

# CPSC-354 Report

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## Abstract

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# 1 Introduction

## 2 Week by Week

### 2.1 Week 1

#### 2.1.1 HW 1 - MU Puzzle

The MU puzzle comes from the book *Gödel, Escher, Bach*. You start with the string **MI** and the goal is to turn it into **MU** by following four rules:

1. If a string ends with **I**, you can add a **U** at the end.
2. If a string starts with **M**, you can copy everything after the M.
3. If you see **III**, you can change it to **U**.
4. If you see **UU**, you can delete it.

The puzzle is about seeing if you can reach **MU** by only using these rules. It is not really about the letters themselves, but about how rules control what strings you can or cannot make.

#### 2.1.2 HW1

The MU puzzle comes from the book *Gödel, Escher, Bach*. You start with the string **MI** and the goal is to turn it into **MU** by following four rules:

1. If a string ends with **I**, you can add a **U** at the end.
2. If a string starts with **M**, you can copy everything after the M.
3. If you see **III**, you can change it to **U**.
4. If you see **UU**, you can delete it.

At first, I tried small derivations. For example:

$$\text{MI} \Rightarrow \text{MIU} \Rightarrow \text{MIUIU}$$

or duplicating I's:

$$\text{MI} \Rightarrow \text{MII} \Rightarrow \text{MIII}$$

From **MIII**, I can replace **III** with **U**, giving **MUI**, but not **MU**. Every time, an extra **I** is left over, and there is no rule that deletes a single **I**.

**Invariant Argument.** Let  $\#I(w)$  denote the number of **I**'s in string  $w$ . If we track  $\#I(w) \pmod{3}$ , we find:

- Rule 1:  $\#I$  unchanged.
- Rule 2:  $\#I$  doubles. Over  $\mathbb{Z}/3\mathbb{Z}$ ,  $1 \mapsto 2$ ,  $2 \mapsto 1$ , never 0.
- Rule 3: Removes 3 **I**'s, leaving the remainder mod 3 unchanged.
- Rule 4: Deletes **U**'s only, so  $\#I$  unchanged.

We start with **MI**, which has  $\#I = 1$ . This is congruent to 1  $(\bmod 3)$ . Because no rule ever makes  $\#I \equiv 0 \pmod{3}$ , it is impossible to reach a string with  $\#I = 0$ .

**Conclusion.** The target MU has  $\#I = 0$ , which is divisible by 3. Since that is unreachable from MI, the puzzle is unsolvable. As a student, the cool part here is that the solution isn't about brute-force trying rules—it's about spotting a hidden invariant (the number of I's mod 3) that blocks the path completely.

## 2.2 Week 2

### 2.2.1 HW2 - Abstract Rewriting Systems (ARS) Properties

**Problem.** Consider the following list of Abstract Rewriting Systems (ARSSs).

1.  $A = \emptyset$ .
2.  $A = \{a\}$  and  $R = \emptyset$ .
3.  $A = \{a\}$  and  $R = \{(a, a)\}$ .
4.  $A = \{a, b, c\}$  and  $R = \{(a, b), (a, c)\}$ .
5.  $A = \{a, b\}$  and  $R = \{(a, a), (a, b)\}$ .
6.  $A = \{a, b, c\}$  and  $R = \{(a, b), (b, b), (a, c)\}$ .
7.  $A = \{a, b, c\}$  and  $R = \{(a, b), (b, b), (a, c), (c, c)\}$ .

**Task.** Draw a picture for each ARS above (nodes = elements of  $A$ , arrows = pairs in  $R$ ). Then determine whether each ARS is *terminating*, *confluent*, and whether it has *unique normal forms*.

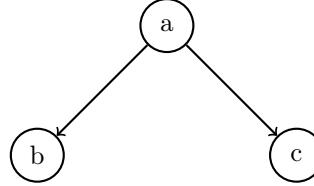
**ARS 1:**  $A = \emptyset$  (no elements to draw)



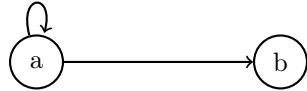
**ARS 2:**  $A = \{a\}$ ,  $R = \emptyset$



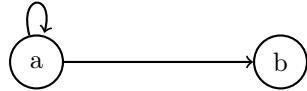
**ARS 3:**  $A = \{a\}$ ,  $R = \{(a, a)\}$

