Assignment # 8 + 9

8.3

8.3.1)

A.) $\sum_{k=-1}^{4} k^2 = 1+0+1+4+9+16 = 31$

B.) $\sum_{k=0}^{4} 2^k = 1+2^1+2^2+2^3+2^4 = 31$

8.3.2)

A.) $\sum_{n=-2}^{7} n^5$ B.) $\sum_{n=-2}^{5} n$

8.4

8.4.1)

A.) Let $n=3$,

$$\sum_{j=1}^{3} j^2 = 1^2+2^2+3^2 = 14$$

$$\frac{3(3+1)(2(3)+1)}{6} = \frac{3\cdot 4\cdot 7}{6} = 14$$

} equivalent $\therefore$ $P(3)$ is true

B.) $P(k)$ is:

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

8.4.2)

A.) Let $p(n)$: $1^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

$\therefore$ $p(1)$: $1^3 = \frac{1^2(1+1)^2}{4} \Rightarrow 1 \equiv 1$ $\therefore$ $p(1)$ is true

Assume $p(k)$ is true,

where $P(k)$: $1^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$

↳ using this,

proving $P(k+1)$: $1^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^3(k+2)^3}{4}$

↳ using LHS + $p(k)$:

↳ $\frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k+1\right) = \frac{(k+1)^2(k+2)^2}{4} \equiv$ R.H.S

$\therefore$ $P(k+1)$ is true $\therefore$ $P(n)$ is true for all $n \in \mathbb{N}$

8.4 cont

8.4.3)

A. Base case: when $n = 2$, (using induction)

$3^2 > 2^2 + 2^2 \Rightarrow 9 > 8$ $\therefore$ base case holds

Inductive hypo.

Let $p(k)$ be true, where $p(k)$: $3^k > 2^k + k^2$.

Inductive step

prove $p(k+1)$: $3^{k+1} > 2^{k+1} + (k+1)^2$,

LHS: $3(3^k)$, using this,

$3^{k+1} > (2^k + k^2) \cdot 3$

$(2+1)(2^k + k^2)$

$= 2^{k+1} + 2k^2 + 2^k + k^2$

RHS: $2^{k+1} + k^2 + 2k + 1$

since $2^{k+1} + 2k^2 + 2^k + k^2 > 2^{k+1} + k^2 + 2k + 1$

by transitivity, and the former is less than 3^{k+1}

$\therefore p(k+1)$ is T!, for all $k \in \mathbb{N}$, $p(k)$ holds.

8.5

8.5.1)

A.) Base case ($n=1$): $\frac{3^{2(1)} - 1}{4} = \frac{8}{4} = 2$ $\therefore$ $p(1)$ holds

inductive hypo: Assume $p(k)$ is true: $3^{2k} - 1 = 4 \cdot x$

Inductive step: Prove $p(k+1)$ is true

$3^{2(k+1)} - 1 = 9(9^k) - 1 = 9(9^k) - 1 - 8 + 8$

$= 9(9^k - 1) + 8 = 9 \cdot 4x + 8 = 4(9x + 2)$

$\therefore$ since its a multiple of 4, it is divisible by 4

$\therefore P(k+1)$ is true $\therefore$ for all $n \in \mathbb{N}$, $P(k)$ holds

8.5 cont

8.5.3)

A.) Base case: since $C_0 = 5$ (given)
$C_0 = 5^{2^0} = 5^1 = 5$ ∴ base case holds

Inductive hypo: Assume results holds for $n = k$
and $k > 0$ ie $C_k = 5^{2^k} = (C_{k-1})^2$

Inductive step: prove $n = k+1$
$\therefore C_{k+1} = (C_{k+1-1})^2 = C_k^2 = (5^{2^k})^2$
$\therefore C_{k+1} = 5^{2^k} \cdot 5^{2^k} = 5^{2^k \cdot 2} = 5^{2^{k+1}}$

∴ $P(k+1)$ holds since it follows same form

∴ for $n \in \mathbb{N}$, $P(n)$ holds for all n

8.6

8.6.2)

A.) f_n satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$
∴ the aux eq is $x^2 = x + 1$, where
$x = \frac{1 \pm \sqrt{5}}{2}$

using thm.s, $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$ for some $A + B$

