

Inference and Expected Value

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Abstract

The article examines the algorithm of expected value. Focusing first on how poor model design and the controversies about statistical inference can shape expected value. Concluding with an argument for the inevitability of Bayesian methods.

Expectations: The Workhorse

There is an algorithm beloved by bureaucrats. The unsung hero of administrators and accountants. An algorithm both ubiquitous and under appreciated. It's pivotal for nearly every project and informs the actions of tech giants and policy makers the world over. It is only mildly hyperbolic to say that understanding this formula unlocks wealth and power.

It lies at the heart of online A/B testing, all policy analysis, sound business strategy and poker play.

$$EV(O)_p = p_1\$(o_1) + p_2\$(o_2) + \dots + p_k\$(o_k)$$

The expected financial value of a random process is just the sum of the dollar outcomes weighted by their probabilities. Outcomes can vary from deals of cards, to customer transactions and election results.¹ Aside from the mercenary possibilities, Pascal can argue sincerely that such considerations compel belief in God. But the formula, glossed as a rule for rational action, merits your attention for more mundane wagers too. The meaning of probability is not an idle concern if you intend to maximise your expected value. While statistics are often tortured to rubber stamp decisions and probabilities are abused to fit policy prescriptions with false precision, the crisp clarity of the rule has an enduring appeal that promises to sift the murky swamps of Big Data. It's a scalpel that anyone can wield to parse the syntax of statistical jargon and carve answers from an abstract space of probabilities.

Two sides of a Coin

Probability has a dual aspect. On one reading it refers to the long run tendency of a random process, on another the probability is construed as the degree of belief in an outcome. On the first (frequentist) interpretation a probability distribution has certain fixed theoretical characteristics, as in a uniform probability

¹In the jargon O is a random variable which can be realised as any of the outcomes $o_j : 1 \leq j \leq k$

distribution of a fair coin where all outcomes are equally likely. On the second (Bayesian) reading the characteristics of the probability distribution are learned from the data. These approaches are united by the Law of Large numbers which states that as the size of our sample increases our sample average will converge to the expected realisation of the theoretical process.

$$\frac{1}{N} \sum_{i=1}^N O_i \text{ converges to } E(O) \text{ as } N \text{ approaches } \infty$$

In the below plot we have fixed a Poisson distribution with a mean of 4.5 and can see three examples of how consecutive averaging from the increasing sample sizes results in a closer and closer convergence to the (true) population mean.

