UC Berkeley
Department of Statistics
Fall 2016

STAT 210A: Introduction to Mathematical Statistics

Problem Set 3

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Note: All measure-theoretic niceties about conditioning on measure-zero sets, almost-sure equality vs. actual equality, "all functions" vs. "all measurable functions," etc. are disregarded for the time being.

1. U-estimable functions

Assume that the model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ has complete sufficient statistic $T(X)$, which can take exactly d distinct values (all with positive probability, $d < \infty$).

(a) Let $\mathcal{G}$ denote the set of real-valued U-estimable functions $g(\theta)$. Show that $\mathcal{G}$ is a vector space of dimension d. (Hint: use completeness)

(b) If $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$, show that $g(\theta)$ is U-estimable if and only if it is a polynomial of degree at most n.

2. Poisson UMVU estimation

Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{Pois}(\theta)$ and consider estimating

$$g(\theta) = e^{-\theta} = \mathbb{P}_\theta(X_1 = 0)$$

(a) Find the UMVU estimator for $g(\theta)$ by Rao-Blackwellizing $1\{X_1 = 0\}$. (Hint: start by showing that $(X_1, \dots, X_n) \sim \text{Multinom}(t, (n^{-1}, \dots, n^{-1}))$ given $\sum_{i=1}^n X_i = t$).

(b) Find the UMVU estimator for $g(\theta)$ directly, using the power series method from class.

(c) If $\theta = 1$, numerically compute the efficiency of the UMVU (relative to the CRLB) for $n = 1, 10, 100$.

3. UMVUE for uniforms

Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{U}[0, \theta]$ and consider estimating $\beta = g(\theta)$ where $g(\cdot)$ is a differentiable function on $(0, \infty)$. Show that the UMVUE of β is given by

$$h(X) \quad = \quad g(X_{(n)}) + \frac{X_{(n)} g'(X_{(n)})}{n},$$

where $X_{(n)} = \max\{X_1, \dots, X_n\}$.

4. Fisher information for scale families

Consider a scale family

$$p_\theta(x) = \frac{1}{\theta} p\left(\frac{x}{\theta}\right), \quad \theta > 0.$$

where p is some fixed density function with respect to the Lebesgue measure.

(a) Show that the Fisher information of a single observation X is given by

$$J(\theta) = \frac{1}{\theta^2} \int_{-\infty}^{\infty} \left[\frac{y p'(y)}{p(y)} + 1\right]^2 p(y) dy.$$

(b) Show that the Fisher information of a single observation X about $\xi = \log\theta$ is independent of θ.

5. HCR inequality for exponential location family

Suppose that $X_1, \dots, X_n$ are i.i.d. with a single observation having density

$$p_\theta^{(1)}(x) = e^{\theta - x}\, 1\{x \geq \theta\}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

Note that this family does not have common support and the log-likelihood is certainly not differentiable w.r.t. θ at $\theta = x_{(1)}$.

(a) Show that (in general) the HCR inequality still applies when the support of $p_{\theta+\varepsilon}$ is contained in the support of p_θ, and conclude that we can therefore still apply the HCR inequality in this problem for $\varepsilon \geq 0$ (Note: to alleviate any existential angst about dividing 0 by 0, we can formally define the function $L(x) = p_{\theta+\varepsilon}(x)/p_\theta(x)$ if the denominator is positive and $L(x) = 1$ if the numerator and denominator are both 0)

(b) If $\delta(X)$ is an unbiased estimator of θ, optimize the HCR bound to show that

$$\mathrm{Var}_\theta(\delta(X)) \geq \frac{a^2}{(e^a - 1)n^2},$$

where a solves

$$\frac{2}{a} - \frac{e^a}{e^a - 1} = 0.$$

(c) Prove that $\delta(X) = \min_i X_i - \frac{1}{n}$ is unbiased and has variance $1/n^2$. (Note: this is much better than the $1/n$ scaling predicted by the CRLB, which does not apply here)