Derivation of the linear Boltzmann equation without cut-off starting from particles

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Abstract

We provide a rigorous derivation of the linear Boltzmann equation without cut-off starting from a system of particles interacting via a potential with infinite range as the number of particles N goes to infinity under the Boltzmann-Grad scaling. The main difficulty in our context is that, due to the infinite range of the potential, a non-integrable singularity appears in the angular collision kernel, making no longer valid the single-use of Lanford's strategy. Our proof relies then on a combination of Lanford's strategy, of tools developed recently by Bodineau, Gallagher and Saint-Raymond to study the collision process, and of new duality arguments to study the additional terms associated to the long-range interaction, leading to some explicit weak estimates.

1 Physical and historical contexts

In kinetic theory, a gas is described as a physical system constituted of a large number of small particles. The point of view adopted is a statistical one. The fundamental model is the evolution equation for the density of particles of a sufficiently rarefied gas first obtained by Boltzmann in 1872. One of its interesting aspects can be found in the fact that Boltzmann's kinetic equation can be used as an intermediate step in the transition between atomistic and continuous models for gas dynamics as it is mentioned in the famous sixth problem of Hilbert. Consequently, the problem of the derivation of kinetic transport equations from systems of particles is an issue which has been widely studied in the literature, especially in the context of the Boltzmann equation. The historical result in this field is due to Lanford [12] in the case of hard-spheres. He proved the convergence in the low density limit (only for short times). His proof has been recently improved by means of quantitative estimates by Gallagher, Saint-Raymond and Texier [8] and by Pulvirenti, Saffirio and Simonella [14] in the case of hard-spheres and short-range potentials. More recently, Bodineau, Gallagher and Saint-Raymond have been able in [7] to extend this result to any time interval $[0, t]$ with $t \ll \log \log N$ and overcome the difficulty of the short time validity in the particular case of a fluctuation around equilibrium, thereby enabling to reach a diffusive limit. Note that previous results without rate concerning the validity in the large of the linear Boltzmann equation were available (see [15, 13]).

However, so far the question of the convergence in the case of long-range potentials is still open. Indeed, for a long time Grad's cut-off assumption, which consists in postulating that the collision kernel is integrable with respect to the angular variable (see [9]), was crucial to work even at the level of the kinetic equation. The problem is that, in the case of infinite range forces, whatever the decay at infinity the huge amount of grazing collisions produces a non integrable singularity in the "angular collision kernel". However, recently several breakthroughs have been made regarding the Cauchy theory for this singular equation inciting us to reconsider this context. Let us mention the work of Alexandre on the Boltzmann linear equation [1], the results of Alexandre and Villani about the existence of renormalized solutions with defect measure [5] or more recently the series of papers of Alexandre et al [2, 3, 4] and Gressman and Strain [10] in the context of a perturbation around the equilibrium.

2 The hard spheres case

We start by explaining what tools are usually used to prove those types of results in the hard spheres case. In the following, we will point out that, because of the singularity in our case, this method has to be adapted and completed in order to obtain our result.

2.1 Formal derivation

We are interested in describing at the mesoscopic level the behavior of a gas constituted of N particles. We denote the positions by $X_N = (x_1, \dots, x_N)$ and the velocities by $V_N = (v_1, \dots, v_N)$. We will consider X_N in $(\mathbf{R}^d)^N$ and V_N in $(\mathbf{R}^d)^N$ where $\mathbf{R}^d$ is the d-dimensional torus. We denote $Z_N := (z_1, \dots, z_N)$ where $z_i := (x_i, v_i)$ for $1 \leq i \leq N$. With a slight abuse we say that Z_N belongs to $\mathbf{R}^{dN} \times \mathbf{R}^{dN}$ if X_N belongs to $\mathbf{R}^{dN}$ and V_N to $\mathbf{R}^{dN}$.

The microscopic model is given by : for $i \in [\![1, N]\!]$,

$$\begin{cases} \dfrac{dx_i}{dt} &= v_i, \quad x_i, v_i \in \mathbf{R}^d \\ \dfrac{dv_i}{dt} &= 0, \quad |x_i(t) - x_j(t)| > \varepsilon, \end{cases}$$

and

$$\begin{aligned} v_i' &= v_i + ((v_j - v_i) \cdot \nu)\nu \\ v_j' &= v_j - ((v_j - v_i) \cdot \nu)\nu \end{aligned}$$

if $|x_i - x_j| = \varepsilon$.

