1 Finite Volume Method with staggered grid

1.1 Continuity Equation

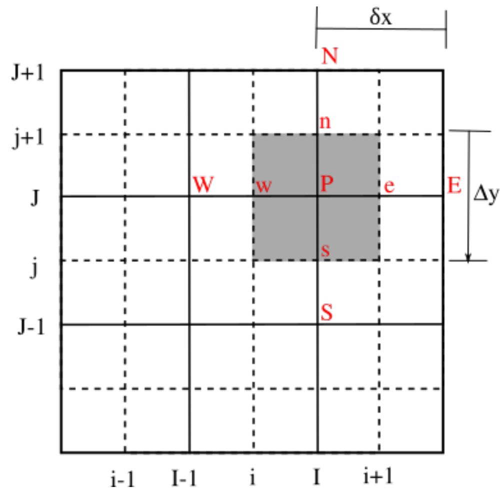

Figure 1:

From Fig. 1, areas $A_e = A_w = \Delta y$, $A_n = A_s = \Delta x$ and $\Delta\Omega$ the cell volume.

$$\frac{1}{\beta}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{1}{\beta\Delta t}\int_t^{t+\Delta t}\frac{\partial p}{\partial t}\Delta\Omega dt + \int_s^n\int_w^e \frac{\partial u}{\partial x}dxdy + \int_s^n\int_w^e \frac{\partial v}{\partial y}dxdy = 0 \tag{2}$$

$$\frac{1}{\beta\Delta t}\int_t^{t+\Delta t}\frac{\partial p}{\partial t}\Delta\Omega dt = \frac{\Delta\Omega}{\Delta t}(p_P^{n+1} - p_P^n) \tag{3}$$

$$\int_s^n\int_w^e \frac{\partial u}{\partial x}dxdy = \Delta y u_e - \Delta y u_w \tag{4}$$

$$\int_s^n\int_w^e \frac{\partial v}{\partial y}dxdy = \Delta x u_n - \Delta x u_s \tag{5}$$

$$\boxed{\frac{1}{\beta}\frac{\Delta\Omega}{\Delta t}(p_P^{n+1} - p_P^n) + \Delta y(u_e - u_w) + \Delta x(u_n - u_s) = 0} \tag{6}$$

1.2 u-Momentum Equation

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (vu)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7}$$

$$\begin{aligned}\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{\partial u}{\partial t}\Delta\Omega dt + \int_s^n\int_w^e \frac{\partial (uu)}{\partial x}dxdy + \int_s^n\int_w^e \frac{\partial (vu)}{\partial y}dxdy = \\ -\int_s^n\int_w^e \frac{\partial p}{\partial x}dxdy + \int_s^n\int_w^e \frac{\partial}{\partial x}\left(\frac{1}{Re}\frac{\partial u}{\partial x}\right)dxdy + \int_s^n\int_w^e \frac{\partial}{\partial y}\left(\frac{1}{Re}\frac{\partial u}{\partial y}\right)dxdy\end{aligned} \tag{8}$$

$$\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{\partial u}{\partial t}\Delta\Omega dt = \frac{\Delta\Omega}{\Delta t}(u_P^{n+1} - u_P^n) \tag{9}$$

$$\int_s^n\int_w^e \frac{\partial (uu)}{\partial x}dxdy = u_e\Delta y\frac{u_E + u_P}{2} - u_w\Delta y\frac{u_P + u_W}{2} \tag{10}$$

$$\int_s^n\int_w^e \frac{\partial (vu)}{\partial y}dxdy = v_n\Delta x\frac{u_N + u_P}{2} - v_s\Delta x\frac{u_P + u_S}{2} \tag{11}$$

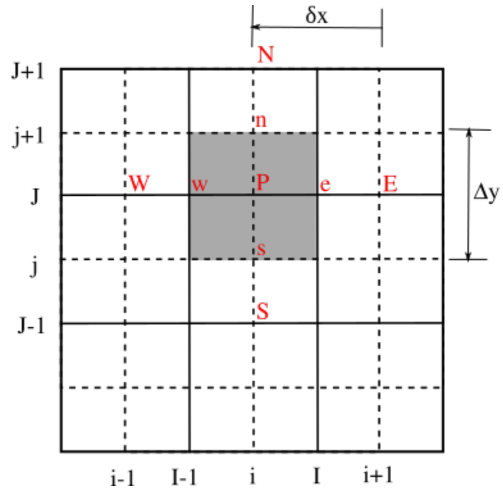

Figure 2:

