Linear Independence

By examining a homogeneous linear system from the perspective of a vector equation $x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$ instead of a matrix equation $A\mathbf{x} = \mathbf{0}$, the relationship of the vectors in the underlying set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors can be determined.

Definition (Linear Independence, Linear Dependence).

1) An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \in \mathbb{R}^n$ is **linearly independent** if the equation $x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$ has *only* the trivial solution $\mathbf{x} = \mathbf{0}$.

2) An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \in \mathbb{R}^n$ is **linearly dependent** if there exists weights $x_1, \ldots, x_k$, not all zero, such that $x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$.

A *linear dependence relation* is a vector equation $x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$ that, when at least one weight $x_i \neq 0$, generalizes the possible relations of linear dependence among a linearly dependent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \in \mathbb{R}^n$.

Example. Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set and, if possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, then the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ and no free variables. Row operations on the associated augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$ show that x_3 is a free variable.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, each nonzero value of x_3 determines a nontrivial solution of the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, and hence the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. To find a possible linear dependence relation between $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, the reduced echelon form of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$ can be used to obtain general solutions for x_1, x_2, and x_3.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. So if $x_3 = 5$, then $x_1 = 10$, $x_2 = -5$ and, by substituting these values into the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, a possible linear dependence relation between $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is obtained.

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

■

Linear Independence of Matrix Columns

Recall that for a given matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, the matrix equation $A\mathbf{x} = \mathbf{0}$ can be expressed as the vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ and, therefore, each linear dependence relation among the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in A corresponds to a nontrivial solution of the matrix equation $A\mathbf{x} = \mathbf{0}$ if such a relation exists.

Fact (Linear Independence of Matrix Columns).

The columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of a matrix A are linearly independent if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution $\mathbf{x} = \mathbf{0}$.

Example. Are the columns of the matrix A linearly independent?

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

Row reducing the augmented matrix $[A \quad \mathbf{0}]$ of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ shows that x_1, x_2, and x_3 are basic variables.

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

Therefore, since $[A \quad \mathbf{0}]$ has no free variables, the corresponding system has only the trivial solution $\mathbf{x} = \mathbf{0}$ and, thus, the columns of A are linearly independent.

■

Linear Independence of Sets of One or Two Vectors

Consider the set $\{\mathbf{v}\}$ containing only one vector. Observe that if $\mathbf{v} = \mathbf{0}$, then infinitely many nontrivial solutions to the vector equation $x_1\mathbf{v} = x_1\mathbf{0} = \mathbf{0}$ exist and thus, the zero vector $\mathbf{0}$ linearly dependent. Also observe that if $\mathbf{v} \neq \mathbf{0}$, then the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution and, therefore, a set $\{\mathbf{v}\}$ containing only one vector is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

Example. Determine if the following set of vectors are linearly independent.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

a. Notice that because $\mathbf{v}_2 = 2\mathbf{v}_1$, the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are related through the vector equation $-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ which, since a nonzero weight $x_1 = -2$ exists that makes the equation true, implies that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

b. Suppose that x_1 and x_2 satisfy the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$. If $x_1 \neq 0$ so $\mathbf{v}_1 \neq \mathbf{0}$, then $\mathbf{v}_1 = (-x_2/x_1)\mathbf{v}_2$. But since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of one another, the result $\mathbf{v}_1 = (-x_2/x_1)\mathbf{v}_2$ is impossible and, therefore, $x_1 = x_2 = 0$. Thus, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. ■

Theorem (Linear Independence of the Set $\{\mathbf{v}_1, \mathbf{v}_2\}$).

A set of *two* vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly *dependent* if at least one of the vectors is a scalar multiple of the other. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly *independent* if neither of the vectors is a scalar multiple of the other.

Linear Independence of Sets of Two or More Vectors

Theorem (Characterization of Linearly Dependent Sets).

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors is linearly *dependent* if and only if at least *one* of the vectors $\mathbf{v}_n \in S$ is a linear combination of the other vectors. In particular, if S is linearly *dependent* and $\mathbf{v}_1 \neq \mathbf{0}$, then there exists a vector $\mathbf{v}_n \in S$, where $n > 1$, such that $\mathbf{v}_n$ is a linear combination of the preceding vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} \in S$.

Example. Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ defined below. Describe $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ and explain why a vector $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$ if and only if the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Observe that since $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of one another, the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and thus $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is a plane in $\mathbb{R}^3$. Suppose that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then there exists some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ that is a linear combination of the preceding vectors because $\mathbf{u} \neq \mathbf{0}$. Notice that since $\mathbf{v}$ is not a scalar multiple (that is, a linear combination) of $\mathbf{u}$, $\mathbf{w}$ must be a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and, therefore, $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$. ■

Consider the $m \times n$ matrix A with columns $\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$. If $m < n$, then will be more variables $x_1, \ldots, x_n$ than equations (that is, more columns than rows) in the augmented matrix $[A \quad \mathbf{0}]$ and, therefore, $[A \quad \mathbf{0}]$ must have a free variable. Thus, in such a case the corresponding homogeneous linear system $A\mathbf{x} = \mathbf{0}$ will have a nontrivial solution, which makes the columns $\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ of A linearly dependent.

Theorem (Sets with More Vectors than Entries).

Any set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \in \mathbb{R}^m$ is linearly *dependent* if $m < n$. That is, if S contains more vectors than there are entries in each vector, then S is linearly dependent.

Suppose that the set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \in \mathbb{R}^n$ contains the zero vector $\mathbf{0}$. By ordering the vectors such that $\mathbf{v}_1 = \mathbf{0}$, it can be shown that there exists a nonzero weight $x_1 = 1$ such that $1\mathbf{v}_1 + \cdots + 0\mathbf{v}_m = \mathbf{0}$, which makes S linearly dependent.

Theorem (Linear Dependence of a Set Containing $\mathbf{0}$).

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \in \mathbb{R}^n$ contains the zero vector $\mathbf{0}$, then S is linearly dependent.