The Liouville equation satisfied by the N-particle distribution function f_N is

$$\partial_t f_N + \sum_{i=1}^{N} v_i . \nabla_{x_i} f_N = 0 \tag{2.1}$$

on $\mathcal{D}_\varepsilon^N := \{(x_1, v_1, \dots, x_N, v_N) \in \mathbf{R}^{2dN} | \forall i \neq j, |x_i - x_j| > \varepsilon\}$ with the boundary condition $f_N(t, Z_N^{\text{in}}) = f_N(t, Z_N^{\text{out}})$.
We denote the marginals of order s of f_N by $f_N^{(s)}(t, Z_s)$ and we define them as follows

$$f_N^{(s)}(t, Z_s) := \int f_N(t, Z_N) dz_{s+1} \dots dz_N. \tag{2.2}$$

Let us take an interest in the first marginal $f_N^{(1)}$. Applying Green's formula, we obtain the following equation :

$$\partial_t f_N^{(1)}(t, Z_1) + v_1 \cdot \nabla_{x_1} f_N^{(1)}(t, Z_1) = (N-1)\varepsilon^{d-1} \int_{\mathbf{S}^{d-1} \times \mathbf{R}^d} \left[f_N^{(2)}(t, x_1, v_1', x_1 + \varepsilon\nu, v_2') - f_N^{(2)}(t, x_1, v_1, x_1 - \varepsilon\nu, v_2) \right] ((v_2 - v_1) \cdot \nu)_+ d\nu dv_2. \tag{2.3}$$

The low density limit corresponds to a scaling called the Boltzmann-Grad scaling. It satisfies $N\varepsilon^{d-1} = 1$. Basically, this corresponds to a situation where the transport and collisions are at the same level. If we pass to the limit $N \to \infty$ in (2.3), we obtain

$$\partial_t f(t, Z_1) + v_1 \cdot \nabla_{x_1} f(t, Z_1) = \int_{\mathbf{S}^{d-1} \times \mathbf{R}^d} \left[f^{(2)}(t, x_1, v_1', x_1, v_2') - f^{(2)}(t, x_1, v_1, x_1, v_2) \right] ((v_2 - v_1) \cdot \nu)_+ d\nu dv_2 \tag{2.4}$$

Thus, if $f^{(2)}(t, Z_2) = f(t, z_1) f(t, z_2)$ then (2.4) is exactly the Boltzmann equation. Yet, this last point would mean that the particles 1 and 2 are independent. Of course, for N fixed, if

it is true at time 0, it won't be true at time t since the spheres interact. But the idea is to say that it will actually be true asymptotically in N. This is the key notion of what is called propagation of chaos.
The result obtained is the following and is originally due to Lanford:

Theorem 2.1 ([12, 8])**.** *Consider a system of N particles interacting as hard-spheres of diameter ε.*
Let $f_0 : \mathbf{R}^{2d} \to \mathbf{R}^+$ be a continuous density of probability such that

$$\|f_0 \exp(\frac{\beta}{2}|v|^2)\|_{L^\infty(\mathbf{R}^d_x \times \mathbf{R}^d_v)} < \exp(-\mu)$$

for some $\beta > 0$, $\mu \in \mathbf{R}$.
Assume that the N particles are initially identically distributed according to f_0 and "independent" (meaning the correlations vanish asymptotically). Then, there exists some $T^* > 0$ (depending only on β and μ) such that, in the Boltzmann-Grad limit $N \to \infty$, $N\varepsilon^{d-1} = 1$, the distribution function of the particles converges uniformly on $[0, T^*] \times \mathbf{R}^{2d}$ to the solution of the Boltzmann equation.

2.2 Lanford's strategy

A general strategy consists in using the Green's formula to obtain the following system of equations for $s < N$ which is called the BBGKY hierarchy:

$$(\partial_t + \sum_{i=1}^{s} v_i.\nabla_{x_i}) f_N^{(s)}(t, Z_s) = (\tilde{\mathcal{C}}_{s,s+1} f_N^{(s+1)})(t, Z_s) \tag{2.5}$$

on $\mathcal{D}^s_\varepsilon$ with the operator $\tilde{\mathcal{C}}_{s,s+1}$ defining the collision term as follows

$$\begin{aligned}
&(\tilde{\mathcal{C}}_{s,s+1} f_N^{(s+1)})(Z_s) \\
&:= (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} f_N^{(s+1)}(\dots, x_i, v_i^*, \dots, x_i + \varepsilon\nu, v_{s+1}^*) \left((v_{s+1} - v_i).\nu\right)_+ d\nu dv_{s+1} \\
&-(N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} f_N^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon\nu, v_{s+1}) \left((v_{s+1} - v_i).\nu\right)_- d\nu dv_{s+1}
\end{aligned} \tag{2.6}$$

where $\mathbf{S}^{d-1}$ denotes the unit sphere in $\mathbf{R}^d$, and v_i^* and v_{s+1}^* stand for the pre-collisional velocities for the particles i and $s+1$. Mild solutions of the hierarchy can then be defined by Duhamel's formula:

$$f_N^{(s)}(t) = \mathcal{T}_s(t) f_N^{(s)}(0) + \int_0^t \mathcal{T}_s(t - t_1)\tilde{\mathcal{C}}_{s,s+1} f_N^{(s+1)}(t_1) dt_1 \tag{2.7}$$

where we denote by $\mathcal{T}_s$ the group associated with free transport in $\mathcal{D}^s_\varepsilon$ with specular reflection on the boundary. The key point of Lanford's proof is the iterated Duhamel formula in order to express solutions of the BBGKY hierarchy in terms of a series of operators applied to the initial marginals:

$$f_N^{(s)}(t) = \sum_{n=0}^{N-s} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathcal{T}_s(t-t_1)\tilde{\mathcal{C}}_{s,s+1}\mathcal{T}_{s+1}(t_1-t_2)\tilde{\mathcal{C}}_{s+1,s+2}\dots\mathcal{T}_{s+n}(t_n) f_N^{(s+n)}(0) dt_n \dots dt_1. \tag{2.8}$$