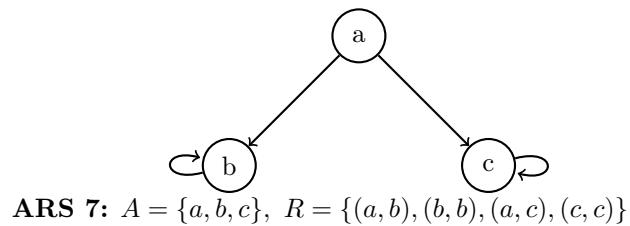
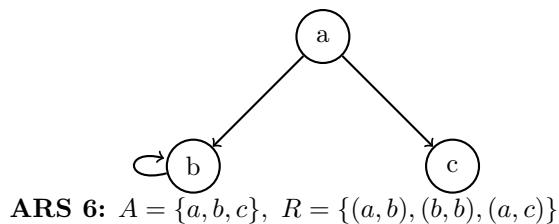


**ARS 4:**  $A = \{a, b, c\}$ ,  $R = \{(a, b), (a, c)\}$



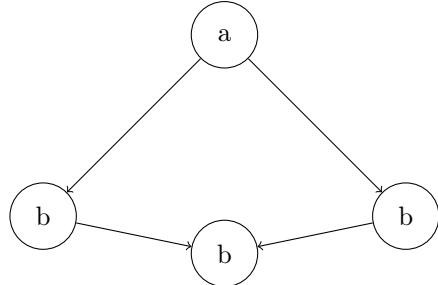
**ARS 5:**  $A = \{a, b\}$ ,  $R = \{(a, a), (a, b)\}$



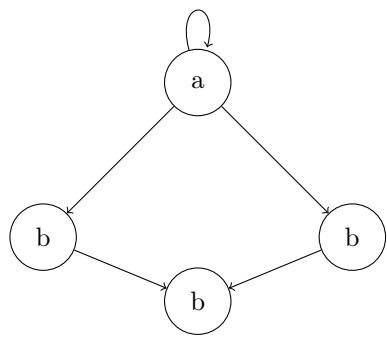


ARS	Terminating	Confluent	Has Unique Normal Forms
1	X	X	X
2	X	X	X
3		X	
4	X		X
5		X	X
6			
7			

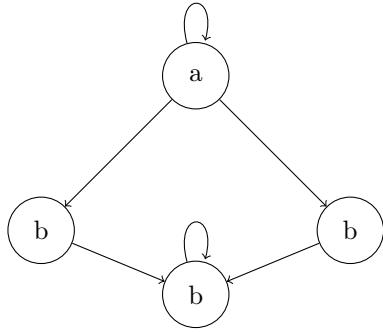
1. Confluent: True, Terminating: True, Unique Normal Forms: False



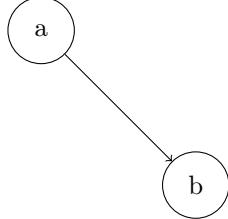
2. Confluent: True, Terminating: False, Unique Normal Forms: True



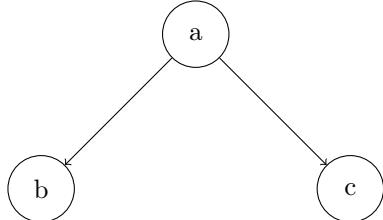
3. Confluent: True, Terminating: False, Unique Normal Forms: False



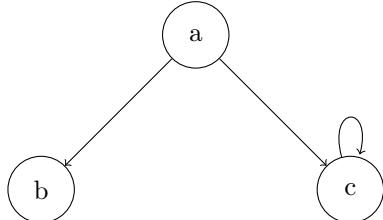
4. Confluent: False, Terminating: True, Unique Normal Forms: True



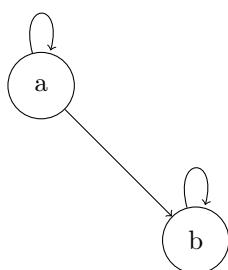
5. Confluent: False, Terminating: True, Unique Normal Forms: False



6. Confluent: False, Terminating: False, Unique Normal Forms: True



7. Confluent: False, Terminating: False, Unique Normal Forms: False



## 2.3 Week 3

### 2.3.1 HW 3 - Exercise 5: Reduction

**Exercise 5.**

*Rules:*

$$ab \rightarrow ba, \quad ba \rightarrow ab, \quad aa \rightarrow \epsilon, \quad b \rightarrow \epsilon$$

*Sample reductions:*

$$abba \rightarrow bbaa \rightarrow baa \rightarrow aa \rightarrow \epsilon$$

$$bababa \rightarrow aaabbb \rightarrow aabbb \rightarrow abbb \rightarrow a$$

*Non-termination:* The rules  $ab \rightarrow ba$  and  $ba \rightarrow ab$  form an infinite loop:

$$ab \rightarrow ba \rightarrow ab \rightarrow ba \rightarrow \dots$$

*Non-equivalent strings:*  $a$  and  $\epsilon$  are not equivalent, since a single  $a$  cannot be eliminated.