$$\int_s^n \int_w^e \frac{\partial p}{\partial x} dxdy = \frac{p_e - p_w}{\Delta x} \Delta\Omega \tag{12}$$

$$\int_s^n \int_w^e \frac{\partial}{\partial x}\left(\frac{1}{Re}\frac{\partial u}{\partial x}\right) dxdy = \frac{1}{Re}\Delta y \frac{u_E - u_P}{\delta x_e} - \frac{1}{Re}\Delta y \frac{u_P - u_W}{\delta x_w} \tag{13}$$

$$\int_s^n \int_w^e \frac{\partial}{\partial y}\left(\frac{1}{Re}\frac{\partial u}{\partial y}\right) dxdy = \frac{1}{Re}\Delta x \frac{u_N - u_P}{\delta x_n} - \frac{1}{Re}\Delta x \frac{u_P - u_S}{\delta x_s} \tag{14}$$

Defining:

$$F_f = A_f u_f \qquad \text{and} \qquad D_f = A_f \frac{\Gamma}{\delta x_f} \tag{15}$$

Where f could be n, s, e, w and Γ could be $1/Re$ for momentum or $1/Pe$ for energy eq.

$$\boxed{\frac{\Delta\Omega}{\Delta t}(u_P^{n+1} - u_P^n) + F_e u_e - F_w u_w + F_n u_n - F_s u_s =}$$

$$\boxed{-\frac{p_e - p_w}{\Delta x}\Delta\Omega + D_e(u_E - u_P) - D_w(u_P + u_W) + D_n(u_N - u_P) - D_s(u_P + u_S)} \tag{16}$$

From QUICK (Fig. 3), for momentum in x direction:

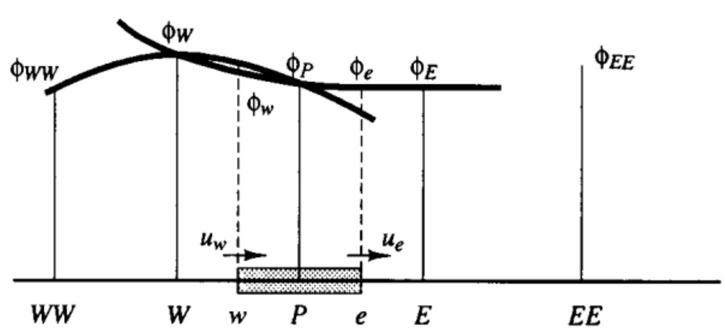

Figure 3:

When $u_w > 0$:

$$u_w = \frac{6}{8}u_W + \frac{3}{8}u_P - \frac{1}{8}u_{WW} \tag{17}$$

When $u_e > 0$:

$$u_e = \frac{6}{8}u_P + \frac{3}{8}u_E - \frac{1}{8}u_W \tag{18}$$

When $u_s > 0$:

$$u_s = \frac{6}{8}u_S + \frac{3}{8}u_P - \frac{1}{8}u_{SS} \tag{19}$$

When $u_n > 0$:

$$u_n = \frac{6}{8}u_P + \frac{3}{8}u_N - \frac{1}{8}u_S \tag{20}$$

$$\begin{aligned}\frac{\Delta\Omega}{\Delta t}(u_P^{n+1} - u_P^n) + F_e\left(\frac{6}{8}u_P + \frac{3}{8}u_E - \frac{1}{8}u_W\right) - F_w\left(\frac{6}{8}u_W + \frac{3}{8}u_P - \frac{1}{8}u_{WW}\right)\\ + F_n\left(\frac{6}{8}u_P + \frac{3}{8}u_N - \frac{1}{8}u_S\right) - F_s\left(\frac{6}{8}u_S + \frac{3}{8}u_P - \frac{1}{8}u_{SS}\right) =\\ -\frac{p_e - p_w}{\Delta x} + D_e(u_E - u_P) - D_w(u_P - u_W) + D_n(u_N - u_P) - D_s(u_P - u_S)\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\Delta\Omega}{\Delta t}(u_P^{n+1} - u_P^n) + \left(D_e + \frac{6}{8}F_e + D_w - \frac{3}{8}F_w + D_n + \frac{6}{8}F_n + D_s - \frac{3}{8}F_s\right)u_P =\\ -\frac{p_e - p_w}{\Delta x} + \left(D_w + \frac{1}{8}F_e + \frac{6}{8}F_w\right)u_W + \left(D_e - \frac{3}{8}F_e\right)u_E + \left(D_s + \frac{1}{8}F_n + \frac{6}{8}F_s\right)u_S + \left(D_n - \frac{3}{8}F_n\right)u_N +\\ \left(-\frac{1}{8}F_w\right)u_{WW} + \left(-\frac{1}{8}F_s\right)u_{SS}\end{aligned} \tag{22}$$