The Boltzmann series expansion is obtained by taking the formal limit. The asymptotic expression for the collision operator is given by

$$\begin{aligned}
&(\tilde{\mathcal{C}}^0_{s,s+1} g^{(s+1)})(Z_s) \\
&:= \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} g^{(s+1)}(\dots, x_i, v_i^*, \dots, x_i, v_{s+1}^*) \left((v_{s+1} - v_i).\nu\right)_+ d\nu dv_{s+1} \\
&- \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} g^{(s+1)}(\dots, x_i, v_i, \dots, x_i, v_{s+1}) \left((v_{s+1} - v_i).\nu\right)_- d\nu dv_{s+1}
\end{aligned} \tag{2.9}$$

and this leads to the following expression for the Boltzmann series

$$g^{(s)}(t) = \sum_{n\geq 0}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \mathcal{T}_s^0(t-t_1)\tilde{\mathcal{C}}^0_{s,s+1}\mathcal{T}^0_{s+1}(t_1-t_2)\tilde{\mathcal{C}}^0_{s+1,s+2}\dots\mathcal{T}^0_{s+n}(t_n)g^{(s+n)}(0)dt_n\dots dt_1 \tag{2.10}$$

where we denote by $\mathcal{T}_s^0$ the free flow of s particles on $\mathbf{R}^{2ds}$.

Two steps are then necessary to prove Lanford's result:

- a uniform short time bound for the series expansion associated with the BBGKY hierarchy and the Boltzmann equation,
- the term by term convergence.

Let us be more precise and introduce the notion of pseudo-trajectory to explain the strategy of convergence. We introduce the following notation

$$\tilde{\mathcal{C}}_{s,s+1} = \sum_{i=1}^{s}\tilde{\mathcal{C}}^{+,i}_{s,s+1} - \tilde{\mathcal{C}}^{-,i}_{s,s+1} \tag{2.11}$$

where

$$\left(\tilde{\mathcal{C}}^{\pm,i}_{s,s+1}f_N^{(s+1)}\right)(Z_s) := (N-s)\varepsilon^{d-1}\int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} f_N^{(s+1)}(\dots,x_i,v_i',\dots,x_i+\varepsilon\nu,v'_{s+1})$$
$$((v_{s+1}-v_i).\nu)_\pm\, d\nu dv_{s+1} \tag{2.12}$$

with $v_i'=v_i^*$ and $v'_{s+1}=v^*_{s+1}$ for $\tilde{\mathcal{C}}^{+,i}_{s,s+1}$ and $v_i'=v_i$ and $v'_{s+1}=v_{s+1}$ for $\tilde{\mathcal{C}}^{-,i}_{s,s+1}$. We can do the same for $\tilde{\mathcal{C}}^0_{s,s+1}$.

Definition 2.2. We call elementary terms in the series the following elements

$$\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \mathcal{T}_s(t-t_1)\tilde{\mathcal{C}}^{j_1,m_1}_{s,s+1}\mathcal{T}_{s+1}(t_1-t_2)\tilde{\mathcal{C}}^{j_2,m_2}_{s+1,s+2}\dots\mathcal{T}_{s+n}(t_n)f_N^{(s+n)}(0)dt_n\dots dt_1 \tag{2.13}$$

with $(j_1,j_2,\dots,j_n)\in\{+,-\}$ and $m_i\in\{1,2,\dots,s+i-1\}$.

Each elementary term has a geometric interpretation as an integral over some pseudo-trajectory.

Definition 2.3. We call pseudo-trajectory associated with the BBGKY hierarchy and the elementary term

$$\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \mathcal{T}_s(t-t_1)\tilde{\mathcal{C}}^{j_1,m_1}_{s,s+1}\mathcal{T}_{s+1}(t_1-t_2)\tilde{\mathcal{C}}^{j_2,m_2}_{s+1,s+2}\dots\mathcal{T}_{s+n}(t_n)f_N^{(s+n)}(0)dt_n\dots dt_1 \tag{2.14}$$

the following description of the evolution of the positions and the velocities:

- We start at time t with s particles with the configuration $Z_s\in\mathbf{R}^{ds}\times\mathbf{R}^{ds}$. We denote by $\hat{\Psi}_s$ the backward s-particle flow. For $u\in[t_1,t]$, $Z_s(u):=\hat{\Psi}_s(u)Z_s$.
- The first collision operator $\tilde{\mathcal{C}}^{j_1,m_1}_{s,s+1}$ is interpreted as the adjunction at time t_1 of a new particle at $x_{m_1}(t_1)+\varepsilon\nu_{s+1}$ for a deflection angle $\nu_{s+1}\in\mathbf{S}^{d-1}$ and a velocity $v_{s+1}\in\mathbf{R}^d$, provided that there is no overlap. Depending on the sign of j_1, it means that the particle is added in incoming ($j_1=-$) or outgoing ($j_1=+$) collision configuration. In that last case, the particles will follow a backward scattering as it can be seen in the next item.
- Then Z_{s+1} evolves according to the backward $s+1$-particles flow $\hat{\Psi}_{s+1}$ during the time interval $[t_2,t_1]$ starting at t_1 from

