Utility Theory and the Representation of Preference

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This article investigates the expected utility model of rational choice. We begin by discussing two interpretations of probability and examining the role of expectation in probability modelling before then discussing the the useful properties of utility functions. We then continue on to two representation theorems which formalise the connection between probability and utility theory. Will we see that the nature of these theorems, coupled with the indeterminacy of statistical inference, is a brittle foundation for predictive economic models of consumer choice.

Expectations: The Workhorse

There is an algorithm beloved by bureaucrats. The unsung hero of administrators and accountants. An algorithm both ubiquitous and under appreciated. It's pivotal for nearly every project and informs the actions of tech giants and policy makers the world over. It is only mildly hyperbolic to say that understanding this formula unlocks wealth and power. The algorithm lies at the heart of online A/B testing, all policy analysis, sound business strategy and poker play.

$$EV(O)_p = p_1u(o_1) + p_2u(o_2) + ... + p_ku(o_k)$$

The expected financial value of a random process is just the sum of the utility (typically dollar outcomes) weighted by their probabilities. Outcomes can vary from deals of cards, to customer transactions and election results.[1] Aside from the mercenary possibilities, Pascal can argue sincerely that such considerations compel belief in God. But the formula, glossed as a rule for rational action, merits your attention for more mundane wagers too. The meaning of probability is not an idle concern if you intend to maximise your expected value. While statistics are often tortured to rubber stamp decisions and probabilities are abused to fit policy prescriptions with false precision, the crisp clarity of the rule has an enduring appeal that promises to sift the murky swamps of Big Data. It's a scalpel that anyone can wield to parse the syntax of statistical jargon and carve answers from an abstract space of probabilities. "What's my expected return?" - a simple question, with a surprisingly complex answer.

In this article we will delve into the meaning of the (EV) formula by (i) discussing two **interpretations of the probability values** p_i and how expectation features in statistical models. Showing (ii) how the two interpretations prompt different **styles of inference** (Bayesian

[1] In the jargon O is a random variable which can be realised as any of the outcomes $o_j : 1 \leq j \leq k$

and Frequentist) and different distributions of expected value. Following this we will consider (iii) the notion of **subjective utility** and the role of $u(o_i)$ in optimisation problems and (iv) two **representation theorems** which purport to account for how an agent's preferences relate to rational decision over expected value. Finally, we'll conclude with a brief contrast to how **customer segmentation** models implement these considerations to make predictions about consumer price sensitivity in a modern on-line marketplace.

The narrative threads through esoteric issues of philosophy, statistics, decision theory and the practicalities of machine learning - moving from agent-based models and inference from small samples to prediction at scale across cohorts of agents. There is a pivot point in the journey where an explanatory model of rational expected action is substituted for a more obscure black-box. The tendency is common where profit trumps all other priorities, but the loss of understanding often amounts to a longer term net-loss. This dynamic enforces a kind of inescapable see-saw motion where the consumer modelling exercise goes through a constant feedback loop, - a good model (informal or formal) of customer utility feeds a better a predictive model of customer action. When the latter fails we go the back to the utility curves because it is (if not reliably predictively) a rich and deeply explanatory model of human action.

Probability: Two sides of a Coin

Probability emerged slowly and with a dual aspect. On one tradition it refers to the long run tendency of a random process, on another probability is construed as the degree of belief in an outcome. On the first (frequentist) interpretation a probability distribution has certain fixed theoretical characteristics, as in a uniform probability distribution of a fair coin where all outcomes are equally likely. On the second (Bayesian) reading the characteristics of the probability distribution are learned from the data. The controversy centres around the fact that it's unclear how a frequentist could ascribe probabilities to unique events. Without appeal to a large set of observations (or known theoretical distribution) the claim that an event appears in frequently or infrequently is not well defined. Consequently tabulations of probability appear inappropriate for claims of unique or rare events. On the other hand, the Bayesian is content to ascribe probabilities to any all partial beliefs no matter how specific.[2] For the Bayesian, the probability calculus is a set of edicts about how to rationally manage and modulate your beliefs, so it's acceptable to have a probabilistic belief in rare cases so long as you update those prob-

[2] For example: $p(o) =_{def}$ the probability that you will hit your head tomorrow at 2.00 o'clock

abilities with new data when available. These approaches are united by the Law of Large numbers which states that as the size of our sample increases our sample average will converge to the expected realisation of the theoretical process.

$$\frac{1}{N}\sum_{i=1}^{N} O_i \text{ converges to } E(O) \text{ as } N \text{ approaches } \infty$$

In this plot we have fixed a Poisson distribution with a mean of 4.5 and can see three examples of how consecutive averaging from the increasing sample sizes results in a closer and closer convergence to the (true) population mean.

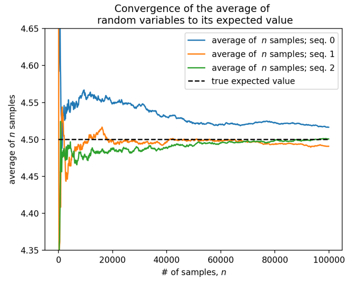


Figure 1: Convergence with large samples

This is fundamental to the interpretation of probability. Given a game with fixed and fair odds we see that repeated play will converge because of characteristics which govern the process. Dice are the paradigm example. In the wild we never know the characteristics which cause the observed spread of outcomes, but such is the influence of gambling on probability, that we assume there is a stable pattern to be gamed. Partially this is pragmatic. The maths is more tractable if we can assume a well behaved underlying process, and the results are compelling. The Normal (Bell Curve) distribution, the Poisson distribution the Bernoulli distribution (to name a few) are all rightly famous. Their shapes are characteristics of innumerable random processes.

