \mathbb{R}^n \to \mathbb{R}$ is a concave C^1 function, and f, g satisfy a quadratic growth condition in x.

Applications in Stochastic Control : Stochastic maximum principle

We define the generalized Hamiltonian
$H : [0, T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ by

$$H(t, x, a, y, z) = b(x, a) \cdot y + \mathrm{tr}(\sigma(x, a) \cdot z) + f(t, x, a),$$

and we assume that H is differentiable in x with derivative denoted by $D_x H$. We consider for each $\alpha \in A$, the BSDE, called the adjoint equation:

$$-dY_t = D_x H(t, X_t, \alpha_t, Y_t, Z_t)dt - Z_t dB_t, \quad Y_T = D_x g(X_T).$$

Applications in Stochastic Control : Stochastic maximum principle

Theorem

Let $\hat{\alpha} \in A$ and $\hat{X}$ be the associated controlled diffusion. Suppose that there exists a solution $(\hat{Y}, \hat{Z})$ to the associated BSDE such that

$$H(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a \in A} H(t, \hat{X}_t, a, \hat{Y}_t, \hat{Z}_t), \quad 0 \leq t \leq T, \ a.s.$$

and
$(x, a) \to H(t, x, a, \hat{Y}_t, \hat{Z}_t)$ is a concave function,
for all $t \in [0, T]$. Then $\hat{\alpha}$ is an optimal control, i.e.

$$J(\hat{\alpha}) = \sup_{\alpha \in A} J(\alpha).$$

The end

To summarise...