$$\begin{aligned} Z_{s+1}(t_1) &= (\{z_j(t_1)\}_{j\neq m_1},(x_{m_1}(t_1),v_{m_1}(t_1)),(x_{m_1}(t_1)+\varepsilon\nu_{s+1},v_{s+1})) \quad if\ j_1=-\\ &= (\{z_j(t_1)\}_{j\neq m_1},(x_{m_1}(t_1),v^*_{m_1}(t_1)),(x_{m_1}(t_1)+\varepsilon\nu_{s+1},v^*_{s+1})) \quad if\ j_1=+. \end{aligned} \tag{2.15}$$

- We iterate this procedure by adding a particle labeled $s+i$ at time t_i at $x_{m_i}(t_i)+\varepsilon\nu_{s+i}$ for a deflection angle $\nu_{s+i}\in\mathbf{S}^{d-1}$ and a velocity $v_{s+i}\in\mathbf{R}^d$, provided that there is no overlap. The evolution of Z_{s+i} follows the flow of the backward $s+i$-particles flow $\hat{\Psi}_{s+i}$ during the time interval $[t_{i+1},t_i]$ starting at t_i from

$$\begin{aligned} Z_{s+i}(t_i) &= (\{z_j(t_i)\}_{j\neq m_i},(x_{m_i}(t_i),v_{m_i}(t_i)),(x_{m_i}(t_i)+\varepsilon\nu_{s+i},v_{s+i})) \quad if\ j_i=- \\ &= (\{z_j(t_i)\}_{j\neq m_i},(x_{m_i}(t_i),v^*_{m_i}(t_i)),(x_{m_i}(t_i)+\varepsilon\nu_{s+i},v^*_{s+i})) \quad if\ j_i=+. \end{aligned} \tag{2.16}$$

The elementary term can then be rewritten as follows

$$\varepsilon^{(d-1)n}(N-s)(N-s-1)\dots(N-s-n+1)\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}dt_n\dots dt_1$$
$$\int_{(\mathbf{S}^{d-1}\times\mathbf{R}^d)^n}d\nu_{s+1}\dots\nu_{s+n}dv_{s+1}\dots dv_{s+n}\mathbf{1}_{\{no\ overlap\}}\prod_{i=1}^n((v_{s+i}-v_{m_i}(t_i)).\nu_{s+i})\,f_N^{0(s+n)}(Z_{s+n}(0)) \tag{2.17}$$

where $Z_{s+n}(0)$ is the pseudo-trajectory at time 0.

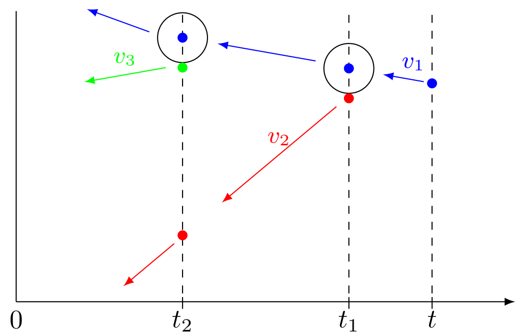

FIGURE 1. Representation of a pseudo-trajectory associated with the term $\displaystyle\int_0^t\int_0^{t_1}\mathcal{T}_1(t-t_1)\mathcal{C}_{1,2}\mathcal{T}_2(t_1-t_2)\mathcal{C}_{2,3}\mathcal{T}_3(t_2)f_N^{(3)}(0)dt_2dt_1$ for the BBGKY hierarchy.

We then give the definition of the two notions of collision and recollision.

Definition 2.4. We call a collision the creation of a particle in the process described above and a recollision the event when two particles collide in the flow $\hat{\Psi}_{s+i}$, with $0\le i\le N-s$.

Note that the pseudo-trajectories do not involve physical particles but are a geometric interpretation of the iterated Duhamel formula in terms of a branching process flowing backward in time and determined by

- the collision times $T:=(t_1,\dots,t_n)$ which are interpreted as branching times,
- the labels of the particles involved in the collisions $m:=(m_1,\dots,m_n)$ from which branching occurs and such that $1\le m_i\le s+i-1$ for all i,
- the coordinate of the initial particles Z_s at time t,
- the velocities $v_{s+1},\dots,v_{s+n}$ in $\mathbf{R}^d$ and deflection angles $\nu_{s+1},\dots,\nu_{s+i}\in\mathbf{S}^{d-1}$ for each additional particle.

