The Dimension of a Vector Space

Suppose that the vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and let the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \in V$, where $p > n$. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ from V to $\mathbb{R}^n$ produces the linearly *dependent* set $\{[\mathbf{x}_1]_{\mathcal{B}}, \ldots, [\mathbf{x}_p]_{\mathcal{B}}\} \in \mathbb{R}^n$ because $p > n$. That is, the set $\{[\mathbf{x}_1]_{\mathcal{B}}, \ldots, [\mathbf{x}_p]_{\mathcal{B}}\} \in \mathbb{R}^n$ is linearly dependent because it contains more vectors p than there are entries n in each vector. Therefore, there exists weights $\{c_1, \ldots, c_p\}$, not all zero, such that $c_1 [\mathbf{x}_1]_{\mathcal{B}} + \cdots + c_p [\mathbf{x}_p]_{\mathcal{B}} = \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^n$ and, since the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a linear transformation, it follows that $c_1 [\mathbf{x}_1]_{\mathcal{B}} + \cdots + c_p [\mathbf{x}_p]_{\mathcal{B}} = [c_1\mathbf{x}_1 + \cdots + c_p\mathbf{x}_p]_{\mathcal{B}} = \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^n$.

Theorem (Set Containing More Vectors Than Basis).

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \in V$ where $p > n$ must be linearly dependent.

As a corollary to the preceding theorem, it follows that if a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every linearly *independent* set in $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \in V$ must have no more than n vectors because, if $p > n$, then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ produces a linearly dependent set $\{[\mathbf{x}_1]_{\mathcal{B}}, \ldots, [\mathbf{x}_p]_{\mathcal{B}}\} \in \mathbb{R}^n$, which is isomorphic to the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \in V$.

Let $\mathcal{B}_1 = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{B}_2 = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ be bases of the vector space V. Observe that since $\mathcal{B}_1$ is a basis and $\mathcal{B}_2$ is linearly independent in V, $\mathcal{B}_2$ has no more than n vectors (that is, $p \leq n$) by the preceding theorem. Also observe that since $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is linearly independent in V, $\mathcal{B}_2$ has at least n vectors (that is, $p \geq n$) and, therefore, $\mathcal{B}_2$ must contain exactly n vectors (that is, $p = n$).

Theorem (Cardinality of Vector Space Bases).

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V must consist of exactly n vectors.

Recall that by the Spanning Set Theorem, if a vector space $V = \text{Span}\{S\}$, where $V \neq \{\mathbf{0}\}$, then there exists a subset $U \subseteq S$ such that U is a basis for V.

Definition (Dimension).

If the vector space $V = \text{Span}\{S\}$ for some finite set S, then V is *finite-dimensional*, and the **dimension** of V, denoted by $\dim(V)$, is the number of vectors in a basis for V. If $V = \{\mathbf{0}\}$, then $\dim(V) = 0$. If the vector space $V \neq \text{Span}\{S\}$ for some finite set S, then V is *infinite-dimensional.*

Note. The standard basis $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$ contains n vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and hence $\dim(\mathbb{R}^n) = n$. The standard basis $\xi = \{1, t, \ldots, t^n\}$ for $\mathbb{P}_n$ contains $n + 1$ polynomials and hence $\dim(\mathbb{P}_n) = n + 1$.

Example. Find the $\dim(H)$, where the subspace H is defined as follows.

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

First notice that expressing $\langle a-3b+6c, 5a+4d, b-2c-d, 5d \rangle \in H$ in parametric vector form shows that $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

Observe that $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2 \neq c\mathbf{v}_1$, and $\mathbf{v}_3 = -2\mathbf{v}_2$, so by the Spanning Set Theorem, the vector $\mathbf{v}_3$ can be removed from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ to obtain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ which still spans H. Finally, observe that since $\mathbf{v}_4 \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent and is hence a basis for H. Thus, $\dim(H) = 3$. ■

Subspaces of a Finite-Dimensional Space

Let U be a subspace of a finite-dimensional vector space V, and let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be any linearly independent set in U. If $U = \mathrm{Span}\{S\}$, then S is a basis for U and, if otherwise, there exists a vector $\mathbf{u}_{n+1} \in U$ such that $\mathbf{u}_{n+1} \notin \mathrm{Span}\{S\}$. That is, if $U \neq \mathrm{Span}\{S\}$, then there exist a vector $\mathbf{u}_{n+1} \in U$ such that $\mathbf{u}_{n+1} \neq c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ and hence $\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}\}$ is also linearly independent, because no vector $\mathbf{u}_{n+1} \in \{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}\}$ is a linear combination of the preceding vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Therefore, the set S can be repeatedly expanded to a larger linearly independent set $S_{ex} \in U$ as long as $U \neq \mathrm{Span}\{S_{ex}\}$. But recall that the number of vectors in a linearly independent S_{ex} can never exceed $\dim(V)$, so eventually, $U = \mathrm{Span}\{S_{ex}\}$ and hence S_{ex} will be a basis for U, where $\dim(U) \leq \dim(V)$.

Theorem (Expanding a Linearly Independent Set to a Basis).

Let U be a finite-dimensional subspace of a finite-dimensional vector space V. Any linearly independent set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \in U$ can be expanded, if necessary, to form a basis for U, where $\dim(U) \leq \dim(V)$.

Consider the vector space V with $\dim(V) = p$, where $p \geq 1$. By the preceding theorem, a linearly independent set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ can be expanded into a basis $\mathcal{B}$ for V, but since $\dim(V) = p$, $\mathcal{B}$ must contain exactly p elements and hence S must already be a basis $\mathcal{B}$ for V. Suppose that $V = \mathrm{Span}\{S\} = \mathrm{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$. Observe that since $\dim(V) = p \geq 1$, the Spanning Set Theorem implies that a subset $S' \subseteq S$ is a basis of V and, since $\dim(V) = p$, the set S' must contain p vectors, which implies that $S' = S$.

Theorem (The Basis Theorem).

Let V be a vector space, where $\dim(V) = p$ and $p \geq 1$. Any linearly independent set $S \in V$ of exactly p elements is automatically a basis for V. Any set S of exactly p elements, where $V = \text{Span}\{S\}$, is automatically a basis for V.

The Dimensions of Nul(A) and Col(A)

Recall that the pivot columns of an $m \times n$ matrix A form a basis for Col(A), so if A has k pivot columns, then $\dim[\text{Col}(A)] = k$. Also recall that if the matrix equation $A\mathbf{x} = \mathbf{0}$ has k free variables, then row reducing the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ will produce a linearly independent set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ such that $\text{Nul}(A) = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, which hence makes $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ a basis for Nul(A) and $\dim[\text{Nul}(A)] = k$.

Fact (The Dimensions of Nul(A) and Col(A)).

If the matrix equation $A\mathbf{x} = \mathbf{0}$ has k free variables, then $\dim[\text{Nul}(A)] = k$ and, if the matrix A has k pivot columns, then $\dim[\text{Col}(A)] = k$.

Example. Find $\dim[\text{Nul}(A)] = k$ and $\dim[\text{Col}(A)] = k$, where A is defined as

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

By row reducing the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to echelon form, the number of free variables in the matrix equation $A\mathbf{x} = \mathbf{0}$ and the number of pivot columns in A can be determined as follows.

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variables are x_2, x_4, and x_5 and, therefore, $\dim[\text{Nul}(A)] = 3$. Columns 1 and 2 of the matrix A are the only pivot columns, and hence $\dim[\text{Col}(A)] = 2$. ■