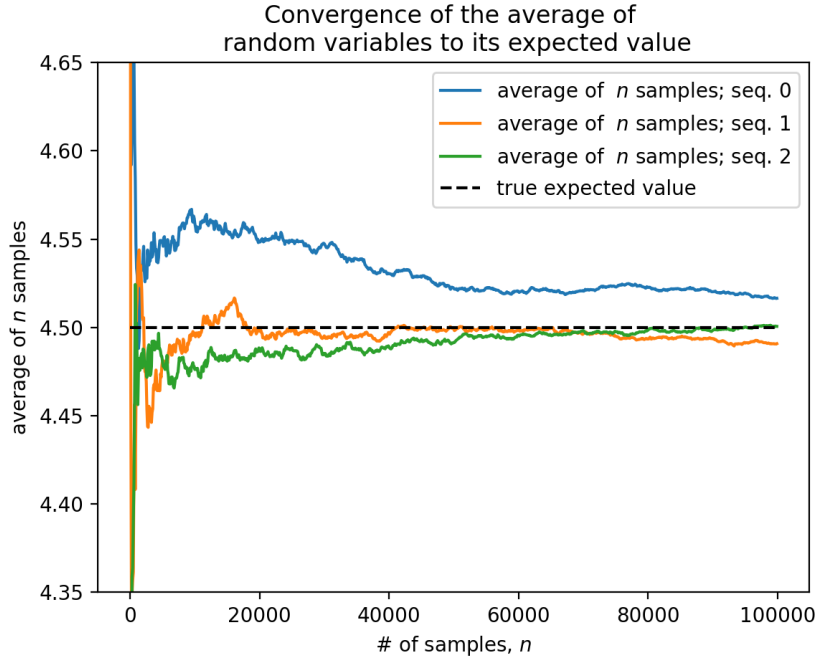


Figure 1: Convergence with large samples

This is fundamental to the interpretation of probability. Given a game with fixed and fair odds we see that repeated play will converge because of characteristics which govern the process. Dice are the paradigm example. In the wild we never know the characteristics which cause the observed spread of outcomes, but such is the influence of gambling on probability, that we assume there is a stable pattern to be gamed. Partially this is pragmatic. The maths is more tractable if we can assume a well behaved underlying process. The results are compelling. The Normal (Bell Curve) distribution, the Poisson distribution the Bernoulli distribution (to name a few) are all rightly famous. Their shapes are characteristics of innumerable random processes. But the paradigm clouds the fact that in practice we start on the left side of the law of large numbers (with samples) and we often start with small numbers.

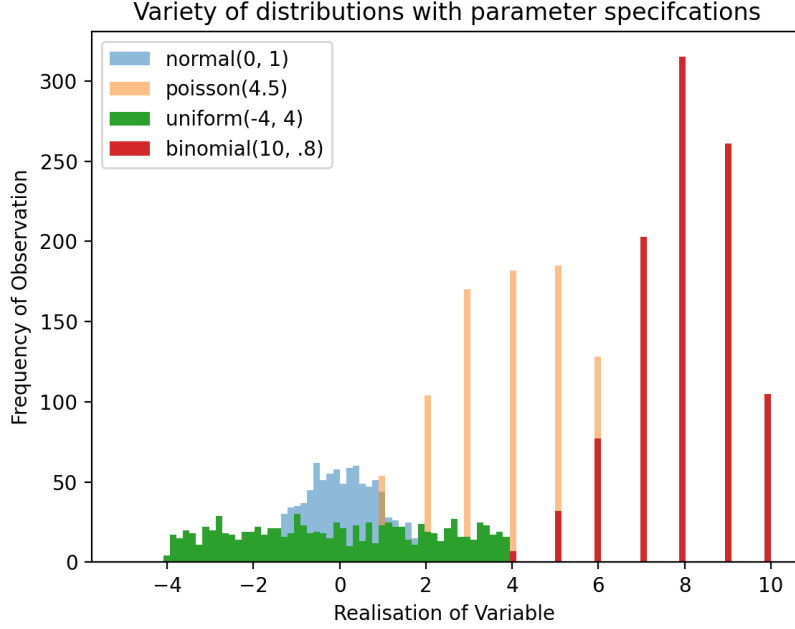


Figure 2: Some theoretical distributions with parameters

Well behaved probability distributions are rare beasts; a tiny fraction of the world's arbitrary menagerie. The fundamental question in probability is not whether probability is a measure of belief or frequency - it is whether we can safely assume that the underlying process adheres to a known model? If so, we can rely on the structure of the model's theoretical distribution to inform inference. If not we are better learning what we can from the sample - trusting to wide confidence intervals and worst scenario planning. The challenge is knowing which scenario we're in.

Models, Errors and Sampling

When your only tool is a hammer, then everything is a nail. When one model won't do, use two. This is roughly the approach adopted when models fail. The problem is especially vivid in forecasting; projected death rates, body counts and stock prices are all subject to sudden shocks. A basic regression model tries to predict an outcome Y as linear function of X :

$$Y \sim \text{const} + \beta X + \epsilon$$

where ϵ is a random variable representing the error (or noise). A modest notational device for disaster. While const, β are parameters estimated by an optimisation process to ensure the equation fits the data as neatly as possible. In the below graph we have a series characterised by change. After the first shock we can refit the model so that the line tracks well with the evolving data. After the second shock we try another refit, but the range the and variance of the data makes our basic model a poor fit i.e. the data no longer exhibits

a linear relationship. This presents three examples of error in the modelling process: (i) forecasts fail for the reason that it's also difficult to identify (in the moment) those changepoints in the data which reflect structural change, (ii) the linearity assumptions that go into the model are sound, but the parameters need be re-estimated based on the new data