Definition 2.5. We call pseudo-trajectory associated with the Boltzmann hierarchy and the elementary term

$$\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\mathcal{T}^0_s(t-t_1)\tilde{\mathcal{C}}^{0,j_1,m_1}_{s,s+1}\mathcal{T}^0_{s+1}(t_1-t_2)\tilde{\mathcal{C}}^{0,j_2,m_2}_{s+1,s+2}\dots\mathcal{T}^0_{s+n}(t_n)f_N^{(s+n)}(0)dt_n\dots dt_1 \tag{2.18}$$

the following description of the evolution of the positions and the velocities:

we start at time t with s particles with the configuration $Z_s^0 \in \mathbf{R}^{ds} \times \mathbf{R}^{ds}$. The $(s+k)^{th}$ particle is added at $x^0_{m_k}(t_k)$ with a velocity $v_{s+k} \in \mathbf{R}^d$. Then Z^0_{s+k} evolves according to the backward free flow denoted by $\hat{\Psi}^0_{s+k}$ during the time interval $[t_{k+1}, t_k]$ until the next creation, starting from

$$\begin{aligned} Z^0_{s+k}(t_k) &= \left(\{z_j^0(t_k)\}_{j\neq m_k}, (x^0_{m_k}(t_k), v_{m_k}(t_k)), (x^0_{m_k}(t_k), v_{s+k})\right) \quad \text{if } j_k = - \\ &= \left(\{z_j^0(t_k)\}_{j\neq m_k}, (x^0_{m_k}(t_k^+), v^*_{m_k}(t_k^+)), (x^0_{m_k}(t_k^+), v^*_{s+k})\right) \quad \text{if } j_k = +. \end{aligned} \tag{2.19}$$

The elementary term can then be rewritten as follows

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \dots dt_1$$

$$\int_{(\mathbf{S}^{d-1}\times\mathbf{R}^d)^n} d\nu_{s+1}\dots\nu_{s+n} dv_{s+1}\dots dv_{s+n} \prod_{i=1}^n \left((v_{s+i} - v_{m_i}(t_i)).\nu_{s+i}\right) g^{0(s+n)}(Z^0_{s+n}(0)) \tag{2.20}$$

where $Z^0_{s+n}(0)$ is the Boltzmann pseudo-trajectory at time 0.

Remark 2.1. *(i) The notion of collision is defined similarly as previously as the creation of a particle in the above process. Nevertheless, in the case of the pseudo-trajectory associated with the Boltzmann hierarchy, the particles are points and no recollision occurs in the branching process.*

(ii) We stress that, at time t, $Z_s = Z_s^0$.

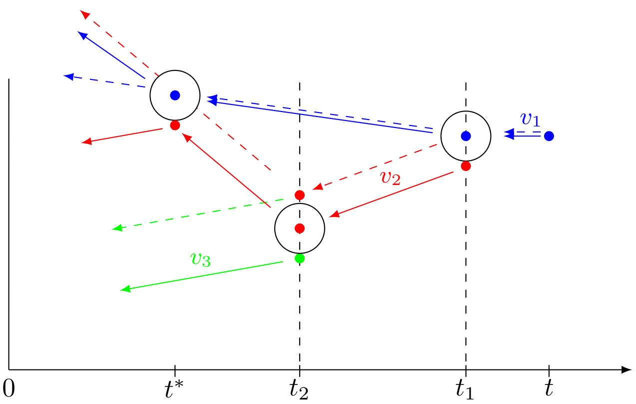

FIGURE 2. An example of a recollision between particles 1 and 2 at time t^*.

The key point to prove the convergence is actually to prove that the pseudo-trajectories associated with both series can be coupled precisely. Indeed, the differences between the BBGKY series and the Boltzmann series are the prefactors $(N-s)\varepsilon^{d-1}$, the micro-translation $x_i + \varepsilon\nu$ when a particle is created in the BBGKY pseudo-trajectory and most importantly the absence of recollisions in the case of the Boltzmann pseudo-trajectories. The two first points are easily dealt with by passing to the limit. The main concern of the proof is then to deal with the third one and prove that outside a geometrical ensemble of vanishing measure, no recollision occurs either for the BBGKY pseudo-trajectories.

2.3 Discussion of Lanford's result

Unfortunately, this result is actually only valid for short times. Indeed, it is due to the fact that, in Lanford's strategy, the bounds for the series are obtained by ignoring the compensation between gain and loss terms. As mentionned in the previous section, this difficulty is overcome by adopting a linear setting when considering a tagged particle in a gas at equilibrium, see [15, 13] and more recently [7]. The key point of this last proof is to exploit the maximum principle and establish global uniform a priori bounds for the distribution of particles, and more generally for all finite order marginals of the N-particle distribution.

Let us go back to the long-range interaction case. The issue is the following, no matter how decreasing the potential is taken, a non-integrable singularity in the angular collision kernel appears due to the huge amount of grazing collisions. By grazing collisions, we mean collisions with a very large impact parameter, the impact parameter being the distance of closest approach if the two particles move freely (so concretely, grazing collisions involve colliding particles which are barely deviated).

