Row Reduction and Echelon Forms

In the definitions that follow, a *nonzero* row or column in a matrix is a row or column that contains at least one nonzero entry, and the *leading entry* of a row is the leftmost nonzero entry in a nonzero row.

Definition (Echelon Form, Reduced Echelon Form).

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are *above* any rows of all zeros.
2. Each leading entry of a row is in a column *to the right* of the leading entry of the row *above* it.
3. All entries in a column *below* a leading entry are zeros.

$$\begin{bmatrix} a_1 & a_2 & a_3 & b \\ 0 & a_2 & a_3 & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**).

1. The leading entry in each nonzero row is *1*.
2. Each leading entry (*1*) is the *only nonzero entry* in its column.

$$\begin{bmatrix} 1 & 0 & a_3 & b \\ 0 & 1 & a_3 & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An *echelon matrix* (respectively, *reduced echelon matrix*) is a matrix that is in echelon form (respectively, reduced echelon form). Any nonzero matrix A may be *row reduced* – that is, transformed through Gaussian elimination using the elementary row operations – into *more than one* matrix in echelon form using different sequences of row operations. However, the reduced echelon form of a matrix A that is obtained through Gaussian elimination is *unique.*

Theorem (Uniqueness of the Reduced Echelon Form).

Each matrix is row equivalent to *one and only one* reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix B, then B is called an *echelon form* (or *row echelon form*) of A. If a matrix A is row equivalent to a reduced echelon matrix B, then B is called *the reduced echelon* (or *the reduced row echelon form*) form of A.

Pivot Positions

When the elementary row operations are used on a matrix A to produce an echelon form of A, further row operations to obtain the reduced echelon form of A do *not* change the positions of the leading entries. Therefore, since the reduced echelon form of A is unique, the leading entry of each row R_n in A will always be in the *same position* in any echelon form obtained from A.

Definition (Pivot, Pivot Position, Pivot Column).

1) A **pivot** is a nonzero number in a pivot position that is used as needed to create zeros through the elementary row operations.
2) A **pivot position** in a matrix A is a location in A that corresponds to a leading entry 1 in the *reduced echelon form* of A.
3) A **pivot column** in a matrix A is a column in A that contains a pivot position.

Example. Row reduce matrix A to echelon form. Find the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

The top of the leftmost nonzero column is the first pivot position and, therefore, a nonzero entry, or *pivot*, must be placed in this position. R_4 contains a leading entry of 1, so R_1 and R_4 can be interchanged.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

The pivot in $R_{1,1}$ can now be used to create zeros (eliminate the leftmost entry of each row) in the first pivot column. This is done by replacing each row below R_1 with the sum of itself and R_1 scaled – the result is the following matrix. The pivot position in R_2 must be as far to the left as possible – therefore, the entry 2 in $R_{2,2}$ is chosen to be the next pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

The pivot in $R_{2,2}$ can now be used to create zeros in the second pivot column. This can be done by replacing R_3 with $R_3 = -\frac{5}{2}R_2 + R_3$ and replacing R_4 with $R_4 = \frac{3}{2}R_2 + R_4$. The result is the following matrix.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

R_1 and R_2 cannot be used to create a leading entry in column 3 because doing so would change the echelon arrangement of the leading entries that have already been produced. However, R_3 and R_4 can be interchanged to produce a leading entry in column 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix A is now in echelon form and thus reveals that columns 1, 2, and 4 are pivot columns.

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The Row Reduction Algorithm

The *row reduction algorithm* consists of 5 steps. Steps 1-4 produce a matrix in echelon form, and the 5^{th} step produces a matrix in reduced echelon form.

Algorithm (Row Reduction).

Step 1. Begin with the leftmost nonzero column. This is the pivot position. The pivot position is at the top.

Step 2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 3. Use row replacement operations to create zeros in all positions below the pivot.

Step 4. Cover (or ignore) the row containing the pivot position and cover all rows (if any) above it. Apply steps 1-3 to the sub-matrix that remains. Repeat this process until there are no more nonzero rows to modify.

Step 5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 through a scaling operation.

The combination of steps 1-4 is called the **forward phase** of the Row Reduction algorithm. The 5^{th} step, which produces the unique reduced echelon form, is called the **backward phase** of the Row Reduction algorithm.