*Equivalence classes:* Order does not matter (due to swapping).  $b$ 's vanish.  $aa \rightarrow \epsilon$  ensures only the *parity* of the number of  $a$ 's matters.

$$I(w) = \#a(w) \bmod 2 \in \{0, 1\}$$

Thus there are exactly two equivalence classes:

$$\{w \mid \#a(w) \equiv 0 \pmod{2}\} \mapsto \epsilon$$

$$\{w \mid \#a(w) \equiv 1 \pmod{2}\} \mapsto a$$

*Modified terminating system:*

$$ab \rightarrow ba, \quad aa \rightarrow \epsilon, \quad b \rightarrow \epsilon$$

Termination follows from length and inversion-count measures.

*Specification:* The algorithm computes the *parity of the number of a's*, ignoring  $b$ 's.

**Exercise 5b.**

*Rules:*

$$ab \rightarrow ba, \quad ba \rightarrow ab, \quad aa \rightarrow a, \quad b \rightarrow \epsilon$$

*Sample reductions:*

$$abba \rightarrow bbaa \rightarrow baa \rightarrow aa \rightarrow a$$

$$bababa \rightarrow aaabbb \rightarrow aabbb \rightarrow abbb \rightarrow a$$

*Non-termination:* As before, infinite swapping is possible.

*Non-equivalent strings:*  $\epsilon$  and  $a$  are not equivalent: all  $b$ 's vanish, and any positive number of  $a$ 's reduces to  $a$ .

*Equivalence classes:* Order does not matter.  $b$ 's vanish.  $aa \rightarrow a$  collapses any positive number of  $a$ 's to a single  $a$ .

$$J(w) = \begin{cases} 0 & \text{if } \#a(w) = 0 \\ 1 & \text{if } \#a(w) \geq 1 \end{cases}$$

Thus there are exactly two equivalence classes:

$$\{w \mid \#a(w) = 0\} \mapsto \epsilon$$

$$\{w \mid \#a(w) \geq 1\} \mapsto a$$

*Modified terminating system:*

$$ab \rightarrow ba, \quad aa \rightarrow a, \quad b \rightarrow \epsilon$$

This terminates and yields unique normal forms.

*Specification:* The algorithm computes whether the input contains at least one  $a$ , ignoring all  $b$ 's.

## 2.4 Week 4

### 2.4.1 HW4 - Termination

For the definition of a *measure function*, see our notes on rewriting and, in particular, on termination.

**HW 4.1.** Consider the following algorithm (Euclid's algorithm for the greatest common divisor):

```
while b != 0:
    temp = b
    b = a mod b
    a = temp
return a
```

**Conditions.** Assume inputs  $a, b \in \mathbb{N}$  with  $b \geq 0$  and the usual remainder operation, i.e. for  $b > 0$  we have  $0 \leq a \bmod b < b$ . (If  $b = 0$ , the loop is skipped and the algorithm terminates immediately.)

**Measure function.** Define

$$\mu(a, b) := b \in \mathbb{N}.$$

**Proof of termination.** If the loop guard holds ( $b \neq 0$ ), one iteration maps the state  $(a, b)$  to

$$(a', b') = (b, a \bmod b).$$

By the property of the remainder,

$$0 \leq b' = a \bmod b < b = \mu(a, b).$$

Thus  $\mu$  strictly decreases on every loop iteration and is bounded below by 0. Since  $(\mathbb{N}, <)$  is well-founded, no infinite descending chain

$$\mu(a_0, b_0) > \mu(a_1, b_1) > \mu(a_2, b_2) > \dots$$

exists. Hence only finitely many iterations are possible; the loop terminates and the algorithm halts.  $\square$

**HW 4.2.** Consider the following fragment of merge sort:

```
function merge_sort(arr, left, right):
    if left >= right:
        return
    mid = (left + right) / 2
    merge_sort(arr, left, mid)
    merge_sort(arr, mid+1, right)
    merge(arr, left, mid, right)
```

Define

$$\phi(left, right) := right - left + 1.$$

**Claim.**  $\phi$  is a measure function for `merge_sort`.

**Proof.**

- *Well-defined, nonnegative.* For valid indices with  $left \leq right$ , we have  $\phi(left, right) \in \mathbb{N}$  and  $\phi \geq 1$ . If  $left > right$  the function is not called (or  $\phi \leq 0$ , and the base case applies immediately).
- *Base case.* When  $left \geq right$ , the function returns immediately; in this case  $\phi(left, right) \leq 1$ , i.e. there is no further recursion.
- *Strict decrease on recursive calls.* Suppose  $left < right$  and let  $n := \phi(left, right) = right - left + 1 \geq 2$ . With  $mid = \lfloor (left + right)/2 \rfloor$ :

$$\phi(left, mid) = mid - left + 1 \leq \lfloor \frac{n}{2} \rfloor < n,$$

$$\phi(mid + 1, right) = right - (mid + 1) + 1 = right - mid \leq \lceil \frac{n}{2} \rceil < n.$$

Thus both recursive calls strictly reduce the measure.

