An Asymptotic Preserving scheme for the diffusion limit of a stochastic linear kinetic equation

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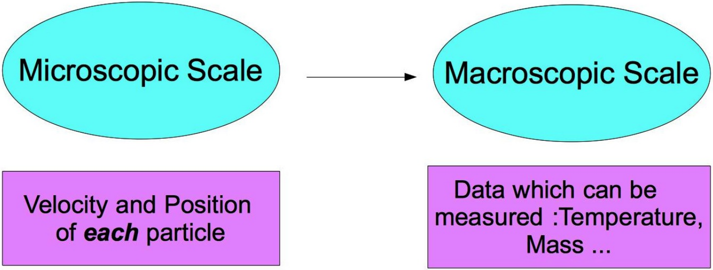

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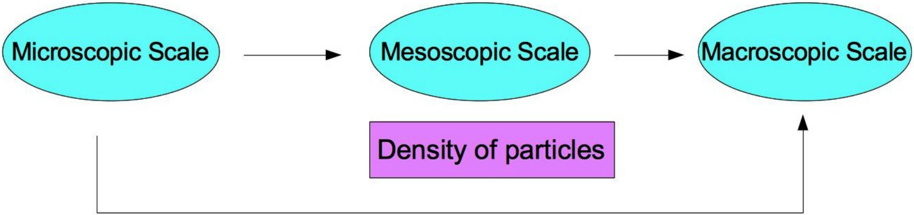

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Aim = **Numerical scheme** adapted to the passage from the **mesoscopic scale** to the **macroscopic one** by a diffusive limit for a **stochastic linear kinetic equation**.

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Aim = A **numerical approximation** of the **kinetic equation** which **converges** to a **numerical approximation** of the **macroscopic equation** when passing to the limit.

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$\Longrightarrow$ Construct a **scheme** for which h is **indepedent** of ε.

Context

A stochastic linear kinetic equation

$$df + \frac{1}{\varepsilon} v \partial_x f dt = \frac{\sigma}{\varepsilon^2} \mathcal{L} f dt + f \circ Q dW_t$$

whith $t \in [0, T]$, $x \in \mathbf{T}$, $v \in [-1, 1]$ and dW_t a **cylindrical Wiener process**.

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$$dW_t = \sum_{k \geq 0} e_k d\beta_k(t)$$

whith $(\beta_k)_{k \geq 0}$ **independent real-valued Brownian motions**, $(e_k)_{k \geq 0}$ an **orthonormal basis** of the Hilbert space $L^2(\mathbf{T})$, Q a **linear self-adjoint operator** on $L^2(\mathbf{T})$ such that

$$\sum_{k \geq 0} \|Q e_k\|^2_{L^\infty_x} < +\infty,$$

σ a **function** which satisfies $0 < \sigma_m \leq \sigma(x) \leq \sigma_M$ for every x.

The linear operator

$$\mathcal{L}f(v) = \int_{-1}^{1} s(v,v')(f(v') - f(v))dv',$$

with s such that $0 < s_m \leq s(v,v') \leq s_M$ for every $v, v' \in [-1,1]$, s satisfies $\displaystyle\int_{-1}^{1} s(v,v')dv' = 1$ and is symmetric: $s(v,v') = s(v',v)$.

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$\mathcal{L}$ satisfies the following **properties**:

- $\Pi(\mathcal{L}\phi) = 0$ for every $\phi \in L^2([-1,1])$.
- The null space of $\mathcal{L}$ is $\mathcal{N}(\mathcal{L}) = \{\phi = \Pi\phi\}$.
- The rank of $\mathcal{L}$ is $\mathcal{R}(\mathcal{L}) = \mathcal{N}^{\perp}(\mathcal{L}) = \{\phi \text{ s.t. } \Pi\phi = 0\}$.
- $\mathcal{L}$ admits a pseudo inverse from $\mathcal{N}^{\perp}(\mathcal{L})$ onto $\mathcal{N}^{\perp}(\mathcal{L})$ denoted by $\mathcal{L}^{-1}$.