(General) Solutions to Linear Systems

When the Row Reduction algorithm is applied to the augmented matrix of a linear system, the algorithm will lead directly to an *explicit description* of the system's solution set $\{x_{1,}\dots,x_n\}$. Let matrix (1) be the augmented matrix of a linear system that has been changed into the equivalent *reduced* echelon form.

$$\begin{array}{ccccc} x_1 & & +ax_3 & = b \\ & x_2 & +ax_3 & = b \\ & & 0 & = 0 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & a_3 & b \\ 0 & 1 & a_3 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

In a *reduced echelon matrix*, the variables that correspond to the pivot columns of the matrix are called **basic variables**. Any variable in a *reduced echelon matrix* that is *not* a basic variable is called a **free variable**. Therefore, in matrix (1), the basic variables are x_1 and x_2, while the free variable is x_3.

If a linear system is *consistent*, the general solution set of the system can be explicitly described by solving the *reduced echelon form* of the system for the basic variables in terms of the free variables – this operation is possible because in reduced echelon form, each basic variable is only placed in *one* equation.

A *free variable* has no limitations and can represent any value it needs to represent. Once the value of a free variable has been chosen, the free variable can be used to solve the reduced echelon form of the linear system. Therefore, every solution to the reduced echelon form of a linear system is determined by the free variable. So the solution of linear system in matrix (1) is determined by the value of the free variable x_3.

$$\begin{cases} x_1 = b - ax_3 \\ x_2 = b - ax_3 \\ x_3 \text{ is free} \end{cases} \tag{2}$$

Example. Find the general solution of the linear system of equations whose augmented matrix has been row reduced to the following echelon matrix.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

The given matrix is in echelon form, but a matrix needs to be in reduced echelon form before solving for the basic variables. Therefore, the row reduction algorithm should be performed. Let the symbol $\sim$ before a matrix indicate that the matrix is row equivalent to the matrix preceding it.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1, x_2, and x_5. Thus, the basic variables are x_2 and x_4, so the general solution is:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

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Parametric Descriptions of a Solution Set

A **parametric description** of a system's solution set describes a solution set in terms of *parametric* equations, where the free variables act as parameters. Solving a system of linear equations amounts to finding a parametric description of the solution set or determining that the solution set is empty.

To be consistent, the (arbitrary) convention is to always use free variables as the parameters when describing a solution set. If a system is inconsistent, the solution set is *empty*, even when the system has free variables – in such a case, the solution has *no* parametric representation.

Back Substitution

If a linear system has been transformed into an echelon form, but *not* reduced echelon form, a computer program will solve the system using **back substitution** method, rather than computing the reduced echelon form. In method of back substitution, the last equation of a system is solved first to obtain the value of x_n, and x_n is then substituted into the next equation to obtain the value of x_{n-1}, and so on, until the system has been solved.

Existence and Uniqueness Questions

If the matrix of a linear system has been row reduced into echelon form, the echelon matrix can be used to answer two fundamental questions about a system's solution set. Recall that a linear system can be solved using back substitution, so if there is no equation such as $0 = b$, the *existence* of a solution is verified. Also recall that free variables have no limitations, so if a matrix has been row reduced and contains free variables, the solution is *not unique.*

Remark. When a linear system is in echelon form and contains no equation of the form $0 = b$, where $b \neq 0$, every nonzero equation contains a basic variable with a nonzero coefficient. If the basic variables are completely determined (with no free variables), there exists a unique solution. If at least one of the basic variables can be expressed in terms of one or more free variables, there exists infinitely many solutions – that is, one solution for each value of the free variables.

Theorem (Existence and Uniqueness).

A linear system is consistent *if and only if* the rightmost column of the augmented matrix is *not* a pivot column – that is, *if and only if* an echelon form of the augmented matrix has *no* row of the following form, where $b \neq 0$.

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The Existence and Uniqueness Theorem, together with the Row Reduction Algorithm, can be used to find and describe *all* solutions of a linear system.

Strategy (Using Row Reduction to Solve a Linear System).

1. Write the augmented matrix of the system.
2. Use the Row Reduction Algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. Stop if there is no solution, go to the next step if otherwise.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.