Since  $\phi$  maps each call to a natural number that strictly decreases along every recursive edge and is bounded below, there are no infinite descending chains. By well-founded induction on  $\phi$ , all recursive calls terminate.  $\square$

## 2.5 HW 5

### 2.5.1 Workout: Step-by-step $\alpha/\beta$ -reductions

**Problem.** Evaluate

$$(\lambda f. \lambda x. f(f x)) (\lambda f. \lambda x. f(f(f x))).$$

**Notation.** We use  $\rightsquigarrow_{\beta}$  for a single  $\beta$ -reduction step and “ $\alpha$ ” to indicate a capture-avoiding renaming of bound variables.

**Intuition.** The term  $\lambda f. \lambda x. f(f x)$  applies a function twice (*iterate-2*). The term  $\lambda f. \lambda x. f(f(f x))$  applies a function three times (*iterate-3*). Applying iterate-2 to iterate-3 yields iterate-9.

### Derivation.

$$\begin{aligned}
& (\lambda f. \lambda x. f(f x)) (\lambda f. \lambda x. f(f(f x))) \\
& \rightsquigarrow_{\beta} \lambda x. [(\lambda f. \lambda x. f(f(f x))) ((\lambda f. \lambda x. f(f(f x))) x)] \quad (\text{substitute } f := \lambda f. \lambda x. f(f(f x)) \text{ into } \lambda x. f(f x)) \\
& \stackrel{\alpha}{=} \lambda x. (\lambda f. \lambda y. f(f(f y))) ((\lambda f. \lambda u. f(f(f u))) x) \quad (\text{rename bound } x\text{'s to } y, u \text{ to avoid shadowing}) \\
& \rightsquigarrow_{\beta} \lambda x. (\lambda f. \lambda y. f(f(f y))) (\lambda u. x(x(x u))) \quad (\text{apply } \beta \text{ to } (\lambda f. \lambda u. f(f(f u))) x) \\
& \rightsquigarrow_{\beta} \lambda x. \lambda y. F(F(F y)) \quad \text{with } F := \lambda u. x(x(x u)) \quad (\text{apply } (\lambda f. \lambda y. f(f(f y))) F) \\
& = \lambda x. \lambda y. F(F(F y)) \\
& = \lambda x. \lambda y. F(F(x(x(x y)))) \quad (\text{since } F y = x(x(x y))) \\
& = \lambda x. \lambda y. F(x(x(x(x(x(x y)))))) \quad (\text{apply } F \text{ again; adds 3 more } x\text{'s: total 6}) \\
& = \lambda x. \lambda y. x(x(x(x(x(x(x(x(x y)))))))) \quad (\text{apply } F \text{ a third time; +3 more: total 9}).
\end{aligned}$$

### Normal form.

$$\boxed{\lambda x. \lambda y. \underbrace{x(x(x(x(x(x(x(x(x(x(x y)))))))))}_{9 \text{ applications of } x}}$$

So the result is the *iterate-9* operator: given  $x$  and  $y$ , it applies  $x$  to  $y$  nine times.

## 2.6 HW 6

### 2.6.1 Computing fact 3 via a Fixed Point Combinator

We use the computation rules

$$\begin{aligned}
\mathbf{fix} \ F & \rightarrow (F \ (\mathbf{fix} \ F)) && (\mathbf{fix}) \\
\mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 & \rightarrow ((\lambda x. e_2) e_1) && (\mathbf{let}) \\
\mathbf{let} \ \mathbf{rec} \ f = e_1 \ \mathbf{in} \ e_2 & \rightarrow \mathbf{let} \ f = (\mathbf{fix} \ (\lambda f. e_1)) \ \mathbf{in} \ e_2 && (\mathbf{let} \ \mathbf{rec})
\end{aligned}$$

and the usual  $\beta$ -reduction  $((\lambda x. e) v) \rightarrow e[x := v]$ , plus base computation rules

$$0 = 0 \rightarrow \mathbf{True}, \quad n > 0 \Rightarrow (n = 0) \rightarrow \mathbf{False}, \quad \mathbf{if} \ \mathbf{True} \ \mathbf{then} \ A \ \mathbf{else} \ B \rightarrow A, \quad \mathbf{if} \ \mathbf{False} \ \mathbf{then} \ A \ \mathbf{else} \ B \rightarrow B.$$

**Abbreviation.** Let

$$F \equiv \lambda f. \lambda n. \mathbf{if} \ (n = 0) \ \mathbf{then} \ 1 \ \mathbf{else} \ n * f(n - 1).$$

Then  $\mathbf{fact} \equiv \mathbf{fix} \ F$ .