Examples

The one-group transport equation

For

$$\mathcal{L}f = \int_{-1}^{1} \frac{1}{2}(f(v') - f(v))dv' = \Pi f - f,$$

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$$df + \frac{1}{\varepsilon} v \partial_x f dt = \frac{\sigma}{\varepsilon^2}(\Pi f - f)dt + f \circ Q dW_t.$$

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The telegraph equation

With the velocity set $\{-1, 1\}$, dv the discrete Lebesgue measure, we denote $f(t,x,1) := p(t,x)$ and $f(t,x,-1) := q(t,x)$. For $\sigma = 1$, the **equation** becomes

$$\begin{cases} dp + \frac{1}{\varepsilon}\partial_x p dt = \frac{1}{\varepsilon^2}(\frac{p+q}{2} - p)dt + p \circ Q dW_t \\ dq - \frac{1}{\varepsilon}\partial_x q dt = \frac{1}{\varepsilon^2}(\frac{p+q}{2} - q)dt + q \circ Q dW_t. \end{cases}$$

The diffusion limit

The macroscopic equation

$$d\rho + \partial_x(\kappa\partial_x\rho)dt = \rho \circ Q dW_t$$

with $\kappa(x) = \dfrac{\Pi(v\mathcal{L}^{-1}v)}{\sigma(x)}$.

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$$f(t,x,v) = f_0(t,x,v) + \varepsilon f_1(t,x,v) + \varepsilon^2 f_2(t,x,v) + \dots$$

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Identifying the **same power** of ε :

$$\begin{aligned} \sigma\mathcal{L}f_0 = 0 \quad &\Rightarrow \quad f_0 = \Pi f_0 = \rho_0, \\ v\partial_x f_0 = \sigma\mathcal{L}f_1 \quad &\Rightarrow \quad f_1 = \mathcal{L}^{-1}(\frac{1}{\sigma}v\partial_x\rho_0) \end{aligned}$$

$$df_0 = \sigma\mathcal{L}f_2 dt + f_0 \circ QdW_t - v\partial_x\mathcal{L}^{-1}(\frac{1}{\sigma}v\partial_x\rho_0)dt.$$

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We apply Π and obtain

$$d\rho_0 = \rho_0 \circ QdW_t - \partial_x\left(\Pi\left[v\mathcal{L}^{-1}\left(\frac{v}{\sigma}\right)\right]\partial_x\rho_0\right)dt.$$

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$$d(\rho + \varepsilon g) + \frac{1}{\varepsilon} v \partial_x (\rho + \varepsilon g) dt = \frac{\sigma}{\varepsilon^2} \mathcal{L}(\rho + \varepsilon g) dt + (\rho + \varepsilon g) \circ Q dW_t. \tag{1}$$

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We apply Π and obtain

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We do (1) - (2) and obtain

$$dg + \frac{1}{\varepsilon}(I - \Pi)(v \partial_x g) dt = \frac{\sigma}{\varepsilon^2} \mathcal{L} g dt + g \circ Q dW_t - \frac{1}{\varepsilon^2} v \partial_x \rho dt.$$

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The micro-macro formulation

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Advantages = The decomposition **only uses basic properties** of the collision operator common to most of kinetic equations, approach which works for **both diffusive and hydrodynamical limits**.

Examples

The one group transport equation

$$\begin{cases} d\rho + \partial_x \Pi(vg)dt = \rho \circ QdW_t \\ dg + \dfrac{1}{\varepsilon}(I-\Pi)(v\partial_x g)dt = -\dfrac{\sigma}{\varepsilon^2} g dt + g \circ QdW_t - \dfrac{1}{\varepsilon^2} v \partial_x \rho dt. \end{cases}$$

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The telegraph equation

We denote $g = (\alpha, \gamma)$.

$$\begin{cases} d\rho + \partial_x \dfrac{\alpha - \gamma}{2} dt = \rho \circ QdW_t \\ d\alpha + \dfrac{1}{\varepsilon}\partial_x \dfrac{\alpha + \gamma}{2} dt = -\dfrac{1}{\varepsilon^2}\alpha dt + \alpha \circ QdW_t - \dfrac{1}{\varepsilon^2}\partial_x \rho dt \\ d\gamma - \dfrac{1}{\varepsilon}\partial_x \dfrac{\gamma + \alpha}{2} dt = -\dfrac{1}{\varepsilon^2}\gamma dt + \gamma \circ QdW_t + \dfrac{1}{\varepsilon^2}\partial_x \rho dt. \end{cases}$$

Formal analytical limit

- $dg + \frac{1}{\varepsilon}(I - \Pi)(v\partial_x g)dt = \frac{\sigma}{\varepsilon^2}\mathcal{L}g dt + g \circ Q dW_t - \frac{1}{\varepsilon^2} v\partial_x \rho dt$

$$\Rightarrow g = \mathcal{L}^{-1}\left(\frac{1}{\sigma} v\partial_x \rho\right) + \mathcal{O}(\varepsilon).$$

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- $d\rho + \partial_x \Pi(vg)dt = \rho \circ QdW_t$

$$\Rightarrow d\rho + \partial_x(\kappa\partial_x\rho)dt = \rho \circ QdW_t + \mathcal{O}(\varepsilon)$$

with $\kappa(x) = \dfrac{\Pi(v\mathcal{L}^{-1}v)}{\sigma(x)}$.

