Vector Spaces and Subspaces

A **vector space** is a nonempty set V of objects called *vectors* on which two operations, called *addition* and *scalar multiplication*, are defined. The operations of addition and scalar multiplication are subject to the ten vector space axioms and must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars c and d.

Axiom (The Vector Space Axioms).

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in V and let c and d be scalars.

1. The sum of u and v, denoted by $\mathbf{u} + \mathbf{v}$, is in V.

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

4. There is a zero vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

5. For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

9. $c(d\mathbf{u}) = (cd)\mathbf{u}$

10. $1\mathbf{u} = \mathbf{u}$

Using the Vector Space Axioms, it can be shown that both the zero vector $\mathbf{0} \in V$ and the vector $-\mathbf{u}$, called the *negative* of $\mathbf{u}$, are unique. It can also be shown that the properties $0\mathbf{u} = \mathbf{0}$, $c\mathbf{0} = \mathbf{0}$, and $-\mathbf{u} = (-1)\mathbf{u}$ hold for each $\mathbf{u} \in V$ and scalar c.

Example. For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials with $\deg(\mathbf{p}) \leq n$ consists of all polynomials of the form (1), where the coefficients $a_0, \ldots, a_n$ and variable t are real numbers. If $\mathbf{p}(t) = a_0 \neq 0$, then $\deg(\mathbf{p}) = 0$, and if all the coefficients $a_0, \ldots, a_n$ are zero, then $\mathbf{p}$ is called the *zero polynomial*, which acts as $\mathbf{0}$ in Axiom (4).

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \tag{1}$$

If $\mathbf{p}$ is given by equation (1) and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$, then the sum $\mathbf{p} + \mathbf{q}$ is defined by $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (a_0 + b_0) + (a_1 + b_1)\, t + \cdots + (a_n + b_n)\, t^n$, and the scalar multiple $c\mathbf{p}$ is defined by $(c\mathbf{p})(t) = c\mathbf{p}(t) = ca_0 + (ca_1)\, t + \cdots + (ca_n)\, t^n$. The definitions of $\mathbf{p} + \mathbf{q}$ and $c\mathbf{p}$ respectively satisfy Axioms (1) and (6) because $\mathbf{p} + \mathbf{q}$ and $c\mathbf{p}$ are polynomials of deg $\leq n$ and are thus elements of $\mathbb{P}_n$. Observe that Axiom (5) is satisfied in the fact that $(-1)\mathbf{p}$ acts as the negative of $\mathbf{p}$. Axioms (2), (3) and (7)-(10) follow from the properties of real numbers. ■

Subspaces

When a vector space U consists of some appropriate subset of vectors from a larger vector space V, the vector space U is said to be a *subspace* of the vector space V. In such a case, the subspace U only needs to satisfy three of the ten Vector Space Axioms, as the rest are inherited from the vector space V.

Definition (Subspace).

A **subspace** of a vector space V is a subset $U \subseteq V$ that has three properties.

a. The zero vector $\mathbf{0} \in V$ is also $\mathbf{0} \in U$.

b. The subspace U is *closed* under vector addition. That is, for each vector $\mathbf{u}, \mathbf{v} \in U$, the sum $\mathbf{u} + \mathbf{v} \in U$.

c. The subspace U is *closed* under scalar multiplication. That is, for each vector $\mathbf{u} \in U$ and scalar c, the scalar multiple $c\mathbf{u} \in U$.

The properties (a), (b), and (c) of subspaces correspond –respectively – to Axioms (4), (1), and (6) of vector spaces and therefore guarantee that a subspace U of a vector space V is itself a *vector space*. Axioms (2), (3), and (7)-(10) are automatically inherited from V in that they apply to all elements of V, and hence also apply to all elements of U. Observe that if $\mathbf{u} \in U$, then $(-1)\mathbf{u} \in U$ by property (c) and, since $(-1)\mathbf{u} = -\mathbf{u}$, Axiom (5) is therefore satisfied.

Example. The set U defined below is a subset of $\mathbb{R}^3$ that "looks" and "acts" like $\mathbb{R}^2$, although it is logically distinct from $\mathbb{R}^2$. Show that U is a subset of $\mathbb{R}^3$.

$$U = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Observe that if $U = \{\langle s\ t\ 0\rangle : s, t \in \mathbb{R}\}$, then $\mathbf{0} \in U$ because there exists s and t (that is, $s = t = 0$) such that $\langle s\ t\ 0\rangle = \mathbf{0} \in \mathbb{R}^3$. Also observe that U must be closed under vector addition and scalar multiplication because both operations produce vectors $\langle s\ t\ 0\rangle$ whose third entry is zero, and thus belong to U. Therefore, U satisfies the three properties of subspaces and is hence a subspace of $\mathbb{R}^3$. ■

Note. In general, if a plane in $\mathbb{R}^3$ does not pass through the origin, then the plane does not contain the zero vector $\mathbf{0} \in \mathbb{R}^3$ and hence is not a subspace of $\mathbb{R}^3$. Similarly, if a line in $\mathbb{R}^2$ does not pass through the origin, then the line does not contain the zero vector $\mathbf{0} \in \mathbb{R}^2$ and hence is not a subspace of $\mathbb{R}^2$.

Subspaces Spanned by a Set of Vectors

Consider the vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$, where V is a vector space, and suppose that $U = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Observe that since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$, the zero vector $\mathbf{0} \in U$ and hence U satisfies property (a) of subspaces. Also observe that by vector space Axioms (2), (3), and (8), the addition of vectors in U can be expressed by the following, where $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$ and $\mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ are arbitrary vectors in U.

$$\mathbf{u} + \mathbf{w} = (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) = (s_1 + t_1)\,\mathbf{v}_1 + (s_2 + t_2)\,\mathbf{v}_2$$

Furthermore, observe that by Axioms (7) and (8) of vector spaces, any scalar multiplication of vectors in U can be expressed as follows, where $c \in \mathbb{R}$.

$$c\mathbf{u} = c\,(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\,\mathbf{v}_1 + (cs_2)\,\mathbf{v}_2$$

Therefore, since the zero vector $\mathbf{0} \in U$, and since U is closed under vector addition and scalar multiplication (that is, for vectors in $U = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, vector addition and scalar multiplication can be expressed as linear combinations of $\mathbf{v}_1, \mathbf{v}_2 \in V$), U is a subset of V, and hence a vector space itself.

Theorem (The Subspace Spanned By $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$).

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V, and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is called the *subspace spanned* by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example. Consider the set U defined below. Show that U is a subspace of $\mathbb{R}^4$.

$$U = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

If the arbitrary vector in U is expressed as the sum of two column vectors in parametric vector form, then an arbitrary vector in U is of the following form.

$$\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} : \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the set $U = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and hence is a subspace of $\mathbb{R}^4$. ■