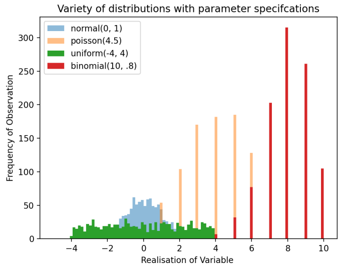


Figure 2: Some theoretical distributions with parameters

But the paradigm clouds the fact that in practice we start on the left side of the law of large numbers (with samples) and we often start with small numbers and unknown number of data-generating processes. Well behaved probability distributions are rare beasts; a tiny fraction of the world's arbitrary menagerie. The fundamental question in probability is not whether probability is a measure of belief or frequency - it is whether we can safely assume that the underlying process adheres to a known model? If so, we can rely on the structure of the model's theoretical distribution to inform inference. If not we are better learning what we can from the sample - trusting to wide confidence intervals and worst scenario planning. But these apparently esoteric questions of philosophy are inescapable when we need to make a decision. What are your expectations based on? How do they figure in our choices, and can we use them to improve our outcomes?

Models, Errors and Sampling

Statistical models are often complex, but at heart they're mostly machines for figuring out expected values of a statistical process.

A basic regression model tries to predict an outcome Y as linear function of X:

$$Y = const + \beta X + \epsilon$$

where ϵ is a random variable representing the error (or noise). A modest notational device for disaster. While $const, \beta$ are parameters estimated by an optimisation process to ensure the equation fits the data as neatly as possible. In (Figure 3) below we have a series characterised by change. After the first shock we can refit the model so that the line tracks well with the evolving data. After the second shock we try another refit, but the range the and variance of the data makes our basic model a poor fit i.e. the data no longer exhibits a linear relationship. This presents three examples of error in the modelling process: (i) forecasts fail for the reason that's it's also difficult to identify (in the moment) those changepoints in the data which reflect structural change, (ii) the linearity assumptions that go into the model are sound, but the parameters need be re-estimated based on the new data and (iii) the third linear model is simply a terrible match for the pattern in the data.

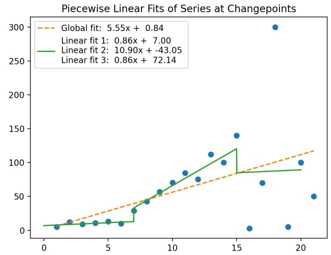


Figure 3: Three samples with starkly different parametrisations

Every model is a guess as to the implicit order in apparent noise. Sometimes there is no order, and other other times the patterns is too subtle for a dumb model to capture. In practice you never really know whether a single new error stems from a misfit but appropriate model or an entirely inappropriate model. As we increase our number of sample fits we hope to better approximate the true linear process (if any) generating the data. Imagine now that the data points in Figure 3 are repeatedly re-speckled over the canvas. We can refit a new model for each set of scattered data points and each refit gives us a new sample values for $const, \beta$. If the underlying data generating process is stable, then the parameter fits will converge to the correct values of $const, \beta$; correct in the sense that they can be used to draw the line of best fit for the data. A statistically stable process is one that can be modelled with errors ϵ normally distributed around 0, so that the model will be *correct on average* because $E(\epsilon) = 0$. Our predictions will overshoot in some cases but on the whole the errors up and down will cancel each other out. Forecasting with the parameters of best fit minimises our forecast errors because the fluctuations are stable about the centre of the line. These are the required assumptions for a process to exhibit the tendency of regression towards the mean. If they're not met, we will see poor parameter estimates and wild swings away from the linear path.

Improper Assumptions and Skewed Expectations

The code below builds two sampling distributions based on different underlying processes. One in which the errors are independent, normally distributed around 0 the other in which the errors are correlated in a sine-wave like pattern, increasing and decreasing periodically. This is akin to difference between measuring error when predicting the heights of randomly sampled people versus predicting the sales volumes on randomly selected days of the week. A random sample of daily sales risks clumping weekends together and skewing the expected values. No such risk exists when sampling from independent individuals.

```
### Build True Models
N = 100000
X = random.uniform(0, 20, N)
independent_err = random.normal(0, 10, N)
corr_err = random.uniform (0, 10) +  sin(np.linspace(0, 10*pi, N)) +
sin(np.linspace(0, 5*pi, N))**2 +  sin(np.linspace(1, 6*pi, N))**2

Y_corr = -2 + 3.5 * X + corr_err
Y = -2 + 3.5 * X + independent_err

population = pd.DataFrame({'X': X,  'Y': Y, 'Y_corr': Y_corr})

### Sample from Data
### and build smaller models

fits = DataFrame(columns=['iid_const', 'iid_beta', 'corr_const',
 'corr_beta'])

for i in range(0, 10000):
    sample = population.sample(n=100,
    replace=True)
    Y = sample['Y']; X = sample['X']
    Y_corr = sample['Y_corr']
    X = add_constant(X)
    iid_model = OLS(Y, X)
    results = iid_model.fit()
    corr_model = OLS(Y_corr, X)
    results_2 = corr_model.fit()
    row = [results.params[0],  results.params[1],
           results_2.params[0], results_2.params[1]]
    fits.loc[len(fits)] = row
fits.boxplot()
```

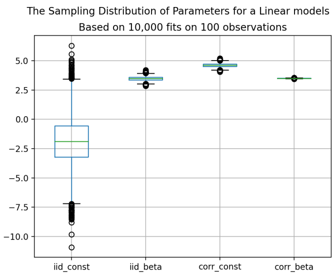


Figure 4: The expected realisations for $\beta, const$ with different errors structures

In the case with independent errors the expected value for our parameter estimates match almost exactly the true values of the process. In the second model with correlated errors the parameter estimate for our constant is 4.9 which is significantly different from the true value of -2, and will lead to systematically skewed predictions. Statistical models are just algebraic equations where we use regular sampling to solve for Y from X. Because Y is also a random variable the regression model encodes a conditional expectation result.

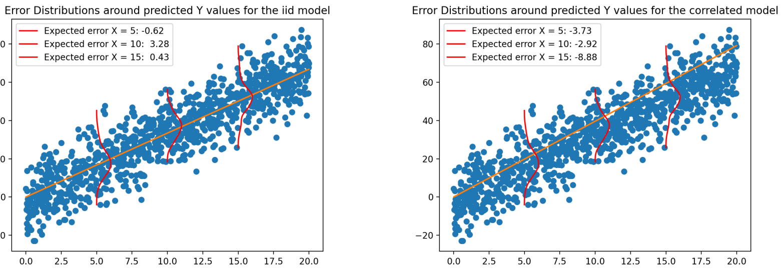


Figure 5: Error Distribution for the two models on a random sample of 1000 observations

$$E(Y_i | X_i = x)$$

For fixed values of X, the predictions are spread in a pattern enforced by the various ways we can realise the linear function with estimates for β and *const*.The consequent point predictions for Y are always approximate, skewed by the sample data as much as by poor choices in model design. So too any measures of expected value based on these models.