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$$\frac{g^{n+1} - g^n}{\Delta t} = -\frac{1}{\varepsilon^2} g^n$$

$$\Rightarrow g^{n+1} = \left(1 - \frac{\Delta t}{\varepsilon^2}\right) g^n = \left(1 - \frac{\Delta t}{\varepsilon^2}\right)^{n+1} g^0 .$$

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$\Rightarrow g^{n+1} = \left(\dfrac{\varepsilon^2}{\varepsilon^2 + \Delta t}\right) g^n \Rightarrow$ Stability issues independent of the size ε.

Stratonovich to Itô yields

The micro-macro formulation

$$\begin{cases} d\rho + \partial_x \Pi(vg)dt = \rho Q dW_t + \dfrac{1}{2}\rho \displaystyle\sum_{k\geq 0}(Qe_k)^2 dt \\ dg + \dfrac{1}{\varepsilon}(I-\Pi)(v\partial_x g)dt = \dfrac{\sigma}{\varepsilon^2}\mathcal{L}g dt + gQdW_t + \dfrac{1}{2} g \displaystyle\sum_{k\geq 0}(Qe_k)^2 dt - \dfrac{1}{\varepsilon^2} v\partial_x \rho dt. \end{cases}$$

Implicitation of the collision term, **Upwind discretization** of $(I-\Pi)(v\partial_x g)$, **centered approximations** of $\partial_x \Pi(vg)$ and $v\partial_x \rho$.

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Notations:

- Time: $t_n = n\Delta t$ with Δt time step.
- Space: Two staggered grids of step Δx with nodes $x_i = i\Delta x$ and $x_{i+\frac{1}{2}} = (i+\frac{1}{2})\Delta x$ extended by periodicity.

$$\rho_i^n \approx \rho(t_n, x_i) \text{ and } g_{i+\frac{1}{2}}^n(v) \approx g(t_n, x_{i+\frac{1}{2}}, v)$$

Numerical scheme

$$\begin{cases} \rho_i^{n+1} = & \rho_i^n - \Delta t \Pi \left(v \dfrac{g_{i+\frac12}^{n+1} - g_{i-\frac12}^{n+1}}{\Delta x} \right) + \rho_i^n \left(\dfrac{\Delta t}{2} \displaystyle\sum_{k \geq 0} (b_{ik})^2 + \sqrt{\Delta t} \sum_{k \geq 0} b_{ik} \xi_k^{n+1} \right) \\ g_{i+\frac12}^{n+1} = & g_{i+\frac12}^n - \dfrac{\Delta t}{\varepsilon \Delta x} (I - \Pi) \left(v^+ \left(g_{i+\frac12}^n - g_{i-\frac12}^n \right) + v^- \left(g_{i+\frac32}^n - g_{i+\frac12}^n \right) \right) \\ & - \dfrac{\sigma_{i+\frac12}}{\varepsilon^2} \mathcal{L} g_{i+\frac12}^{n+1} \Delta t + g_{i+\frac12}^n \left(\dfrac12 \Delta t \displaystyle\sum_{k \geq 0} (b_{i+\frac12,k})^2 + \sqrt{\Delta t} \sum_{k \geq 0} b_{i+\frac12,k} \xi_k^{n+1} \right) \\ & - \dfrac{1}{\varepsilon^2} v \dfrac{\rho_{i+1}^n - \rho_i^n}{\Delta x} \Delta t \end{cases}$$

with $v^+ = \max(v, 0)$ and $v^- = \min(v, 0)$, $(\xi_k^n)_{n \geq 1, k \geq 0}$ i.i.d. with a normal distribution, $b_{ik} := Q e_k(x_i)$ and $b_{i+\frac12,k} := Q e_k(x_{i+\frac12})$.

Formal numerical limit

- $g_{i+\frac{1}{2}}^{n+1} = \frac{1}{\sigma_{i+\frac{1}{2}}} \mathcal{L}^{-1} \left(v \frac{\rho_{i+1}^n - \rho_i^n}{\Delta x} \right) + \mathcal{O}(\varepsilon)$

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$$\Rightarrow \rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(\kappa_{i+\frac{1}{2}}\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x} - \kappa_{i-\frac{1}{2}}\frac{\rho_i^n - \rho_{i-1}^n}{\Delta x}\right)$$
$$+\rho_i^n\left(\frac{1}{2}\Delta t\sum_{k\geq 0}(b_{ik})^2 + \sqrt{\Delta t}\sum_{k\geq 0} b_{ik}\xi_k^{n+1}\right) + \mathcal{O}(\varepsilon)$$

