The Inverse of a Matrix

Recall that the multiplicative *inverse* of a nonzero real number c is $c^{-1} = 1/c$, where $c^{-1}c = cc^{-1} = 1$. An $n \times n$ matrix A is said to be **invertible** if there exists an $n \times n$ matrix A^{-1} called the **inverse** of A such that $AA^{-1} = A^{-1}A = I$, where I is the $n \times n$ identity matrix. Suppose that there exists $n \times n$ matrices B and C such that $AB = BA = I$ and $AC = CA = I$. Observe that, through the identity and associative properties of matrix multiplication, B and C can be expressed by $C = CI = C(AB) = (CA)B = IB = B$. Therefore, if the $n \times n$ matrix A is invertible, then A^{-1} is *unique*.

Example. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then $B = A^{-1}$ because

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ BA &= \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

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In the following theorem, the quantity $ad - bc$ is called the *determinant* of the 2×2 matrix A and is denoted by $\det(A) = ad - bc$. Proof omitted.

Theorem (Inverse of a 2×2 Matrix).

Consider the 2×2 matrix A below. If $\det(A) \neq 0$, then A is invertible and A^{-1} is defined as follows. If $\det(A) = 0$, then A is not invertible.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The Solution Vector $\mathbf{x} = A^{-1}\mathbf{b}$

Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, where A is an invertible $n \times n$ matrix and the vector $\mathbf{b} \in \mathbb{R}^n$. Observe that if $\mathbf{x} = A^{-1}\mathbf{b}$ in the matrix-vector product $A\mathbf{x}$, then $A\mathbf{x}$ can be expressed as follows, and hence $\mathbf{x} = A^{-1}\mathbf{b}$ is a *solution* to $A\mathbf{x} = \mathbf{b}$.

$$A\mathbf{x} = A\left(A^{-1}\mathbf{b}\right) = \left(AA^{-1}\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}$$

Also observe that for each vector $\mathbf{b} \in \mathbb{R}^n$, the solution $\mathbf{x} = A^{-1}\mathbf{b}$ must be *unique* because, if $\mathbf{x}$ is any solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, multiplying both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1} shows that $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \implies I\mathbf{x} = A^{-1}\mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$. Thus, if A is an invertible $n \times n$ matrix, then the solution vector $\mathbf{x}$ of the matrix equation $A\mathbf{x} = \mathbf{b}$ can be found by computing $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem (The Solution Vector $\mathbf{x} = A^{-1}\mathbf{b}$).

If A is an invertible $n \times n$ matrix, then for each vector $\mathbf{b} \in \mathbb{R}^n$, the matrix equation

$A\mathbf{x} = \mathbf{b}$ has the unique solution vector $\mathbf{x} = A^{-1}\mathbf{b}$.

Example. Consider the following linear system and it's corresponding coefficient matrix A. Find and use A^{-1} to solve the given linear system.

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned} \qquad A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

First note that $\det(A) = 3(6) - 4(5) = -2 \neq 0$, so A^{-1} is defined by the following.

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Since the given linear system is equivalent to the matrix equation $A\mathbf{x} = \mathbf{b}$, the inverse of the coefficient matrix A^{-1} and the vector $\mathbf{b}$ can be used to compute $\mathbf{x}$.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

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Properties of Invertible Matrices

If A is an invertible $n \times n$ matrix, then it's inverse A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$ because, by the definition of an invertible matrix, A satisfies $A^{-1}A = AA^{-1} = I$.

Theorem (The Inverse of A^{-1}).

If A is an invertible matrix, then A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

Suppose that A and B are invertible $n \times n$ matrices. Then, by the definition of an invertible matrix, the matrix AB must also be *invertible* where $(AB)^{-1} = B^{-1}A^{-1}$ because multiplying AB by $B^{-1}A^{-1}$ on both sides produces the identity matrix I.

$$\begin{aligned} (AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = AA^{-1} = I \\ \left(B^{-1}A^{-1}\right)(AB) &= B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I \end{aligned}$$

Theorem (The Inverse of AB).

If A and B are invertible $n \times n$ matrices, then the matrix AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. That is, $(AB)^{-1}$ is the product of the inverses A^{-1} and B^{-1} in reverse order.

Recall that for two matrices A and B, the transpose of AB is $(AB)^T = B^TA^T$. Therefore, if A is an invertible $n \times n$ matrix, then by the definition of an invertible matrix, A^T is also *invertible* whereby $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ because multiplying A^T by $\left(A^{-1}\right)^T$ on both sides produces the identity matrix I.

$$\begin{aligned} \left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T = I^T = I \\ A^T\left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T = I^T = I \end{aligned}$$

Theorem (The Inverse of A^T).

