UC Berkeley
Department of Statistics
Fall 2016

STAT 210A: Introduction to Mathematical Statistics

Problem Set 10

Student: Ivana Malenica **Due:** Tuesday, Nov. 22

Note: All measure-theoretic niceties about conditioning on measure-zero sets, almost-sure equality vs. actual equality, "all functions" vs. "all measurable functions," etc. are disregarded for the time being.

1. Uniform Convergence

Take $K = [0, 1]$, let W_n, $n \geq 1$, be random functions taking values in $C(K)$, and let f be a fixed function in $C(K)$. Consider the following conjecture: If $\|W_n - f\|_\infty \xrightarrow{p} 0$ as $n \to \infty$, then $\int_0^1 W_n(t)\,dt \xrightarrow{p} \int_0^1 f(t)\,dt$. Is this conjecture true or false? If true, give a proof; if false, give a counterexample.

2. Lognormal MLE

If $Z \sim N(\mu, \sigma^2)$, then $X = e^Z$ has the lognormal distribution with parameters μ and σ^2. In some situations a threshold γ, included by taking

$$X = \gamma + e^Z,$$

may be desirable, and in this case X is said to have the three-parameter lognormal distribution with parameters γ, μ, and σ^2. Let data $X_1, \ldots, X_n$ be i.i.d. from this three-parameter lognormal distribution.

(a) Find the common marginal density for the X_i.

(b) Suppose the threshold γ is known. Find the maximum likelihood estimators $\hat{\mu} = \hat{\mu}(\gamma)$ and $\hat{\sigma}^2 = \hat{\sigma}^2(\gamma)$ of μ and σ^2. (Assume $\gamma < X_{(1)}$.)

(c) Let $\ell(\gamma, \mu, \sigma^2)$ denote the log-likelihood function. The maximum likelihood estimator for γ, if it exists, will maximize $\ell(\gamma, \hat{\mu}(\gamma), \hat{\sigma}^2(\gamma))$ over γ. Determine

$$\lim_{\gamma \uparrow X_{(1)}} \ell(\gamma, \hat{\mu}(\gamma), \hat{\sigma}^2(\gamma)).$$

Hint: Show first that as $\gamma \uparrow X_{(1)}$,

$$\hat{\mu}(\gamma) \sim \frac{1}{n} \log(X_{(1)} - \gamma), \quad \text{and } \hat{\sigma}^2(\gamma) \sim \frac{n-1}{n^2} \log^2(X_{(1)} - \gamma),$$

where the notation $f(\gamma) \sim g(\gamma)$ means $f(\gamma)/g(\gamma) \to 1$.

3. Censored Poisson

Suppose that Y_i, $i = 1, \ldots, n$ are drawn i.i.d. from a Pois(θ) distribution for $\theta \in (0, \infty)$, but that we only observe the censored quantities

$$X_i = \begin{cases} 0 & \text{if } Y_i = 0 \\ 1 & \text{if } Y_i > 0. \end{cases}$$

(a) Under what conditions does an MLE $\hat{\theta}$ of θ based on $(X_1, \ldots, X_n)$ exist? Compute the probability that the $\hat{\theta}$ exists as a function of θ and n. What happens as $n \to +\infty$?

(b) Define the modified estimator

$$\delta(X) \quad = \quad \begin{cases} \hat{\theta}(X) & \text{when the MLE exists} \\ 1 & \text{otherwise.} \end{cases}$$

Prove that $\sqrt{n}(\delta - \theta) \Rightarrow N(0, e^{\theta} - 1)$.

4. MLE Under Reparameterization

Given a family $\mathcal{P} = \{p_\theta(x) : \theta \in \Theta\}$ of density functions, let $\widehat{\theta}$ denote the MLE of θ. Suppose h is a one-to-one function from Θ to $h(\Theta)$. Setting $\eta = h(\theta)$, and $q_\eta(x) = p_{h^{-1}(\eta)}(x)$, define the family $\mathcal{Q} = \{q_\eta(x) : \eta \in h(\Theta)\}$, meaning that we have reparameterized the model using η.

(a) Show that the MLE $\hat{\eta}$ of η is $h(\widehat{\theta})$. (This result shows that MLEs are invariant under one-to-one reparameterizations.)

(b) If $\hat{\theta}$ is consistent for θ, then must $\hat{\eta}$ be consistent for η? Prove or give a counterexample.

5. Superefficiency Revisited

Recall the superefficient estimator for θ given $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\theta, 1)$, from the last problem set:

$$\delta_n = \begin{cases} \bar{X}_n & |\bar{X}_n| > n^{-1/4} \\ 0 & |\bar{X}_n| \leq n^{-1/4} \end{cases}, \quad \text{where } \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Show that, pointwise in θ, as $n \to \infty$,

$$n \, \mathrm{MSE}(\delta_n; \theta) \to 1\{\theta \neq 0\},$$

but that the convergence is not uniform in θ; in fact,

$$\sup_{\theta \in \mathbb{R}} |n \, \mathrm{MSE}(\delta_n; \theta) - 1\{\theta \neq 0\}| \to \infty$$

(Hint: consider the sequence $\theta_n = n^{-1/4}$)