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$$\Rightarrow \rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left(\kappa_{i+\frac{1}{2}} \frac{\rho_{i+1}^n - \rho_i^n}{\Delta x} - \kappa_{i-\frac{1}{2}} \frac{\rho_i^n - \rho_{i-1}^n}{\Delta x} \right) \\ + \rho_i^n \left(\frac{1}{2} \Delta t \sum_{k \geq 0} (b_{ik})^2 + \sqrt{\Delta t} \sum_{k \geq 0} b_{ik} \xi_k^{n+1} \right) + \mathcal{O}(\varepsilon)$$

$\Rightarrow$ the usual **3-points stencil explicit scheme** for **the diffusion equation** with $\kappa_{i+\frac{1}{2}} = -\dfrac{\Pi(v\mathcal{L}^{-1}v)}{\sigma_{i+\frac{1}{2}}}$.

The one-group transport equation

Numerical scheme for the kinetic equation:

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{\varepsilon \Delta x}\left(v^+(f_i^n - f_{i-1}^n) + v^-\left(f_{i+1}^n - f_i^n\right)\right) + \frac{1}{\varepsilon^2}(\Pi f_i^n - f_i^n) + f_i^n \left(\frac{1}{2}\Delta t \sum_{k \geq 0} (b_{ik})^2 + \sqrt{\Delta t} \sum_{k \geq 0} b_{ik} \xi_k^{n+1} \right)$$

The one-group transport equation

Numerical scheme for the kinetic equation:

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{\varepsilon \Delta x}\left(v^+(f_i^n - f_{i-1}^n) + v^-\left(f_{i+1}^n - f_i^n\right)\right)$$
$$+ \frac{1}{\varepsilon^2}(\Pi f_i^n - f_i^n) + f_i^n \left(\frac{1}{2}\Delta t \sum_{k\geq 0}(b_{ik})^2 + \sqrt{\Delta t}\sum_{k\geq 0} b_{ik}\xi_k^{n+1}\right)$$

Numerical (Crank-Nicholson) scheme for the diffusion equation:

$$\rho_i^{n+1} = \rho_i^n - \frac{\kappa}{2}\frac{\Delta t}{\Delta x^2}\left(\rho_{i+1}^{n+1} - 2\rho_i^{n+1} + \rho_{i-1}^{n+1}\right) - \frac{\kappa}{2}\frac{\Delta t}{\Delta x^2}\left(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n\right)$$
$$+ \rho_i^n \left(\frac{1}{2}\Delta t \sum_{k\geq 0}(b_{ik})^2 + \sqrt{\Delta t}\sum_{k\geq 0} b_{ik}\xi_k^{n+1}\right)$$

with $\kappa = \Pi(v\mathcal{L}^{-1}v) = \Pi(-v^2) = -1/3.$

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with $\kappa = \Pi(v\mathcal{L}^{-1}v) = \Pi(-v^2) = -1/3.$

• For the simulation, discretization in velocities $\Rightarrow \kappa \approx -1/2\sum_{j=1}^{N} v_j^2 \Delta v.$

Simulations

• **Initial data**: $f_0(x,v) = (1 + \cos(2\pi x + \pi))$.
• **Space domain** $[0,1]$ discretized with $N = 200$ points, periodic boundary conditions.
• **Noise** of the form $\displaystyle\sum_{k\in\mathbf{Z}} \frac{1}{|k|+1} \left(\cos(kx) + \sin(kx)\right) d\beta_k$.
• **Comparison of three regimes**: kinetic for $\varepsilon = 1$, intermediate for $\varepsilon = 10^{-2}$, diffusion for $\varepsilon = 10^{-8}$.
• 100 **realizations** for each time $\Rightarrow$ comparison of the mean.

Kinetic regime

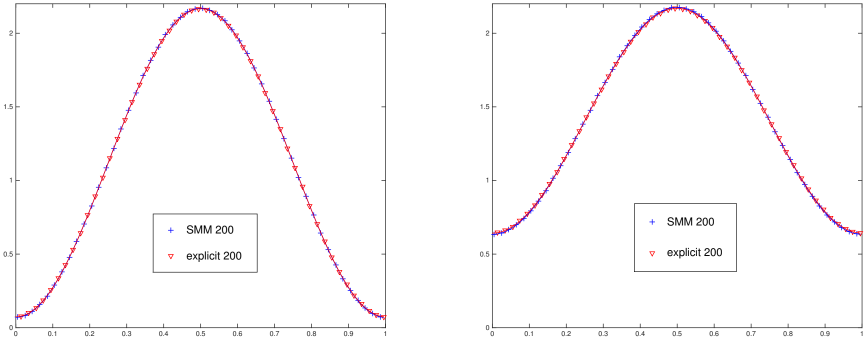


Figure: one-group transport equation $\varepsilon = 1$: comparison between SMM and explicit schemes (200 grid points): $t = 0.1$ (left), $t = 0.3$ (right).