**Goal.** Evaluate

$$\mathbf{let} \ \mathbf{rec} \ \mathbf{fact} = \lambda n. \mathbf{if} \ (n = 0) \ \mathbf{then} \ 1 \ \mathbf{else} \ n * \mathbf{fact}(n - 1) \ \mathbf{in} \ \mathbf{fact} \ 3.$$

```

let rec fact =  $\lambda n. \dots$  in fact 3
→ let fact = (fix ( $\lambda f. \lambda n. \dots$ )) in fact 3           (let rec)
→ (( $\lambda$ fact. fact 3) (fix F))                                (let)
→ (fix F) 3                                              ( $\beta$ )
→ (F (fix F)) 3                                         (fix)
→ (( $\lambda f. \lambda n. \text{if } (n = 0) \text{ then } 1 \text{ else } n * f(n - 1)$ ) (fix F)) 3   (def. of F)
→ ( $\lambda n. \text{if } (n = 0) \text{ then } 1 \text{ else } n * (\text{fix } F)(n - 1)$ ) 3          ( $\beta$ )
→ if (3 = 0) then 1 else 3 * (fix F)(2)                      ( $\beta$ )
→ 3 * (fix F)(2)                                           (arith. and if-False)

```

Now expand (fix F) 2:

```

(fix F)2 → (F(fix F))2                               (fix)
→ if (2 = 0) then 1 else 2 * (fix F)(1)            (def. F,  $\beta$ )
→ 2 * (fix F)(1)                                    (if-False)

```

Expand (fix F)1:

```

(fix F)1 → (F(fix F))1                               (fix)
→ if (1 = 0) then 1 else 1 * (fix F)(0)            (def. F,  $\beta$ )
→ 1 * (fix F)(0)                                    (if-False)

```

Expand (fix F)0:

```

(fix F)0 → (F(fix F))0                               (fix)
→ if (0 = 0) then 1 else 0 * (fix F)(-1)          (def. F,  $\beta$ )
→ 1                                                 (if-True)

```

Unwinding:

$$1 * (\text{fix } F)0 \rightarrow 1 * 1 \rightarrow 1, \quad 2 * (\text{fix } F)1 \rightarrow 2 * 1 \rightarrow 2, \quad 3 * (\text{fix } F)2 \rightarrow 3 * 2 \rightarrow 6.$$

fact 3 $\rightarrow^*$ 6
--------------------------

## 2.7 Week 7

### 2.7.1 HW7 - Parse Trees for Arithmetic Expressions

Using the context-free grammar:

```

Exp → Exp '+' Exp1
Exp1 → Exp1 '*' Exp2
Exp2 → Integer
Exp2 → '(' Exp ')',
Exp → Exp1
Exp1 → Exp2

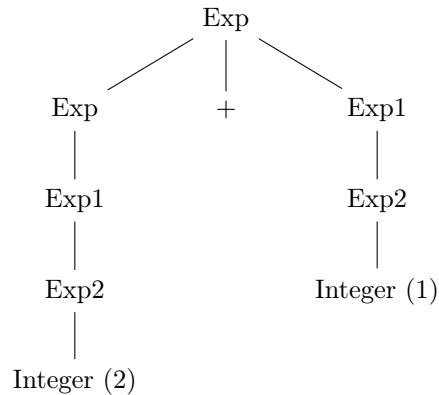
```

Write out the derivation trees (parse trees) for the following strings:

$2+1$ ,  $1+2*3$ ,  $1+(2*3)$ ,  $(1+2)*3$ ,  $1+2*3+4*5+6$

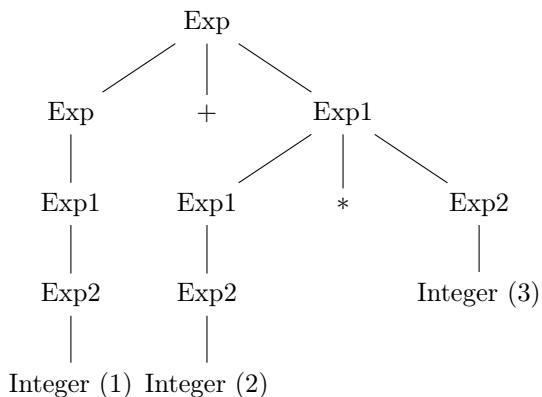
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**1. Parse tree for  $2 + 1$**