Small Worlds and Statistical Inference

Prediction is a visceral need. It precedes probability in any order of analysis. Without some regularity between X, Y we can only interpret their collisions as timorous noise. Only when there is a better than arbitrary correlation between X, Y will we think to ascribe a measurable probability to their association. So whether we view probabilities as a measure of credibility or frequency, the focus is always on a process which in reality reveals a pattern under repetition. We're loath to apply a probability model of any kind without some base inductive evidence. Models then are deliberate simplifications of complexity, lego-like versions of the world in which we bash parts together to see what reliably sticks. Forget about the pieces we don't own, count the pairs of blocks that match, wedge, smash, click-into-place or break;

draw out the ratio of success and the spread of outcomes. The expected revenue depends on both this uncertainty of output and the finer points of statistical inference.

Frequentism: Inference from Expected Frequency

Count the number heads in a series of 5 successive coin flips, then repeat the process 1000 times and you'll arrive a proportion which characterises that process. If it's a fair coin the long run expected result will be half the number of your coin flips. If the coin is weighted you might have as few as 0. This is the binomial distribution, and it really shines when you're trying to gauge fairness. If a process is biased, the distribution will be skewed. We can use this fact for inference. Consider a dispute over whether the game was rigged.

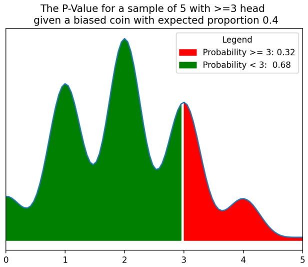


Figure 6: The Binomial Distribution

$$H_0: \text{ true proportion of heads } = 0.4$$

$$H_1: \text{ true proportion of heads } = 0.5$$

Take $(H0)$ as given then if we observe a sequence:

$$(3in5): H, H, H, T, T$$

what does it say about the possibility that we're being hustled? If the coin is biased, then the count of heads in repeated sampling will reflect a clear bias. For any new data we can check if the data is consistent with the data generated by the biased coin. The pattern of reasoning is straightforward (i) make some assumptions about the structure of the random process under investigation, (ii) tease out the consequences of these assumptions (iii) evaluate the incoming data against these consequences to see if you need to revise your assumptions. The frequentist asks, does the data looks weird given the assumed shape of our probability distribution?

In this instance the shape of the binomial distribution defined by a 0.4 biased coin allows for significantly greater than 5% chance for observing the above sequence. So we do not have enough reason to reject $(H0)$ at the traditional threshold. By design the assumed distribution builds in characteristics of long-run variance of the process, and the slim threshold for rejection is designed to minimise incorrect rejections of $(H0)$. With a low number of observations the sample distribution is unlikely to be properly centred around a stable point. This makes even small p-value thresholds unreliable. Elections are an example of repeated process which (arguably) offer a repeated choice, but political dynamics are such to undermine the statistical properties of an apparent binary choice. One voter is not easily exchangeable with another, a sample poll in Texas is not the same as one in New York but might be similar to Florida. We cannot blindly take a sample poll to imply the spread or volatility of a population, and with low samples it's hard to justify any kind of inference from expectation.

Bayesian Inference: Inference to Expectated Value

If instead we use probability to calibrate our beliefs, then we can be more explicit in our assessment of $(H0), (H1)$. Let's assume that our prior beliefs about whether the game is rigged is 50/50. Then we evaluate the two hypothesis using Bayes's rule for incorporating our prior belief and the data. The Bayesian asks whether our hypothesis is a good explanation of the data compared to alternatives. How, upon observing the data, should we view our hypothesis?

$$\overset{posterior}{p(H_i|Data)} = \frac{\overset{prior}{p(H_i)}\overset{liklihood}{p(Data|H_i)}}{\underset{evidence}{\sum_{i=1}^{i=K} p(Data|H_i)p(H_i)}}$$

where $1 \leq i \leq K$ spans the ways in which the data could have been realised across all competing hypotheses. Then, in our toy example, we have:

$$\frac{p(H_1|3in5)}{p(H_0|3in5)} = \frac{\frac{.5 \cdot .23}{.5 \cdot .32 + .5 \cdot .23}}{\frac{.5 \cdot .32}{.5 \cdot .32 + .5 \cdot .23}} = \frac{.57}{.42}$$

which would lead us to infer that the coin was fair.[3] The really radical move in the Bayesian setting is that you're allowed to ascribe a probability to any event regardless of whether there is any long-run sequence to observe. You may know nothing about your opponent or the coin, but for Bayesians this is no bar to assigning suspicion in the form of expected probability ,so long as you act in accordance with the axioms of probability. It's this freedom which can seem arbitrary and unmotivated, but in practice probabilities are rarely ascribed without reason.

Big Data and Expected Returns

Neither the Bayesian or Frequentist analysis ends with these simple calculations, both would continue to probe the limits of each hypothesis. We'd have to consider things like sample size and sensitivity testing, model performance and cost of errors, appropriateness of the priors. The point is just that there are reasons for dispute. This example shows the heart of the conflict in the dual aspect of probability. There is enough latitude in the manner in which we set up a probability model that the mathematics can

[3] In practice, the denominator of the Bayesian formula is usually an integral over a continuous space of possible realisations of our hypothetical rate. Historically this presented computational difficulties for Bayesian inference. Today these integrals are solved by Monte-Carlo style simulations.

yield apparently inconsistent results. The frequentist evaluation of our biased coin is very sensitive to the choice of hypotheses, while the Bayesian approach is influenced by the choice of prior. Why set up a significance test against assumed cheating rather than assumed fairness? Why attribute equal weight to both hypotheses? Why use a 5% threshold if you're concerned about systematic cheating? Both offer strategies for managing uncertainty, but both approaches come with baggage, that not even Big data can solve.