The standard form for discretised equations

$$\frac{\Delta\Omega}{\Delta t}(u_P^{n+1} - u_P^n) + a_P u_P = -\frac{p_e - p_w}{\Delta x} + a_W u_W + a_E u_E + a_S u_S + a_N u_N + a_{WW} u_{WW} + a_{SS} u_{SS} \tag{23}$$

Where

$$a_P = D_e + \frac{6}{8}F_e + D_w - \frac{3}{8}F_w + D_n + \frac{6}{8}F_n + D_s - \frac{3}{8}F_s \tag{24}$$

$$a_W = D_w + \frac{1}{8}F_e + \frac{6}{8}F_w \tag{25}$$

$$a_E = D_e - \frac{3}{8}F_e \tag{26}$$

$$a_S = D_s + \frac{1}{8}F_n + \frac{6}{8}F_s \tag{27}$$

$$a_N = D_n - \frac{3}{8}F_n \tag{28}$$

$$a_{WW} = -\frac{1}{8}F_w \tag{29}$$

$$a_{SS} = -\frac{1}{8}F_e \tag{30}$$

Repeating for other velocities directions and signals:

$$\boxed{\frac{\Delta\Omega}{\Delta t}(u_P^{n+1} - u_P^n) + a_P u_P = -\frac{p_e - p_w}{\Delta x}\Delta\Omega + a_W u_W + a_E u_E + a_S u_S + a_N u_N + a_{WW} u_{WW} + a_{SS} u_{SS} + a_{EE} u_{EE} + a_{NN} u_{NN}} \tag{31}$$

$$a_P = a_W + a_E + a_S + a_N + a_{WW} + a_{EE} + a_{SS} + a_{NN} + F_e - F_w + Fn - F_s \tag{32}$$

$$a_W = D_w + \frac{6}{8}\alpha_w F_w + \frac{1}{8}\alpha_e F_e + \frac{3}{8}(1 - \alpha_w)F_w \tag{33}$$

$$a_E = D_e - \frac{3}{8}\alpha_e F_e - \frac{6}{8}(1 - \alpha_e)F_e - \frac{1}{8}(1 - \alpha_w)F_w \tag{34}$$

$$a_S = D_s - \frac{6}{8}\alpha_s F_s + \frac{1}{8}\alpha_n F_n - \frac{3}{8}(1-\alpha_s)F_s \tag{35}$$

$$a_N = D_n - \frac{3}{8}\alpha_n F_n - \frac{6}{8}(1-\alpha_n)F_n - \frac{1}{8}(1-\alpha_s)F_s \tag{36}$$

$$a_{WW} = -\frac{1}{8}F_w \tag{37}$$

$$a_{EE} = \frac{1}{8}(1-\alpha_e)F_e \tag{38}$$

$$a_{SS} = -\frac{1}{8}F_s \tag{39}$$

$$a_{NN} = \frac{1}{8}(1-\alpha_n)F_n \tag{40}$$

where

$$\begin{array}{llllll} \alpha_w = 1 & \text{for} & F_w > 0 & \text{and} & \alpha_e = 1 & F_e > 0 \\ \alpha_w = 0 & \text{for} & F_w < 0 & \text{and} & \alpha_e = 0 & F_e < 0 \end{array}$$

$$\begin{array}{llllll} \alpha_s = 1 & \text{for} & F_s > 0 & \text{and} & \alpha_n = 1 & F_n > 0 \\ \alpha_s = 0 & \text{for} & F_s < 0 & \text{and} & \alpha_n = 0 & F_n < 0 \end{array}$$

