Eigenvalues and Eigenvectors

An *eigenvector* of an $n \times n$ matrix A is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, called an *eigenvalue* of A, which describes how $\mathbf{x}$ changes when left-multiplied by A. Geometrically, an eigenvalue λ of an $n \times n$ matrix A describes whether an eigenvector $\mathbf{x}$ is contracted, dilated, reversed, or left unchanged when left-multiplied by A. In the context of linear transformations, an eigenvalue λ of an $n \times n$ matrix A describes how an eigenvector $\mathbf{x}$ changes under the action of A in the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Definition (Eigenvector, Eigenvalue).

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, and the scalar λ is an **eigenvalue** of A if there exists a nontrivial solution $\mathbf{x}$ to $A\mathbf{x} = \lambda\mathbf{x}$. If there exists a nontrivial solution $\mathbf{x}$ to $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector *corresponding* to the eigenvalue λ.

Example. Are $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ eigenvectors of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

Observe that $A\mathbf{u}$ can be expressed as $A\mathbf{u} = -4\mathbf{u}$, and thus $\mathbf{u}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -4$. Also observe that because $A\mathbf{v} \neq \lambda\mathbf{v}$ for any scalar λ, it follows that $\mathbf{v}$ is *not* an eigenvector of A.

$$\begin{aligned} A\mathbf{u} &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u} \\ A\mathbf{v} &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix} \end{aligned}$$

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Eigenvalues and Eigenspace

Consider the $n \times n$ matrix A and scalar λ. By definition, the scalar λ is an eigenvalue of A if and only if there exists a nontrivial solution $\mathbf{x}$ (that is, an eigenvector $\mathbf{x}$ corresponding to λ) to the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$. Observe that if $A\mathbf{x} = \lambda\mathbf{x}$, then $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$, and hence it follows that $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$. Therefore, a scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if there exists a nontrivial solution $\mathbf{x}$ to the homogeneous matrix equation $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ or, equivalently, if and only if the augmented matrix $\begin{bmatrix} A - \lambda I & \mathbf{0} \end{bmatrix}$ has at least one free variable. Notice that the set of all solutions to $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ is $\operatorname{Nul}(A - \lambda I) = \{\mathbf{x} \in \mathbb{R}^n : (A - \lambda I)\,\mathbf{x} = \mathbf{0}\}$, and hence $\operatorname{Nul}(A - \lambda I)$ is a subspace of $\mathbb{R}^n$ called the *eigenspace* of A corresponding to the eigenvalue λ, which consists of $\mathbf{0}$ and all eigenvectors corresponding to λ.

Example. Show that $\lambda = 7$ is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, and then find the eigenvectors corresponding to $\lambda = 7$.

The scalar $\lambda = 7$ is an eigenvalue of A if and only if there exists a nontrivial solution $\mathbf{x}$ to the homogeneous matrix equation $(A - 7I)\mathbf{x} = \mathbf{0}$, where $(A - 7I)$ is defined as

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

Let $\{\mathbf{a}_1, \mathbf{a}_2\}$ be the columns of the matrix $A - 7I$, then $-\mathbf{a}_1 = \mathbf{a}_2$, so $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a linearly dependent set and hence there exists a non trivial solution $\mathbf{x}$ to the matrix equation $(A - 7I)\mathbf{x} = \mathbf{0}$, which makes $\lambda = 7$ and eigenvalue of the matrix A. The eigenvectors corresponding to $\lambda = 7$ can be obtained by row reducing the augmented matrix $\begin{bmatrix} A - 7I & \mathbf{0} \end{bmatrix}$ as follows, which shows that the set of all eigenvectors corresponding to $\lambda = 7$ is $\{\mathbf{x} : x_2 \neq 0\}$.

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Example. Consider the matrix A defined below. An eigenvalue of A is $\lambda = 2$. Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 2$.

