

Stochastic Discrete Velocity Averaging Lemmas and Rosseland Approximation

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(in collaboration with T. Goudon)

Abstract

In this note, we investigate some questions around velocity averaging lemmas, a class of results which ensure the regularity of the “velocity average” $\int f(x, v)\psi(v) d\mu(v)$ when f and $v \cdot \nabla_x f$ both belong to L^p , $p \in [1, \infty)$ and the measured set of velocities $(\mathcal{V}, d\mu)$ satisfy a nondegeneracy assumption. We are interested in the case when the variable v lies in a discrete subset of \mathbb{R}^D .

We present results obtained in collaboration with T. Goudon in [2]. First of all, we provide a rate, depending on the number of velocities, to the defect of $H^{1/2}$ regularity which is reached when v ranges over a continuous set. Second of all, we show that the $H^{1/2}$ regularity holds in expectation when the set of velocities is chosen randomly. We apply this statement to obtain a consistency result for the diffusion limit in the case of the Rosseland approximation.

1 Introduction

The physical context associated with the start of our reflection is the one of the Rosseland approximation. We are interested in the following simple model which can be motivated from radiative transfer theory:

$$\varepsilon \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} \sigma(\rho_\varepsilon)(\rho_\varepsilon - f_\varepsilon) \quad (1.1)$$

where

$$\rho_\varepsilon(t, x) = \int_{\mathcal{V}} f_\varepsilon(t, x, v) d\mu(v),$$

and $\sigma : [0, \infty) \rightarrow [0, \infty)$ is a given nonlinear smooth function. The parameter $0 < \varepsilon \ll 1$ is defined from physical quantities. It is now a well known result that, as it tends to 0, both $f_\varepsilon(t, x, v)$ and $\rho_\varepsilon(t, x)$ converge to $\rho(t, x)$, which satisfies the non linear diffusion equation

$$\partial_t \rho = \nabla_x \cdot \left(A \nabla_x F(\rho) \right), \quad A = \int_{\mathcal{V}} v \otimes v d\mu(v), \quad F(\rho) = \int_0^\rho \frac{dz}{\sigma(z)} \quad (1.2)$$

when $\mathcal{V} = \mathbf{S}^{D-1}$ is associated with the normalized Lebesgue measure, see [3].

Indeed, the entropy dissipation associated with (1.1) provides key estimate which lead to weak compactness. The difficulty is then the passage to the limit in presence of the nonlinearity due to σ . Typically, averaging lemmas are efficient tools to deal with such a problem. Roughly speaking, they can be explained as follows. Let $\mathcal{V} \subset \mathbb{R}^D$, endowed with a measure $d\mu$. We consider a sequence of functions $f_n : \mathbb{R}^D \times \mathcal{V} \rightarrow \mathbb{R}$. We assume that

- a) $(f_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^D \times \mathcal{V})$,
- b) $(v \cdot \nabla_x f_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^D \times \mathcal{V})$.

Given $\psi \in C_c^\infty(\mathbb{R}^D)$, we denote the *velocity average*

$$\rho_n[\psi](x) = \int_{\mathcal{V}} f_n(x, v) \psi(v) d\mu(v).$$

Then, $(\rho_n[\psi])_{n \in \mathbb{N}}$ is bounded in the Sobolev space $H^{1/2}(\mathbb{R}^D)$. Thus, it is relatively compact in $L^2_{\text{loc}}(\mathbb{R}^D)$, by virtue of the standard Rellich's theorem. This basic result has been improved in many directions: L^2 can be replaced by the L^p framework, at least with $1 < p < \infty$, and we can relax b) by allowing derivatives with respect to v and certain loss of regularity with respect to x ; see, among others, [7, 9, 14]. Time derivative or force terms can be considered as well, see, additionally to the above-mentioned references, [4].

Going back to the problem of the Rosseland approximation, we note that it is possible to conclude to the convergence with the use of the averaging lemmas. Beyond that, such an argument plays a crucial role in the stunning theory of “renormalized solutions” of the Boltzmann equation [6], and more generally for proving the existence of solutions to non linear kinetic models like in [5]. It is equally a crucial ingredient for the analysis of hydrodynamic regimes, which establish the connection between microscopic models and fluid mechanics systems; for the asymptotic of the Boltzmann equation to the incompressible Navier–Stokes system, which needs a suitable L^1 version of the average lemma [10], we refer the reader to [11, 15, 17]. Finally, it is worth pointing out that averaging lemma can be used to investigate the regularizing effects of certain PDE (convection-diffusion and elliptic equations, nonlinear conservation laws, etc) [16].

