Systems of Linear Equations

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form of equation (1), where b and the **coefficients** a_n are real or complex numbers and the **subscript** $n \in \mathbb{Z}^+$.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

A **system of linear equations**, or **linear system**, is a collection of one or more linear equations involving the *same* variables $x_1, \ldots, x_n$. A **solution** of a linear system is a list of numbers $(s_1, \ldots, s_n)$ that, when substituted for the variables $x_1, \ldots, x_n$, makes each linear equation a true statement. The **solution set** of a linear system is the set of all possible solutions.

Two linear systems are **equivalent** if they have the *same* solution set – that is, each solution of the first linear system is a solution of the second linear system, and each solution of the second linear system is a solution of the first.

Solutions to a System of Linear Equations

Finding the solution set of a linear system in *two* variables comes down to finding where the two lines *intersect*. The graphs of both linear equations in (2) are lines, denoted by l_1 and l_2. A pair of numbers (x_1, x_2) satisfies *both* linear equations in the system if and only if the point (x_1, x_2) lies on both lines l_1 and l_2.

$$\begin{aligned} a_1x_1 + a_2x_2 &= b \\ a_1x_1 + a_2x_2 &= b \end{aligned} \tag{2}$$

However, two lines need not intersect at a single point. Lines l_1 and l_2 could be parallel, or coincide and hence intersect at every point, which leads to the following fact about linear systems:

Fact (Solutions to a System of Linear Equations).

A system of linear equations either has

1. no solution
2. exactly one solution
3. infinitely many solutions

A linear system is said to be **consistent** if it has either one solution or infinitely many solutions, and **inconsistent** if it has no solution.

Matrix Notation

A **matrix** is a rectangular array of numbers that is used to compactly record the essential information of a linear system – consider the following linear system:

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 = b \\ a_1x_1 + a_2x_2 + a_3x_3 = b \\ a_1x_1 + a_2x_2 + a_3x_3 = b \end{aligned} \tag{3}$$

The **coefficient matrix**, or **matrix of coefficients**, of (3) is a matrix (4) that consists of the coefficients a_n of each variable x_n aligned in columns.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{bmatrix} \tag{4}$$

The **augmented matrix** of (3) is a matrix (5) that consists of the coefficient matrix and an additional column that contains the constants b, where b is the constant from the right side of the linear equations.

$$\begin{bmatrix} a_1 & a_2 & a_3 & b \\ a_1 & a_2 & a_3 & b \\ a_1 & a_2 & a_3 & b \end{bmatrix} \tag{5}$$

The *size* of a matrix represents the number of rows m and columns n it contains. Let $m, n \in \mathbb{Z}^+$ then an $\boldsymbol{m} \times \boldsymbol{n}$ **matrix** is a rectangular array of numbers with m rows and n columns.

Solving a Linear System

A linear system can be solved using algorithm called *Gaussian elimination* which replaces a given linear system with an equivalent linear system, with the same solution set, that is easier to solve. Roughly speaking, the elimination process begins by using the x_1 term of the first row R_1 to eliminate the x_1 terms in the other rows $R_2, \ldots, R_n$ and then using the x_2 term from the second row R_2 to eliminate the x_2 terms from the other rows $R_1, R_3, \ldots, R_n$ and so on. Three *elementary row operations* are used simplify a linear system.

Definition (Elementary Row Operations).

1. *Replacement.* Replace a row R_i with the sum of itself and a constant multiple c of another row R_j. That is, $R_i = cR_j + R_i$.
2. *Interchange.* Interchange two rows R_i and R_j. That is, $R_i \leftrightarrow R_j$.
3. *Scale.* Multiply all entries in a row by a constant c, where $c \neq 0$. That is, $R_i = cR_i$.

Example. Solve the linear system represented in the augmented matrix.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

The x_1 term of R_1 can be used to eliminate the x_1 term of R_3 through the replacement operation $R_3 = -5R_1 + R_3$.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{R_3 = -5R_1 + R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

The scaling operating $R_2 = \frac{1}{2}R_2$ can be used to obtain $1x_2$ for R_2 and, subsequently, the term $1x_2$ of R_2 can be used to eliminate the x_2 term of R_3 through the replacement operation $R_3 = -10R_2 + R_3$. Once the x_2 term of R_3 has been eliminated, R_3 can be scaled by $\frac{1}{30}$ for simplification.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -1 & 10 \end{bmatrix} \xrightarrow{R_3 = -10R_2 + R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix} \xrightarrow{R_3 = \frac{1}{30}R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The x_3 term from R_3 can now be used to eliminate the x_3 terms from R_1 and R_2 by replacing R_1 with $R_1 = -1R_3 + R_1$ and replacing R_2 with $R_2 = 4R_3 + R_2$.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 = -1R_3 + R_1} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 = 4R_3 + R_2} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The term $-2x_2$ of R_1 can now be eliminated though the operation $R_1 = 2R_2 + R_1$.

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 = 2R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore, the system has one solution: $(x_1, x_2, x_3) = (1, 0, -1)$.

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Row Equivalency and Reversible Row Operations

Two matrices are **row equivalent** if there is a sequence of elementary row operations that can transform one matrix into the other. The three elementary row operations are *reversible*. If two rows R_i and R_j are interchanged through the operation $R_i \leftrightarrow R_j$, they can be returned to their original positions through another interchange operation $R_i \leftrightarrow R_j$. If a row R_i is scaled by a nonzero constant c through the operation $R_i = cR_i$, the original row can be obtain through a subsequent operation $R_i = (\frac{1}{c})R_i$. If a row R_i is replaced with the sum of itself and a constant multiple c of another row R_j through the operation $R_i = cR_j + R_i$, the original row can be obtained through the operation $R_i = -cR_j + R_i$.

Suppose that the augmented matrix of a linear system has been changed to a new augmented matrix through the three elementary row operations. By considering each row operation performed, it can be seen that each solution $(s_1, \ldots s_n)$ of the original matrix is a solution $(s_1, \ldots s_n)$ of the new matrix.

Theorem (Solution Set of Row Equivalent Augmented Matrices).

If the augmented matrices of two linear systems are row equivalent, then the two linear systems have the same solution set.

Existence and Uniqueness Questions

A solution set for a linear system will contain either no solutions, one solution, or infinitely many solutions – to determine which possibility is true for a particular system, the two following questions are asked.

Question (Two Fundamental Questions about a Linear System).

1. Is the system consistent? That is, does at least one solution exist?

2. If a solution exists, it is the *only* one? That is, is the solution *unique*?

Example. Determine if the system is consistent.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

This is the same system from Example 1. Suppose that the necessary row operations have been performed to obtain the triangular form:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

At this point, the value of x_3 is known – if the value of x_3 were to be substituted into R_2, the value of x_2 could be determined and, by substituting x_3 and x_2 into

R_1, the value of x_1 could be determined. Therefore, a solution exists and the system is consistent.

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