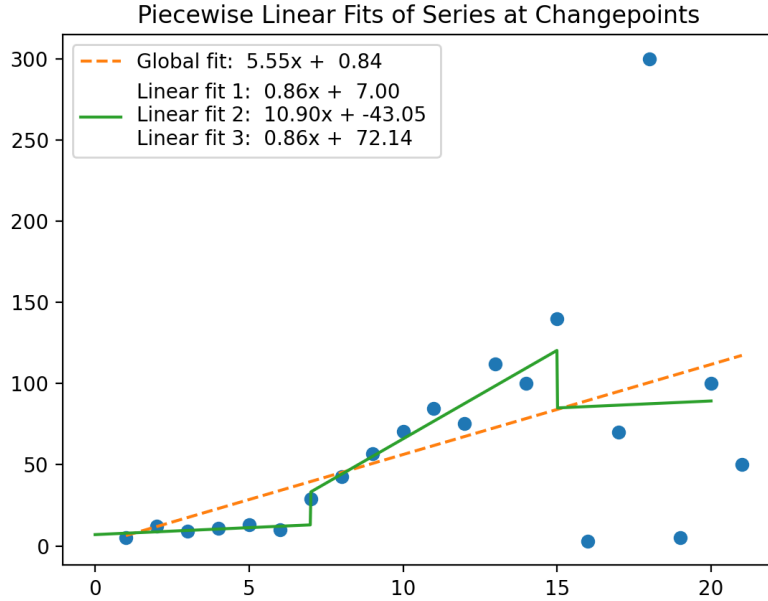


Figure 3: Three samples with starkly different parametrisations

and (iii) the third linear model is simply a terrible match for the pattern in the data. Every model is a guess as to the implicit order in apparent noise. Sometimes there is no order, and other other times the patterns is too subtle for a dumb model to capture. In practice you never really know whether a new error stems from a misfit but appropriate model or an entirely inappropriate model. But as we increase our number of sample fits we hope to better approximate the true linear process (if any) generating the data.

This warrants a note about errors. Imagine now that the data points on our previous graph are repeatedly re-speckled over the canvas. We can refit a new model for each set of scattered data points and each refit gives us a new sample values for $const, \beta$. If the underlying data generating process is stable, then the parameter fits will converge to the correct values of $const, \beta$; correct in the sense that they can be used to draw the line of best fit for the data. A statistically stable process is one that can be modelled with errors ϵ normally distributed around 0, so that the model will be *correct on average*. Our predictions will overshoot in some cases but on the whole the errors up and down will cancel each other out. Forecasting with the parameters of best fit minimises our forecast errors because the fluctuations are stable about the centre of the line. These are assumptions required for a process to exhibit the tendency of

regression towards the mean. If they're not met, we will see poor parameter estimates and wild swings away from the linear path.

The code below builds two sampling distributions based on different underlying processes. One in which the errors are independent, normally distributed around 0 the other in which the errors are correlated in a sine-wave like pattern, increasing and decreasing periodically. This is akin to difference between measuring error when predicting the heights of randomly sampled people versus predicting the sales volumes on randomly selected days of the week. A random sample of daily sales risks clumping weekends together and skewing the expected values. No such risk exists when sampling from independent individuals.

```
#### Build True Models
N = 100000
X = np.random.uniform(0, 20, N)
independent_err = random.normal(0, 10, N)
corr_err = random.uniform(0, 10) +
sin(np.linspace(0, 10*np.pi, N)) +
sin(np.linspace(0, 5*np.pi, N))*2 +
sin(np.linspace(1, 6*np.pi, N))*2

Y_corr = -2 + 3.5 * X + corr_err
Y = -2 + 3.5 * X + independent_err

population = pd.DataFrame({'X': X,
                           'Y': Y, 'Y_corr': Y_corr})

#### Sample from Data
#### and build smaller models

fits = DataFrame(columns=['iid_const',
                          'iid_beta', 'corr_const',
                          'corr_beta'])

for i in range(0, 10000):
    sample = population.sample(n=100,
                               replace=True)
    Y = sample['Y']; X = sample['X']
    Y_corr = sample['Y_corr']
    X = sm.add_constant(X)
    iid_model = sm.OLS(Y, X)
    results = iid_model.fit()
    corr_model = sm.OLS(Y_corr, X)
    results_2 = corr_model.fit()
    row = [results.params[0],
           results.params[1],
           results_2.params[0],
           results_2.params[1]]
    fits.loc[len(fits)] = row
```

```
fits.boxplot()
```

