Inner Product, Length, and Orthogonality

Consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, both expressed as $n \times 1$ matrices. Observe that since $\mathbf{u}$ is an $n \times 1$ matrix, it follows that $\mathbf{u}^T$ is a $1 \times n$ matrix, and hence the matrix product $\mathbf{u}^T\mathbf{v}$, called the *inner product* of $\mathbf{u}$ and $\mathbf{v}$ defined by equation (1), is a 1×1 matrix which is expressed as some scalar $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \sum_{i=1}^n u_i v_i$ because $\mathbf{u}^T\mathbf{v}$ is a linear combination of the columns in $\mathbf{u}^T$ using the corresponding entries in $\mathbf{v}$ as weights.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n \tag{1}$$

Example. Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix} 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = (2)(3) + (-5)(2) + (-1)(-3) = -1$$
$$\mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T\mathbf{u} = \begin{bmatrix} 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = (3)(2) + (2)(-5) + (-3)(-1) = -1$$

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Properties of the Inner Product

The commutativity property $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, as shown in the preceding example, holds for all vectors in $\mathbb{R}^n$ because if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then by the definition of the inner product, it follows that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \sum_{i=1}^n u_i v_i = \sum_{i=1}^n v_i u_i = \mathbf{v}^T\mathbf{u} = \mathbf{v} \cdot \mathbf{u}$. Observe that if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then by the definition of the inner product, the transpose property $(A + B)^T = A^T + B^T$, and right-distributive law $(A + B)C = AC + BC$, the inner product $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} + \mathbf{v})^T\mathbf{w} = \left(\mathbf{u}^T + \mathbf{v}^T\right)\mathbf{w} = \mathbf{u}^T\mathbf{w} + \mathbf{v}^T\mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ and therefore $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and c is a scalar, then by definition, the inner product $(c\mathbf{u}) \cdot \mathbf{v} = (c\mathbf{u})^T\mathbf{v}$. Observe that since $(cA)^T = cA^T$ and $c(AB) = (cA)B = A(cB)$ for any scalar c, it follows that the inner product $(c\mathbf{u}) \cdot \mathbf{v} = (c\mathbf{u})^T\mathbf{v} = c\left(\mathbf{u}^T\mathbf{v}\right) = c(\mathbf{u} \cdot \mathbf{v})$ and, again, by the property $c(AB) = (cA)B = A(cB)$, it also follows that $\mathbf{u} \cdot (c\mathbf{v}) = \mathbf{u}^T(c\mathbf{v}) = c\left(\mathbf{u}^T\mathbf{v}\right) = c(\mathbf{u} \cdot \mathbf{v})$ and hence $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalars c. Finally, observe that if $\mathbf{u} \in \mathbb{R}^n$, then $\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^T\mathbf{u} = \sum_{i=1}^n u_i u_i = \sum_{i=1}^n u_i^2$ and, since $u_i^2 \geq 0$ for all u_i, it follows that $\mathbf{u} \cdot \mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathbb{R}^n$. Also observe that $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\sum_{i=1}^n u_i^2 = 0$, in which case $u_1 = \cdots = u_n = 0$ and, therefore, the inner product $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Theorem (Properties of the Inner-Product).

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar.

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$

Note. Properties (b) and (c) of the inner product can be combined to produce (2).

$$(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1 (\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p (\mathbf{u}_p \cdot \mathbf{w}) \tag{2}$$

The Length of a Vector

If $\mathbf{v} \in \mathbb{R}^n$ where $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, then $\mathbf{v} \cdot \mathbf{v} \geq 0$ and, therefore, it follows that $\sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$ and hence the *length* of $\mathbf{v}$, denoted by $\|\mathbf{v}\|$, is defined.

Definition (Length of a Vector).

The **length** of $\mathbf{v} \in \mathbb{R}^n$ is a nonnegative scalar $\|\mathbf{v}\|$ defined by the following.

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2} \quad \text{where} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

Note. Geometrically, $\mathbf{v} \in \mathbb{R}^n$ has an initial point at $\mathbf{0} \in \mathbb{R}^n$ and terminal point at $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, so $\|\mathbf{v}\| = \sqrt{(v_1 - 0)^2 + \cdots + (v_n - 0)^2} = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and, therefore, the geometric notion of $\|\mathbf{v}\|$ coincides with the general notion of $\|\mathbf{v}\|$.

Observe that if $\mathbf{v} \in \mathbb{R}^n$ and c is a scalar, then $\|c\mathbf{v}\|^2 = (c\mathbf{v}) \cdot (c\mathbf{v}) = c^2\mathbf{v} \cdot \mathbf{v} = c^2\|\mathbf{v}\|^2$ and thus since $\|c\mathbf{v}\|^2 = c^2\|\mathbf{v}\|^2$, it follows that $\sqrt{\|c\mathbf{v}\|^2} = \sqrt{c^2\|\mathbf{v}\|^2}$ and $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$. That is, for any $\mathbf{v} \in \mathbb{R}^n$ and scalar c, the length of $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$. A *unit vector* is a vector $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ can be *normalized* to obtain a unit vector $\mathbf{u} \in \mathbb{R}^n$ in the *same direction* as $\mathbf{v}$ by letting $\mathbf{u} = (1/\|\mathbf{v}\|)\,\mathbf{v}$ because $\|\mathbf{u}\| = \big\|(1/\|\mathbf{v}\|)\,\mathbf{v}\big\| = (1/\|\mathbf{v}\|)\,\|\mathbf{v}\| = 1$ and, to verify that $\|\mathbf{u}\| = 1$, it suffices to show that $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 1$ because if $\|\mathbf{u}\| = 1$, then $\|\mathbf{u}\|^2 = 1$.

