# The Characteristic Equation

Let $A$ be the $2\times 2$ matrix defined below, then $\{\lambda : (A-\lambda I)\mathbf{x} = \mathbf{0} \text{ and } \mathbf{x} \neq \mathbf{0}\}$ is the set of all eigenvalues of $A$. That is, the set of all scalars $\lambda$ that satisfy the matrix equation $(A-\lambda I)\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is a nontrivial solution to $(A-\lambda I)\mathbf{x} = \mathbf{0}$, is the set of all eigenvalues of $A$. Recall that by the Invertible Matrix Theorem, a matrix $A-\lambda I$ is invertible if and only if the matrix equation $(A-\lambda I)\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ and, hence, each $\lambda$ in $\{\lambda : (A-\lambda I)\mathbf{x} = \mathbf{0} \text{ and } \mathbf{x} \neq \mathbf{0}\}$ can be identified by finding each $\lambda$ such that the matrix $A-\lambda I$ is *non*-invertible because, if $A-\lambda I$ is non-invertible, then $(A-\lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

Observe that the matrix $A-\lambda I$ fails to be invertible when $\det(A-\lambda I) = 0$, and hence the eigenvalues of $A$ are scalars $\lambda$ that satisfy $\det(A-\lambda I) = 0$ and, since $\det(A-\lambda I) = (a-\lambda)(d-\lambda) - bc$, it follows that the eigenvalues of $A$ are scalars $\lambda$ that satisfy $(a-\lambda)(d-\lambda)-bc = 0$. Thus, $\det(A-\lambda I)$ can be used to transform the matrix equation $(A-\lambda I)\mathbf{x} = \mathbf{0}$ into the scalar equation $\det(A-\lambda I) = 0$ in $\lambda$, called the *characteristic equation* of $A$, which can then be used to explicitly define $\{\lambda : (A-\lambda I)\mathbf{x} = \mathbf{0} \text{ and } \mathbf{x} \neq \mathbf{0}\}$, the set of all eigenvalues of $A$.

**Fact** (Characteristic Equation)

A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the **characteristic equation** $\det(A-\lambda I) = 0$.

**Example.** Find the characteristic equation of the matrix $A$ defined below.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First notice that $A$ is upper triangular, so $\det(A-\lambda I)$ is defined as follows.

$$\det(A-\lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

Therefore, the characteristic equation of $A$ is $(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$, $(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$, $(\lambda-5)^2(\lambda-3)(\lambda-1) = 0$ or, by expanding the product, the characteristic equation of $A$ is $\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$. ■

## The Characteristic Polynomial

If $A$ is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial $p(\lambda)$ with $\deg(p) = n$ called the **characteristic polynomial** of $A$ and, if a scalar $\lambda$ is an eigenvalue of $A$, then the algebraic *multiplicity* of $\lambda$ is the number of times $\lambda$ appears as a root of $p(\lambda) = 0$. That is, the algebraic multiplicity of an eigenvalue $\lambda$ is the power $m$ of the term $\lambda^m$ or $(c - \lambda)^m$ in $p(\lambda) = 0$, where $c$ is a constant.

**Example.** Let $p(\lambda) = \lambda^6 - 4\lambda^5 - 12\lambda^4$ be the characteristic polynomial of a $6 \times 6$ matrix $A$. Find each eigenvalue of $A$ and it's multiplicity.

Observe that $p(\lambda) = \lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4 (\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda - 6)(\lambda + 2)$, and hence by the characteristic equation $\lambda^4(\lambda - 6)(\lambda + 2) = 0$, it follows that the eigenvalues of $A$ are $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = -2$ with multiplicities 4, 1, and 1, respectively. ■

## Matrix Similarity

Two $n \times n$ matrices $A$ and $B$ are *similar* if there exists an invertible $n \times n$ matrix $P$ such that $P^{-1}AP = B$ and $A = PBP^{-1}$, where the matrix transformation $A \mapsto P^{-1}AP$ is called a *similarity transformation*.

**Theorem** (Similar Matrices)

If the $n \times n$ matrices $A$ and $B$ are similar, then $A$ and $B$ have the same characteristic polynomial $p(\lambda)$ and hence also have the same eigenvalues $\lambda_1, \ldots, \lambda_p$, which have the same multiplicities.

*Proof.* Observe that if $A$ and $B$ are similar, then there exists an invertible $n \times n$ matrix $P$ such that $B = P^{-1}AP$, and thus the matrix $B - \lambda I$ can be expressed by $B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$ and, by the multiplicative property of determinants $\det(AB) = \det(A) \cdot \det(B)$, it follows that $\det(B - \lambda I)$ can be expressed by the following.

$$\det(B - \lambda I) = \det\left[P^{-1}(A - \lambda I)P\right] = \det\left(P^{-1}\right) \cdot \det(A - \lambda I) \cdot \det(P)$$

Observe that since $\det(P^{-1}) \cdot \det(P) = \det(P^{-1}P) = \det(I) = 1$, it follows from the preceding equation that $\det(B - \lambda I) = \det(A - \lambda I)$ and, therefore, the matrices $A$ and $B$ have the same characteristic polynomial and eigenvalues, which have the same multiplicities. □