What is Big Data? There is some hype over the term, but it's not magic, it's just bigger sample sizes for specific questions. So far, mostly questions of consumer preference. Websites and apps collect traffic and log interactions. Your details are captured and pulled into vast aggregates of consumer data. I can route and re-route your trajectory across an online environment. Applying the same pressures to tens of thousands of others, we can trace out how the topology of particular sites throw up speed bumps on the customer's journey. Imagine we're running a website which aims to funnel customers through to a number of different purchase plans. The historic patterns are relatively stable with only 10% of customers dropping out of our conversion funnel on a daily basis. We can sample actions online (Figure 7) under differing pressures with a view to evaluating expected values of repeated coercive prompts.

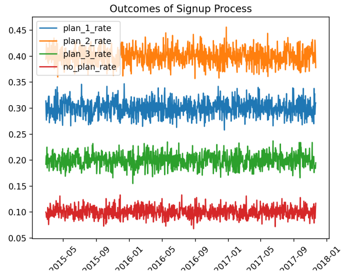


Figure 7: Stable long run Sign Ups

Assume the particular values for each plan then the expected value of customer journey is just: $p_1\$(o_1) + p_2\$(o_2) + p_3\$(o_3) + p_4\$(o_4) = .3 * \$10 + .4 * \$7 + .2 * \$12 + .1 * \$0 = \$8.20$. Now imagine there was a change to the website and we observe the following pattern (Figure 8) for the next 20 days:

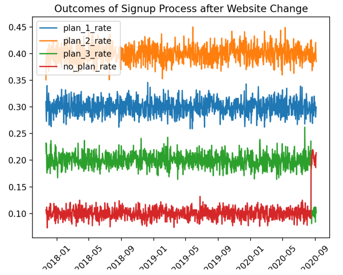


Figure 8: Abrupt increase in dropouts

What is the new expected value? From the frequentist point of view the macro distributional properties haven't significantly changed. But given what we know about the change to the website it would be foolish to assume such a static distributional assumption. Looking only at the small sample of new data, the variance will be large and the estimates of rates of sign-up for each plan will be unstable. Following the Bayesian paradigm we can condition our expectations on the new data, the old data or all the data we get slightly different results. The below graph illustrates the spread in values expected revenue calculated on different slices of our data using Bayes Rule. Using a large number of observations, the influence of our priors are minimal and washed out by the data, giving us a strong point estimate with low variance stable around 8.2, but since the recent data involves a step change, we might be better off ignoring the old data.

But we can also see that if we condition our expectations only on the new data with different priors drawn from the past data or hope, we can positively bias our expectations.

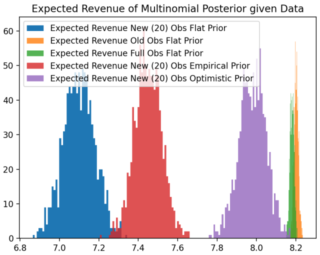


Figure 9: Expected revenue differs by choice of prior and data

Nearly all substantial decisions are made with small samples in circumstances where past behaviour is not a guide. Past behavioural patterns are exactly what we're trying to avoid or change. If you want to know whether the change on your website will drive a material change in financial revenue, you won't have long run patterns to rely on and it's an open question on how to weight the new data. All models smuggle in a host of statistical assumptions and these can be range from reasonable to absurd. Even when reasonable they're only supported by large sample sizes, and most questions of interest are driven by novelty that short circuits appeal to robust patterns of history. Reasoning from small samples is common, best done with caution and plenty of caveats, but better reason than not. Expectations should be modified accordingly.

The Stakes: From Utility to Indifference

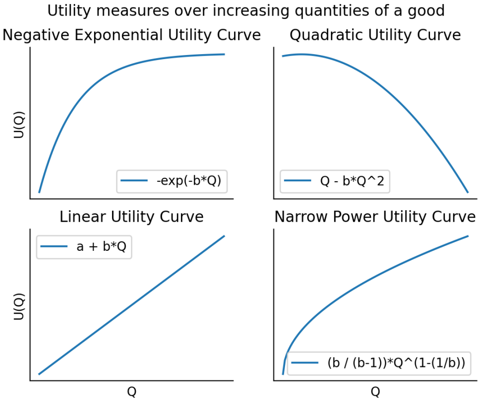


Figure 10: Consumer attitudes with differently satisfied appetites for a good

Too much of a good thing often tends to the bad. So we dabble, sample and share. In pursuit of variety we swap our goods, shunning stale options in favour of the novel exchange. For a given good we can differ in our appetites but it's relatively straightforward to find the point where one more donut is one too many. While it can be a bit unclear how we should measure utility, once we've decided on a metric the mathematical characteristics are meaningful. We can infer aspects of your attitudes towards acquisition and enthusiasm for donuts. In most cases we're interested not just in your pursuit of pastries, but how you'd be willing to trade for those pastries.

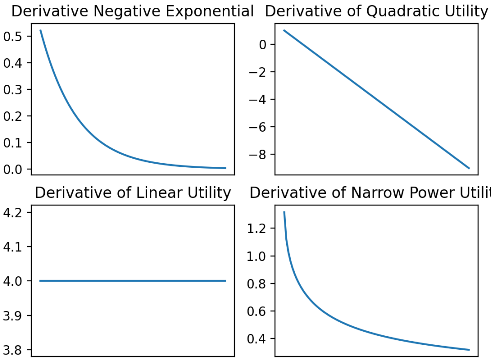


Figure 11: The Rates of Change of personal Utility

The possibility of coordinated compromise lies at the core of maximising subjective utility. We seek competitive advantage for our own produce to balance the cost owed to the skills of others. At the limit some trades do not admit any admixture of goods. Not all babies can be cut in half. In most cases though a consumer will try to optimise their bundle of goods over an entire marketplace, preserving enough on one key good; money, to remain liquid.

$$u(\mathbf{g}) = f(g_0, g_1 ... g_n)$$

There are number of ways we can specify a utility function, but a typical example is the Cobb-Douglas function.