Kinetic regime

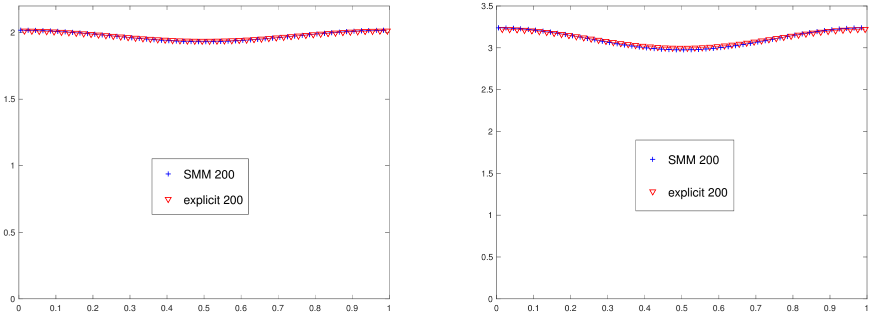


Figure: one-group transport equation $\varepsilon = 1$: comparison between SMM and explicit schemes (200 grid points): $t = 0.6$ (left), $t = 1$ (right).

Intermediate regime

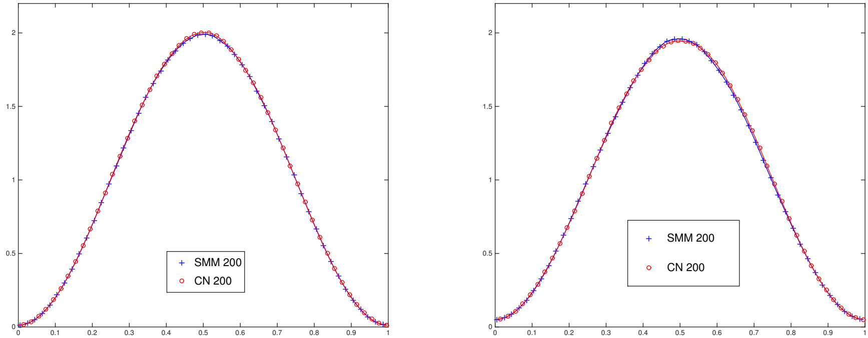


Figure: one-group transport equation $\varepsilon = 10^{-2}$: comparison between SMM and CN schemes (200 grid points): $t = \varepsilon/10$ (left), $t = 4\varepsilon/10$ (right).

Intermediate regime

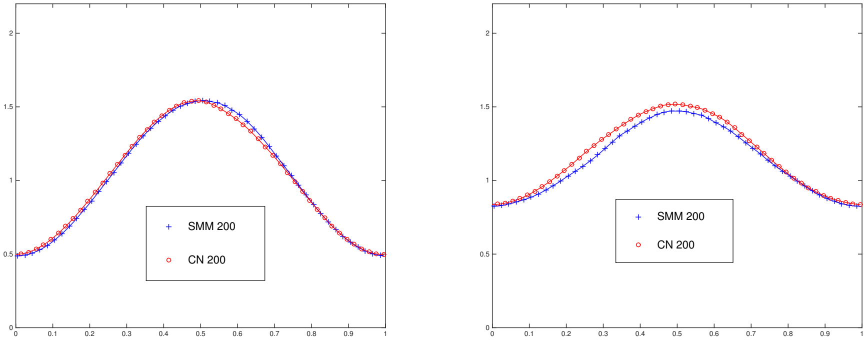


Figure: one-group transport equation $\varepsilon = 10^{-2}$: comparison between SMM and CN schemes (200 grid points): $t = 0.05$ (left), $t = 0.1$ (right).