Base case: $f_0 = 0$ + $f_1 = 1$, using these values,
$A + B = 0$
$B = -A$

Inductive step: $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$ (f_{n+1})
" " $- A\left(\frac{1-\sqrt{5}}{2}\right) = 1$ $(\because B = -A)$
$A(\sqrt{5}) = 1 \Rightarrow A = \frac{1}{\sqrt{5}}$ ∴ $B = -\frac{1}{\sqrt{5}}$

substituting $A + B$ into f_n,
$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

∴ since we have a equality, for $n \in \mathbb{N}$, $f(n)$ holds.

8.8

8.8.2)

A.) Base case: $\varepsilon \in A^*$

Recur case: if $w \in A^*$, then:

1. $aw \in A^*$

2. $bw \in A^*$

8.8.4)

A.) Base case: $\varepsilon \in S$

recurs. case: if $x \in S$, then:

1. $x_0 \in S$ (if x has an even # of 1s)

2. $x_1 \in S$ (if x has an odd # of 1s)

8.8.6)

-5 base case

A) $S = \{-18, 13, -8, -3, 2, 7, 12, 17\}$

+5

8.8.7)

A.) $S = \{\lambda, a, \lambda b, \lambda ba, ab, aba, \lambda bb, \lambda bba, \lambda bab\}$

8.9

8.9.2)

A.) Base case: 'a' true because it fulfills all cond.

Induc. step: using struc ind, prove statement holds for xb & xa

$S = \{a, aa, \ldots\}$

- Rule 1: Is true, bc since x begins w/ 'a' then xb also begins w/ a

- rule 2: is true since x begins w/ 'a', xa also begins w/ 'a'

∴ by princ. of structural ind., all strings in S satisfy the property ∴ for all $x \in S$, all str start w/ 'a'

8.10

8.10.1)

A.)
```
def sumCubes(n):
    if n == 0:
        return 0
    else:
        return(sumCubes(n-1) + n**3)
```

8.10.2)

A.)
```
def sumOfOdds(n):
    if n == 1:
        return 1
    else:
        return (2*n-1) + sumOfOdds(n-1)
```

8.11

8.11.1)

A.) Base case: n=1, which returns 1 ∴ true

Ind. hypo: Assume f(k) is correct

ind step: prove f(k+1), n = k+1,

∴ s := sumCubes (k) ∴ using ,

Algorithm returns $n^3 + s$, which equals

$(k+1)^3 + s$, which proves f(k+1) ∴

by induction, $n \in \mathbb{N}$, f(n) holds

9.1

9.1.1)

A.) True

B.) false

9.1.2)

A.) 1, 2, 3, 4, 6, 8, 12, 24

B.) 1, 2, 3, 4, 6, 9, 12, 18, 36

9.1.3)

A.) 4

B.) 68

C.) 1

D.) -68

9.2

9.2.1)

A.) $46^{30} \bmod 9 = 46^{30 \bmod 9} = 46^{3} \bmod 9 = 1$

B.) $38^{7} \bmod 3 = 38^{7 \bmod 3} = 38^{1} \bmod 3 = 2$

C.) $-4469 \bmod 7 = 4$

9.2.3)

A.) $46^{10} \bmod 7 = (46^{10 \bmod 7}) \bmod 7 = 46^{3} \bmod 7 = 1$

9.2.5)

A.)

remainder (after mod 11)	values
0	{0, -110, 1111}
1	{232}
6	{17, -93}
9	{-57, 108, 130}

9.3

9.3.2)

A.) $\gcd(532, 15435) = 7$

$532 = 2^{2} \cdot 7 \cdot 19$

$15435 = 3^{2} \cdot 5 \cdot 7^{3}$

B.) $\gcd(648, 1083) = 3$

$648 = 2^{3} \cdot 3^{4}$

$1083 = 3 \cdot 19^{2}$

C.) $\text{lcm}(532, 1083) = 30324$

$532 = 2^{2} \cdot 7 \cdot \cancel{19}$

$1083 = 3 * 19^{2}$

$\rightarrow 2^{2} \cdot 3 \cdot 7 \cdot 19^{2}$