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

First note that $\mathrm{Nul}\,(A - 2I) = \{\mathbf{x} \in \mathbb{R}^n : (A - 2I)\,\mathbf{x} = \mathbf{0}\}$ is the eigenspace corresponding to the eigenvalue $\lambda = 2$. Since $\lambda = 2$ is an eigenvalue of A, there exists a nontrivial solution $\mathbf{x}$ to the matrix equation $(A - 2I)\,\mathbf{x} = \mathbf{0}$, and hence the augmented matrix $\begin{bmatrix} A - 2I & \mathbf{0} \end{bmatrix}$ has one or more free variables, which can be used to explicitly describe the spanning set of $\mathrm{Nul}\,(A - 2I)$.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} A - 2I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution $\mathbf{x}$ of $(A - 2I)\,\mathbf{x} = \mathbf{0}$ is defined as follows, and, since the spanning set $\{\mathbf{u}, \mathbf{v}\}$ contains nonzero vectors, it follows that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and hence a basis for $\mathrm{Nul}\,(A - 2I)$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : \mathbf{u} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

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Eigenvalues of Triangular Matrices

Suppose that A is an 3×3 upper triangular matrix, then it follows that $A - \lambda I$ is of the form (1). Recall that by definition, a scalar λ is an eigenvalue of A if and only if the matrix equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$ or, equivalently, if and only if the augmented matrix $\begin{bmatrix} A - \lambda I & \mathbf{0} \end{bmatrix}$ has at least one free variable.

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix} \tag{1}$$

Observe that since A is triangular, $A - \lambda I$ is also triangular, and thus the augmented matrix $\begin{bmatrix} A - \lambda I & \mathbf{0} \end{bmatrix}$ has a free variable if and only if at least one entry along the main diagonal of $A - \lambda I$ is 0. That is, $\begin{bmatrix} A - \lambda I & \mathbf{0} \end{bmatrix}$ has a free variable if and only if $A - \lambda I$ has an entry $a_{ij} - \lambda = 0$ and, such a case only occurs when $a_{ij} = \lambda$, where $i = j$. Thus, it follows that for a triangular matrix A, the eigenvalues of A are the entries $a_{11}, \ldots, a_{nn}$ along the main diagonal of A.

Theorem (Eigenvalues of a Triangular Matrix).

The eigenvalues of a triangular matrix are the entries on it's main diagonal.

Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly *dependent* set of eigenvectors that correspond to *distinct* eigenvalues $\{\lambda_1, \ldots, \lambda_r\}$ of an $n \times n$ matrix A. Notice that since $\mathbf{v}_1 \neq \mathbf{0}$ (all eigenvectors are nonzero), and since $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent, there must exist some vector in $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ that is a linear combination of the preceding vectors. Let i be the smallest index such that $\mathbf{v}_{i+1}$ is a linear combination of the preceding linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$, then it follows that there exists scalars $\{c_1, \ldots, c_i\}$ such that $c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i = \mathbf{v}_{i+1}$. Recall that each eigenvector in $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ corresponds to a distinct eigenvalue in $\{\lambda_1, \ldots, \lambda_r\}$, and hence by multiplying both sides of $c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i = \mathbf{v}_{i+1}$ by A and using the fact that $A\mathbf{v}_p = \lambda_p\mathbf{v}_p$ for each p, then it can be shown that if $c_1A\mathbf{v}_1 + \cdots + c_iA\mathbf{v}_i = A\mathbf{v}_{i+1}$, then it follows that $c_1\lambda_1\mathbf{v}_1 + \cdots + c_i\lambda_i\mathbf{v}_i = \lambda_{i+1}\mathbf{v}_{i+1}$ and, observe that by multiplying both sides of $c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i = \mathbf{v}_{i+1}$ by λ_{i+1} and subtracting the result from the equation $c_1\lambda_1\mathbf{v}_1 + \cdots + c_i\lambda_i\mathbf{v}_i = \lambda_{i+1}\mathbf{v}_{i+1}$, the following can be shown.

$$c_1 (\lambda_1 - \lambda_{i+1}) \mathbf{v}_1 + \cdots + c_i (\lambda_i - \lambda_{i+1}) \mathbf{v}_i = \mathbf{0}$$

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is linearly independent, and thus the weights $c_1 (\lambda_1 - \lambda_{i+1}), \ldots, c_i (\lambda_i - \lambda_{i+1})$ in equation ( ) must all be zero. Since every eigenvalue in $\{\lambda_1, \ldots, \lambda_r\}$ is distinct, it follows that $\lambda_i - \lambda_{p+1} \neq 0$ and therefore each $c_i = 0$. TBF

Theorem (Eigenvectors and Distinct Eigenvalues).

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ are eigenvectors that correspond to distinct eigenvalues $\{\lambda_1, \ldots, \lambda_r\}$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Eigenvectors and Difference Equations