# Eigenvectors and Linear Transformations

Let $V$ be a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ so $\dim(V) = n$, let $W$ be a vector space with basis $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ so $\dim(W) = m$, and let $T : V \to W$ be a linear transformation. Thus for any $\mathbf{v} \in V$, there exist unique scalars $b_1, \ldots, b_n$ such that $\mathbf{v} = b_1\mathbf{b}_1 + \cdots + b_n\mathbf{b}_n$, so $[\mathbf{x}]_{\mathcal{B}} = \left\langle b_1, \ldots, b_n \right\rangle \in \mathbb{R}^n$ and, since $T : V \to W$ is a linear transformation, it follows that $T(\mathbf{v}) = T\left(b_1\mathbf{b}_1 + \cdots + b_n\mathbf{b}_n\right) = b_1 T\left(\mathbf{b}_1\right) + \cdots + b_n T\left(\mathbf{b}_n\right)$, where $T(\mathbf{v}) \in W$ and, by the linearity of the coordinate mapping $T(\mathbf{v}) \mapsto [T(\mathbf{v})]_{\mathcal{C}}$ from $W$ onto $\mathbb{R}^m$, it follows from $T(\mathbf{v}) = T\left(b_1\mathbf{b}_1 + \cdots + b_n\mathbf{b}_n\right) = b_1 T\left(\mathbf{b}_1\right) + \cdots + b_n T\left(\mathbf{b}_n\right)$ that $[T(\mathbf{v})]_{\mathcal{C}} = b_1\left[T\left(\mathbf{b}_1\right)\right]_{\mathcal{C}} + \cdots + b_n\left[T\left(\mathbf{b}_n\right)\right]_{\mathcal{C}}$. Observe that since each $\left[T\left(\mathbf{b}_i\right)\right]_{\mathcal{C}} \in \mathbb{R}^m$, it follows that the vector equation $[T(\mathbf{v})]_{\mathcal{C}} = b_1\left[T\left(\mathbf{b}_1\right)\right]_{\mathcal{C}} + \cdots + b_n\left[T\left(\mathbf{b}_n\right)\right]_{\mathcal{C}}$ can be expressed as a matrix equation $[T(\mathbf{v})]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$, where $M$ is the *matrix for $T$ relative to $\mathcal{B}$ and $\mathcal{C}$*, defined by $M = \left[\left[T\left(\mathbf{b}_1\right)\right]_{\mathcal{C}} \quad \cdots \quad \left[T\left(\mathbf{b}_n\right)\right]_{\mathcal{C}}\right]$.

## Linear Transformations from $V$ to $V$

Let $V$ be a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $T : V \to V$ be a linear transformation, then the *matrix for $T$ relative to $\mathcal{B}$* is a matrix $[T]_{\mathcal{B}} = \left[[T(\mathbf{b}_1)]_{\mathcal{B}} \quad \cdots \quad [T(\mathbf{b}_n)]_{\mathcal{B}}\right]$ that satisfies the matrix equation $[T(\mathbf{v})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$ for all $\mathbf{v} \in V$.

## Linear Transformations on $\mathbb{R}^n$