Coordinate Systems

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for the vector space V, and let $\mathbf{x}$ be an arbitrary element of V. Since $\mathcal{B}$ is a basis for V, the set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent and $V = \text{Span}\{\mathcal{B}\} = \text{Span}\,\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, and since $\mathcal{B}$ spans V, there exists a set of scalars $\{c_1, \ldots, c_n\} \in \mathbb{R}$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ for each $\mathbf{x} \in V$. Suppose the the arbitrary $\mathbf{x} \in V$ can also be expressed by $\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n$ for suitable scalars $\{d_1, \ldots, d_n\} \in \mathbb{R}$. Observe that by subtracting $\mathbf{x} - \mathbf{x}$ (the two representations of $\mathbf{x}$), the following is obtained.

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\,\mathbf{b}_1 + \cdots + (c_n - d_n)\,\mathbf{b}_n$$

The set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent, so each scalar $c_j - d_j$ in the set $\{c_1 - d_1, \ldots, c_n - d_n\}$ must be zero. That is, $c_j = d_j$ for $1 \leq j \leq n$ and, therefore, the set of scalars $\{c_1, \ldots, c_n\}$ that satisfy $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ must be *unique*.

Theorem (The Unique Representation Theorem).

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V. Then for each $\mathbf{x} \in V$, there exists a unique set of scalars $\{c_1, \ldots, c_n\}$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V, and let $\mathbf{x} \in V$. If the weights $\{c_1, \ldots, c_n\}$ satisfy the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, then $\{c_1, \ldots, c_n\}$ are the *coordinates* of $\mathbf{x}$ *relative* to $\mathcal{B}$, and the vector $[\mathbf{x}]_\mathcal{B} \in \mathbb{R}^n$ defined by $[\mathbf{x}]_\mathcal{B} = \langle c_1, \ldots, c_n\rangle$ is the *coordinate vector* of $\mathbf{x}$ *relative* to $\mathcal{B}$.

Definition ($\mathcal{B}$-Coordinates of $\mathbf{x}$).

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for the vector space V and $\mathbf{x} \in V$. The **coordinates of x with respect to** $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $\{c_1, \ldots, c_n\}$ that satisfy the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Example. Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \langle 1, 0\rangle$ and $\mathbf{b}_2 = \langle 1, 2\rangle$. Suppose an $\mathbf{x} \in \mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_\mathcal{B} = \langle -2, 3\rangle$. Find $\mathbf{x}$.

The $\mathcal{B}$-coordinates of $\mathbf{x}$ given in $[\mathbf{x}]_\mathcal{B} = \langle -2, 3\rangle$ can be used as follows to construct $\mathbf{x}$ from vectors in $\mathcal{B}$.

$$\mathbf{x} = (-2)\mathbf{b}_1 + 3\mathbf{b}_2 = (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

■

Coordinates in $\mathbb{R}^n$

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a *fixed* basis for $\mathbb{R}^n$. Then for each $\mathbf{x} \in \mathbb{R}^n$, the $\mathcal{B}$-coordinate vector $[\mathbf{x}]_\mathcal{B}$ consists of unique scalars $\{c_1, \dots, c_n\}$ that satisfy the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$. Observe that if $P_\mathcal{B} = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$, then $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ can be represented by the matrix equation (1). Thus, $[\mathbf{x}]_\mathcal{B}$ can be obtained by reducing the augmented matrix $\begin{bmatrix}P_\mathcal{B} & \mathbf{x}\end{bmatrix}$ to row echelon form.

$$\mathbf{x} = P_\mathcal{B}[\mathbf{x}]_\mathcal{B} \tag{1}$$

The matrix $P_\mathcal{B}$ is called the *change-of-coordinates matrix* from $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ because left-multiplication by $P_\mathcal{B}$ transforms $[\mathbf{x}]_\mathcal{B}$ into $\mathbf{x}$. That is, the matrix $P_\mathcal{B}$ transforms the $\mathcal{B}$-coordinates of $\mathbf{x}$ into the standard $\mathcal{E}$-coordinates of $\mathbf{x}$. Also observe that since the columns of $P_\mathcal{B}$ form a basis for $\mathbb{R}^n$, $P_\mathcal{B}$ is invertible by the Invertible Matrix Theorem and hence $[\mathbf{x}]_\mathcal{B}$ can be obtained through the following matrix equation, whereby left-multiplication of $P_\mathcal{B}$ converts $\mathbf{x}$ into $[\mathbf{x}]_\mathcal{B}$. That is, in the context of linear transformations, $P_\mathcal{B}^{-1}$ produces the one-to-one coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$.

$$[\mathbf{x}]_\mathcal{B} = P_\mathcal{B}^{-1}\mathbf{x} \tag{2}$$

Example. Consider the vectors $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_\mathcal{B}$, the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

The $\mathcal{B}$-coordinates $\{c_1, c_2\}$ of $\mathbf{x}$ satisfy the following matrix and vector equations.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Reducing the augmented matrix $\begin{bmatrix}P_\mathcal{B} & \mathbf{x}\end{bmatrix}$ to row echelon form shows that the $\mathcal{B}$-coordinates $\{c_1, c_2\} = \{3, 2\}$ and hence $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$. Thus $[\mathbf{x}]_\mathcal{B} = \langle 3, 2\rangle$. ■