However, the discussion above hides the fact that we need some assumptions on the measured set of velocities $(\mathcal{V}, d\mu)$ in order to obtain the regularization property of the velocity averaging. Roughly speaking, we need “enough” directions v when we consider the derivatives in b). More technically, the compactness statement holds provided for any $0 < R < \infty$ we can find $C_R > 0$, $\delta_0 > 0$, $\gamma > 0$ such that for $0 < \delta < \delta_0$ and $\xi \in \mathbb{S}^{N-1}$, we have

$$\text{meas}\left(\left\{v \in \mathcal{V} \cap B(0, R), |v \cdot \xi| \leq \delta\right\}\right) \leq C_R \delta^\gamma.$$

This assumption appears in many statements about regularity of the velocity averages; as far as we are only interested in compactness issue, it can be replaced by the more intuitive assumption (see e. g. [8, Th. 1 in Lect. 3]): for any $\xi \in \mathbb{S}^{N-1}$ we have

$$\text{meas}\left(\left\{v \in \mathcal{V} \cap B(0, R), v \cdot \xi = 0\right\}\right) = 0. \quad (1.3)$$

Clearly these assumptions are satisfied when the measure $d\mu$ is absolutely continuous with respect to the Lebesgue measure (with, for the sake of concreteness, $\mathcal{V} = \mathbb{R}^D$ or $\mathcal{V} = \mathbb{S}^{D-1}$). However, they fail for models based on a discrete set of velocities. For instance let $\mathcal{V} = \{v_1, \dots, v_N\}$, with $v_j \in \mathbb{R}^D$, and $d\mu(v) = \frac{1}{N} \sum_{j=1}^N \delta(v - v_j)$; it suffices to pick $\xi \in \mathbb{S}^{N-1}$ orthogonal to one of the v_j 's to contradicts (1.3).

This is precisely this framework that we wish to investigate. Actually, despite the absence of gain of regularity in the discrete case, when the discrete velocities come from a discretization grid of the whole space, the averaging lemma can be recovered asymptotically letting the mesh step go to 0, as shown in [13], motivated by the convergence analysis of numerical schemes for the Boltzmann equation.

The results we obtained in [2] aim at investigating further these issues. The paper is organized as follows : in section 2, we start by precisizing the analysis of [13] and obtain a rate on the defect to the $H^{1/2}$ regularity of the velocity average, depending on the mesh size. We give along a sketch of the proof. In section 3, we state stochastic versions of the averaging lemmas. We are still working with a finite number of velocities on bounded sets; however, choosing the velocities randomly, the “compactifying” property of assumption b) can be restored by dealing with the expectation of $\rho_n[\psi]$. This is a natural way to involve “enough velocities”, by looking at a large set of realizations of the discrete velocity grid. Some ideas of the proof will be given. Finally, in section 4, we go back to the Rosseland approximation and explain a consistency result with a random discretization of the velocity variable.

2 Discrete Velocities and Averaging Lemma in the Deterministic Case

As mentioned above, though the averaging lemma fails for discrete velocity models, Mischler established in [13] that the compactness of velocity averages is recovered asymptotically when we refine a velocity grid in order to recover a continuous velocity model. Here, we wish to quantify the defect of compactness when the number of velocities is finite and fixed. This is the aim of the following claim which shows that the macroscopic density $\rho[\psi]$ “belongs to $H^{1/2}(\mathbb{R}^D) + O(\frac{1}{\sqrt{N}})L^2(\mathbb{R}^D)$ ”.

Proposition 2.1. *Let $N \in \mathbb{N} \setminus \{0\}$ and define*

$$A_N = \left(\frac{1}{N} \mathbb{Z} \right)^D \cap [-0.5, 0.5]^D.$$

Let $f, g \in L^2(\mathbb{R}^D \times A_N)$ satisfy for all $k \in \mathbb{Z}^D$,

$$v_k \cdot \nabla_x f(x, v_k) = g(x, v_k). \quad (2.1)$$

We suppose that the L^2 norm of f and g is bounded uniformly with respect to N . Then, for all $\psi \in C_c^\infty(\mathbb{R}^D)$, the macroscopic quantity

$$\rho[\psi](x) = \frac{1}{(N+1)^D} \sum_k f(x, v_k) \psi(v_k)$$

can be split as $\rho[\psi](x) = \Theta[\psi](x) + \frac{1}{\sqrt{N}} \Delta[\psi](x)$ where $\Theta[\psi]$ and $\Delta[\psi]$ are bounded uniformly with respect to N in $H^{1/2}(\mathbb{R}^D)$ and $L^2(\mathbb{R}^D)$ respectively.

