The Matrix of a Linear Transformation

If T is a *linear* transformation defined by $T : \mathbb{R}^n \to \mathbb{R}^m$, then T can be expressed by a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$, where the $m \times n$ matrix A is the *standard matrix* of T. That is, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, and the standard matrix A can be defined explicitly by observing how T acts on the columns of the identity matrix $I = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$. Observe that since $I\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, it follows from the linearity of T that $T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n)$, and hence $T(\mathbf{x})$ can also be expressed by $T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}\mathbf{x} = A\mathbf{x}$.

Theorem (Standard Matrix for the Linear Transformation T).

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix A, called the **standard matrix** of T, such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The standard matrix of T is the $m \times n$ matrix A whose jth column is the vector $T(\mathbf{e}_j) \in \mathbb{R}^m$, where $\mathbf{e}_j \in \mathbb{R}^n$ is the jth column of the identity matrix I.

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

Example. Find the standard matrix A of the transformation $T(\mathbf{x}) = 3\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

Observe that $T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, so $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. ■

Existence and Uniqueness Questions

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ if for each $\mathbf{b} \in \mathbb{R}^m$, there exists at least one $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$. That is, if the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.

Definition (Onto Linear Transformation).

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of *at least* one $\mathbf{x} \in \mathbb{R}^n$.

Consider again the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. If each $T(\mathbf{x}) \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$, then T is *one-to-one*. That is, in the context of uniqueness, if $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or no solution at all, then T is one-to-one.

Definition (One-to-One Linear Transformation).

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^m$.