So the strategy consisting in ignoring the cancellations between the gain and the loss terms no longer works, even for a short time. Indeed, separating the gain and the loss terms no longer makes sense since the two integrals diverge in

$$\int\int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} f(v^*)f(v_1^*)b(v-v_1^*,\nu)dv_1d\nu - \int\int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} f(v)f(v_1)b(v-v_1,\nu)dv_1d\nu$$

So, the key point will be to adopt the linear setting mentioned above and especially to separate the contribution of the long-range interaction from the one of the "moderate-range" interaction by introducing a truncation parameter R. The "moderate-range" interaction part should be treated using exactly the same strategy as the one explained above in the hard-spheres case introducing some fictitious boundary at distance $R\varepsilon$ (see [11]). The long-range interaction part should be treated as an additional remainder term which will vanish in the limit provided that R goes to ∞. The main difficulty in that last term (which does not appear in the case of a short-range potential) is due to the presence of derivatives acting on the marginals. The strategy will be not to iterate on terms involving derivatives and to adopt a weak approach making the derivatives act on test functions. It is actually the core of our proof to develop new duality arguments to study those additional terms and then establish some weak estimates.

3 The case of an infinite range potential

3.1 Statement of the result

We are interested in a potential which satisifies the following assumption :

Assumption 3.1.1. $\Phi : \mathbf{R}^d\backslash\{0\} \to \mathbf{R}_+^*$ *is a radial, nonnegative, nonincreasing function which goes to zero at infinity and presents a singularity in* 0. *Moreover,* $\nabla\Phi$ *is a Lipschitz function with fast decay.*

The framework is the following: we examine a small perturbation around the equilibrium of a fixed number of particles. For the sake of simplicity, we initially perturb only one particle (which will be labeled by 1) with respect to the position x_1 of the tagged particle. In order to do so, we consider initial data of the form

$$f_N^0(Z_N) := M_{N,\beta}(Z_N)\rho^0(x_1) \tag{3.1}$$

where ρ^0 is a continuous density of probability on $\mathbf{T}^d$ and $M_{N,\beta}$ is the Gibbs measure defined as follows: for $\beta > 0$ given,

$$M_{N,\beta}(Z_N) := \frac{1}{\overline{\mathcal{Z}}_N}\left(\frac{\beta}{2\pi}\right)^{dN/2}\exp(-\beta H_N(Z_N)) \tag{3.2}$$

with $H_N(Z_N) := \sum_{1\leq i\leq N}\frac{1}{2}|v_i|^2 + \sum_{1\leq i<j\leq N}\Phi(\frac{x_i-x_j}{\varepsilon})$ and

$$\overline{\mathcal{Z}}_N := \int_{\mathbf{T}^{dN}\times\mathbf{R}^{dN}}\left(\frac{\beta}{2\pi}\right)^{dN/2}\exp(-\beta H_N(Z_N))dZ_N. \tag{3.3}$$

Theorem 3.1 ([6]). *We consider the initial distribution* f_N^0 *defined in* (3.1) *describing the state of a tagged particle in a background of* $N-1$ *particles at equilibrium. Under Assumptions 3.1.1 on the potential, for* $t>0$ *an arbitrary time, the distribution* $f_N^{(1)}(t,x,v)$ *of the tagged particle converges in* $\mathcal{D}'(\mathbf{T}^d\times\mathbf{R}^d)$ *when* N *goes to* ∞ *under the Boltzmann-Grad scaling* $N\varepsilon^{d-1}=1$ *to* $M_\beta(v)h(t,x,v)$ *where* $h(t,x,v)$ *is the solution of the linear Boltzmann equation without cut-off*

$$\partial_t h + v.\nabla_x h = -\int\int[h(v)-h(v^*)]M_\beta(v_1)b(v-v_1,\nu)dv_1d\nu \tag{3.4}$$

with initial data $\rho^0(x_1)$ *and where* $M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right)$, $\beta > 0$. *The cross-section* b *has a non-integrable singularity depending implicitly on* Φ.

3.2 Ideas of the proof

Due to the presence of the long-range potential, we artificially truncate the potential by considering truncated marginals $\tilde{f}^{(s)}_{N,R}$ defined as follows

$$\tilde{f}^{(s)}_{N,R}(t, Z_s) := \int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \prod_{\substack{1\le i\le s\\ s+1\le j\le N}} \mathbf{1}_{\{|x_i-x_j|>R\varepsilon\}} dZ_{(s+1,N)} \tag{3.5}$$

where $dZ_{(s+1,N)} := dz_{s+1}dz_{s+2}\dots dz_N$.

We consider Λ^R a smooth function such that

$$\Lambda^R(x) = \begin{cases} 1 & if\ |x| > R \\ 0 & if\ |x| < R-1. \end{cases}$$

We will denote $\Phi^>(x) := \Phi(x)\Lambda^R(x)$ and $\Phi^<(x) := \Phi(x)(1-\Lambda^R(x))$.