$$u(\mathbf{g}) = g_0^{\alpha_0} g_1^{\alpha_1} ... g_n^{\alpha_n}$$

Then taking the case of two goods $g1, g2$ we can in this particular case determine an indifference curve where you would be willing to exchange quantities of $g1$ for an agreeable amount of $g2$. The task it to express the value of a given good as priced in terms of the other goods. Set

$$u(\mathbf{g}) = \lambda = g_1^{\frac{1}{2}} g_2^{\frac{1}{2}} = (g_1 g_2)^{\frac{1}{2}} = \sqrt{g_1 g_2}$$

$$\Rightarrow \lambda^2 = g_1 g_2 \Rightarrow \frac{\lambda^2}{g_2} = g_1$$

Using this formula we can express how the quantities of fair exchange vary based on a fixed utility value. This is not to say that these curves represent an actual or objective fair price, just that when measured in terms of utility these are mappings of quantities of good we would be happy to exchange. Your view of a fair price is encoded in your utility theory. It's at this point when utility theory can be said to verge on empirical science.If we can model your preferences as a utility function characteristic of some general attitude toward acquisition, we might also hope to able to predict future trades.

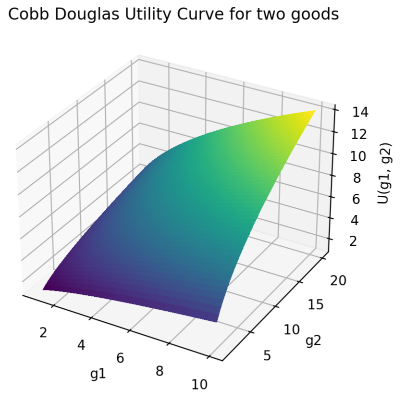


Figure 12: A consumers utility curve for combinations of two goods

Optimising Utility

A further complication arises when we try to factor for a consumer's budget. The shape of the Cobb-Douglas function shows that the utility surface is constantly increasing with our rate acquisition. So without any constraints the consumer would not achieve satisfaction, but continue like glutton. But add a budget constraint and we need to find the maximum point at which an indifference curve insects with our budgetary line. Instead of solving the equation:

$$\text{Find } g_1, g_2 \text{ such that } u(g_1, g_2) = \lambda$$

we need to solve a constrained optimisation problem:

$$\text{maximise } u(g_1, g_2) \text{ subject to } cost(g_1, g_2) = \lambda$$

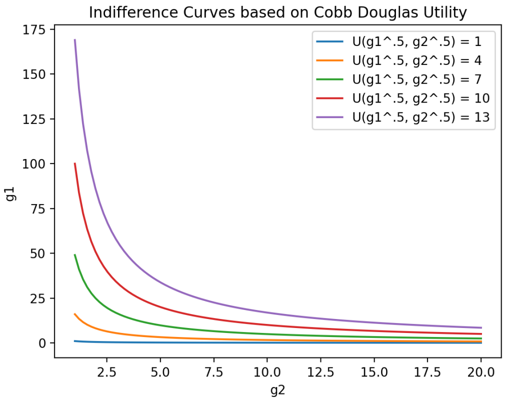


Figure 13: A range of indifference curves without budget constraints.

This style of problem can be approached with the method of Lagrange multipliers. If we let:

$$L = g_1^{\frac{1}{2}} g_2^{\frac{1}{2}} - \lambda(2g_1 + 3g_2 - 40)$$

where 2 and 3 are the unit cost of the respective goods, and 40 is our total budget. While λ is our Lagrangian multiplier. This term will be used to re-express the algebra of our equation as a function of the consumer's capacity to spend. We can discover where utility is maximised when the gradient of the curve can be set to zero. This is the theory behind the "hill climbing" algorithms of gradient descent.

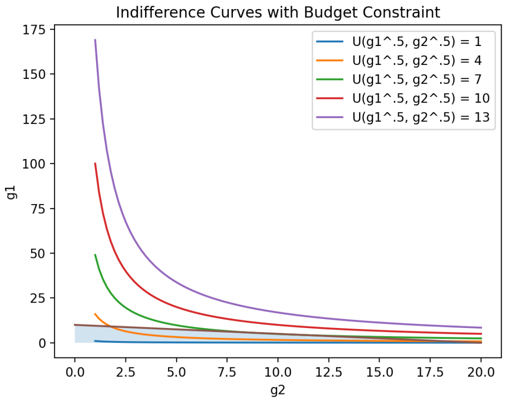


Figure 14: A range of indifference curves with budget constraints.

When the curvature of the "slope" has plateaued i.e. is zero, then we've reached a maximum or minimum in the multivariate space of the function. As before we want to use this fact to express the implicit function of g_1 in terms of g_2, but this time including the constraints.

Box .1: Lagrangian Multiplier

$$\nabla L = dL/d\mathbf{g} = \left(\frac{\partial u(\mathbf{g})}{\partial g_1} \quad , \frac{\partial u(\mathbf{g})}{\partial g_2} \right) = \left(\tfrac{1}{2} g_1^{-\frac{1}{2}} g_2^{\frac{1}{2}} - 2\lambda \quad , \tfrac{1}{2} g_2^{-\frac{1}{2}} g_1^{\frac{1}{2}} - 3\lambda \right) = \mathbf{0}$$

$$\Rightarrow \lambda = \frac{1}{4} g_1^{-\frac{1}{2}} \sqrt{g_2} = \frac{4}{25} g_2^{-\frac{1}{2}} \sqrt{g_1}$$

$$\Rightarrow (\frac{1}{4})^2 \frac{1}{g_1} g_2 = (\frac{4}{25})^2 \frac{1}{g_2} g_1 \Rightarrow (\frac{1}{4})^2 g_2 = (\frac{4}{25})^2 \frac{1}{g_2} g_1^2 \Rightarrow (\frac{1}{4})^2 g_2^2 = (\frac{4}{25})^2 g_1^2$$

$$\Rightarrow g_2 = \frac{16}{25} g_1$$

The same pattern holds for cases with more than two goods. We can express the value of given good g_n in terms of a function $f(g_1, ...g_{n-1})$. Then substituting this value into our constraint we get:

$$2g_1 + 3(\frac{16}{25})g_1 = 40 \Rightarrow 2g_1 + 1.92g_1 = 40 \Rightarrow 3.92g_1 = 40$$

Proving the optimial settings are $g_1^* = 10.20$ and $g_2^* = 6.52$ and $\lambda^* = 0.20$

This proof shows how we triangulate a consumer's view of any good as expressed through the medium of their utility function. But the method of Lagrangian multipliers is more than a mere algebraic trick. We can interpret the λ term as the rate of change of the consumer's utility as a function of the cost. The proof is a little more involved, but the significance of this interpretation should be obvious. If we knew our consumer's adhered to a particular style of utility function we could model how price-changes would impact their returns to utility and select for maximum profit.