Example. Let $\mathbf{v} = \langle 1, -2, 2, 0 \rangle$. Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$.

First note that $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$, and hence $\|\mathbf{v}\| = \sqrt{9} = 3$. Therefore, $\mathbf{v}$ can be normalized to obtain the unit vector $\mathbf{u} = (1/\|\mathbf{v}\|)\,\mathbf{v}$ defined below.

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}\mathbf{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

Observe that $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (0)^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0 = 1$ and thus since $\sqrt{\|\mathbf{u}\|^2} = \|\mathbf{u}\| = 1$, it follows that $\mathbf{u} = \langle 1/3, -2/3, 2/3, 0 \rangle$ is a unit vector in the same direction as $\mathbf{v}$. ■

Example. Let $W = \text{Span}\{\mathbf{x}\}$ be a subspace of $\mathbb{R}^2$, where $\mathbf{x} = \big\langle 2/3, 1 \big\rangle$. Find a unit vector $\mathbf{z}$ that is a basis for W.

First note that since $W = \text{Span}\{\mathbf{x}\}$, it follows that $W = \{c\mathbf{x} : c \in \mathbb{R}\}$ and hence any nonzero $\mathbf{x} \in W$ is a basis for W. Let $\mathbf{y} = 3\mathbf{x} = \big\langle 2, 3 \big\rangle$, then $\|\mathbf{y}\|^2 = 2^2 + 3^2 = 13$ and $\|\mathbf{y}\| = \sqrt{13}$, so $\mathbf{y}$ can be normalized to obtain a unit vector $\mathbf{z}$ as follows.

$$z = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$$

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Distance in $\mathbb{R}^n$

Recall that if $a, b \in \mathbb{R}$, then the *distance* between a and b in $\mathbb{R}$ (the real number line) is $|a - b|$. The notion of distance in $\mathbb{R}$ is analogous to the notion of distance in $\mathbb{R}^n$.

Definition (Distance Between Two Vectors).

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the **distance** between $\mathbf{u}$ and $\mathbf{v}$ is $\text{dist}(\mathbf{u}, \mathbf{v}) = \big\|\mathbf{u} - \mathbf{v}\big\|$. That is, the distance between the vectors $\mathbf{u}$ and $\mathbf{v}$ is the length of the vector $\mathbf{u} - \mathbf{v}$.

Example. Find $\text{dist}(\mathbf{u}, \mathbf{v})$, where $\mathbf{u} = \big\langle 7, 1 \big\rangle$ and $\mathbf{v} = \big\langle 3, 2 \big\rangle$.

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \therefore \ \text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

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Orthogonal Vectors

The notion of *orthogonality* in $\mathbb{R}^n$ is analogous to the notion of *perpendicularity* in Euclidean geometry. If the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ each form a line which intersect one other at $\mathbf{0} \in \mathbb{R}^n$, then the two lines formed by $\mathbf{u}$ and $\mathbf{v}$ are geometrically perpendicular if and only if $\text{dist}(\mathbf{u}, \mathbf{v}) = \text{dist}(\mathbf{u}, -\mathbf{v})$ or $\big[\text{dist}(\mathbf{u}, \mathbf{v})\big]^2 = \big[\text{dist}(\mathbf{u}, -\mathbf{v})\big]^2$. Observe that since $\big[\text{dist}(\mathbf{u}, \mathbf{v})\big]^2 = \|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$ and $\big[\text{dist}(\mathbf{u}, -\mathbf{v})\big]^2 = \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$, properties (a) and (b) of the inner product can be used to express $\big[\text{dist}(\mathbf{u}, \mathbf{v})\big]^2$ and $\big[\text{dist}(\mathbf{u}, -\mathbf{v})\big]^2$ as follows.

$$\begin{aligned}
\big[\text{dist}(\mathbf{u}, \mathbf{v})\big]^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) + (-\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot (-\mathbf{v}) + (-\mathbf{v}) \cdot \mathbf{u} + (-\mathbf{v}) \cdot (-\mathbf{v}) \\
&= \|\mathbf{u}\|^2 + \|-\mathbf{v}\|^2 + 2\mathbf{u} \cdot (-\mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} \qquad (3) \\
\big[\text{dist}(\mathbf{u}, -\mathbf{v})\big]^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v} \qquad (4)
\end{aligned}$$

By equations (3) and (4), it follows that $\big[\text{dist}(\mathbf{u}, \mathbf{v})\big]^2 = \big[\text{dist}(\mathbf{u}, -\mathbf{v})\big]^2$ if and only if $2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$ or, equivalently, $\big[\text{dist}(\mathbf{u}, \mathbf{v})\big]^2 = \big[\text{dist}(\mathbf{u}, -\mathbf{v})\big]^2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Definition (Orthogonality).