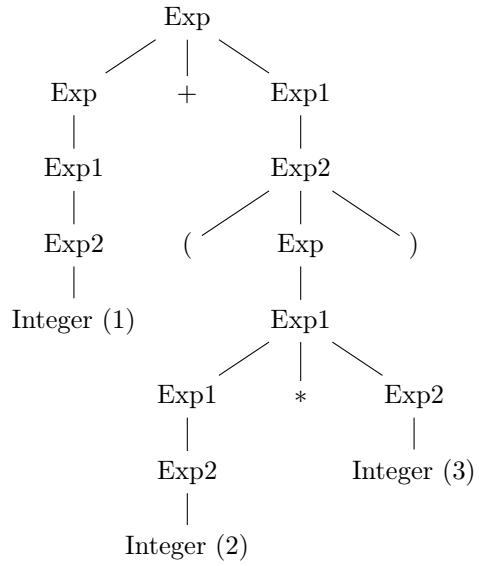
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**2. Parse tree for  $1 + 2 * 3$**



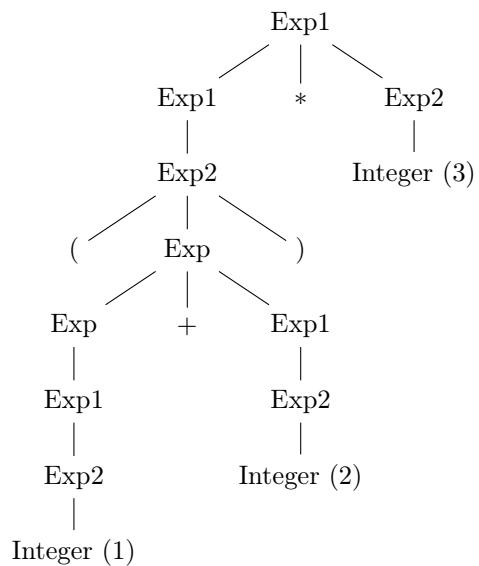
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**3. Parse tree for  $1 + (2 * 3)$**



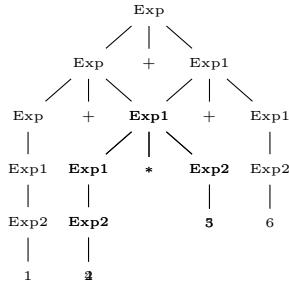

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4. Parse tree for  $(1 + 2) * 3$



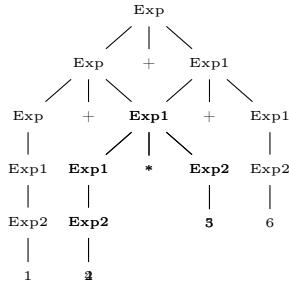

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5. Parse tree for  $1 + 2 * 3 + 4 * 5 + 6$




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## 5. Parse tree for $1 + 2 * 3 + 4 * 5 + 6$




---

### Notes:

- Each derivation strictly follows the grammar rules.
- Multiplication has higher precedence than addition, as enforced by the separation of Exp and Exp1.
- Parentheses in examples (3) and (4) override default precedence by forcing evaluation order.

## 2.8 Week 7

### 2.8.1 HW 8 Natural Numbers Game

#### Problem: 5

Prove that

$$a + (b + 0) + (c + 0) = a + b + c$$

for all natural numbers  $a, b, c \in \mathbb{N}$ .

#### Solution in Lean:

```
example (a b c : ℕ) : a + (b + 0) + (c + 0) = a + b + c := by
  rw [add_zero]
  rw [add_zero]
  rfl
```

#### Explanation:

- The lemma `add_zero` states that  $x + 0 = x$  for any natural number  $x$ .
- The first `rw [add_zero]` simplifies  $b + 0$  to  $b$ .

- The second `rw [add_zero]` simplifies  $c + 0$  to  $c$ .
- Finally, `rfl` (reflexivity) completes the proof since both sides are identical.

**Proof.** On the natural numbers, addition is defined so that  $x + 0 = x$  for every  $x$  (the identity law for 0), and the rest of addition is built recursively. Applying the identity law with  $x = b$  gives  $b + 0 = b$ , and with  $x = c$  gives  $c + 0 = c$ . Substituting these equalities into the left-hand side yields

$$a + (b + 0) + (c + 0) = a + b + c.$$

Both sides are now the same expression, so the equality holds.  $\square$

### Problem: 6

Prove that

$$a + (b + 0) + (c + 0) = a + b + c$$

for all natural numbers  $a, b, c \in \mathbb{N}$ , but this time tell Lean to simplify the  $c + 0$  term first.