Two Theorems: From Indifference to Utility

If we can elicit preference statements from our consumer we construct a utility curve. First observe the preferences expressed by consumer decisions and then map the maximal and least preferred options to convenient polarities. For instance:

$$g_1 \succ g_2 \succ g_3 \succ g_4 \succ g_5$$

where:

$$u(g_1) = 0 \text{ and } u(g_5) = 1$$

then each of the intermediary options can be measured in the interval between 0 and 1. However there are an infinite number of simple ordinal mappings that would work, and a strict ordering does nothing to convey the degree of feeling associated with each option. But we can calibrate utility scales based on decisions made about offered bets. Each individual good can be assessed against a simple win-loss lottery between the two most extreme outcomes. If the consumer is indifferent between the sure prospect of the good and a fixed odds lottery on their most preferred outcome, they've implicitly weighed their utility of the good.

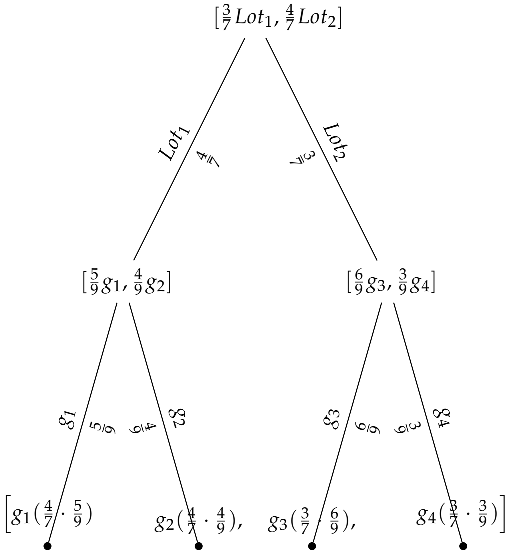
$$\forall i \exists p : g_i \sim [p \cdot g_1, (1-p) \cdot g_5] \rightarrow u(g_i) = p$$

So whenever we are indifferent between a sure thing and a win-loss lottery over the best and worst outcomes we have implicitly chosen the utility of the of good on a 0-1 scale. In this manner we can construct a utility curve across the entire range of options.

Representation: Decision Under Risk

The most famous result in decision theory is von Neumann and Morgenstern's Representation theorem. It shows using the technique discussed above how expressed preferences (which adhere to certain axioms of rationality) can track with a utility measure. As such they can be interpreted as an agent's attempt to maximise their expected utility. But the theorem is limited to decisions over well-defined lotteries, and as such makes a poor model for general choice under uncertainty where the odds are approximate.

Figure 15: Compound lotteries as probability trees


$[\frac{3}{7}Lot_1, \frac{4}{7}Lot_2]$
Lot_1
$\frac{4}{7}$
$\frac{3}{7}$
Lot_2
$[\frac{5}{9}g_1, \frac{4}{9}g_2]$
$[\frac{6}{9}g_3, \frac{3}{9}g_4]$
g_1
$\frac{5}{9}$
$\frac{4}{9}$
g_2
g_3
$\frac{6}{9}$
$\frac{3}{9}$
g_4
$\Big[g_1(\frac{4}{7} \cdot \frac{5}{9})$
$g_2(\frac{4}{7} \cdot \frac{4}{9}),$
$g_3(\frac{3}{7} \cdot \frac{6}{9}),$
$g_4(\frac{3}{7} \cdot \frac{3}{9})\Big]$

Theorem 1 (vNM's Representation Theorem) *If an individual i's preference relation $\succeq$ is transitive, complete and satisfies:*

1. *(Continuity): $\forall g_1, g_2, g_3 : (g_1 \succeq g_2 \succeq g_3) \rightarrow \exists v \in [0,1] \land g_2 \sim_i [vg_1, (1-v)g_3]_{Lot}$*
2. *(Monotonicity): If $v_1, v_2 \in [0,1]$ and $g_1 \succ g_2$ then $\Big([v_1 g_1, (1-v_1)g_2]_{Lot} \succeq [v_2 g_1, (1-v_2)g_2]_{Lot}\Big) \Leftrightarrow v_1 \geq v_2$*
3. *(Reduction of Compound lotteries): Each compound lottery $[q_1 Lot_{p_1}, ..., q_n Lot_{p_n}]$ reduces to a simple lottery where each good $(1, ..k)$ is weighted across all branches of the nested decision tree $[(q_1 p_1^k + q_2 p_2^k ... + q_n p_n^k)g_k ...., (q_1 p_1^{k-1}, ...)g_{k-1} + ...(q_1 p_1^1, ...)g_1]_{Lot}$ by the usual rules of probability for branching compound events*

such that $\widehat{Lot} \sim Lot$

4. *(Independence) If* $\widehat{Lot} = [q_1 Lot_1, ..., q_j Lot_j ... q_n Lot_n]$ *and* $L_j \sim M$, *then* $\widehat{Lot} \sim \widehat{Lot}' = [q_1 Lot_1, ..., q_j M ... q_n Lot_n]$

then $\exists u_p : [g_1, ...g_n] \mapsto Val$ *where* $u_p(Lot) = p_1 u(g_1) + ... + p_k u(g_k) = E(u_p(Lot))$ *and* $u(g_1) \geq u(g_2) \Leftrightarrow g_1 \succeq g_2$ *so that u represents* $\succeq$ *unique up to a positive linear transformation.*

[4]

For a well defined and fixed probability function p over the goods $g_1.....g_n$ the above axioms of rationality are sufficient to define a sensible utility function based on an agent's expressed preferences. The thought gives hope to the idea that you would be able to predict an individual's actions in any environment where you knew both their preferences and the objective probabilities at play. This is the basic model for understanding poker play - the probabilities are generally known and it just remains to determine the game theoretical dynamics.