Sketch of the proof. The beginning of the proof consists in applying the Fourier transform to (2.1). Then for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^D$, we get

$$\xi \cdot v_k \hat{f}(\xi, v_k) = (-i) \hat{g}(\xi, v_k).$$

We denote

$$F(\xi) := \left(\frac{1}{(N+1)^D} \sum_k |\hat{f}(\xi, v_k)|^2 \right)^{1/2}, \quad G(\xi) := \left(\frac{1}{(N+1)^D} \sum_k |\hat{g}(\xi, v_k)|^2 \right)^{1/2}.$$

By assumption, we have $F, G \in L_\xi^2$. Following the standard arguments, we pick $\delta > 0$ and we split

$$\begin{aligned} \hat{\rho}[\psi](\xi) &= \frac{1}{(N+1)^D} \sum_k \hat{f}(\xi, v_k) \psi(v_k) \\ &= \frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| < \delta |\xi|} \hat{f}(\xi, v_k) \psi(v_k) + \frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| \geq \delta |\xi|} \hat{f}(\xi, v_k) \psi(v_k). \end{aligned}$$

Using Cauchy–Schwarz inequality and (2.1), we obtain the two following bounds

$$\left| \frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| < \delta |\xi|} \hat{f}(\xi, v_k) \psi(v_k) \right| \leq \|\psi\|_\infty F(\xi) \left(\frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| < \delta |\xi|} 1 \right)^{1/2} \quad (2.2)$$

and

$$\left| \frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| \geq \delta |\xi|} \hat{f}(\xi, v_k) \psi(v_k) \right| \leq \|\psi\|_\infty G(\xi) \left(\frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| \geq \delta |\xi|} \frac{1}{|\xi \cdot v_k|^2} \right)^{1/2}. \quad (2.3)$$

Then, the only terms left to control are

$$\left(\sum_{|\xi \cdot v_k| < \delta |\xi|} 1 \right) \text{ and } \left(\sum_{|\xi \cdot v_k| \geq \delta |\xi|} \frac{1}{|\xi \cdot v_k|^2} \right).$$

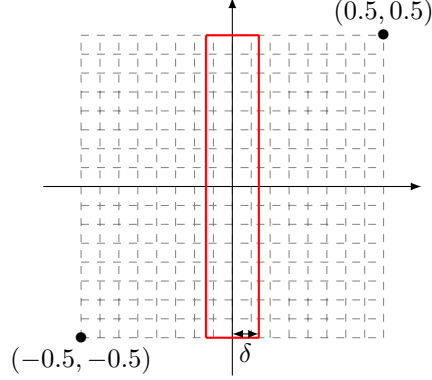


Figure 1: The delimited area corresponds to $|\xi \cdot v_k| < \delta|\xi|$ for ξ colinear to e_1 .

We start by dealing with the particular case of a ξ aligned with an axis (see Figure 1). On each horizontal line we find $2\lfloor \delta N \rfloor + 1$ velocities such that $|\xi \cdot v_k| < \delta|\xi|$, where $\lfloor s \rfloor$ stands for the integer part of s . Thus, since there is $(N+1)^{D-1}$ such lines on the domain A_N , we obtain

$$\sum_{|\xi \cdot v_k| < \delta|\xi|} 1 = (2\lfloor \delta N \rfloor + 1)(N+1)^{D-1} = 2\left(\delta + \frac{1}{N}\right)(N+1)^D.$$

Coming back to (2.2), we arrive at

$$\left| \frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| < \delta|\xi|} \hat{f}(\xi, v_k) \psi(v_k) \right| \leq C F(\xi) \sqrt{\delta + \frac{1}{N}}.$$

where $C > 0$ is a generic constant which does not depend on N and ξ .

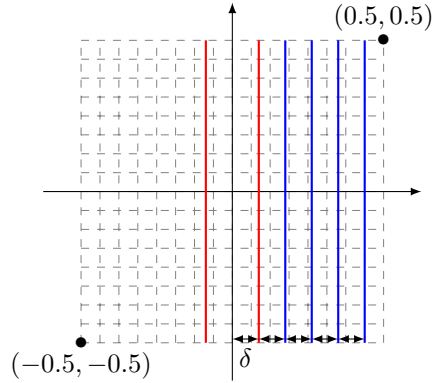


Figure 2: Splitting of the velocity space in strips of width δ . This space being symmetric, we only deal with the part corresponding to positive abscissae.

The second term is handled quite similarly using the additional idea of splitting the velocity space in strips of width δ (see Figure 2). We denote by S_p the p -th strip delimited by the straight lines $x = p\delta$ and $x = (p+1)\delta$. Each velocity on the strip S_p satisfies $p\delta \leq v_k^1 \leq (p+1)\delta$. Moreover, given a strip S_p , we cannot find more than $\lfloor \delta N \rfloor + 1$ abscissae in the strip and there

is $(N+1)^{D-1}$ lines in the domain. It follows that

$$\sum_{|\xi \cdot v_k| \geq \delta |\xi|} \frac{1}{|\xi \cdot v_k|^2} \leq \frac{1}{|\xi|^2} 2 \left(\sum_{p \geq 1} \frac{1}{p^2} \right) \frac{1}{\delta} \left(1 + \frac{1}{\delta N} \right) (N+1)^D.$$