1.3 v-Momentum Equation

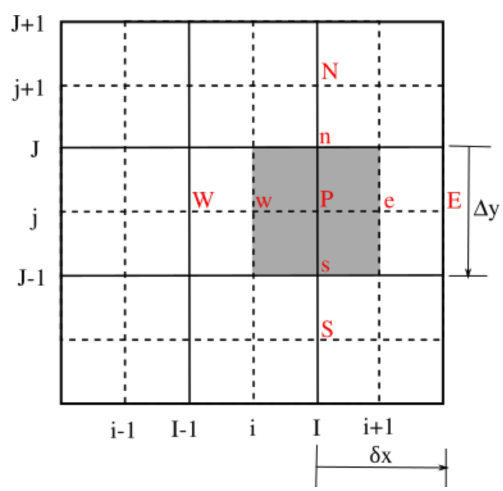

Figure 4:

$$\frac{\partial(v)}{\partial t} + \frac{\partial(uv)}{\partial x} + \frac{\partial(vv)}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + F_g \tag{41}$$

$$\begin{aligned} &\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{\partial v}{\partial t}\Delta\Omega dt + \int_s^n\int_w^e \frac{\partial(uv)}{\partial x}dxdy + \int_s^n\int_w^e \frac{\partial(vv)}{\partial y}dxdy = \\ &-\int_s^n\int_w^e \frac{\partial p}{\partial y}dxdy + \int_s^n\int_w^e \frac{\partial}{\partial x}\left(\frac{1}{Re}\frac{\partial v}{\partial x}\right)dxdy + \int_s^n\int_w^e \frac{\partial}{\partial y}\left(\frac{1}{Re}\frac{\partial v}{\partial y}\right)dxdy + \int_s^n\int_w^e F_g dxdy \end{aligned} \tag{42}$$

$$\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{\partial v}{\partial t}\Delta\Omega dt = \frac{\Delta\Omega}{\Delta t}(v_P^{n+1} - v_P^n) \tag{43}$$

$$\int_s^n \int_w^e \frac{\partial(uv)}{\partial x} dxdy = u_e \Delta y \frac{v_E + v_P}{2} - u_w \Delta y \frac{v_P + v_W}{2} \tag{44}$$

$$\int_s^n \int_w^e \frac{\partial(vv)}{\partial y} dxdy = v_n \Delta x \frac{v_N + v_P}{2} - v_s \Delta x \frac{v_P + v_S}{2} \tag{45}$$

$$\int_s^n \int_w^e \frac{\partial p}{\partial y} dxdy = \frac{p_n - p_s}{\Delta y} \Delta\Omega \tag{46}$$

$$\int_s^n \int_w^e \frac{\partial}{\partial x} \left(\frac{1}{Re} \frac{\partial v}{\partial x} \right) dxdy = \frac{1}{Re} \Delta y \frac{v_E - v_P}{\delta y_e} - \frac{1}{Re} \Delta y \frac{v_P - v_W}{\delta y_w} \tag{47}$$

$$\int_s^n \int_w^e \frac{\partial}{\partial y} \left(\frac{1}{Re} \frac{\partial u}{\partial y} \right) dxdy = \frac{1}{Re} \Delta x \frac{v_N - v_P}{\delta y_n} - \frac{1}{Re} \Delta x \frac{v_P - v_S}{\delta y_s} \tag{48}$$

$$\int_s^n \int_w^e F_g dxdy = F_g \Delta\Omega \tag{49}$$

Where

$$F_g = \frac{1}{Fr} \left(1 - \frac{1}{(T_n + T_s)/2} \right)$$

Defining:

$$F_f = A_f v_f \qquad \text{and} \qquad D_f = A_f \frac{\Gamma}{\delta x_f} \tag{50}$$

Where f could be n, s, e, w and Γ could be $1/Re$ or $1/Pe$.

$$\boxed{\frac{\Delta\Omega}{\Delta t}(v_P^{n+1} - v_P^n) + F_e v_e - F_w v_w + F_n v_n - F_s v_s =}$$
$$\boxed{-\frac{p_n - p_s}{\Delta y} \Delta\Omega + D_e(v_E - v_P) - D_w(v_P + v_W) + D_n(v_N - v_P) - D_s(v_P + v_S) + F_g \Delta\Omega} \tag{51}$$

Applying QUICK (Fig. 3):

$$\boxed{\frac{\Delta\Omega}{\Delta t}(v_P^{n+1} - v_P^n) + a_P v_P = -\frac{p_n - p_s}{\Delta x} \Delta\Omega + a_W u_W + a_E u_E + a_S u_S + a_N u_N + a_{WW} u_{WW} + a_{SS} u_{SS} + a_{EE} u_{EE} + a_{NN} u_{NN} + F_g \Delta\Omega} \tag{52}$$