Diffusion regime

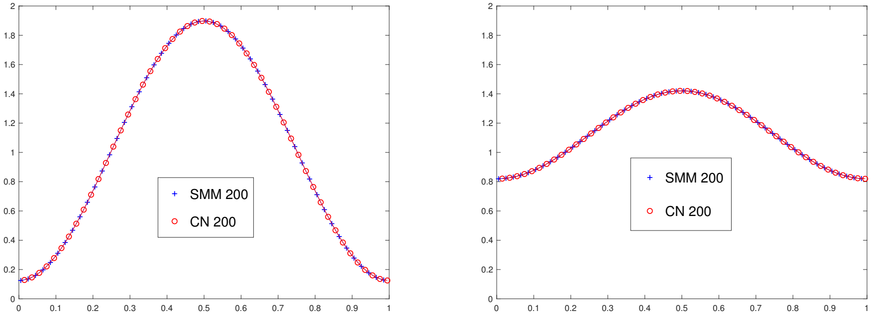
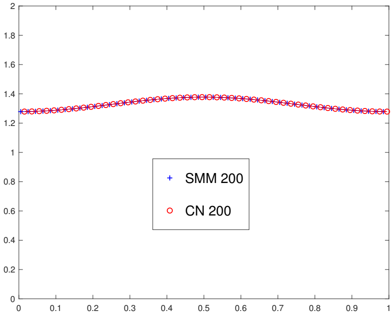


Figure: one-group transport equation $\varepsilon = 10^{-8}$: comparison between SMM and CN schemes (200 grid points): $t = 0.01$ (left), $t = 0.1$ (right).

Diffusion regime



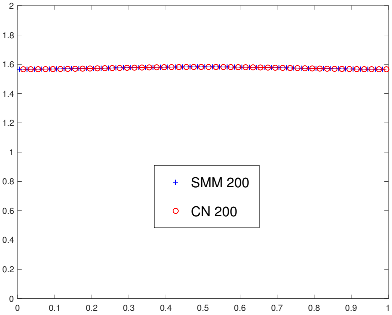


Figure: one-group transport equation $\varepsilon = 10^{-8}$: comparison between SMM and CN schemes (200 grid points): $t = 0.25$ (left), $t = 0.4$ (right).

Stability analysis

The telegraph equation with a **one dimensional Brownian motion** for noise:

$$\begin{cases} d\rho + \partial_x \dfrac{\alpha - \gamma}{2} dt = \rho \circ d\beta(t) \\ d\alpha + \dfrac{1}{\varepsilon} \partial_x \dfrac{\alpha + \gamma}{2} dt = -\dfrac{1}{\varepsilon^2} \alpha dt + \alpha \circ d\beta(t) - \dfrac{1}{\varepsilon^2} \partial_x \rho dt \\ d\gamma - \dfrac{1}{\varepsilon} \partial_x \dfrac{\gamma + \alpha}{2} dt = -\dfrac{1}{\varepsilon^2} \gamma dt + \gamma \circ d\beta(t) + \dfrac{1}{\varepsilon^2} \partial_x \rho dt. \end{cases}$$

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We denote $j := \frac{1}{2\varepsilon}(p-q) = \frac{1}{2}(\alpha-\gamma)$. The **micro-macro scheme** rewrites

$$\begin{cases} \rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(j_{i+\frac{1}{2}}^{n+1} - j_{i-\frac{1}{2}}^{n+1}\right) + \rho_i^n \left(\frac{\Delta t}{2} + \sqrt{\Delta t}\,\xi^{n+1}\right) \\ j_{i+\frac{1}{2}}^{n+1} = j_{i+\frac{1}{2}}^n + \dfrac{\Delta t}{2\varepsilon\Delta x}\left[j_{i+\frac{3}{2}}^n - 2j_{i+\frac{1}{2}}^n + j_{i-\frac{1}{2}}^n\right] \\ \qquad\qquad - \dfrac{\Delta t}{\varepsilon^2} j_{i+\frac{1}{2}}^{n+1} + j_{i+\frac{1}{2}}^n \left(\dfrac{\Delta t}{2} + \sqrt{\Delta t}\,\xi^{n+1}\right) - \dfrac{1}{\varepsilon^2}\Delta t \left(\dfrac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\right) \end{cases}$$

Stability analysis

Theorem

There exist constants $C(T)$, Δt_0, Δx_0 and ε_0 such that for all $\Delta t \leq \Delta t_0$, $\Delta x \leq \Delta x_0$ and $\varepsilon \leq \varepsilon_0$ satisfying the ***CFL condition***

$$\Delta t \leq \frac{1}{2}\left(\frac{\Delta x^2}{2} + \varepsilon \Delta x\right)$$

then we have

$$\mathbb{E}\left[\sum_i (\rho_i^n)^2 + (\varepsilon j_{i+\frac{1}{2}}^n)^2\right] \leq C(T)\,\mathbb{E}\left[\sum_i (\rho_i^0)^2 + (\varepsilon j_{i+\frac{1}{2}}^0)^2\right]$$

for every n.