Representation: Decision Under Uncertainty

There is an altogether different view of what we're doing when we pursue expected utility. Again, we may try to elicit an agent's utility function from their preferences, but in addition we can set up the axioms of rationality so as to derive a probability function based on the expressed desires. This is a radically Bayesian in approach to what's going when we act to maximise expected utility.

From Neutrality to Desire

One of the issues with eliciting a utility curve with appeals to bets over lotteries stems from the stigma associated with gambling. An alternative approach, more in the spirit of Bayesian philosophy is to try to elicit the desirability of a prospect by situating it between two polarities and repeatedly seeking a third prospect, midpoint between the two, which is half as desirable by construction. The method, originally proposed by Frand Ramsey relies on the idea that we express preferences over a boolean algebra of propositions and we can gauge utility by appeal to an "Ethically Neutral" proposition *Neutral* - one which if it exists is such that for all other prospects α we're utterly indifferent between:

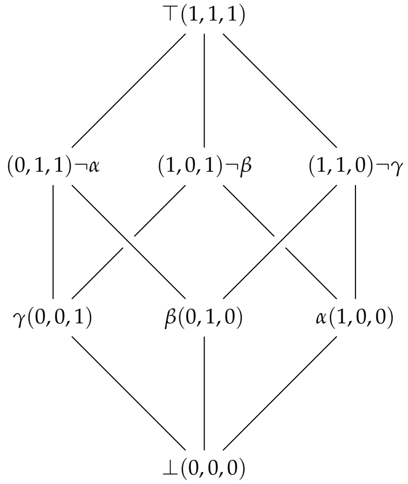
$$(Neutral \wedge \alpha) \sim \alpha \sim (\neg Neutral \wedge \alpha)$$

. The idea is that we can gauge desire by offers of repeatedly refined contracts based on an ethically neutral proposition. This sequence

[4] We follow the example in:

of offers can be used to construct a utility curve. This is Bayesian in spirit because it allows for the expression of a probability measure for any proposition even if there is no known probability distribution over the considered outcomes.

Figure 16: Boolean Algebra of Propositions



First observe that

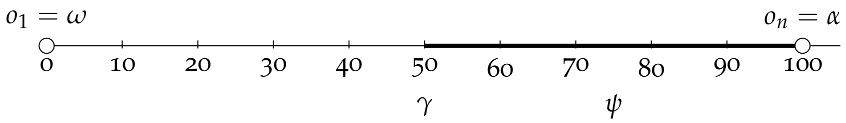
$$\begin{aligned}
&[\alpha \text{ if } Neutral, \omega \text{ if } \neg Neutral]_{contract_1}\\
&\sim [\omega \text{ if } Neutral, \alpha \text{ if } \neg Neutral]_{contract_2}\\
&\Rightarrow u(contract_1) = u(contract_2)\\
&\Rightarrow EU(contract_1)\\
&= u(\alpha)p(Neutral) + u(\omega)(1 - p(Neutral))\\
&= u(\omega)p(Neutral) + u(\alpha)(1 - p(Neutral))\\
&= EU(contract_2)\\
&\Leftrightarrow p(Neutral) = 0.5
\end{aligned} \tag{1}$$

Then we can take any two extremes $u(\alpha) = 1, u(\omega) = 0$ and we can use our test for indifference to situate any third proposition γ on a desirability scale since:

$$\begin{aligned}
&[\gamma \text{ if } Neutral, \gamma \text{ if } \neg Neutral]_{contract_1}\\
&\sim [\alpha \text{ if } Neutral, \omega \text{ if } \neg Neutral]_{contract_2}\\
&\Leftrightarrow EU(contract_2) = u(alpha)\frac{1}{2} + u(omega)\frac{1}{2} = .5\\
&= u(\gamma) = EU(contract_1)
\end{aligned} \tag{2}$$

Repeating this step we can find a contract on a sure-thing ψ for which we're indifferent between:

$$\begin{aligned}
&[\psi \text{ if } Neutral, \psi \text{ if } \neg Neutral]_{contract_1}\\
&\sim [\alpha \text{ if } Neutral, \gamma \text{ if } \neg Neutral]_{contract_2}\\
&\Rightarrow u(\psi) = .75\\
&...etc
\end{aligned} \tag{3}$$



Repeating this process indefinitely we can refine our utility scale as exactly as we please, by repeatedly finding prospects with a utility precisely on the mid-point the between two poles. If these measures adhere to certain basic constraints of rationality regarding consistency of utility we can show how a Bayesian agent can be seen to maximise their expected utility when making decisions under uncertainty. But unlike the Von Neumann representation theorem for a Bayesian perspective the probability function over prospects is not unique, we can have multiple pairs $\langle p, u \rangle$ which are representative of an individual's preference ordering $\succeq$ without converging on the particular probabilities ascribed by one individual. This is precisely the content of the following theorem.

Theorem 2 (Bolker Representation Theorem) *Let $\mathbb{B} = \langle \Omega, \succeq \rangle$ be Bolker structure if Ω is an atomless Boolean algebra and $\models$ forms an implication relation over Ω, while $\succeq$ is complete, transitive, continuous over $\Omega \setminus \bot$ and the following hold:*

1. *(Impartiality) Suppose $\alpha \sim \beta$ and $\exists\gamma(\neg(\gamma \sim \alpha))$ such that $\alpha \wedge \gamma = \bot = \beta \wedge \gamma$ and $\alpha \vee \gamma \sim \beta \vee \gamma$ Then $\forall\gamma(\alpha \vee \gamma \sim \beta \vee \gamma)$*
2. *(Averaging) If $\alpha \wedge \beta = \bot$ then $\alpha \succeq \beta \Leftrightarrow \alpha \succeq \alpha \vee \beta \succeq \beta$*