Thus, we deduce from (2.3) that

$$\left| \frac{1}{(N+1)^D} \sum_{|\xi \cdot v_k| \geq \delta |\xi|} \hat{f}(\xi, v_k) \psi(v_k) \right| \leq C G(\xi) \frac{1}{|\xi| \sqrt{\delta}} \left(1 + \frac{1}{\delta N} \right)^{1/2}.$$

We conclude that

$$|\hat{\rho}[\psi](\xi)| \leq C \left(F(\xi) \sqrt{\delta + \frac{1}{N}} + G(\xi) \frac{1}{|\xi| \sqrt{\delta}} \left(1 + \frac{1}{\delta N} \right)^{1/2} \right). \quad (2.4)$$

holds when ξ is aligned to the axis. The general case is actually not much difficult to prove using the following splitting of the space. It is left to the reader.

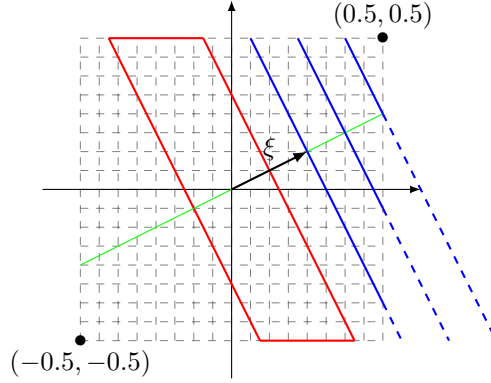


Figure 3: The area corresponding to $|\xi \cdot v_k| \leq \delta |\xi|$ is delimited as previously. The complementary set is split into strips of width δ .

Therefore, we have established that for all $\xi \neq 0$, we get (2.4) for all $\delta > 0$. We take

$$\delta = \frac{1}{|\xi|} \mathbf{1}_{\{N \geq |\xi|\}} + \frac{1}{N} \mathbf{1}_{\{N < |\xi|\}}$$

and we denote

$$\Theta_N(\xi) := \hat{\rho}[\psi](\xi) \mathbf{1}_{\{N \geq |\xi|\}}, \quad \Delta_N(\xi) := \hat{\rho}[\psi](\xi) \mathbf{1}_{\{N < |\xi|\}}.$$

We finally deduce from (2.4)

$$|\xi| \Theta_N(\xi)^2 \leq C(G^2(\xi) + F^2(\xi)),$$

and

$$\Delta_N^2(\xi) \leq \frac{C}{N} (F^2(\xi) + G^2(\xi)),$$

which are also satisfied when $\xi = 0$. This concludes the proof by assumption on f and g . \square

3 Stochastic Discrete Velocity Averaging Lemmas

In this section, we now deal with random discrete velocities. Indeed, we can expect to make the defect vanish when taking the expectation of the velocity averages. This is indeed the case as shown in the following statement.

Theorem 3.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $V_1, \dots, V_{\mathcal{N}}$ be i.i.d. random variables, distributed according to the continuous uniform distribution on $[-0.5, 0.5]^D$. We set*

$$d\mu = \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \delta(v = V_k).$$

Let $f, g \in L^2(\mathbb{R}^D \times \mathbb{R}^D \times \Omega, dx d\mu(v) d\mathbb{P})$ satisfy for all $x \in \mathbb{R}^D$, $\omega \in \Omega$, and $k \in \{1, \dots, \mathcal{N}\}$

$$V_k \cdot \nabla_x f(x, V_k) = g(x, V_k). \quad (3.1)$$

Then, for all $\psi \in C_c^\infty(\mathbb{R}^D)$, the macroscopic quantity

$$\rho[\psi](x) := \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} f(x, V_k) \psi(V_k) = \int_{\mathbb{R}^D} f(x, v) \psi(v) d\mu(v)$$

satisfies $\mathbb{E}\rho[\psi] \in H^{1/2}(\mathbb{R}^D)$ (and it is bounded in this space if the L^2 norm of f and g is bounded uniformly with respect to \mathcal{N}).

The proof is actually straightforward using the same strategy as in the previous section. The difference with what has been done before appears when counting the velocities to bound the terms involved in the macroscopic quantity. Denoting M_p the number of velocities in the p -th strip (see Fig. 2), with the distribution adopted for the velocities, M_p actually follows a binomial distribution of parameters \mathcal{N} and δ . This leads to

$$|\mathbb{E}\hat{\rho}[\psi](\xi)| \leq C \left(F(\xi) \sqrt{\delta} + \frac{G(\xi)}{|\xi| \sqrt{\delta}} \right)$$

and applying this inequality with $\delta = \frac{G(\xi)}{|\xi| F(\xi)}$, it leads to

$$|\mathbb{E}\hat{\rho}[\psi](\xi)| \leq C \sqrt{F(\xi) G(\xi)} \frac{1}{\sqrt{|\xi|}}.$$

which concludes the proof by using the assumptions on f and g .