Proof:

We denote $J^n_{i+\frac{1}{2}} = \varepsilon j^n_{i+\frac{1}{2}}$, $\mu = \frac{\Delta t}{\varepsilon \Delta x}$ and $\lambda = \frac{1}{1+\Delta t/\varepsilon^2}$.

The scheme becomes

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \mu\left(J^{n+1}_{j+\frac{1}{2}} - J^{n+1}_{j-\frac{1}{2}}\right) + \rho_j^n\left(\frac{\Delta t}{2} + \sqrt{\Delta t}\,\xi^{n+1}\right) \\ J^{n+1}_{j+\frac{1}{2}} = \lambda\Bigg(J^n_{j+\frac{1}{2}}(1 + \dfrac{\Delta t}{2} + \sqrt{\Delta t}\,\xi^{n+1}) \\ \qquad\qquad\qquad\qquad + \dfrac{\mu}{2}\left[J^n_{j+\frac{3}{2}} - 2J^n_{j+\frac{1}{2}} + J^n_{j-\frac{1}{2}}\right] - \mu\left(\rho^n_{j+1} - \rho^n_j\right)\Bigg) \end{cases}$$

Proof:

We denote $J^n_{i+\frac12} = \varepsilon j^n_{i+\frac12}$, $\mu = \frac{\Delta t}{\varepsilon \Delta x}$ and $\lambda = \frac{1}{1+\Delta t/\varepsilon^2}$.

The scheme becomes

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \mu \left(J^{n+1}_{j+\frac12} - J^{n+1}_{j-\frac12} \right) + \rho_j^n \left(\frac{\Delta t}{2} + \sqrt{\Delta t}\, \xi^{n+1} \right) \\ J^{n+1}_{j+\frac12} = \lambda \Big(J^n_{j+\frac12} (1 + \dfrac{\Delta t}{2} + \sqrt{\Delta t}\, \xi^{n+1}) \\ \qquad\qquad\qquad + \dfrac{\mu}{2} \left[J^n_{j+\frac32} - 2J^n_{j+\frac12} + J^n_{j-\frac12} \right] - \mu \left(\rho^n_{j+1} - \rho^n_j \right) \Big) \end{cases}$$

We take ρ_j^n and $J^n_{j+\frac12}$ on the form of elementary waves $\rho_j^n = \rho^n(\varphi) e^{ij\varphi}$ and $J^n_{j+\frac12} = J^n(\varphi) e^{i(j+\frac12)\varphi}$.

$$\begin{cases} \rho^{n+1} = \rho^n (1 + \dfrac{\Delta t}{2} + \sqrt{\Delta t}\, \xi^{n+1}) - 2i\mu J^{n+1} \sin\theta \\ J^{n+1} = \lambda \left(J^n \left(1 + \dfrac{\Delta t}{2} + \sqrt{\Delta t}\, \xi^{n+1} - 2\mu \sin^2\theta \right) - 2i\mu \sin\theta \rho^n \right) \end{cases}$$

with $\theta = \dfrac{\varphi}{2}$.

This rewrites as $\begin{pmatrix}\rho^{n+1}\\ J^{n+1}\end{pmatrix} = A_{n+1}\begin{pmatrix}\rho^{n}\\ J^{n}\end{pmatrix}$ with

$$\begin{pmatrix} 1+\frac{\Delta t}{2}+\sqrt{\Delta t}\,\xi^{n+1}-4\mu^2\lambda\sin^2\theta & -i(1+\frac{\Delta t}{2}+\sqrt{\Delta t}\,\xi^{n+1}-2\mu\sin^2\theta)2\lambda\mu\sin\theta \\ -2i\mu\lambda\sin\theta & (1+\frac{\Delta t}{2}+\sqrt{\Delta t}\,\xi^{n+1}-2\mu\sin^2\theta)\lambda. \end{pmatrix}$$

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A_{n+1} is a **stochastic perturbation** of the matrix $\widetilde{A}$ appearing in the deterministic version :

$$\widetilde{A} = \begin{pmatrix} 1 - 4\mu^2\lambda \sin^2\theta & -i(1 - 2\mu \sin^2\theta)2\lambda\mu \sin\theta \\ -2i\mu\lambda \sin\theta & (1 - 2\mu \sin^2\theta)\lambda \end{pmatrix}.$$

$\Rightarrow A_{n+1} = \widetilde{A} + \sqrt{\Delta t}\xi^{n+1}\widetilde{B} + \Delta t\,\widetilde{C}$,

with $\widetilde{B} = \widetilde{B}(\mu\lambda, \mu^2\lambda, \sin\theta)$ and $\widetilde{C} = \widetilde{C}(\mu\lambda, \mu^2\lambda, \sin\theta)$.