Applying Green's formula in a similar way as in [8], we obtain the following BBGKY hierarchy

$$\begin{aligned}
&\partial_t \tilde{f}^{(s)}_{N,R} + \sum_{i=1}^{s} v_i.\nabla_{x_i}\tilde{f}^{(s)}_{N,R} - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s} \nabla\Phi^<\big(\frac{x_i-x_j}{\varepsilon}\big).\nabla_{v_i}\tilde{f}^{(s)}_{N,R} \\
&\quad = \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s} \nabla\Phi^>\big(\frac{x_i-x_j}{\varepsilon}\big).\nabla_{v_i}\tilde{f}^{(s)}_{N,R} \\
&\quad + \frac{(N-s)}{\varepsilon}\sum_{i=1}^{s}\int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}} \nabla\Phi\big(\frac{x_i-x_{s+1}}{\varepsilon}\big).\nabla_{v_i} f_N(t,Z_N) \prod_{\substack{1\le l\le s\\ s+1\le k\le N}} \mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}} dZ_{(s+1,N)} \\
&\qquad\qquad + \mathcal{C}_{s,s+1}\tilde{f}^{(s+1)}_{N,R} + \mathcal{C}_{s,s+1}\overline{f}^{(s+1)}_{N,R}
\end{aligned} \tag{3.6}$$

where for $g_{s+1} : \mathbf{T}^{d(s+1)} \times \mathbf{R}^{d(s+1)} \to \mathbf{R}$

$$\begin{aligned}
\mathcal{C}_{s,s+1}g_{s+1}(Z_s) = (N-s)\sum_{i=1}^{s}\int_{S_{R\varepsilon}(x_i)\times\mathbf{R}^d} &\left(\prod_{\substack{j=1\\ j\neq i}}^{s} \mathbf{1}_{|x_j-x_{s+1}|>R\varepsilon}\right) \nu^{s+1,i}.(v_{s+1}-v_i) \\
&g_{s+1}(Z_{s+1})d\sigma_i(x_{s+1})dv_{s+1}
\end{aligned} \tag{3.7}$$

with $\nu^{s+1,i} = \dfrac{x_{s+1}-x_i}{|x_{s+1}-x_i|}$, $d\sigma_i$ is the surface measure on $S_{R\varepsilon} := \{x \in \mathbf{T}^d, |x-x_i| = R\varepsilon\}$ and

$$\begin{aligned}
\overline{f}^{(s+1)}_{N,R}(t, Z_{s+1}) := \int_{\mathbf{T}^{d(N-(s+1))}\times\mathbf{R}^{d(N-(s+1))}} & f_N(t, Z_N) \\
&\left(\prod_{\substack{1\le k\le s\\ s+2\le l\le N}} \mathbf{1}_{|x_k-x_l|>R\varepsilon}\right)\left(1-\prod_{j=s+2}^{N}\mathbf{1}_{|x_j-x_{s+1}|>R\varepsilon}\right) dZ_{(s+2,N)}.
\end{aligned} \tag{3.8}$$

We denote by $H^<_s$ the s-particle Hamiltonian defined as follows

$$H^<_s(Z_s) := \sum_{1\le i\le s}\frac{1}{2}|v_i|^2 + \sum_{1\le i<j\le s}\Phi^<\left(\frac{x_i-x_j}{\varepsilon}\right) \tag{3.9}$$

and we notice that $H^<_s$ depends on ε and R.

Mild solutions of the BBGKY hierarchy are thus defined by Duhamel's formula

$$
\begin{aligned}
\tilde f^{(s)}_{N,R}(t,Z_s) = \mathcal{S}_s(t)\tilde f^{(s)}_{N,R}(0,Z_s)&\\
+\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s}\int_0^t \mathcal{S}_s(t-t_1)&\left[\nabla\Phi^{>}(\frac{x_i-x_j}{\varepsilon}).\nabla_{v_i}\tilde f^{(s)}_{N,R}\right](t_1,Z_s)dt_1\\
+\frac{(N-s)}{\varepsilon}\sum_{i=1}^{s}\int_0^t \mathcal{S}_s(t-t_1)&\Bigg[\int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}}\nabla\Phi(\frac{x_i-x_{s+1}}{\varepsilon}).\nabla_{v_i}f_N\\
&\prod_{\substack{1\le l\le s\\ s+1\le k\le N}}\mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}}dZ_{(s+1,N)}\Bigg](t_1,Z_s)dt_1\\
+\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}&\tilde f^{(s+1)}_{N,R}(t_1,Z_s)dt_1\\
+\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}&\overline{f}^{(s+1)}_{N,R}(t_1,Z_s)dt_1
\end{aligned}
\tag{3.10}
$$

denoting by $\mathcal{S}_s$ the group associated with the solution operator

$$
\mathcal{S}_s(t): f\in\mathcal{C}^0(\mathbf{T}^{ds}\times\mathbf{R}^{ds};\mathbf{R})\mapsto f(\Psi_s(-t,.))\in\mathcal{C}^0(\mathbf{T}^{ds}\times\mathbf{R}^{ds};\mathbf{R}) \tag{3.11}
$$

where $\Psi_s(t)$ is the s-particle Hamiltonian flow associated with $H_s^<$. We notice that $\mathcal{S}_s$ depends on ε and R.

Before explaining the iteration strategy, let us point out four possible obstacles to the convergence:

- the very long-range interactions,
- clusters (or multiple simultaneous interactions),
- the presence of recollisions,
- a super-exponential collision process.

The strategy will be to iterate Duhamel's formula on a term where none of those four situations happens. The other terms where at least one of those four situations happen will give remainders and we will prove that they vanish in the limit.