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Note. Observe that since $\mathbf{0} \cdot \mathbf{v} = \mathbf{0}^T \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}^n$, it follows from the definition of orthogonality that $\mathbf{0}$ is orthogonal to *every* vector $\mathbf{v} \in \mathbb{R}^n$.

The *Pythagorean Theorem* follows directly from the derivation of equation (4) and the definition of orthogonality, as $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem (The Pythagorean Theorem).

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Orthogonal Complements

If a vector $\mathbf{z}$ is orthogonal to every vector $\mathbf{w} \in W$ (that is, if $\mathbf{z} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$), where W is a subspace of $\mathbb{R}^n$, then $\mathbf{z}$ is *orthogonal* to W, and the set of all $\mathbf{z}$ that are orthogonal to W, denoted by $W^\perp$, is the *orthogonal complement* of W. That is, the orthogonal complement of W is the set $W^\perp = \{\mathbf{z} : \mathbf{z} \cdot \mathbf{w} = 0, \mathbf{w} \in W \subseteq \mathbb{R}^n\}$.

Example. Let W be a plane through $\mathbf{0} \in \mathbb{R}^3$ and let L be a line through $\mathbf{0} \in \mathbb{R}^3$, where W and L are perpendicular. Observe that since W and L are perpendicular subspaces of $\mathbb{R}^n$, it follows that if $\mathbf{z} \in L$ and $\mathbf{w} \in W$, where $\mathbf{z}$ and $\mathbf{w}$ are nonzero, then every line segment $\overline{\mathbf{0z}} \in L$ is perpendicular to every line segment $\overline{\mathbf{0w}} \in W$ or, equivalently, $\mathbf{z} \cdot \mathbf{w} = 0$ for all $\mathbf{z} \in L$ and $\mathbf{w} \in W$. Therefore, it also follows that $L = \{\mathbf{z} : \mathbf{z} \cdot \mathbf{w} = 0, \mathbf{w} \in W \subseteq \mathbb{R}^n\} = W^\perp$ and $W = \{\mathbf{w} : \mathbf{w} \cdot \mathbf{z} = 0, \mathbf{z} \in L \subseteq \mathbb{R}^n\} = L^\perp$. ■

Suppose that $\mathbf{x} \cdot \mathbf{v}_i = 0$ for all $\mathbf{v}_i \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, where $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Observe that since every $\mathbf{w} \in W$ can be expressed as $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ with suitable scalars $c_1, \ldots, c_p$ it follows from equation (2) that the inner product $\mathbf{w} \cdot \mathbf{x}$ can be expressed as $\mathbf{w} \cdot \mathbf{x} = (c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) \cdot \mathbf{x} = c_1 (\mathbf{v}_1 \cdot \mathbf{x}) + \cdots + c_p (\mathbf{v}_p \cdot \mathbf{x})$ and, since $\mathbf{x} \cdot \mathbf{v}_i = 0$ for all $\mathbf{v}_i \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, it follows that $\mathbf{w} \cdot \mathbf{x} = c_1 (0) + \cdots + c_p (0) = 0$ and, thus, if $\mathbf{x} \cdot \mathbf{v}_i = 0$ for all $\mathbf{v}_i \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, where $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{x} \in W^\perp$. That is, if $\mathbf{x}$ is orthogonal to every vector in a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans a vector space W, then $\mathbf{x}$ is orthogonal to W or, equivalently, $\mathbf{x} \in W^\perp$.

Fact (Orthogonal Complement of W).

1) A vector $\mathbf{x} \in W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans W. That is, $\mathbf{x} \in W^\perp$ if and only if $\mathbf{x} \cdot \mathbf{v}_i = 0$ for all $\mathbf{v}_i \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, where $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

2) $W^\perp$, the orthogonal compliment of W, is a subspace of $\mathbb{R}^n$.

Proof. That $W^\perp$ is a subspace of $\mathbb{R}^n$ □

Remark. Recall that if A is an $m \times n$ matrix, then each row in A has n entries and hence can be expressed as a vector $\mathbf{r}_i \in \mathbb{R}^n$, where the *row space* of A, denoted by $\text{Row}(A)$, is the set $\text{Row}(A) = \text{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ and, since the rows of A are the columns of A^T, it follows that $\text{Col}(A^T) = \text{Row}(A)$.

Theorem (Orthogonal Complements of Fundamental Subspaces).

Let A be an $m \times n$ matrix. Then $\big[\text{Row}(A)\big]^\perp = \text{Nul}(A)$ and $\big[\text{Col}(A)\big]^\perp = \text{Nul}(A^T)$. That is, the orthogonal complement of $\text{Row}(A)$ is $\text{Nul}(A)$, and the orthogonal complement of $\text{Col}(A)$ is $\text{Nul}(A^T)$.