# Linearly Independent Sets and Bases

Recall that an indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in V$ is *linearly independent* if the vector equation $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution $c_1 = 0, \ldots, c_p = 0$. Also recall that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in V$ is *linearly dependent* if there exists weights $c_1, \ldots, c_p$, not all zero, that satisfy $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ and, in such a case, $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ is called the *linear dependence relation* among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

**Theorem** (Linearly Dependent Set of $\geq 2$ Vectors).

An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, where $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$, where $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$. That is, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, where $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if $\mathbf{v}_j = c\mathbf{v}_1 + \cdots + c\mathbf{v}_{j-1}$, where $j > 1$.

The general vector space $V$ may contain vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ that are not $n$-tuples (an ordered list on $n$ numbers $v_1, \ldots, v_n$ in a column) and, in such a case, the homogeneous vector equation $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ cannot be expressed by the matrix equation $A\mathbf{x} = \mathbf{0}$, because the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ cannot be expressed as the columns of a matrix $A$. Therefore, the definition of linear dependence and the preceding theorem must be used to study a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in V$, since the matrix equation $A\mathbf{x} = \mathbf{0}$ cannot always be studied.

**Example.** Let $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$. Then the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent in $\mathbb{P}$ because $\mathbf{p}_3 = 4\mathbf{p}_1 - \mathbf{p}_2$. ■

**Definition** (Basis).

Let $U$ be a subspace of the vector space $V$. The set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\} \in V$ is a **basis** of $U$ if $\mathcal{B}$ is linearly independent and $U = \operatorname{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$.

Every vector space $V$ is a subspace of itself, so if $U = V$, then the *basis* of $V$ is the linearly independent set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\} \in V$, where $V = \operatorname{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix $I_n$. Then the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** of $\mathbb{R}^n$.

**Example.** Consider $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

There are three vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which, themselves, are in $\mathbb{R}^3$, so the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ is $3 \times 3$. Performing two row replacement operations on $A$ shows that $A$ has three pivot positions, and hence is invertible. Thus, the columns of $A$ form a basis of $\mathbb{R}^3$ because by the Invertible Matrix Theorem, the columns of $A$ are linearly independent and span $\mathbb{R}^3$, since $A$ is invertible. ■

## The Spanning Set Theorem

Consider the sets $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in V$ and $U = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Suppose that $S$ is linearly dependent. Then by rearranging the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in S$, the vector $\mathbf{v}_p$ can be expressed by $\mathbf{v}_p = a_1\mathbf{v}_1 + \cdots + a_{p-1}\mathbf{v}_{p-1}$. Also consider the arbitrary vector $\mathbf{x} \in U$, which can be expressed by $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p$ for suitable scalars $c_1, \ldots, c_p$. Observe below that by substituting the expression for $\mathbf{v}_p$ into the expression for $\mathbf{x}$, the arbitrary vector $\mathbf{x} \in U$ can be expressed as a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$, and hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans $U$, because $\mathbf{x}$ is an arbitrary element of $U$.

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \\ &= (c_1 + \cdots + c_pa_1)\,\mathbf{v}_1 + \cdots + (c_{p-1} + \cdots + c_pa_{p-1})\,\mathbf{v}_{p-1} \end{aligned}$$

If $S$ is linearly dependent and $U = \text{Span}\{S\} \neq \{\mathbf{0}\}$, then $S$ contains two or more vectors and, therefore, vectors in $S$ can repeatedly be removed until $S$ is linearly independent and hence a basis for $U$.

**Theorem** (The Spanning Set Theorem).

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in V$ and let $U = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If some vector $\mathbf{v}_j \in S$ is a linear combination of the remaining vectors in $S$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ that is formed by removing $\mathbf{v}_j$ from $S$ still spans $U$. That is, $U = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$.

b. If $U \neq \{\mathbf{0}\}$, then some subset of $S$ is a basis for $U = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

**Example.** Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, where $U = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, and then find a basis for the subspace $U$.

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

First notice that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$ for all scalars $c_1, c_2 \in \mathbb{R}$ and, therefore, every vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is also an element of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Let $\mathbf{x}$ be an arbitrary vector in $U$, then $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Observe that since $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, the vector $\mathbf{v}_3$ can be substituted into $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ to express $\mathbf{x}$ as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\,(5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3)\,\mathbf{v}_1 + (c_2 + 3c_3)\,\mathbf{v}_2 \end{aligned}$$

Thus, $\mathbf{x} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and hence every vector in $U$ is also an element of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, indicating that $U = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Therefore, since the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, it follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $U$. ■

## Bases for Nul($A$) and Col($A$)

Recall that for a given $m \times n$ matrix $A$, solving the matrix equation $A\mathbf{x} = \mathbf{0}$ amounts the explicitly defining $\text{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ in terms of it's spanning set (the parametric vector equation, or linear combination, that describes the general solution to $A\mathbf{x} = \mathbf{0}$), which is always linearly independent when it contains nonzero vectors. In such a case, it follows that the linearly independent spanning set of Nul($A$) produces a *basis* for Nul($A$).

**Example.** Find a basis for Col($B$), where $B$ is defined as follows.

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

First recall that $\text{Col}(B) = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\}$. Observe that in the given matrix $B$, each non-pivot column is a linear combination of one or more pivot columns, where $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$. By the Spanning Set Theorem, the vectors $\mathbf{b}_2$ and $\mathbf{b}_4$ can be removed from $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\}$ to produce the set $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$, which still spans Col($B$). Let $S$ be defined as follows.

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since $\mathbf{b}_1 \neq \mathbf{0}$ and no vector $\mathbf{b}_j \in S$ is a linear combination of the preceding vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{j-1}$, the set $S$ is linearly independence and is hence a basis for Col($B$).■

**Theorem** (The Basis of Column Space).

The pivot columns of a matrix $A$ form a basis for Col($A$).

*Proof.* TO BE WRITTEN □