We can also adapt this stochastic averaging lemma to the case when the variable v lies on the sphere, which is more adapted to our model which belongs to the radiative transfer theory.

Theorem 3.2. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $V_1, \dots, V_{\mathcal{N}}$ be i.i.d. random variables, distributed according to the continuous uniform distribution on \mathbb{S}^{D-1} . We set*

$$d\mu = \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \delta(v = V_k).$$

Let $f, g \in L^2(\mathbb{R}^D \times \mathbb{R}^D \times \Omega, dx d\mu(v) d\mathbb{P})$ satisfy for all $x \in \mathbb{R}^D$, $\omega \in \Omega$, and $k \in \{1, \dots, \mathcal{N}\}$

$$V_k \cdot \nabla_x f(x, V_k) = g(x, V_k).$$

Then, for all $\psi \in C_c^\infty(\mathbb{S}^{D-1})$, the macroscopic quantity

$$\rho[\psi](x) := \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} f(x, V_k) \psi(V_k) = \int_{\mathbb{R}^D} f(x, v) \psi(v) d\mu(v)$$

satisfies $\mathbb{E}\rho[\psi] \in H^{1/2}(\mathbb{R}^D)$.

The results can be extended to the L^p cases for $1 < p < \infty$ by using an interpolation argument as in [9, Theorem 2].

Corollary 3.3. *In Theorems 3.1 and 3.2, we assume that f and g belong to $L^p(\mathbb{R}^D \times \mathcal{V} \times \Omega, dx d\mu(v) d\mathbb{P})$ for some $1 < p < \infty$, with \mathcal{V} either \mathbb{R}^D or \mathbb{S}^{D-1} . Then $\mathbb{E}\rho[\psi]$ lies in the Sobolev space $W^{s,p}(\mathbb{R}^D)$ with $0 < s < \min(1/p, 1 - 1/p) < 1$.*

We can equally extend the compactness statement to the L^1 framework, by following [10].

Corollary 3.4. *We consider a random set of velocities defined as in Theorem 3.1 or 3.2. Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be two sequences of functions defined on $\mathbb{R}^D \times \mathcal{V} \times \Omega$ such that*

- i) $\{f_n, n \in \mathbb{N}\}$ *is a relatively weakly compact set in $L^1(\mathbb{R}^D \times \mathcal{V} \times \Omega, dx d\mu(v) d\mathbb{P})$,*
- ii) $\{g_n, n \in \mathbb{N}\}$ *is bounded in $L^1(\mathbb{R}^D \times \mathcal{V} \times \Omega, dx d\mu(v) d\mathbb{P})$,*
- iii) *we have $V_k \cdot \nabla_x f_n(x, V_k) = g_n(x, V_k)$.*

Then $\mathbb{E}\rho_n[\psi](x) = \mathbb{E} \int f_n(x, v) \psi(v) d\mu(v)$ lies in a relatively compact set of $L^1(B(0, R))$, for any $0 < R < \infty$ (for the strong topology).

4 Application to the Rosseland Approximation

We go back to the problem of the Rosseland Approximation and more precisely we are interested in the asymptotic behavior of the solutions of (1.1). Using our previous stochastic averaging lemma, we actually are able to establish the following result.

Theorem 4.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $V_1, \dots, V_{\mathcal{N}}$ be i.i.d. random variables distributed according to the continuous uniform law on \mathcal{V} . Then, we obtain a set $\mathcal{V}_{\mathcal{N}}$ of $2\mathcal{N}$ velocities in \mathcal{V} by setting $V_{\mathcal{N}+j} = -V_j$, for all $j \in \{1, \dots, \mathcal{N}\}$. We denote the associated discrete measure on \mathcal{V} by*

$$d\mu_{\mathcal{N}}(v) = \frac{1}{2\mathcal{N}} \sum_{k=1}^{2\mathcal{N}} \delta(v = V_k).$$

Let $f_\varepsilon|_{t=0} = f_\varepsilon^0 \geq 0$ satisfy

$$\sup_{\varepsilon > 0, \mathcal{N} \in \mathbb{N}} \left(\mathbb{E} \int_{\mathbb{R}^D} \int_{\mathcal{V}} (1 + \varphi(x) + |\ln f_\varepsilon^0|) f_\varepsilon^0 d\mu_{\mathcal{N}}(v) dx + \|f_\varepsilon^0\|_{L^\infty(\Omega \times \mathbb{R}^D \times \mathcal{V})} \right) = M_0 < +\infty. \quad (4.1)$$

