Vector Equations

A **column vector**, or simply **vector**, is a matrix that has *one column*. The set of all vectors that have *two* entries is denoted by $\mathbb{R}^2$ (r-two), where $\mathbb{R}$ is the set of real numbers and the exponent 2 represents the number of entries that vectors in $\mathbb{R}^2$ contain. Consider the vector $\mathbf{u}$ in (1), where u_1 and u_2 are real numbers.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are **equal** *if and only if* their corresponding entries (u_1, u_2) and (v_1, v_2) are equal. Therefore, two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are only equal if and only if $u_1 = v_1$ and $u_2 = v_2$, because vectors in $\mathbb{R}^2$ are *ordered pairs* of real numbers. The **sum** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ is the vector $\mathbf{u} + \mathbf{v}$ defined by (2) that is obtained by adding the corresponding entries of $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \tag{2}$$

Given a vector $\mathbf{u}$ in $\mathbb{R}^2$, the **scalar multiple** of $\mathbf{u}$ by a **scalar** c is the vector $c\mathbf{u}$ (3) that is obtained by multiplying each entry in $\mathbf{u}$ by c.

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \tag{3}$$

Geometric Descriptions of $\mathbb{R}^2$

Consider a rectangular coordinate system in a plane region. Each point in the plane region is defined by an ordered pair $(a, b) \in \mathbb{R}$, so a column vector (4) can be identified by a geometric point (a, b) and, therefore, $\mathbb{R}^2$ may be regarded as the set of all points in the plane region.

$$\begin{bmatrix} a \\ b \end{bmatrix} = (a, b) \tag{4}$$

The geometric visualization of a column vector such as (4) is represented by a directed line segment that extends from the origin of the plane region $(0, 0)$ to the point (a, b), where the origin $(0, 0)$ corresponds to the *zero vector* $\mathbf{0}$.

Theorem (Parallelogram Rule for Addition).

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in a plane region, then $\mathbf{u} + \mathbf{v}$ corresponds to the *fourth* vertex of a parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$.

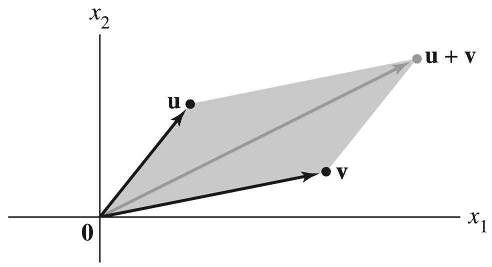

Figure 1: Parallelogram Rule for Addition

Vectors in $\mathbb{R}^3$

A vector in $\mathbb{R}^3$ is a 3×1 column matrix (5) with *three* entries. Geometrically, vectors in $\mathbb{R}^3$ are represented by and *ordered triple* (a, b, c) in a three-dimensional coordinate system, with an arrow extending from the origin to the said points.

$$\begin{bmatrix} a \\ b \\ b \end{bmatrix} = (a, b, c) \tag{5}$$

Vectors in $\mathbb{R}^n$

If $n \in \mathbb{Z}^+$, then $\mathbb{R}^n$ denotes the set of all ordered lists, or ordered n-tuples, of numbers $n \in \mathbb{R}$. Therefore, a vector in $\mathbb{R}^n$, or n-vector, is an n-tuple of n real numbers and is represented by a $n \times 1$ column matrix, such as (6). A vector whose entries are all zero is called a **zero vector** and is denoted by $\mathbf{0}$.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (u_1, u_2, \ldots, u_n) \tag{6}$$

For all vectors in $\mathbb{R}^n$, equality, scalar multiplication, and vector addition are all defined on an entry by entry basis. Therefore, the following algebraic properties of vectors in $\mathbb{R}^n$ can be verified directly from the corresponding algebraic properties of real numbers.

Theorem (Algebraic Properties of $\mathbb{R}^n$).

For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	(5) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
(2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	(6) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
(3) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$	(7) $c(d\mathbf{u})=(cd)\mathbf{u}$
(4) $\mathbf{u}+(-1)\mathbf{u}=\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$	(8) $1\mathbf{u}=\mathbf{u}$

Linear Combinations

In general, a *linear combination* is an expression that is obtained by multiplying each term in a set of terms by a constant and adding the results. Therefore, given the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_i \in \mathbb{R}^n$ and scalars $c_1, \ldots, c_i$, the vector $\mathbf{u}$ defined by (7) is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_i$ with **weights** $c_1, \ldots, c_i$.