If A is an invertible $n \times n$ matrix, then A^T is invertible and $\left(A^T\right)^{-1} = (A^{-1})^T$.

Elementary Matrices

An **elementary matrix** is an $n \times n$ matrix E that is obtained by performing a *single* elementary row operation on an identity matrix I of the same size.

Fact (Row Operations on A by Computation of EA).

When a *single* elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be expressed as EA, where E is the $m \times m$ elementary matrix that is obtained by performing the same row operation on I_m.

Example. Consider the matrices E_1, E_2, E_3, and A defined below. Compute E_1A, E_2A, and E_3A then describe how these products can be obtained by performing elementary row operations on A.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The products E_1A, E_2A, and E_3A are defined as follows.

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix} E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

The operation $R_3 = (-4)R_1 + R_3$ produces E_1A, the operation $R_1 \leftrightarrow R_2$ produces E_2A, and the operation $R_3 = (5)R_3$ produces E_3A. ■

Consider the elementary matrix E_1 that, by definition, is obtained by performing a single elementary row operation on I_n. Observe that since the elementary row operations are reversible, there must exist a row operation that transforms E back into I_n. That is (in the context of elementary matrices), there must exist an elementary matrix E^{-1} such that $E^{-1}E = I_n$ and, since E^{-1} and E correspond to reverse operations, $EE^{-1} = I_n$. Thus, *every* elementary matrix E is invertible.

Fact (The Inverse of the Elementary Matrix E).

Every elementary matrix E is invertible, where E^{-1} is the elementary matrix of the same type that transforms E back into I_n.

Example. Find the inverse of $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

The given elementary matrix E is obtained by performing the elementary row operation $R_3 = -4R_1 + R_3$ on I_n. Therefore, the reverse operation that transforms E back into I_n is $R_3 = 4R_1 + R_3$ and hence E^{-1} is defined as follows.

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

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Suppose that A is an invertible $n \times n$ matrix. Then for each vector $\mathbf{b} \in \mathbb{R}^n$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ and thus A must have a pivot position in every row (that is, A must have n pivot positions). Observe that since A is a *square* matrix, the n pivot positions of A must lie along the main diagonal, which implies that the reduced echelon form of A is the identity matrix I_n and, therefore, also implies that $A \sim I_n$.

If $A \sim I_n$, then each elementary row operation performed in the row reduction of A corresponds to left-multiplication by an elementary matrix E_p and hence there exists elementary matrices $E_1, \ldots, E_p$ such that $E_p \times \cdots \times E_1 A = I_n$.

$$A \sim E_1 A \sim E_2 (E_1 A) \sim \cdots \sim E_p (E_{p-1} \times \cdots \times E_1 A) = I_n$$

Observe that since the product $E_p \times \cdots \times E_1$ of invertible matrices is invertible, the preceding equation leads to the following, which implies that A is invertible because A can be expressed as the inverse of an invertible matrix.

$$\begin{aligned} (E_p \times \cdots \times E_1)^{-1} (E_p \times \cdots \times E_1) A &= (E_p \times \cdots \times E_1)^{-1} I_n \\ A &= (E_p \times \cdots \times E_1)^{-1} \end{aligned}$$

If $A = (E_p \times \cdots \times E_1)^{-1}$, then $A^{-1} = \left[(E_p \times \cdots \times E_1)^{-1}\right]^{-1} = E_p \times \cdots \times E_1$, which implies that A^{-1} results from successively applying $E_1, \ldots, E_p$ to I_n.

Theorem (Row Equivalency of the Invertible Matrix A and I_n).

An $n \times n$ matrix A is invertible if and only if $A \sim I_n$. In such a case, any sequence of elementary row operations that transforms A into I_n also transforms I_n into A^{-1}.

An Algorithm for Finding A^{-1}

Suppose that the $n \times n$ matrix A and identity matrix I_n are placed side-by-side to form the augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$. Observe that if elementary row operations were to be applied to $\begin{bmatrix} A & I_n \end{bmatrix}$, then identical elementary row operations would simultaneously be applied to A and I_n. Therefore, by the preceding theorem, if A is invertible, then there must exist elementary row operations such that $A \sim I_n$ and $I_n \sim A^{-1}$ when applied to $\begin{bmatrix} A & I_n \end{bmatrix}$.

Algorithm (Finding A^{-1}).

Row reduce the augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$. If A is row equivalent to the identity matrix I_n, then $\begin{bmatrix} A & I_n \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$ and if otherwise, A does not have an inverse.