Let f_ε be a solution of the following equation

$$\partial_t f_\varepsilon(t, x, V_j) + \frac{1}{\varepsilon} V_j \cdot \nabla_x f_\varepsilon(t, x, V_j) = \frac{1}{\varepsilon^2} \sigma(\rho_{\varepsilon, \mathcal{N}}) [\rho_{\varepsilon, \mathcal{N}}(t, x) - f_\varepsilon(t, x, V_j)], \quad (4.2)$$

with $\rho_{\varepsilon, \mathcal{N}}(t, x) := \frac{1}{2\mathcal{N}} \sum_{i=1}^{2\mathcal{N}} f_\varepsilon(t, x, V_i)$. We suppose that $\rho \in [0, \infty) \mapsto \sigma(\rho)$ is a nonnegative function such that for any $0 < R < \infty$, there exists $\sigma_(R) > 0$ verifying $0 < 1/\sigma_*(R) \leq \sigma(\rho) \leq \sigma_*(R)$ and $|\sigma'(\rho)| \leq \sigma_*(R)$ for any $0 \leq \rho \leq R$. Then $\mathbb{E}\rho_{\varepsilon, \mathcal{N}}$ converges to $\mathbb{E}\rho_{\mathcal{N}}$ in $L^2((0, T) \times \mathbb{R}^D)$ as ε goes to 0 with $0 < T < \infty$ where $\mathbb{E}\rho_{\mathcal{N}}$ is solution of*

$$\begin{aligned} \partial_t \mathbb{E}\rho_{\mathcal{N}} + \operatorname{div}(\mathcal{J}_{\mathcal{N}}) &= 0, \\ \sigma(\mathbb{E}\rho_{\mathcal{N}}) \mathcal{J}_{\mathcal{N}} &= -\mathbb{E}A_{\mathcal{N}} \nabla_x \mathbb{E}\rho_{\mathcal{N}} + O\left(\frac{1}{\sqrt{\mathcal{N}}}\right), \end{aligned}$$

with $A_{\mathcal{N}}$ the $D \times D$ matrix with random components defined by

$$A_{\mathcal{N}} := \frac{1}{2\mathcal{N}} \sum_{j=1}^{2\mathcal{N}} V_j \otimes V_j,$$

and $\mathbb{E}\rho_{\mathcal{N}}|_{t=0}$ is the weak limit of $\int \mathbb{E}f_\varepsilon^0 d\mu(v)$.

Sketch of the proof. The beginning consists in, after establishing them, using the following entropy estimates :

$$\begin{aligned} \sup_{\varepsilon > 0, \mathcal{N} \in \mathbb{N}} \left\{ \sup_{0 \leq t \leq T} \mathbb{E} \int_{\mathbb{R}^D} \int_{\mathcal{V}} (1 + \varphi(x) + |\ln f_\varepsilon|) f_\varepsilon d\mu_{\mathcal{N}}(v) dx \right. \\ \left. + \|f_\varepsilon\|_{L^\infty(\Omega \times (0, T) \times \mathbb{R}^D \times \mathcal{V})} \right\} \leq C(T) < +\infty \end{aligned} \quad (4.3)$$

and

$$\sup_{\varepsilon > 0, \mathcal{N} \in \mathbb{N}} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_{\mathcal{V}} \frac{\sigma(\rho_{\varepsilon, \mathcal{N}})}{\varepsilon^2} (f_\varepsilon - \rho_{\varepsilon, \mathcal{N}}) \ln \left(\frac{f_\varepsilon}{\rho_{\varepsilon, \mathcal{N}}} \right) d\mu_{\mathcal{N}}(v) dx dt \leq C(T). \quad (4.4)$$

We recognize the Dunford-Pettis Criterion in (4.3) and it yields

$$f_\varepsilon \rightharpoonup f_{\mathcal{N}} \text{ weakly in } L^1(\Omega \times (0, T) \times \mathbb{R}^d \times \mathcal{V}_{\mathcal{N}}). \quad (4.5)$$

Furthermore, we deduce from (4.4) that f_ε behaves like its macroscopic part, i.e. $f_\varepsilon = \rho_{\varepsilon, \mathcal{N}} + \varepsilon g_{\varepsilon, \mathcal{N}}$ with

$$\sup_{\varepsilon > 0, \mathcal{N}} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left| \int_{\mathcal{V}} g_{\varepsilon, \mathcal{N}} d\mu_{\mathcal{N}}(v) \right|^2 dx dt \leq C(T).$$