The Coordinate Mapping

Consider the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ from a vector space V to $\mathbb{R}^n$, where $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V. Since $\mathcal{B}$ is a basis for V, two arbitrary vectors $\mathbf{u}, \mathbf{w} \in V$ can be expressed by $\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ and $\mathbf{w} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n$. Observe that by using vector operations, the vector sum $\mathbf{u} + \mathbf{w}$ can be expressed by $\mathbf{u} + \mathbf{w} = (c_1 + d_1)\,\mathbf{b}_1 + \cdots + (c_n + d_n)\,\mathbf{b}_n$ and hence, it follows by the definition of a $\mathcal{B}$-coordinates vector, that $[\mathbf{u} + \mathbf{w}]_\mathcal{B}$ can be expressed by the following, which implies that the mapping $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ preserves the operation of vector addition.

$$[\mathbf{u} + \mathbf{w}]_\mathcal{B} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_\mathcal{B} + [\mathbf{w}]_\mathcal{B}$$

Also observe that if r is any scalar, then the scalar multiple $r\mathbf{u}$ can be expressed by $r\mathbf{u} = r\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) = \left(rc_1\right)\mathbf{b}_1 + \cdots + \left(rc_n\right)\mathbf{b}_n$ and, again, by the definition of a $\mathcal{B}$-coordinates vector, it follows that $[r\mathbf{u}]_{\mathcal{B}}$ can be expressed by the following, which implies that the mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ preserves the operation of scalar multiplication. Thus, the mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a *linear* transformation.

$$[r\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathbf{u}]_{\mathcal{B}}$$

Theorem (The Mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$).

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Proof. Consider the vectors $\mathbf{u}, \mathbf{w} \in V$, where $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$. Since $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V, the vectors $\mathbf{u}$ and $\mathbf{v}$ can be represented by $\mathbf{u} = p_1\mathbf{b}_1 + \ldots + p_n\mathbf{b}_n$ and $\mathbf{w} = q_1\mathbf{b}_1 + \ldots + q_n\mathbf{b}_n$ and, by the definition of a $\mathcal{B}$-coordinates vector, it follows that $[\mathbf{u}]_{\mathcal{B}}$ and $[\mathbf{w}]_{\mathcal{B}}$ are defined as follows.

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \qquad [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

Observe that since $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$, it also follows that $\{p_1, \ldots, p_n\} = \{q_1, \ldots, q_n\}$ and, therefore, since $\mathbf{u}$ and $\mathbf{w}$ and spanned by the same set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, it also follows that $\mathbf{u} = \mathbf{w}$. Thus, the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation. □

In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an *isomorphism* from V to W. That is, V and W are indistinguishable as vector spaces in that every operation produced in V is accurately produced in W, and vice versa.

Example. Consider the standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of the space $\mathbb{P}_3$ of polynomials with $\deg(\mathbf{p}) \leq 3$. The arbitrary element $\mathbf{p} \in \mathbb{P}_3$ is of the form $\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ and, since $\mathbf{p}$ is expressed as a linear combination of the standard basis vectors, it follows that $[\mathbf{p}]_{\mathcal{B}}$ is defined as follows.

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Thus, the coordinate mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$ is an isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$. ■

Example. Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, and $3+2t$ are linearly dependent in $\mathbb{P}_2$.

First note that the standard basis of $\mathbb{P}_2$ is the set $\mathcal{B} = \{1, t, t^2\}$, and let $\mathbf{p}_1(t) = 1+2t^2$, $\mathbf{p}_2(t) = 4+t+5t^2$, and $\mathbf{p}_3(t) = 3+2t$. The polynomials $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \in \mathbb{P}_2$ are expressed as linear combinations of the standard basis vectors, so the isomorphism $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$ from $\mathbb{P}_2$ onto $\mathbb{R}^3$ shows that $[\mathbf{p}_1]_{\mathcal{B}} = \langle 1, 0, 2\rangle$, $[\mathbf{p}_2]_{\mathcal{B}} = \langle 4, 1, 5\rangle$, and $[\mathbf{p}_3]_{\mathcal{B}} = \langle 3, 2, 0\rangle$. Observe that by letting $\{[\mathbf{p}_1]_{\mathcal{B}}, [\mathbf{p}_2]_{\mathcal{B}}, [\mathbf{p}_3]_{\mathcal{B}}\}$ be the columns of a matrix A, the matrix equation $A\mathbf{x} = \mathbf{0}$ and the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ can be used to verify that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ are linearly dependent in $\mathbb{P}_2$.

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ shows that A has a free variable, and thus the columns $\{[\mathbf{p}_1]_{\mathcal{B}}, [\mathbf{p}_2]_{\mathcal{B}}, [\mathbf{p}_3]_{\mathcal{B}}\}$ are linearly dependent in $\mathbb{R}^3$, which implies that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ are linearly dependent in $\mathbb{P}_2$. ■