# Properties of Determinants

**Theorem** (Row Operations and Determinants).

Let $A$ be a square matrix.

a. If a multiple of one row in $A$ is added to another row (row replacement operation performed on $A$) to produce a matrix $B$, then $\det(B) = \det(A)$.

b. If two rows in $A$ are interchanged to produce $B$, then $\det(B) = -\det(A)$.

c. If one row in $A$ is scaled by $c$ to produce $B$, then $\det(B) = c\det(A)$.

**Example.** Compute $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

Recall that the determinant of a triangular matrix is the product of the diagonal entries. Using this result, $\det(A)$ can be computed by reducing $A$ to echelon form while concurrently using the preceding theorem to examine how $\det(A)$ changes. The first two row replacements in column 1 do not change $\det(A)$, and thus:

$$\det(A) = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

An interchange of rows 2 and 3 reverses the sign of $\det(A)$ and also reduces $A$ to echelon form. Thus $\det(A)$ is defined by the following.

$$\det(A) = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

■

**Example.** Compute $\det(A)$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Observe that property (c) of the preceding theorem can be used to factor 2 (the common multiple) out of row 1, and hence simplify the arithmetic of reducing $A$ to echelon form. The row replacements operations on column 1 that follow the factoring operation do not change $\det(A)$.

$$\det(A) = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

Two more row replacement operations show that $\det(A)$ is defined as follows.

$$\det(A) = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = \det(A) = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= 2 \cdot (1)(3)(-6)(1) = -36$$

■

## Determinants and Invertible Matrices

Consider the $n \times n$ matrix $A$, which has been reduced to an echelon form $U$ through row replacements and row interchanges. If $A$ has undergone $r$ interchanges to produce $U$, then the preceding theorem shows that $\det(A) = (-1)^r \det(U)$. Since $U$ is an echelon matrix, it must be triangular and thus $\det(U) = u_{11} \times \cdots \times u_{nn}$, the product of the diagonal entries $u_{11}, \ldots, u_{nn}$. Notice that if $A$ is invertible, then the entries $u_{ij}$, where $i = j$, are all nonzero pivots, and hence $\det(A) \neq 0$. Also observe that if $A$ is *not* invertible, then at least $u_{nn} = 0$, and hence $\det(A) = 0$.

$$\det(A) = \begin{cases} (-1)^r \cdot (u_{11} \times \cdots \times u_{nn}) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases} \tag{1}$$

Equation (1) serves as a proof of the following theorem, and therefore adds the statement "$\det(A) \neq 0$" to the Invertible Matrix Theorem.

**Theorem** (Determinant of Invertible Matrix).

An $n \times n$ matrix $A$ is invertible if and only if $\det(A) \neq 0$.

Observe that if the columns of an $n \times n$ matrix $A$ are linearly dependent, then $A$ is not invertible and, as a corollary of the preceding theorem, $\det(A) = 0$. Also observe that if the columns of $A^T$ (that is, the rows of $A$) are linearly dependent, then as a corollary of the preceding theorem, $\det(A) = 0$, because if $A^T$ is not invertible, then $A$ is not invertible by the Invertible Matrix Theorem.

**Example.** Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Performing the row replacement operation $R_3 = (2)R_1 + R_3$ on $A$ produces the matrix $B$, which is defined by the following.

$$B = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

The matrix $B$ was obtained by performing a row replacement operation on $A$, and thus $\det(B) = \det(A)$. Rows 2 and 3 of the matrix $B$ are equal, so the rows of $B$ (and the columns of $B^T$) are linearly dependent. Therefore, $B$ is not invertible by the Invertible Matrix Theorem and hence $\det(B) = \det(A) = 0$. ■

**Example.** Compute $\det(A)$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

First note that column 1 can be simplified through the replacement operation $R_4 = R_2 + R_4$, which doesn't change the determinant. Observe that resulting matrix has a first column of mostly zero entries, so a cofactor expansion down column 1 can be used to reduce the size of the determinant as follows.

$$\det(A) = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

The row replacement $R_2 = -3R_1 + R_2$ followed by the interchange $R_2 \leftrightarrow R_3$ produces a triangular determinant, and thus $\det(A)$ is defined by the following.

$$\det(A) = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 2(-15) = -30$$

■