Indeed, we deduce it from the following inequality

$$\begin{aligned} \int_{\mathcal{V}} |f_\varepsilon - \rho_{\varepsilon, \mathcal{N}}| d\mu_{\mathcal{N}}(v) &= \int_{\mathcal{V}} (\sqrt{f_\varepsilon} + \sqrt{\rho_{\varepsilon, \mathcal{N}}}) \left| \sqrt{f_\varepsilon} - \sqrt{\rho_{\varepsilon, \mathcal{N}}} \right| d\mu_{\mathcal{N}}(v) \\ &\leq \left(\int_{\mathcal{V}} (\sqrt{f_\varepsilon} + \sqrt{\rho_{\varepsilon, \mathcal{N}}})^2 d\mu_{\mathcal{N}}(v) \right)^{1/2} \left(\int_{\mathcal{V}} (\sqrt{f_\varepsilon} - \sqrt{\rho_{\varepsilon, \mathcal{N}}})^2 d\mu_{\mathcal{N}}(v) \right)^{1/2} \\ &\leq C \sqrt{\rho_{\varepsilon, \mathcal{N}}} \left(\int_{\mathcal{V}} (f_\varepsilon - \rho_{\varepsilon, \mathcal{N}}) \ln(f_\varepsilon / \rho_{\varepsilon, \mathcal{N}}) d\mu_{\mathcal{N}}(v) \right)^{1/2} \end{aligned}$$

using the Cauchy-Schwarz inequality

$$|\sqrt{b} - \sqrt{a}|^2 = \left| \int_a^b \frac{ds}{2\sqrt{s}} \right|^2 \leq \left| \int_a^b \frac{ds}{4s} \right| \left| \int_a^b ds \right| = \frac{1}{4}(b - a) \ln(b/a).$$

We denote

$$J_{\varepsilon, \mathcal{N}}(t, x) := \frac{1}{2\mathcal{N}} \sum_{i=1}^{2\mathcal{N}} \frac{V_i}{\varepsilon} f_\varepsilon(t, x, V_i), \quad \mathbb{P}_{\varepsilon, \mathcal{N}}(t, x) := \frac{1}{2\mathcal{N}} \sum_{i=1}^{2\mathcal{N}} V_i \otimes V_i f_\varepsilon(t, x, V_i).$$

Integrating (4.2) with respect to the velocity variable v yields

$$\partial_t \rho_{\varepsilon, \mathcal{N}} + \operatorname{div}(J_{\varepsilon, \mathcal{N}}) = 0 \quad (4.6)$$

and multiplying by v and integrating leads to

$$\varepsilon^2 \partial_t J_{\varepsilon, \mathcal{N}} + \operatorname{div}(\mathbb{P}_{\varepsilon, \mathcal{N}}) = -\sigma(\rho_{\varepsilon, \mathcal{N}}) J_{\varepsilon, \mathcal{N}}. \quad (4.7)$$

Since $f_\varepsilon = \rho_{\varepsilon, \mathcal{N}} + \varepsilon g_{\varepsilon, \mathcal{N}}$, we have

$$J_{\varepsilon, \mathcal{N}} = \int v g_{\varepsilon, \mathcal{N}} d\mu_{\mathcal{N}}(v),$$

and

$$\mathbb{P}_{\varepsilon, \mathcal{N}} = \int v \otimes v d\mu_{\mathcal{N}}(v) \rho_{\varepsilon, \mathcal{N}} + \varepsilon \mathbb{K}_{\varepsilon, \mathcal{N}}(t, x)$$

with

$$\mathbb{K}_{\varepsilon, \mathcal{N}}(t, x) := \int v \otimes v g_{\varepsilon, \mathcal{N}}(t, x, v) d\mu_{\mathcal{N}}(v)$$

and using (4.5), we deduce that the sequence $(J_{\varepsilon, \mathcal{N}})_{\varepsilon > 0}$ and the components of $(\mathbb{K}_{\varepsilon, \mathcal{N}})_{\varepsilon > 0}$ are bounded in $L^2(\Omega \times (0, T) \times \mathbb{R}^D)$. Therefore (4.7) can be rewritten

$$\varepsilon (\varepsilon \partial_t J_{\varepsilon, \mathcal{N}} + \operatorname{div}(\mathbb{K}_{\varepsilon, \mathcal{N}})) + A_{\mathcal{N}} \nabla_x \rho_{\varepsilon, \mathcal{N}} = -\nu_{\varepsilon, \mathcal{N}}$$

with $\nu_{\varepsilon, \mathcal{N}} := \sigma(\rho_{\varepsilon, \mathcal{N}}) J_{\varepsilon, \mathcal{N}}$. Passing to the limit, up to subsequences, we are led to

$$\begin{cases} \partial_t \rho_{\mathcal{N}} + \operatorname{div}(J_{\mathcal{N}}) &= 0, \\ A_{\mathcal{N}} \nabla \rho_{\mathcal{N}} &= -\nu_{\mathcal{N}} \end{cases} \quad (4.8)$$