### Solution

```
example (a b c : ℕ) : a + (b + 0) + (c + 0) = a + b + c := by
  rw [add_zero c]
  rw [add_zero b]
  rfl
```

### Explanation:

- The lemma `add_zero x` proves that  $x + 0 = x$ .
- Writing `rw [add_zero c]` explicitly tells Lean to apply this lemma to the term  $c + 0$  first.
- Then `rw [add_zero b]` simplifies  $b + 0$  to  $b$ .
- Finally, `rfl` completes the proof since both sides are identical.

**Result:** The equality holds, and the proof demonstrates how to use precision rewriting in Lean.

**Problem. 7** Prove that for all natural numbers  $n$ ,

$$\text{succ}(n) = n + 1.$$

### Lean solution:

```
theorem succ_eq_add_one (n : ℕ) : succ n = n + 1 := by
  rw [one_eq_succ_zero]
  rw [add_succ]
  rw [add_zero]
  rfl
```

### Explanation:

- We begin by rewriting 1 as  $\text{succ}(0)$  using `one_eq_succ_zero`.
- Next, we apply `add_succ` to expand  $n + \text{succ}(0)$  into  $\text{succ}(n + 0)$ .
- Then, the lemma `add_zero` simplifies  $n + 0$  to  $n$ .
- Finally, `rfl` (reflexivity) confirms that both sides are equal, proving that  $\text{succ}(n) = n + 1$ .

**Problem 8:** Prove that

$$2 + 2 = 4.$$

**Lean solution:**

```
example : 2 + 2 = 4 := by
  rw [two_eq_succ_one]
  rw [add_succ]
  rw [add_one_eq_succ]
  rfl
```

**Explanation:**

- We first rewrite 2 as  $\text{succ}(1)$  using `two_eq_succ_one`.
- Then `add_succ` expands the addition:  $2 + 2 = \text{succ}(1 + 1)$ .
- Next, `add_one_eq_succ` converts  $1 + 1$  into  $\text{succ}(1)$ .
- Finally, `rfl` (reflexivity) confirms both sides are equal, completing the proof that  $2 + 2 = 4$ .

## 2.9 Week9

### 2.9.1 hw9

**Level 5: add\_right\_comm**

**Theorem.** For all natural numbers  $a, b, c$ , we have

$$(a + b) + c = (a + c) + b.$$

This property is known as the *right commutativity of addition*.

**Solution 1 (Using Induction).** We can prove this by performing induction on one of the variables, for example  $c$ .

**Base Case:** Let  $c = 0$ . Then

$$(a + b) + 0 = a + b = (a + 0) + b,$$

where we use the identity property of addition ( $x + 0 = x$  and  $0 + x = x$ ).

**Inductive Step:** Assume the statement holds for some  $c = k$ , i.e.

$$(a + b) + k = (a + k) + b.$$

We must show it holds for  $c = k + 1$ . Then:

$$\begin{aligned} (a + b) + (k + 1) &= ((a + b) + k) + 1 && (\text{by definition of addition}) \\ &= ((a + k) + b) + 1 && (\text{by inductive hypothesis}) \\ &= (a + k) + (b + 1) && (\text{by associativity}) \\ &= (a + (k + 1)) + b && (\text{by definition of addition}). \end{aligned}$$

Thus, by induction,  $(a + b) + c = (a + c) + b$  for all  $c \in \mathbb{N}$ .

**Lean-style Inductive Proof:**

```

theorem add_right_comm (a b c : ℕ) : (a + b) + c = (a + c) + b := by
  induction c with d hd
  case zero =>
    rw [add_zero]
    rw [add_zero]
    rfl
  case succ =>
    rw [add_succ]
    rw [hd]
    rw [add_succ]
    rfl

```

---

**Solution 2 (Without Induction).** We can also prove  $(a + b) + c = (a + c) + b$  *without induction*, by using the results we have already established: the **associativity** and **commutativity** of addition.

**Proof:**

$$\begin{aligned}
 (a + b) + c &= a + (b + c) && (\text{by associativity}) \\
 &= a + (c + b) && (\text{by commutativity}) \\
 &= (a + c) + b && (\text{by associativity}).
 \end{aligned}$$

Hence  $(a + b) + c = (a + c) + b$ .