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i \tag{7}$$

Example. Let $\mathbf{a}_1 = (1, -2, 5)$, $\mathbf{a}_2 = (2, 5, 6)$, and let $\mathbf{b} = (7, 4, -3)$. Determine whether $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether the weights c_1 and c_2 exist such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}$.

First observe that the definitions of scalar multiplication and vector addition can be used to express the vector equation $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}$ as follows.

$$c_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ -2c_1 \\ -5c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 5c_2 \\ 6c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$
$$\implies \begin{bmatrix} c_1 + 2c_2 \\ -2c_1 + 5c_2 \\ -5c_1 + 6c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Recall that two vectors are equal if and only if their corresponding entries are equal. Therefore, the vector equation $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}$ is true if and only if there exist weights c_1 and c_2 such that c_1 and c_2 satisfy the following linear system.

$$\begin{aligned} c_1 + 2c_2 &= \ 7 \\ -2c_1 + 5c_2 &= \ 4 \\ -5c_1 + 6c_2 &= \ 3 \end{aligned}$$

Application of the row reduction algorithm to the augmented matrix of the preceding linear system provides the solution set $\{c_1, c_2\} = \{3, 2\}$.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the vector $\mathbf{b}$ *can* be expressed as a linear combination of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ with weights $c_1 = 3$ and $c_2 = 2$.

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

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Observe that, as shown in the preceding example, a vector equation of the form $c_1\mathbf{a}_1+\cdots+c_n\mathbf{a}_n=\mathbf{b}$ can be expressed by an augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ that can be row reduced to obtain the weights $c_1,\ldots,c_n$.

Fact (Solution Set of a Vector Equation).

A vector equation of the form $c_1\mathbf{a}_1+\cdots+c_n\mathbf{a}_n=\mathbf{b}$ has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

In particular, $\mathbf{b}$ can be generated (or expressed) by a linear combination of $\mathbf{a}_1,\ldots,\mathbf{a}_n$ if and *only* if there exists a solution $(c_1,\ldots,c_n)$ to the linear system that corresponds to the preceding augmented matrix.

Span

The *span* of a fixed set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}\in\mathbb{R}^n$ is the set of all vectors that can be generated, or expressed, as a linear combination of $\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}$.

Definition (Span).

If $\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}\in\mathbb{R}^n$, then the set of *all* linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_i$ is the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) by $\mathbf{v}_1,\ldots,\mathbf{v}_i$, denoted by $\operatorname{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}$. That is, $\operatorname{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}$ is the collection of all vectors that can be written in the following form with $c_1,\ldots,c_i$ scalars.

$$c_1\mathbf{v}_1+\cdots+c_i\mathbf{v}_i$$

If a vector $\mathbf{b}\in\operatorname{Span}\{\mathbf{v}_1\ldots,\mathbf{v}_i\}$, then there exists a solution $\{c_1,\ldots,c_n\}$ to the vector equation $c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n=\mathbf{b}$ or, equivalently, a solution $\{c_1,\ldots,c_n\}$ to the linear system whose corresponding augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$. Note that since $\operatorname{Span}\{\mathbf{v}_1,\ldots\mathbf{v}_i\}$ contains *every* scalar multiple $c_1,\ldots,c_i$ of the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}$, the zero vector $\mathbf{0}\in\operatorname{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_i\}$.

Geometric Description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u},\mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. The set of all scalar multiples of $\mathbf{v}$ is $\operatorname{Span}\{\mathbf{v}\}$, so geometrically, $\operatorname{Span}\{\mathbf{v}\}$ represents the set of all points on a *line* in $\mathbb{R}^3$ that extends (indefinitely) through $\mathbf{v}$ and $\mathbf{0}$. If $\mathbf{u}$ is also a vector in $\mathbb{R}^3$ and not a scalar multiple of $\mathbf{v}$, then the set of all scalar multiples of $\{\mathbf{u},\mathbf{v}\}$ is $\operatorname{Span}\{\mathbf{u},\mathbf{v}\}$, which is geometrically represented by the set of all points on a *plane* region in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$.