UC Berkeley
Department of Statistics
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STAT 210A: Introduction to Mathematical Statistics

Problem Set 9

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Note: All measure-theoretic niceties about conditioning on measure-zero sets, almost-sure equality vs. actual equality, "all functions" vs. "all measurable functions," etc. are disregarded for the time being.

1. Super-Efficient Estimator

Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\theta, 1)$ and consider estimating θ via:

$$\delta_n = \begin{cases} \bar{X}_n & |\bar{X}_n| > n^{-1/4} \\ 0 & |\bar{X}_n| \leq n^{-1/4} \end{cases}, \quad \text{where } \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Show that δ_n has the same asymptotic distribution as $\bar{X}_n$ (which is UMVU) when $\theta \neq 0$, but that $\sqrt{n}(\delta_n - 0) \xrightarrow{p} 0$ if $\theta = 0$.

(Not required, just for fun: is it possible to find a scaling for which δ_n converges to a non-degenerate distribution; i.e. not converging in probability to a constant?)

2. Maximum Likelihood for Uniform

(a) Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Unif}[0, \theta]$. Find the maximum likelihood estimator $\hat{\theta}_n$ for θ and show that its asymptotic distribution is given by:

$$n(\theta - \hat{\theta}_n) \Rightarrow \text{Exp}(\theta) = \frac{1}{\theta} e^{-x/\theta} 1\{x > 0\}.$$

Here the error is of order $\frac{1}{n}$ instead of $\frac{1}{\sqrt{n}}$ as we usually expect, and as is predicted by the theorem from class. Why doesn't the $\frac{1}{n}$ rate of convergence contradict the theorem from class on the asymptotic distribution of the MLE?

(Note: a factor of n appears in the display equation above where we would normally see a factor of $\sqrt{n}$ instead. This is not a typo.)

(b) We showed previously that the UMVU is $\delta_n = \frac{n+1}{n} X_{(n)}$, where $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Find the asymptotic distribution of δ_n. Which asymptotic distribution seems better?

3. Asymptotic Relative Efficiency for Poisson

Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Pois}(\theta) = \frac{\theta^x e^{-\theta}}{x!}$, and we are interested in estimating $\pi(\theta) = \mathbb{P}_\theta(X_i = 0) = e^{-\theta}$.

(a) One natural estimator is the proportion of zeros in the sample,

$$\tilde{\pi}_n = \frac{1}{n} \#\{i \leq n : X_i = 0\}.$$

Find its asymptotic distribution.

(b) Another estimator is the MLE, $\hat{\pi}_n$. Give an explicit formula for $\hat{\pi}_n$ and determine its asymptotic distribution. What is the asymptotic relative efficiency of $\tilde{\pi}_n$ with respect to $\hat{\pi}_n$?

4. Some Maximum Likelihood Estimators

Find the MLE for each model below:

(a) Laplace: $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \frac{1}{2}e^{-|x-\theta|}$.

(b) Binomial: $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Binom}(m, \theta)$. Find the MLE for θ and for the canonical parameter $\eta = \log \frac{\theta}{1-\theta}$.

(c) Gaussian variance: $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\theta, \sigma^2)$. Find (i) the MLE for θ if σ^2 is known, (ii) the MLE for σ^2 if θ is known, and (iii) the MLE for (θ, σ^2) if neither is known.

5. Relaxed Condition for Asymptotic Normality

Generalize the proof from class to show that, if $\hat{\theta}_n$ is a consistent estimator of θ (not necessarily the MLE), and

$$\frac{1}{\sqrt{n}}\ell'(\hat{\theta}_n; X) \xrightarrow{p} 0,$$

then we have the same conclusion, namely that:

$$\sqrt{n}(\hat{\theta}_n - \theta) \Rightarrow N\left(0, \frac{1}{J_1(\theta)}\right).$$

(Note: our proof in class is not quite complete yet. You may use, without proof, the following fact which we have not yet justified: under the conditions of the theorem from class, if $\tilde{\theta}_n \xrightarrow{P_\theta} \theta$, then $\frac{1}{n}\ell''(\tilde{\theta}_n) \xrightarrow{P_\theta} -J_1(\theta)$.)