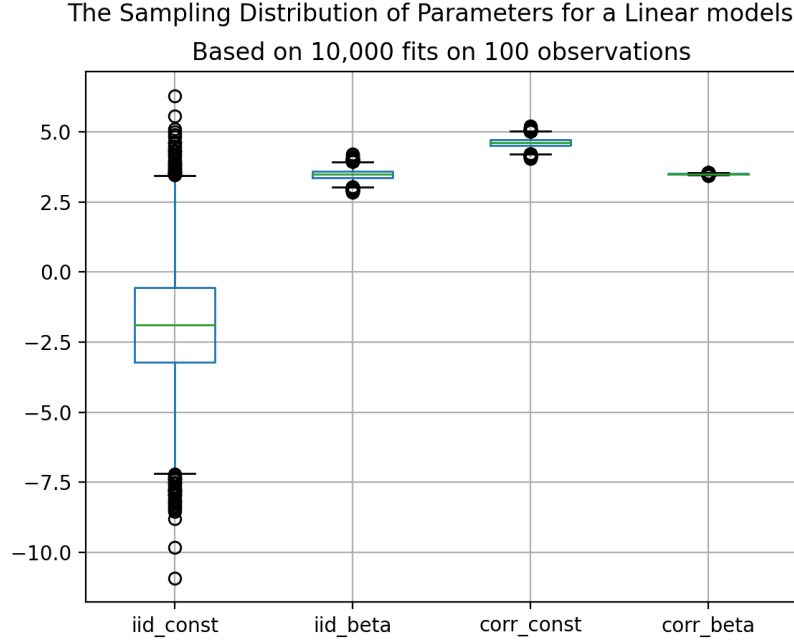


Figure 4: The expected realisations for $\beta, const$ with different errors structures

In the case with independent errors the expected value for our parameter estimates match almost exactly the true values of the process. In the second model with correlated errors the parameter estimate for our constant is 4.9 which is significantly different from the true value of -2, and will lead to systematically skewed predictions. Statistical models are just algebraic equations where we use regular sampling to solve for Y from X . Because Y is also a random variable the regression model encodes a conditional expectation result.

$$E(Y_i|X_i = x)$$

For fixed values of X , the predictions are spread in a pattern enforced by the various ways we can realise the linear function with estimates for β and $const$. The consequent point predictions for Y are always approximate, skewed by the sample data as much as by poor choices in model design. So too any measures of expected value based on these models.

Small Worlds, Inference and Decision

So when does it make sense to use models? In what sense should expected values guide our action? If models are mere heuristics, from where do we pluck the values for

$$p_i : 1 \leq i \leq k$$

in the formula (*EV*). Prediction is a visceral need, it precedes probability in any order of analysis. Without regularity we interpret can only interpret collisions as timorous noise. Only when there is a better than arbitrary correlation between X, Y will we even think to ascribe a measurable probability to their association. So whether we view probabilities as a measure of credibility or frequency, the focus is always on a process which in reality reveals a pattern under repetition. We're loath to apply a probability model of any kind without some base inductive evidence. Models then are deliberate simplifications of complexity, lego-like versions of the world in which we bash parts together to see what reliably sticks.

Innovations in machine learning mean that we can fit models with hundreds, thousands and even billions of parameters, but this increase in scale does not change fact that models are blind to all factors outside their samples. A good model predicts the variance in future data, in part, by ignoring the genuine complexity of reality. Prediction is the only practical test that matters. If the prediction accuracy holds over successive periods we should become increasingly sure that the probabilities implicit in our model are a good measure of objective risk. We can never be certain the model holds over all successive periods robust to changes in external factors. We can however quantify the degree of uncertainty in the model by observing the spread of the distribution. The expected revenue of a process depends on both this uncertainty and the finer points of statistical inference.