1.4 Mixture Fraction

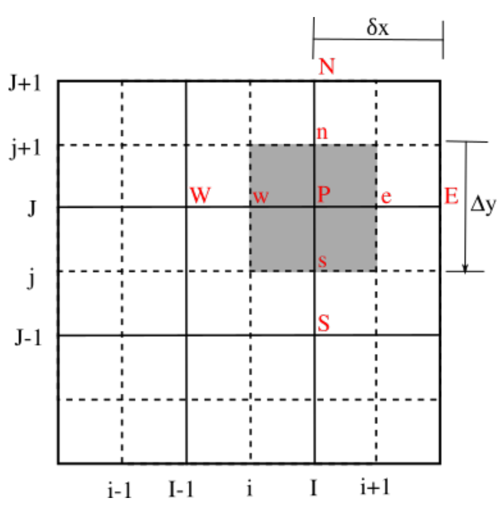

Figure 5:

$$\frac{\partial(Z)}{\partial t}+\frac{\partial(uZ)}{\partial x}+\frac{\partial(vZ)}{\partial y}=\frac{1}{Pe}\left(\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2 Z}{\partial y^2}\right) \tag{53}$$

$$\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{\partial Z}{\partial t}\Delta\Omega dt+\int_s^n\int_w^e\frac{\partial(uZ)}{\partial x}dxdy+\int_s^n\int_w^e\frac{\partial(vZ)}{\partial y}dxdy=\int_s^n\int_w^e\frac{\partial}{\partial x}\left(\frac{1}{Pe}\frac{\partial Z}{\partial x}\right)dxdy+\int_s^n\int_w^e\frac{\partial}{\partial y}\left(\frac{1}{Pe}\frac{\partial Z}{\partial y}\right)dxdy \tag{54}$$

$$\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{\partial Z}{\partial t}\Delta\Omega dt=\frac{\Delta\Omega}{\Delta t}(Z_P^{n+1}-Z_P^n) \tag{55}$$

$$\int_s^n\int_w^e\frac{\partial(uZ)}{\partial x}dxdy=u_e\Delta y\frac{Z_E+Z_P}{2}-u_w\Delta y\frac{Z_P+Z_W}{2} \tag{56}$$

$$\int_s^n\int_w^e\frac{\partial(vZ)}{\partial y}dxdy=v_n\Delta x\frac{Z_N+Z_P}{2}-v_s\Delta x\frac{Z_P+Z_S}{2} \tag{57}$$

$$\int_s^n\int_w^e\frac{\partial}{\partial x}\left(\frac{1}{Pe}\frac{\partial Z}{\partial x}\right)dxdy=\frac{1}{Pe}\Delta y\frac{Z_E-Z_P}{\delta x_e}-\frac{1}{Pe}\Delta y\frac{Z_P-Z_W}{\delta x_w} \tag{58}$$

$$\int_s^n\int_w^e\frac{\partial}{\partial y}\left(\frac{1}{Pe}\frac{\partial Z}{\partial y}\right)dxdy=\frac{1}{Pe}\Delta x\frac{Z_N-Z_P}{\delta y_n}-\frac{1}{Pe}\Delta x\frac{Z_P-Z_S}{\delta y_s} \tag{59}$$

Defining:

$$F_f=A_f v_f \qquad \text{and} \qquad D_f=A_f\frac{\Gamma}{\delta x_f} \tag{60}$$

$$\boxed{\frac{\Delta\Omega}{\Delta t}(Z_P^{n+1}-Z_P^n)+F_eZ_e-F_wZ_w+F_nZ_n-F_sZ_s=D_e(Z_E-Z_P)-D_w(Z_P+Z_W)+D_n(Z_N-Z_P)-D_s(Z_P+Z_S)} \tag{61}$$