Then there is a probability measure and utility (desirability) metric $\langle p, u \rangle$ on Ω such that if the following axioms hold:

- *(A0) $p(\top) = 1$*
- *(A1) $p(\alpha) \geq 0$*
- *(A2) $\alpha \wedge \beta = \bot \rightarrow p(\alpha \vee \beta) = p(\alpha) + p(\beta)$*
- *(A3) $u(\top) = 0$*
- *(A4) $\alpha \wedge \beta = \bot \wedge p(\alpha \vee \beta) \neq 0$ implies* $u(\alpha \vee \beta) = \dfrac{u(\alpha)p(\alpha) + u(\beta)p(\beta)}{p(\alpha \vee \beta)}$

it follows that

$$u(\alpha) \geq u(\beta) \Leftrightarrow \alpha \succeq \beta$$

and there is another such set of functions $\langle p^*, u^* \rangle$ if and only if u^* is a fractional linear transformation of u i.e. $\exists a > 0$ and $\exists c, cu(\alpha) > -1$

$$p^*(\alpha) = p(\alpha) \cdot (cu(\alpha) + 1)$$

$$u^*(\alpha) = \frac{au(\alpha)}{cu(\alpha) + 1}$$

The mathematical machinery used to prove this result is a little more involved, ranging over every possible boolean combination of beliefs measured on three axes: preference, probability and desirability. In addition to the usual probability axioms, (A3) and (A4) tie subjective probability and subjective utility together. The axiom (A3) works to normalise the utility scale so that no sure prospect has any positive utility. This, in a sense, enshrines the requirement that there is only a utility to novel information. While (A4) ensures that the utility of any disjunction is the weighted average of the ways in which it can occur. More importantly it implies:

$$
\begin{aligned}
u(\alpha \vee \neg\alpha) = u(\top) &= p(\alpha)u(\alpha) + p(\neg\alpha)u(\neg\alpha) \\
&= p(\alpha)u(\alpha) + u(\neg\alpha) - p(\alpha)u(\neg\alpha) \\
\Rightarrow u(\top) - u(\neg\alpha) &= p(\alpha)u(\alpha) - p(\alpha)u(\neg\alpha) \\
\Rightarrow p(\alpha) &= \frac{u(\top) - u(\neg\alpha)}{u(\alpha) - u(\neg\alpha)} \text{ if } u(\alpha) \neq u(\neg\alpha)
\end{aligned}
$$

Which confirms how the relationship between probability of a given proposition can be expressed in terms of the desirability or utility of the same proposition and it's negation. This is a view of probability profoundly different from measure of risk we ascribe to poker players. It is not a fixed unique distribution determined by observation across repeated sampling, and consequently much harder to model.

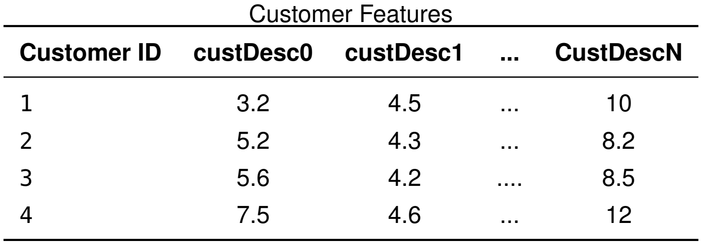
Prospects for a Predictive consumer Utility model

Having arrayed all the ingredients of a consumer utility model and the traditional theorems used to motivate the idea, a few things might cross your mind: (1) This notion of subjective utility would be useful, but seems incredibly hard to pin-down. (2) how for any given individual can we reasonably ascribe either a motivating utility curve or probability measure? (3) Even if we could elicit their preferences, there is no guarantee that the preferences are stable or entirely consistent. (4) Nor is it clear how we could at any kind of scale garner to those expressed preferences. (5) Surmounting all these obstacles, we still have no way to assume they're going to pursue their maximum expected value! In other words, this theory is just that, and the practical constraints make the entire project infeasible. This is all true as far as it goes, but the influence of the above theorems and the utility model of rational choice lies in the perspective it offers, not the recipe it prescribes.

Although there have been some brave attempts to derive customer utility metrics from survey data, the typical approach to modelling customers ignores the subtleties of directly estimating a utility function in favour of a look-alike predictive approach; customer segmentation and product recommendations based on the actions of similar customers in the past.

Segmentation and Purity

There is a wide plurality of segmentation methods which can be applied to the task of classifying both customers and products. These classification schemas are vital inputs for any recommendation algorithm. They cluster individuals based on a wide array of features, which is to say that they simplify the question of expected action? Instead of asking how might Rebecca, (aged between 18-25, from Spain, with a history of frugal purchases) react to a new sales promotion, you can ask about the conversion rate of the young female demographic. Depending on the task and the nature of the clustering algorithm, you might end up bucketing Rebecca with Sven (overweight male, history of lavish spending from Sweden), if for example their historic email open rate were similar. The responsibility of vetting each clustering schema lies with the user of the algorithm, but usually knowledge of the problem is enough to put some kind of context on the structural patterns unearthed by the algorithm.



Customer Features

Customer ID	custDesc0	custDesc1	...	CustDescN
1	3.2	4.5	...	10
2	5.2	4.3	...	8.2
3	5.6	4.2	....	8.5
4	7.5	4.6	...	12

Table 1: *

One of the first steps is to try to parse out our data into the most relevant descriptive features, but in lieu of domain knowledge we can apply apply some data compression techniques such as principle components analysis:

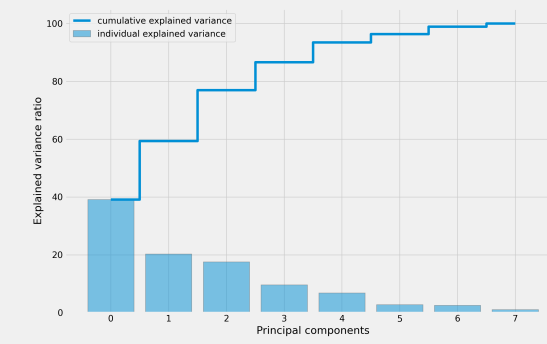


Figure 17: Principle Component Analysis of Customer Features