**Lean-style Non-Inductive Proof:**

```

theorem add_right_comm (a b c : ℕ) : (a + b) + c = (a + c) + b := by
  rw [add_assoc]
  rw [add_comm b c]
  rw [←add_assoc]
  rfl

```

## 2.10 week 10

### 2.10.1 hw10

**Problem: 6** Prove that if  $C \wedge D \rightarrow S$ , then  $C \rightarrow D \rightarrow S$ .

Given:  $h : (C \wedge D) \rightarrow S$

Goal:  $C \rightarrow D \rightarrow S$

Proof:

$C \rightarrow D \rightarrow S$  is shown by constructing a function:

$\lambda c d. h(\langle c, d \rangle)$

**Solution in Lean:**

```
exact fun c d => h ⟨c, d⟩
```

**Problem:7** Prove that if  $h : C \rightarrow D \rightarrow S$ , then  $C \wedge D \rightarrow S$ .

Given:  $h : C \rightarrow D \rightarrow S$ ,

Goal:  $C \wedge D \rightarrow S$ ,

Proof:  $\lambda(cd : C \wedge D). h(cd.\text{left})(cd.\text{right})$

**Solution in Lean:**

```
exact fun cd => h cd.left cd.right
```

**Problem: 8** Prove that if  $(S \rightarrow C) \wedge (S \rightarrow D)$ , then  $S \rightarrow (C \wedge D)$ .

Given:  $h : (S \rightarrow C) \wedge (S \rightarrow D)$ ,

Goal:  $S \rightarrow (C \wedge D)$ ,

Proof:  $\lambda s. \langle h.\text{left } s, h.\text{right } s \rangle$

**Solution in Lean:**

```
exact fun s => ⟨h.left s, h.right s⟩
```

**Problem:9** Prove that if  $R$  (Riffin brings a snack), then  $(S \rightarrow R) \wedge (\neg S \rightarrow R)$ .

Given:  $R$ ,

Goal:  $(S \rightarrow R) \wedge (\neg S \rightarrow R)$ ,

Proof:  $\lambda r. \langle (\lambda _s. r), (\lambda _{\neg s}. r) \rangle$

**Solution in Lean:**

```
exact fun r => ⟨fun _ => r, fun _ => r⟩
```

## 2.11 Week 11

### 2.11.1 hw11

#### Level 9 — Implies a Negation

**Given:**

$$h : P \rightarrow \neg A$$

**Prove:**

$$\neg(P \wedge A)$$

**Reasoning:**

1. Assume  $P \wedge A$  (the opposite of what we want to show).
2. From this, we can extract both  $P$  and  $A$ :

$$p := \text{fst}(P \wedge A), \quad a := \text{snd}(P \wedge A)$$

3. Since  $h : P \rightarrow \neg A$ , applying  $h$  to  $p$  gives  $h(p) : \neg A$  (which means  $A \rightarrow \perp$ ).
4. Applying  $h(p)$  to  $a$  yields a contradiction ( $\perp$ ).
5. Therefore, assuming  $P \wedge A$  leads to  $\perp$ , so we conclude:

$$\neg(P \wedge A)$$

**Lean one-line proof:**

```
exact fun hpa => (h hpa.left) hpa.right
```

## Level 10 — Conjunction Implication

Given:

$$h : \neg(P \wedge A)$$

Goal:

$$P \rightarrow \neg A$$

Reasoning:

1. Assume  $p : P$  (Pippin attends).
2. To prove  $\neg A$ , assume  $a : A$  (there is avocado).
3. From  $p$  and  $a$ , we get  $(p, a) : P \wedge A$ .
4. Applying  $h$  to this gives a contradiction ( $h(p, a) : \perp$ ).
5. Thus  $A$  cannot hold when  $P$  holds, so  $P \rightarrow \neg A$ .

Lean one-line proof:

```
exact fun p a => h ⟨p, a⟩
```

## Level 11 — not\_not\_not

Given:

$$h : \neg\neg\neg A$$

Goal:

$$\neg A$$

Reasoning:

1. To prove  $\neg A$ , assume  $A$ .
2. Then  $A$  implies  $\neg\neg A$  is false, because assuming  $\neg A$  would contradict  $A$ .
3. Thus, if we define  $\lambda na.na(a)$ , this represents a contradiction from  $\neg A$  and  $A$ .
4. Applying  $h$  to this function yields  $\perp$ , confirming  $\neg A$ .

Lean one-line proof:

```
exact fun a => h (fun na => na a)
```

## Level 12 — $\neg$ -Intro Boss

Given:

$$h : \neg(B \rightarrow C)$$

Goal:

$$\neg\neg B$$

Reasoning:

1. To prove  $\neg\neg B$ , assume  $\neg B$  and derive a contradiction.
2. From  $\neg B$ , we can define a function  $f : B \rightarrow C$  by saying: if we had  $b : B$ , we could produce any  $C$  (since  $\neg B$  means  $B \rightarrow \perp$ ).
3. This constructed function  $f$  makes  $h(f)$  yield  $\perp$ , a contradiction.
4. Therefore, assuming  $\neg B$  leads to  $\perp$ , so  $\neg\neg B$  holds.

**Lean one-line proof:**

```
exact fun b => h (fun _ => b)
```

### 3 Essay

### 4 Evidence of Participation

### 5 Conclusion

### References

[BLA] Author, [Title](#), Publisher, Year.