Inference from Expected Frequency

Count the number heads in a series of 5 successive coin flips, then repeat the process 1000 times and you'll arrive a proportion which characterises that process. If it's a fair coin the long run expected result will be half the number of your coin flips. If the coin is weighted you might have as few as 0. This is the binomial distribution, and it really shines when you're trying to gauge fairness. If a process is biased, the distribution will be skewed. We can use this fact for inference. Consider a dispute over whether the game was rigged.

$$H_0 : \text{true proportion of heads} = 0.4$$

$$H_1 : \text{true proportion of heads} = 0.5$$

Take (H_0) as given then if we observe a sequence:

$$(3in5) : H, H, H, T, T$$

, what does this say about the possibility that we're being hustled? If the coin is biased, then the count of heads in repeated sampling will reflect a clear bias. For any new data we can check if the data is consistent with the data generated by the biased coin. The pattern of reasoning is straightforward (i) make some assumptions about the structure of the random process under investigation, (ii) tease out the consequences of these assumptions (iii) evaluate the incoming data against these consequences to see if you need to revise your assumptions. The frequentist asks, does the data looks weird given the assumed shape of our probability distribution?

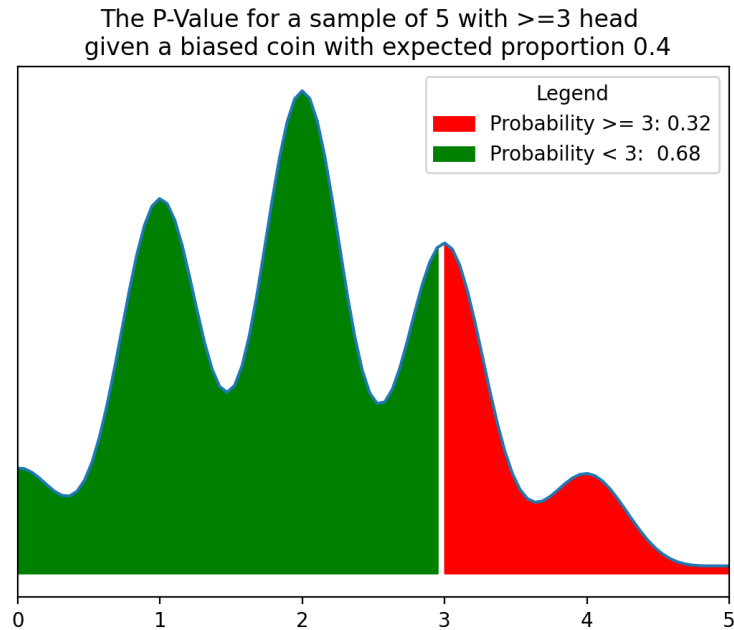


Figure 5: The Binomial Distribution

In this instance the shape of the binomial distribution defined by a 0.4 biased coin allows for significantly greater than 5% chance for observing the above sequence. So we do not have enough reason to reject (H_0) at the traditional threshold. By design the assumed distribution builds in characteristics of long-run variance of the process, and the slim threshold for rejection is designed to minimise errors. Without enough observations the sample distribution is unlikely to be properly centred around a stable point. This makes even small p-value thresholds unreliable. Elections are an example of repeated process which (arguably) offer a repeated choice, but political dynamics are such to undermine the statistical properties of an apparent binary choice. One voter is not easily exchangeable with another, a sample poll in Texas is not the same as one in New York but might be similar to Florida. We cannot blindly take a sample poll to imply the spread or volatility of a population, and with low samples it's hard to justify any kind of inference from expectation.

Inference from Prior Belief

If instead we use probability to calibrate our beliefs, then we can be more explicit in our assessment of $(H_0), (H_1)$. Let's assume that our prior beliefs about whether the game is rigged is 50/50. Then we evaluate the two hypothesis using Bayes's rule for incorporating our prior belief and the data. The Bayesian asks whether our hypothesis is a good explanation of the data compared to alternatives. How, upon observing the data, should we view our hypothesis?

