Introduction to Linear Transformations

A *transformation* is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$ called the *image* of $\mathbf{x}$ under the action of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that $\mathbb{R}^n$ is the *domain* of T and $\mathbb{R}^m$ is the *codomain* (that is, the set of all *possible* values T) and, hence, the subset of $\mathbb{R}^m$ that consists of all $T(\mathbf{x})$ is the *range* of T. An $m \times n$ matrix A can be viewed as an object that *acts* on a vector $\mathbf{x} \in \mathbb{R}^n$ through multiplication to produce a new vector $A\mathbf{x} \in \mathbb{R}^m$. That is, in the context of the matrix equation $A\mathbf{x} = \mathbf{b}$ where A is $m \times n$, it can be said that multiplication by A transforms a vector $\mathbf{x} \in \mathbb{R}^n$ into a vector $\mathbf{b} \in \mathbb{R}^m$.

Matrix Transformations

A **matrix transformation** is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ that is based on the multiplication of a vector $\mathbf{x}$ by an $m \times n$ matrix A, denoted by $\mathbf{x} \mapsto A\mathbf{x}$. That is, for each vector $\mathbf{x} \in \mathbb{R}^n$, the image of $\mathbf{x}$ under the action of T is computed as $T(\mathbf{x}) = A\mathbf{x}$. Since each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$, the *range* of T, that is, the set of all images $T(\mathbf{x})$, is the set of all linear combinations of the columns of A. Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries because to compute $A\mathbf{x}$, a vector $\mathbf{x} \in \mathbb{R}^n$ must have the same number of entries as there are columns in the $m \times n$ matrix A.

Example. Consider the matrix A and vectors $\mathbf{u}$, $\mathbf{b}$, and $\mathbf{c}$ defined below.

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Let the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ such that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.
b. Find a vector $\mathbf{x} \in \mathbb{R}^2$ whose image under T is $\mathbf{b}$.
c. Is there more than one vector $\mathbf{x}$ whose image under T is $\mathbf{b}$?
d. Determine if $\mathbf{c}$ is in the range of the transformation T.

a. $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T is defined by the following.

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

b. If there exists a vector $\mathbf{x} \in \mathbb{R}^2$ whose image under T is $\mathbf{b}$, then there exists a vector $\mathbf{x} \in \mathbb{R}^2$ that satisfies the equation $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ defined by

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

Row reducing the augmented matrix $[A \quad \mathbf{b}]$ shows that $x_1 = 1.5$, $x_2 = -0.5$ and, therefore, the image of $\mathbf{x} = \langle 1.5, -0.5 \rangle$ under T is the vector $\mathbf{b}$.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

c. Any vector $\mathbf{x}$ whose image under T is the vector $\mathbf{b}$ must satisfy the equation $A\mathbf{x} = \mathbf{b}$, as defined in the solution to part (b). Since the reduced echelon form of $[A \quad \mathbf{b}]$ has *no* free variable, the equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution and, therefore, there is exactly one $\mathbf{x}$ whose image under T is $\mathbf{b}$.

d. If the vector $\mathbf{c}$ is in the range of the transformation T, then $\mathbf{c}$ is the image of some vector $\mathbf{x} \in \mathbb{R}^2$. That is, $T(\mathbf{x}) = A\mathbf{x} = \mathbf{c}$ for some vector $\mathbf{x}$. Observe that since $T(\mathbf{x}) = A\mathbf{x}$, part (d) is essentially asking if there *exists* a vector $\mathbf{x}$ whose image under T is $\mathbf{c}$ or, similarly, if there *exists* a vector $\mathbf{x}$ that satisfies the equation $A\mathbf{x} = \mathbf{c}$, which can be determined by row reducing $[A \quad \mathbf{c}]$.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The equation $0 = -35$ in row three shows that the system is inconsistent and, therefore, $\mathbf{c}$ is *not* in the range of the transformation T. ■

Example. Consider the 2×2 matrix A. If the transformation $T(\mathbf{x}) = A\mathbf{x}$ transforms a vector $\mathbf{x} \in \mathbb{R}^2$ into another vector $\mathbf{b} \in \mathbb{R}^2$, then $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a *shear transformation* that, when A acts on $\mathbf{x}$, displaces $\mathbf{x}$ in a fixed direction by an amount proportional to the columns of A. Consider $T(\mathbf{u})$, where $\mathbf{u} = (0, 2)$.

$$T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

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Linear Transformations

A **linear transformation** is a transformation (or mapping) T that preserves the operations of *vector addition* and *scalar multiplication.* Recall that if A is an $m \times n$ matrix, then the matrix-vector product $A\mathbf{x}$ (and hence the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$) have the properties $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v}$ and $A(c\mathbf{u}) = cA\mathbf{u}$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c. Therefore, to preserve the operations of vector addition and scalar multiplication, a linear transformation must have:

Definition (Linear Transformation).

A transformation (or mapping) T is **linear** if:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T.

2) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all vectors $\mathbf{u}$ in the domain of T.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then condition (1) in the preceding definition says that the result $T(\mathbf{u}+\mathbf{v})$ of first adding $\mathbf{u}+\mathbf{v}$ in $\mathbb{R}^n$ and then applying T to $\mathbf{u}+\mathbf{v}$ is the same as applying T to $\mathbf{u}$ and $\mathbf{v}$ and then adding $T(\mathbf{u}) + T(\mathbf{v})$ in $\mathbb{R}^m$. Observe that, as consequence of the two conditions in the preceding definition, if T is a linear transformation, then T has the following properties because, by condition (2) $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$ and, by conditions (1) and (2) together $T(c\mathbf{u}+d\mathbf{v}) = T(c\mathbf{u})+T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Fact (Properties of a Linear Transformation).

If T is a linear transformation, then T has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T and all scalars c and d.

$$T(\mathbf{0}) = \mathbf{0} \qquad T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

Example. For a given scalar r, let the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = r\mathbf{x}$. When $0 \leq r \leq 1$, the transformation T is called a *contraction* and, when $r > 1$, the transformation T is called a *dilation*. Let $r = 3$ and show that T (a dilation) is a linear transformation.

Consider the vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and scalars c_1, c_2. By the given definition of the transformation T, it can be shown that T can be expressed as

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= 3(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = 3c_1\mathbf{v}_1 + 3c_2\mathbf{v}_2 = c_1(3\mathbf{v}_1) + c_2(3\mathbf{v}_2) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) \end{aligned}$$

Therefore, T has the property $cT(\mathbf{u}) + dT(\mathbf{v})$ and is thus a linear combination.

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