1.5 Artificial bulk viscosity

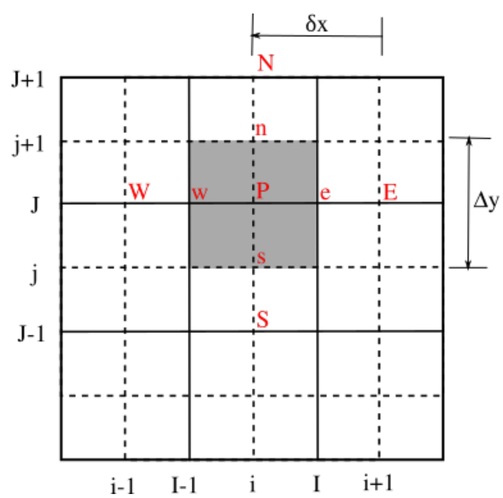

Figure 6:

$$b\nabla\cdot\mathbf{u}=b\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\right] \tag{62}$$

$$\int_s^n\int_w^e\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)dxdy=\Delta y\frac{u_E-u_P}{\delta x_e}-\Delta y\frac{u_P-u_w}{\delta x_w} \tag{63}$$

$$\int_s^n\int_w^e\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)dxdy=\Delta y\left(\frac{\partial v}{\partial y}\right)_e-\Delta y\left(\frac{\partial v}{\partial y}\right)_w=\Delta y\left(\frac{v_{I,j+1}-v_{I,j}}{\partial y_e}\right)-\Delta y\left(\frac{v_{I-1,j+1}-v_{I-1,j}}{\partial y_w}\right) \tag{64}$$

$$b\nabla \cdot \mathbf{u} = b\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)\right] \tag{65}$$

$$\int_s^n \int_w^e \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) dxdy = \Delta x \left(\frac{\partial u}{\partial x}\right)_n - \Delta x \left(\frac{\partial u}{\partial x}\right)_s = \Delta x \left(\frac{u_{I,j+1} - v_{I-1,j+1}}{\partial x_n}\right) - \Delta x \left(\frac{u_{I,j} - u_{I-1,j}}{\partial x_s}\right) \tag{66}$$

$$\int_s^n \int_w^e \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) dxdy = \Delta x \frac{v_N - v_P}{\delta y_n} - \Delta x \frac{v_P - v_S}{\delta y_s} \tag{67}$$

2 Artificial Bulk Viscosity for Artificial Compressibility Method

The time evolution equation for the pressure p, is

$$\partial \mathbf{p}/\partial t = -\rho a^2 \nabla \cdot \mathbf{u} \tag{68}$$

where/in which $\mathbf{u}$ is the fluid velocity, ρ is the fluid density and a is an artificial sound speed. The time evolution of the velocity $\mathbf{u}$ is governed by the unmodified momentum equation,

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} = -\rho^{-1} \nabla p + \nu \nabla^{\mathbf{2}} \mathbf{u} \tag{69}$$

where/in which ν is the kinematic viscosity.

The artificial compressibility method contains no explicit mechanism for damping out the artificial sound waves. These waves simply die out through the combined action of viscosity and the numerical or artificial diffusion associated with the discretization.

Sound waves are compression waves, and the physical effect which damps them most directly is bulk viscosity. It is therefore natural to introduce an artificial bulk viscosity to remove them. The effect of bulk viscosity is to subtract a term proportional to the divergence of the fluid velocity from the thermodynamic pressure in the momentum equation [1]. We therefore replace eq. (69) by

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} = -\rho^{-1} \nabla q + \nu \nabla^{\mathbf{2}} \mathbf{u} \tag{70}$$

where/in which

$$q = p - \rho b \nabla \cdot \mathbf{u} \tag{71}$$

The coefficient ρb is an artificial bulk viscosity, and p is still governed by eq. (68). This modification will not change the steady solution, since $\nabla \cdot \mathbf{u}$ still vanishes in steady state by virtue of eq. (68). The value of b will be chosen to minimize the time needed to reach steady state. On general grounds b will be of order $\Delta x^2/\Delta t$, as this will ensure that the acoustic and diffusional time scales for disturbances of wavelength Δx will be comparable. The artificial damping will work best in problems where physical damping mechanisms, such as shear viscosity, are too small to remove the artificial sound waves efficiently. Thus the present method will generally be more beneficial at higher Reynolds numbers than lower ones.

REFERENCES

[1] Ramshaw, J. D., and V. A. Mousseau. "Accelerated artificial compressibility method for steady-state incompressible flow calculations." Computers fluids 18.4 (1990): 361-367.

[2] POPE, D. N.; GOGOS, G. Numerical simulation of fuel droplet extinction due to forced convection. Combustion and Flame, Elsevier, v. 142, n. 1, p. 89–106, 2005.