where $\nu_{\mathcal{N}}$ is the weak limit as $\varepsilon \rightarrow 0$ of $\nu_{\varepsilon, \mathcal{N}}$, which is a bounded sequence in $L^2(\Omega \times (0, T) \times \mathbb{R}^D)$. It remains to establish a relation between $\nu_{\mathcal{N}}$, $\rho_{\mathcal{N}}$ and $J_{\mathcal{N}}$, or more precisely the

expectation of these quantities. To this end, we are going to use the strong compactness of $\mathbb{E}\rho_{\varepsilon,\mathcal{N}}$ by using the averaging lemma. Indeed, we know that $\mathbb{E}\rho_{\varepsilon,\mathcal{N}}$ belongs to a bounded set in $L^2(0, T; H^{1/2}(\mathbb{R}^D))$; the proof follows exactly the same argument as for Theorem 3.1 taking the Fourier transform with respect to both the time and space variables t, x . However, because of the ε in front of the time derivative, we can not expect a gain of regularity with respect to the time variable. Then, we need to combine this estimate with another argument as follows:

- (i) by using the Weil-Kolmogorov-Fréchet theorem, see [12, Th. 7.56], we deduce from the averaging lemma that

$$\lim_{|h| \rightarrow 0} \left(\sup_{\varepsilon} \int_0^T \int_{\mathbb{R}^D} |\mathbb{E}\rho_{\varepsilon,\mathcal{N}}(t, x+h) - \mathbb{E}\rho_{\varepsilon,\mathcal{N}}(t, x)|^2 dx dt \right) = 0,$$

- (ii) Going back to (4.6), we deduce from above that $\partial_t \mathbb{E}\rho_{\varepsilon,\mathcal{N}} = -\operatorname{div}(\mathbb{E}J_{\varepsilon,\mathcal{N}})$ is bounded, uniformly with respect to ε , in $L^2(0, T; H^{-1}(\mathbb{R}^D))$.

Then, this is enough to deduce that $\mathbb{E}\rho_{\varepsilon,\mathcal{N}}$ strongly converges to $\mathbb{E}\rho_{\mathcal{N}}$ in $L^2((0, T) \times \mathbb{R}^D)$ (see e.g. [1, Appendix B] for a detailed proof).

We conclude noticing that

$$\mathbb{E}J_{\varepsilon,\mathcal{N}} = \mathbb{E} \left(\frac{\nu_{\varepsilon,\mathcal{N}}}{\sigma(\rho_{\varepsilon,\mathcal{N}})} \right) = \frac{\mathbb{E}\nu_{\varepsilon,\mathcal{N}}}{\sigma(\mathbb{E}\rho_{\varepsilon,\mathcal{N}})} + \mathbb{E}r_{\varepsilon,\mathcal{N}},$$

with $r_{\varepsilon,\mathcal{N}} = \left[\nu_{\varepsilon,\mathcal{N}} \left(\frac{1}{\sigma(\rho_{\varepsilon,\mathcal{N}})} - \frac{1}{\sigma(\mathbb{E}\rho_{\varepsilon,\mathcal{N}})} \right) \right]$. Using the strong convergence established above and the assumptions on σ yields

$$\mathbb{E}J_{\varepsilon,\mathcal{N}} \rightharpoonup \mathbb{E}J_{\mathcal{N}} = \frac{\mathbb{E}\nu_{\mathcal{N}}}{\sigma(\mathbb{E}\rho_{\mathcal{N}})} + r_{\mathcal{N}} \quad \text{weakly in } L^2((0, T) \times \mathbb{R}^d)$$

where $\|r_{\mathcal{N}}\|_{L^2((0, T) \times \mathbb{R}^d)} \leq \frac{C}{\sqrt{N}}$. This last point can be easily proved using a classical probabilistic argument.

Furthermore,

$$\mathbb{E}(A_{\mathcal{N}} \nabla_x \rho_{\mathcal{N}}) = -\mathbb{E}\nu_{\mathcal{N}} = -\sigma(\mathbb{E}\rho_{\mathcal{N}}) \mathbb{E}J_{\mathcal{N}} + \sigma(\mathbb{E}\rho_{\mathcal{N}}) r_{\mathcal{N}}$$

and quite similarly we can prove that ,

$$\mathbb{E}(A_{\mathcal{N}} \nabla_x \rho_{\mathcal{N}}) = \mathbb{E}A_{\mathcal{N}} \nabla_x \mathbb{E}\rho_{\mathcal{N}} + s_{\mathcal{N}}$$

where

$$s_{\mathcal{N}} = \mathbb{E}[(A_{\mathcal{N}} - \mathbb{E}A_{\mathcal{N}}) \nabla_x \rho_{\mathcal{N}}]$$

with $s_{\mathcal{N}}$ of order $O(1/\sqrt{N})$ in the $L^2(0, T; H^{-1}(\mathbb{R}^d))$ -norm. □

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