$$p(H_i|Data) = \frac{\overset{prior}{p(H_i)}\overset{likelihood}{p(Data|H_i)}}{\sum_{i=1}^{i=K} \underset{evidence}{p(Data|H_i)p(H_i)}}$$

where $1 \leq i \leq K$ spans the ways in which the data could have been realised across all competing hypotheses. Then we have:

$$\frac{p(H_1|3in5)}{p(H_0|3in5)} = \frac{\frac{.5 \cdot .23}{.5 \cdot .32 + .5 \cdot .23}}{\frac{.5 \cdot .32}{.5 \cdot .32 + .5 \cdot .23}} = \frac{.57}{.42}$$

which would lead us to infer that the coin was fair. The example slightly obscures the difference between Bayesian and frequentist methods. since we're considering gambling odds we want our beliefs to reflect the long run odds and the disagreement appears only in so far as we've allowed for our priors to influence the outcome. However the really radical move in the Bayesian setting is that you're allowed to ascribe a probability to any event regardless of whether there is any long-run sequence to observe. You may know nothing about your opponent or the coin, but for Bayesians this is no bar to assigning suspicion in the form of expected probability ,so long as you act in accordance with the axioms of probability. It's this freedom which can seem arbitrary and unmotivated, but in practice probabilities are rarely ascribed without reason.

Big Data and Expected Returns

Neither style of analysis ends with these calculations, both would continue to probe the limits of each hypothesis. We'd have to consider things like sample size and sensitivity testing, model performance and cost of errors. The point is just that there are reasons for dispute. This example shows the heart of the conflict in the dual aspect of probability. There is enough latitude in the manner in which we set up a probability model that the mathematics can yield apparently inconsistent results.

The frequentist evaluation of our biased coin is very sensitive to the choice of hypotheses, while the Bayesian approach is influenced by the choice of prior. Why set up a significance test against assumed cheating rather than assumed fairness? Why attribute equal weight to both hypotheses? Why use a 5% threshold if you're concerned about systematic cheating? Both offer strategies for managing uncertainty, but both approaches come with baggage, that not even Big data can solve.

What is Big Data? There is some hype over the term, but it's not magic, it's just bigger sample sizes for specific questions. So far, mostly questions of consumer preference. Websites and apps collect traffic and log interactions. Your details are captured and pulled into vast aggregates of consumer data. I can route and re-route your trajectory across an online environment. Applying the same pressures to tens of thousands of others, we can trace out how the topology of particular sites throw up speed bumps on the customer's journey.

Imagine we're running a website which aims to funnel customers through to a number of different purchase plans. The historic patterns are relatively stable with only 10% of customers dropping out of our conversion funnel on a daily

basis. We can sample actions online under differing pressures with a view to evaluating expected values of repeated coercive prompts.

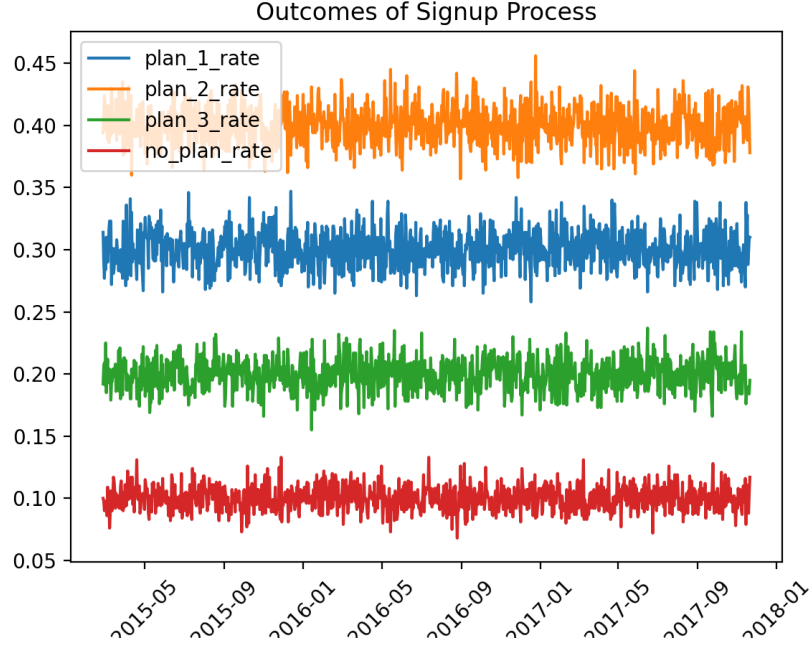


Figure 6: Stable long run Sign Ups

Assume the particular values for each plan then the expected value of customer journey is just: $p_1(o_1) + p_2(o_2) + p_3(o_3) + p_4(o_4) = .3 * \$10 + .4 * \$7 + .2 * \$12 + .1 * \$0 = \8.20 . Now imagine there was a change to the website and we observe the following pattern for the next 20 days:



Figure 7: Abrupt increase in dropouts

What is the new expected value? From the frequentist point of view the macro distributional properties haven't significantly changed. But given what we know about the change to the website it would be foolish to assume such a static distributional assumption. Looking only at the small sample of new data, the variance will be large and the estimates of rates of sign-up for each plan will be unstable. Following the Bayesian paradigm we can condition our expectations on the new data, the old data or all the data we get slightly different results. The below graph illustrates the spread in values expected revenue calculated on different slices of our data using Bayes Rule.

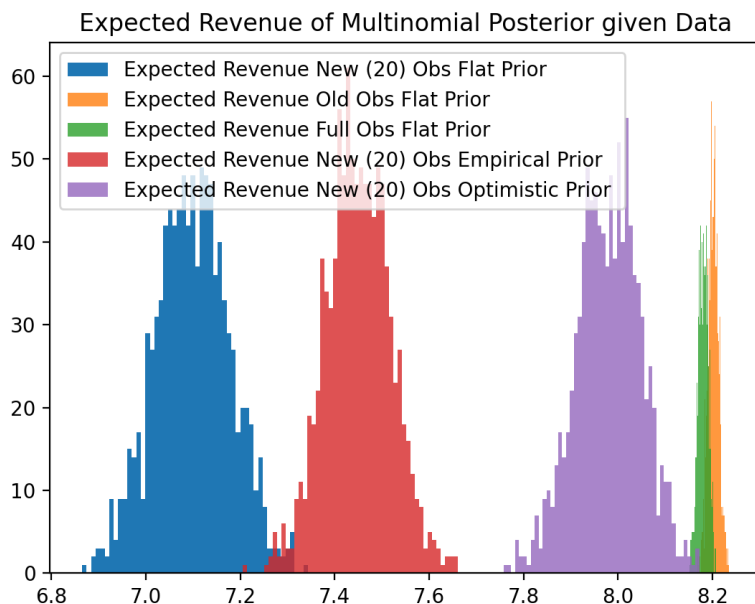


Figure 8: Expected revenue differs by choice of prior and data

Using a large number of observations, the influence of our priors are minimal and washed out by the data, giving us a strong point estimate with low variance stable around 8.2, but since the recent data involves a step change, we might be better off ignoring the old data. But we can also see that if we condition our expectations only on the new data with different priors drawn from the past data or hope, we can positively bias our expectations.

Nearly all substantial decisions are made with small samples in circumstances where past behaviour is not a guide. Past behavioural patterns are exactly what we're trying to avoid or change. If you want to know whether the change on your website will drive a material change in financial revenue, you won't have long run patterns to rely on and it's an open question on how to weight the new data.

Wither Expectation?

All models smuggle in a host of statistical assumptions and these can range from reasonable to absurd. Even when reasonable they're only supported by large sample sizes, and most questions of interest are driven by novelty that short circuits appeal to robust patterns of history.

The weakness of the frequentist account of expected revenue is that it's not clear how probability even relates to small sample investigations, while the Bayesian can be charged with having poorly calibrated expectations based on small samples. The tension is that we'd like to make decisions with probability in mind, but in one case probability statements are not well defined and in the other we risk the bias of arbitrary priors. But this is a false dilemma. When

the evidence is slim, your credence about objective chance is the best and only guide you have.

A fetish for certainty dramatises these gaps as failures. But the error is to make of statistics monolith; although there is an arcane system of tools and symbols common across topics, no sample is large enough to address all questions you could ask. No analyst should ever be so secure in their models to masquerade as an oracle. Reasoning from small samples is common, best done with caution and plenty of caveats, but better reason than not. Expectations should be modified accordingly.