Null Spaces, Column Spaces, Kernel, and Range

The *null space* of an $m \times n$ matrix A, denoted by $\text{Nul}(A)$, is the set of all $\mathbf{x} \in \mathbb{R}^n$ that satisfy the matrix equation $A\mathbf{x} = \mathbf{0}$. In the context of linear transformations, $\text{Nul}(A)$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ that are mapped into the zero vector $\mathbf{0} \in \mathbb{R}^m$ through the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Definition (Null Space).

The **null space** of an $m \times n$ matrix A is the set of all solutions to the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$. That is, $\text{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.

Example. Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $\mathbf{u} \in \text{Nul}(A)$.

The vector $\mathbf{u} \in \text{Nul}(A)$ because $\mathbf{u} \in \mathbb{R}^n$ and, as shown by the following, $A\mathbf{u} = \mathbf{0}$.

$$A\mathbf{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Consider the $m \times n$ matrix A and $\text{Nul}(A)$, the null space of A. Observe that since every homogenous matrix equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution $\mathbf{x} = \mathbf{0}$, the zero vector $\mathbf{0} \in \text{Nul}(A)$, and thus $\text{Nul}(A)$ satisfies property (a) of Subspaces. Also observe that if $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in $\text{Nul}(A)$, then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$ and, by the properties of matrix multiplication, $A(\mathbf{u} + \mathbf{v})$ and $A(c\mathbf{u})$ are also in $\text{Nul}(A)$ because $\text{Nul}(A)$ is closed under both vector addition and scalar multiplication, as shown by the following expressions.

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0} \qquad A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$$

Thus, $\text{Nul}(A)$ satisfies properties (a), (b), and (c) of Subspaces and is hence a subspace of $\mathbb{R}^n$, as well as a vector space itself. So the following has been verified.

Theorem (Null Space of A is a Subspace of $\mathbb{R}^n$).

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the solution set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$, where $A\mathbf{x} = \mathbf{0}$ represents a system of m homogenous linear equations in n variables, is a subspace of $\mathbb{R}^n$.

Example. Let $U = \{a, b, c, d : a - 2b + 5c = d \text{ and } c - a = b\}$. Then by the preceding theorem, U is a subspace of $\mathbb{R}^4$ because U can be expressed as the set of all solutions to the following system of homogenous linear equations.

$$\begin{aligned} a - 2b + 5c - d &= 0 \\ -a - b + c &= 0 \end{aligned}$$

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Explicit Description of Nul(A)

The definition of Nul(A) as the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ defines Nul(A) implicitly, as $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ defines a condition that must be checked. However, recall that solving the matrix equation $A\mathbf{x} = \mathbf{0}$ amounts to explicitly defining the solution set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$, and hence solving $A\mathbf{x} = \mathbf{0}$ also amounts to *explicitly* defining Nul(A) for a given $m \times n$ matrix A.

The Column Space of a Matrix

The *column space* of an $m \times n$ matrix A, denoted by Col(A), is defined explicitly (via linear combinations) as the set of all $\mathbf{b} \in \mathbb{R}^m$ that satisfy the matrix equation $A\mathbf{x} = \mathbf{b}$. Equivalently, Col(A) can be defined as the set of all $A\mathbf{x}$ for some vector $\mathbf{x} \in \mathbb{R}^n$ because $A\mathbf{x}$ represents a linear combination of $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ using the corresponding entries in $\mathbf{x}$ as weights. In the context of linear transformations, Col(A) is the *range* (the set of all $A\mathbf{x}$) of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Definition (Column Space).

The **column space** of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then $\mathrm{Col}(A) = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Equivalently, Col($A$) is the set $\mathrm{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$.

Since $\mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a subspace, it follows from the definition of Col(A) and the fact that the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, that Col($A$) is a subspace of $\mathbb{R}^m$.

Theorem (Column Space is a Subspace of $\mathbb{R}^m$).

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Example. Find a matrix A such that $W = \mathrm{Col}(A)$, where W is defined as follows.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

First note that W can be expressed by a set of linear combinations as follows.

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let $\mathbf{a}_1 = \langle 6, 1, -7 \rangle$ and $\mathbf{a}_2 = \langle -1, 1, 0 \rangle$ be the columns of the matrix A. Then $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ and hence $W = \mathrm{Col}(A)$. ■

Kernel and Range of a Linear Transformation

Subspaces of vector spaces – other than $\mathbb{R}^n$ – can often be described in terms of a linear transformation, instead of a matrix.

Definition (Linear Transformation).

A **linear transformation** T from a vector space V to a vector space W is a rule that assigns to each vector $\mathbf{x} \in V$ a unique vector $T(\mathbf{x}) \in W$, such that

1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$.

2) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and scalars c.

The **kernel** (or **null space**) of a linear transformation $T : V \to W$ is the set of all $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{0}$. That is, $\text{Kernel}(T) = \{\mathbf{x} \in V : T(\mathbf{x}) = \mathbf{0}\}$. If the transformation T arises from the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A, then $\text{Kernel}(T) = \text{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.

Theorem (Kernel of T is a Subspace).

If $T : V \to W$, then the kernel of T is a subspace of V.

The **range** of a linear transformation $T : V \to W$ is the set of all $T(\mathbf{x}) \in W$ for some $\mathbf{x} \in V$. That is, $\text{Range}(T) = \{T(\mathbf{x}) \in W : \mathbf{x} \in V\}$. If the transformation T arises from the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A, then $\text{Range}(T) = \text{Col}(A) = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$.

Theorem (Range of T is a Subspace).

If $T : V \to W$, then the range of T is a subspace of W.

Proof. If T is a linear transformation, then it follows that $T(\mathbf{0}) = \mathbf{0}$, and hence the zero vector $\mathbf{0} \in \text{Range}(T)$. If $\mathbf{w}_1, \mathbf{w}_2 \in \text{Range}(T)$, then there exists $\mathbf{x}_1, \mathbf{x}_2 \in V$ such that $T(\mathbf{x}_1) = \mathbf{w}_1$ and $T(\mathbf{x}_2) = \mathbf{w}_2$. Since T is a linear transformation, $T(\mathbf{x}_1 + \mathbf{x}_2)$ can be expressed as follows, and thus $\text{Range}(T)$ is closed under vector addition.

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) = \mathbf{w}_1 + \mathbf{w}_2$$

If $\mathbf{w} \in \text{Range}(T)$ and $c \in \mathbb{R}$, then there exists $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{w}$ and, since T is a linear transformation, $T(c\mathbf{x}) = cT(\mathbf{x}) = a\mathbf{w}$. Therefore, T is closed under scalar multiplication, and hence satisfies the three properties of subspaces. $\square$