# Cramer's Rule, Volume, & Linear Transformations

For any $n \times n$ matrix $A$ and any $\mathbf{b} \in \mathbb{R}^n$, let $A_i(\mathbf{b})$ be the $n \times n$ matrix that is formed by replacing the $i$th column of $A$ with $\mathbf{b}$. That is, $A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix}$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the columns of an invertible $n \times n$ matrix $A$, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix $I$. Observe that if $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$, then the definition of matrix multiplication can be used to show the following.

$$\begin{aligned} A \cdot I_i(\mathbf{x}) &= A \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} = A_i(\mathbf{b}) \end{aligned}$$

Therefore, since $A \cdot I_i(\mathbf{x}) = A_i(\mathbf{b})$, the multiplicative property of determinants can be used to show that $\det(A) \cdot \det I_i(\mathbf{x}) = \det A_i(\mathbf{b})$ and, since $\det I_i(\mathbf{x}) = x_i$ (through a cofactor expansion along the $i$th row), it can be shown that the unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ has entries given by $x_i = \det A_i(\mathbf{b})/\det(A)$ for $i = 1, \ldots, n$.

**Theorem** (Cramer's Rule).

Let $A$ be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ has entries $x_1, \ldots, x_i, \ldots x_n$ that are given by the following.

$$x_i = \frac{\det A_i(\mathbf{b})}{\det(A)}$$

**Example.** Use Cramer's Rule to solve the following system linear equations.

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$