Let us go back to (3.10). It seems then obvious that we will not iterate the Duhamel formula on the second and third terms of the right-hand side or the last one because they respectively are associated with the long-range interaction part and clusters. Moreover, two of them involve v-derivatives. So those terms will create remainders. The idea will be then to split the fourth term into two terms with one where no recollision happens. Finally on this recollision free term, we will iterate the Duhamel formula. Then, we will proceed as above to choose among the new terms the one involving no long-range interaction, no clusters and being recollision free in which we will again apply the Duhamel formula and so on. We will iterate this procedure a controlled number of times obtaining a term in which one the number of collision is not super exponential and a remainder where it is super exponential.

In order to establish the result, we then have to prove that the main term converges to the one associated with the Boltzmann equation and that the remainders vanish. Let us focus on the remainder associated with the long range part since the other ones will be handled quite similarly to the previous papers.
First, we point out that our iteration strategy is different from the ones developed in the previous papers. Indeed, we choose to do the truncations at each iteration step instead of doing them once all the iterations are done. This choice is linked to our framework. By adopting this method, we can actually handle the remainders associated with the long-range part. They have the following forms :

$$
r^{Pot,a}_{s,m+1}(0,t,Z_s) := \sum_{n=0}^{m}\int_0^{t-n\delta} Q_{s,s+n}(t-t_{n+1})\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s+n}\left[\nabla\Phi^{>}(\frac{x_i-x_j}{\varepsilon}).\nabla_{v_i}\tilde f^{(s+n)}_{N,R}\right](t_{n+1},Z_s)dt_{n+1}
$$

and

$$r_{s,m+1}^{Pot,b}(0,t,Z_s) := \sum_{n=0}^{m} \int_0^{t-n\delta} Q_{s,s+n}(t-t_{n+1})$$
$$\frac{(N-(s+n))}{\varepsilon} \sum_{i=1}^{s+n} \left[\int_{\mathbf{T}^{d(N-(s+n))}\times\mathbf{R}^{d(N-(s+n))}} \nabla\Phi(\frac{x_i - x_{s+n+1}}{\varepsilon}).\nabla_{v_i} f_N \right.$$
$$\left. \prod_{\substack{1\leq l\leq s+n \\ s+n+1\leq k\leq N}} \mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}} dZ_{(s+n,N)} \right] (t_{n+1},Z_s)dt_{n+1}.$$

with

$$Q_{s,s}(t) := \mathcal{S}_s(t)$$

and

$$Q_{s,s+n}(t) := \int_0^{t-\delta}\int_0^{t_1-\delta}\cdots\int_0^{t_{n-1}-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\chi_{H_{s+1}}\left(1-\chi_{geom(s+1)}\right)\chi_{\eta_s+1}\dots$$
$$\dots\mathcal{S}_{s+n-1}(t_{n-1}-t_n)\mathcal{C}_{s+n-1,s+n}\chi_{H_{s+n}}\left(1-\chi_{geom(s+n)}\right)\chi_{\eta_{s+n}}\mathcal{S}_{s+n}(t_n)dt_n\dots dt_1.$$

Because of our strategy, on those terms we know that there is no pathological situations. Indeed, the indicator functions appareating in the operators $Q_{s,s+n}(t)$ prevent this. Thus, it is easy to to pass from $\mathbf{Z_m(t_m)}$ to $\mathbf{z}$ (state of particle 1 at time t) via changes of variables such as

$$\begin{array}{rcl} \mathbf{T}^d\times\mathbf{R}^d\times[0,t-\delta]\times\mathbf{S}^{d-1}\times\mathbf{R}^d\times\cdots\times[0,t_{m-1}-\delta]\times\mathbf{S}^{d-1}\times\mathbf{R}^d & \to & \mathbf{T}^{(m+1)d}\times\mathbf{R}^{(m+1)d} \\ (z,t_1,\nu_2,v_2,\dots,t_m,\nu_{m+1},v_{m+1}) & \mapsto & \tilde{Z}_{m+1} = Z_{m+1}(t_m). \end{array}$$

Actually, because of Assumption (3.1.1), we can say more: z is a Lipschitz function of $(\tilde{x}_1,\tilde{v}_1,\dots,\tilde{x}_{m+1},\tilde{v}_{m+1})$. We can prove it studying the reduced dynamics and using Cauchy-Lipschitz theorem (see [6] for more details). Because of that Lipschitz control associated with the pseudo-trajectories, with some a priori estimates on the truncated marginals, we can obtain a bound of the remainders associated with the long-range part which is controlled by parameters associated with other remainders. We can finally prove that those remainders vanish when passing to the limit.

3.3 Discussion of the result

Our result is for very decreasing potentials and is far from being reached for potentials like inverse power laws for instance. It is purely technical and due to the strategy of Lanford. Indeed, in order to make the other remainders disappear, some parameters have to be taken at a certain order. Yet, they appear in the bound of the remainder associated with the long-range part. In order to make this one converge to 0 with the orders appearing, we have to consider extremely decreasing potential. It seems that we can not hope to do much better by this approach and another method should be developed to handle more reasonable potential.

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