Introduction to Determinants

Consider the $n \times n$ matrix A, whose arbitrary entry is of the form a_{ij}. Let M_{ij} denote the $(n-1) \times (n-1)$ sub-matrix that is obtained by removing the ith row and jth column from A. The *determinant* of an $n \times n$ matrix A, denoted by $\det(A)$, is defined (recursively, in terms of sub-matrix determinants) as the sum of n terms of the form $a_{1j}(-1)^{1+j}\det(M_{1j})$, where $\det(M_{ij})$ is the **minor** of a_{1j}.

Definition (Determinant).

The **determinant** of an $n \times n$ matrix A, where $n \geq 2$, is the sum of n terms of the form $a_{1j}(-1)^{1+j}\det(M_{1j})$, with $\pm$ signs alternating, where the entries $a_{11}, \ldots, a_{1j}, \ldots, a_{1n}$ are obtained from the first row of A.

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det\left(M_{1j}\right)\\ &= a_{11}(-1)^{1+1}\det\left(M_{11}\right) + \cdots + a_{1n}(-1)^{1+n}\det\left(M_{1n}\right)\end{aligned}$$

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

For the given matrix A, $\det(A)$ is defined by the sum $\sum_{j=1}^{3}(-1)^{1+j}a_{1j}\det(M_{1j})$.

$$\begin{aligned}\det(A) = \sum_{j=1}^{3} a_{1j}(-1)^{1+j}\det\left(M_{1j}\right) &= a_{11}\det\left(M_{11}\right) - a_{12}\det\left(M_{12}\right) + a_{13}\det\left(M_{13}\right)\\ &= 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}\\ &= 1(0-2) - 5(0-0) + 0(-4-0) = -2\end{aligned}$$

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Cofactor Expansions

If A is an $n \times n$ matrix, then the **cofactor** of a_{ij} is $C_{ij} = (-1)^{i+j}\det(M_{ij})$ and, using C_{ij}, the determinant $\det(A)$ can be expressed as $\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij}$, in which case $\det(A)$ is defined by a *cofactor expansion* across the ith row of A, or the determinant $\det(A)$ can be expressed as $\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij}$, in which case $\det(A)$ is defined by a *cofactor expansion* down the jth column of A.

Theorem (Determinant by Cofactor Expansion).

The determinant of an $n \times n$ matrix A can be computed via cofactor expansion on the ith row or jth column of A. The cofactor expansion across the ith row of A is defined by $\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$. The cofactor expansion down the jth column of A is defined by $\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$.

Example. Use a cofactor expansion across the 3rd row to compute $\det(A)$.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

The cofactor expansion on the 3rd row of A is defined by the following.

$$\begin{aligned} \det(A) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31}\det(M_{31}) + (-1)^{3+2}a_{32}\det(M_{32}) + (-1)^{3+3}a_{33}\det(M_{33}) \\ &= 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 0 + 2(-1) + 0 = -2 \end{aligned}$$

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When a matrix A contains many zero entries, the preceding theorem can be used to easily compute $\det(A)$ because, if the ith row or jth column is mostly zero entries, then the cofactor expansion on the ith row or jth column will have mostly zero terms, and hence the cofactors for those terms need not be computed.

Example. Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$.

Observe that if the first column of A were to be used for a cofactor expansion, then all cofactors except the first (for a_{11}) need not be calculated.

$$\det(A) = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0C_{21} + 0C_{31} + 0C_{41} + 0C_{51}$$

The 4×4 determinant $\det(M_{11})$ has all zero entries but one, and thus can easily be computed through a cofactor expansion on the first column.

$$\det(A) = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \cdot a_{11} \det(M_{11}) - a_{12} \det(M_{12}) + a_{13} \det(M_{13})$$

$$= 3 \cdot 2 \cdot 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$3 \cdot 2 \cdot 1(0-2) - 5(0-0) + 0(-4-0) = 3 \cdot 2 \cdot (-2) = -12$$

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Theorem (Determinant of a Triangular Matrix).

If A is a triangular $n \times n$ matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.