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$\Rightarrow \widetilde{B}(\lambda\mu, \lambda\mu^2, \sin\theta)$ and $\widetilde{C}(\lambda\mu, \lambda\mu^2, \sin\theta)$ are **uniformly bounded**.

We denote by $\mathcal{F}_n$ the σ-algebra generated by

$$\rho^n, J^n, \xi^n, \rho^{n-1}, J^{n-1}, \xi^{n-1}, \dots, \rho^0, J^0.$$

$\Rightarrow \xi^{n+1}$ independent of $\mathcal{F}_n$ and ρ^n, J^n are $\mathcal{F}_n$-measurable.

$\Rightarrow \mathbb{E}(\xi^{n+1}|\mathcal{F}_n) = 0$, $\mathbb{E}(\rho^n|\mathcal{F}_n) = \rho^n$, $\mathbb{E}(J^n|\mathcal{F}_n) = J^n$.

$$\Rightarrow \mathbb{E}\left[\left\|\begin{pmatrix}\rho^{n+1}\\ J^{n+1}\end{pmatrix}\right\|_2^2 |\mathcal{F}_n\right] = \mathbb{E}\left[\left\|A_{n+1}\begin{pmatrix}\rho^{n}\\ J^{n}\end{pmatrix}\right\|_2^2 |\mathcal{F}_n\right]$$
$$\leq \left\|(\widetilde{A} + \Delta t\widetilde{C})\begin{pmatrix}\rho^{n}\\ J^{n}\end{pmatrix}\right\|_2^2 + \Delta t\left\|\widetilde{B}\begin{pmatrix}\rho^{n}\\ J^{n}\end{pmatrix}\right\|_2^2 \leq \left(\|\widetilde{A}\|_2^2 + L\Delta t\right)\left\|\begin{pmatrix}\rho^{n}\\ J^{n}\end{pmatrix}\right\|_2^2,$$

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$\|\widetilde{A}\|_2^2$ is the **largest eigenvalue** of the matrix $\widetilde{A}^*\widetilde{A}$. Denoting by $\widetilde{T}$ and $\widetilde{D}$ the trace and determinant of this latter matrix, the largest eigenvalue is

$$\frac{\widetilde{T} + \sqrt{\widetilde{T}^2 - 4\widetilde{D}}}{2}$$

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$$1 - \widetilde{T} + \widetilde{D} = 8\lambda\mu^2 X(1 - \lambda + 2\lambda\mu X - 2\lambda\mu^2 X^2 - 2\lambda\mu^2 X)$$

with $X = (\sin\theta)^2$.

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We denote $Q(X) := 1 - \lambda + 2\lambda\mu X - 2\lambda\mu^2 X^2 - 2\lambda\mu^2 X$.

Q **concave** and satisfies $Q(0) = 1 - \lambda > 0$, $Q(1) = 1 - \lambda + 2\lambda\mu - 4\lambda\mu^2 \geq 0$ under **the CFL condition**.

$$\Rightarrow 1 - \widetilde{T} + \widetilde{D} \geq 0 \Rightarrow \|\widetilde{A}\|_2^2 \leq 1.$$

$\square$

Stability in the general case

Theorem

If Δt satisfies the following CFL condition

$$\Delta t \leq \frac{2 s_m \sigma_m \Delta x^2}{2(2+\varepsilon)} + \frac{\varepsilon \Delta x}{2+\varepsilon},$$

then the sequence ρ^n and g^n defined by the scheme satisfy the energy estimate

$$\mathbb{E}\left[\sum_i (\rho_i^n)^2\right] + \varepsilon^2 \mathbb{E}\left[\sum_i \Pi\left((g_{i+\frac{1}{2}}^n)^2\right)\right] \leq C(T)\left(\mathbb{E}\left[\sum_i (\rho_i^0)^2\right] + \varepsilon^2 \mathbb{E}\left[\sum_i \Pi\left((g_{i+\frac{1}{2}}^0)^2\right)\right]\right)$$

for every n with $C(T)$ a constant which only depends on T.

Conclusion and Perspectives

- Diffusion equation recovered when $\varepsilon \to 0 \Rightarrow$ **AP scheme.**
- CFL conditions ensuring the **stability of our scheme**.
- **Numerical results** confirming the **good performances** of the scheme, in particular in the diffusion regime.
- **Perspectives**: Similar technics to study stochastic perturbations of
  - ◇ the non-linear case of the radiative transfer equation,
  - ◇ the Boltzmann equation,
  - ◇ the Vlasov equation,
  - ◇ hydrodynamical limits . . .

Thank you for your attention.