Solution Sets of Linear Systems

A linear system is said to be **homogeneous** if it can be expressed by a matrix equation of the form $A\mathbf{x} = \mathbf{0}$, where A is a $m \times n$ matrix and the vector $\mathbf{0} \in \mathbb{R}^m$. Observe that since a homogeneous linear system is of the form $A\mathbf{x} = \mathbf{0}$, there always exists at least one **trivial** solution $\mathbf{x} = (x_1, \ldots, x_n) = (0, \ldots, 0) = \mathbf{0} \in \mathbb{R}^n$ and, therefore, *all* homogeneous systems are consistent. A **nontrivial** solution is a *nonzero* vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies the matrix equation $A\mathbf{x} = \mathbf{0}$.

Recall that by the Existence and Uniqueness Theorem, a consistent linear system has a *unique* solution if the system's reduced echelon matrix has no free variables and has *infinity many* solutions when the system's reduced echelon matrix has at least one free variable. Therefore, if the homogenous linear system corresponding to the matrix equation $A\mathbf{x} = \mathbf{0}$ has a free variable in the reduced echelon form of $[A \quad \mathbf{0}]$, then there must exist a nontrivial solution vector $\mathbf{x} \in \mathbb{R}^n$.

Fact (Nontrivial Solution of a Homogeneous Linear System $A\mathbf{x} = \mathbf{0}$).

The homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution vector $\mathbf{x} \in \mathbb{R}^n$ if and only if $A\mathbf{x} = \mathbf{0}$ has at least *one* free variable.

Example. Determine if the homogeneous linear system corresponding to the following matrix has a nontrivial solution. Then describe the solution set.

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

Let A be the corresponding matrix of coefficients of $[A \quad \mathbf{0}]$ and let $[A \quad \mathbf{0}]$ be the given augmented matrix, which can be reduced into echelon form as follows.

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that since column 3 is not a pivot column, x_3 is free variable and, thus, the matrix equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions. That is, there exists a nonzero solution vector $\mathbf{x} \in \mathbb{R}^n$ for each choice of x_3. Thus, the solution set $\{x_1, x_2, x_3\}$ can be explicitly described by solving the reduced echelon form of $[A \quad \mathbf{0}]$ below for the basic variables x_1 and x_2 in terms of the free variable x_3.

$$\begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With x_3 free, the reduced echelon form of $[A \quad \mathbf{0}]$ gives $x_1 = \frac{4}{3}x_3$ and $x_2 = 0$. As a vector, the general solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{0}$ is defined by the following.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v} \qquad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

By factoring x_3 out of the expression for the general solution vector $\mathbf{x}$, it is shown that the solution set of $A\mathbf{x} = \mathbf{0}$ is the set of all scalar multiples x_3 of the vector $\mathbf{v}$, where the trivial solution $\mathbf{x} = \mathbf{0}$ is obtained by letting $x_3 = 0$. Geometrically, the solution set is a line – that is, Span$\{\mathbf{v}\}$ – through the origin $\mathbf{0}$ in $\mathbb{R}^3$.

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Parametric Vector Form

Observe, as demonstrated in the preceding example, that since the general solution vector $\mathbf{x}$ of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ can be explicitly expressed in **parametric vector form** (1), that is, as the set of all scalar multiples (free variables) $x_1, \ldots, x_n$ of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the general solution set of $A\mathbf{x} = \mathbf{0}$ can also be expressed as Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for suitable vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

$$\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \tag{1}$$

Therefore, if a homogenous linear system $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution $\mathbf{x} = \mathbf{0}$ (that is, no free variables $x_1, \ldots, x_n$), then the solution set is Span $\{\mathbf{0}\}$, or the origin $\mathbf{0} \in \mathbb{R}^n$. If a homogenous linear system $A\mathbf{x} = \mathbf{0}$ has *one* free variable x_n, then the solution set is Span $\{\mathbf{v}\}$ for some suitable vector $\mathbf{v}$ and hence a *line* through the origin $\mathbf{0} \in \mathbb{R}^n$. If a homogenous linear system $A\mathbf{x} = \mathbf{0}$ has *two or more* free variables $x_1, \ldots, x_n$, then the solution set is Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for suitable vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and hence a *plane* through the origin $\mathbf{0} \in \mathbb{R}^n$.

Solutions of Non-Homogeneous Systems

Algorithm (Solution Set of $A\mathbf{x} = \mathbf{b}$ in Parametric Vector Form).

1. Row reduce the augmented matrix $[A \quad \mathbf{b}]$ to reduced echelon form.
2. Express each basic variable in terms of any free variables that appear in each equation.
3. Write the solution vector $\mathbf{x}$ such that the entries of $\mathbf{x}$ depend on the free variables, if any.
4. Decompose the solution vector $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.