UC Berkeley
Department of Statistics
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STAT 210A: Introduction to Mathematical Statistics

Problem Set 7

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Note: All measure-theoretic niceties about conditioning on measure-zero sets, almost-sure equality vs. actual equality, "all functions" vs. "all measurable functions," etc. are disregarded for the time being.

1. UMAU confidence intervals

We say a confidence interval for $g(\theta)$ is *unbiased* if

$$\mathbb{P}_{\theta_0}(g(\theta_0) \in C(X)) \geq \mathbb{P}_{\theta_0}(g(\theta_1) \in C(X)), \quad \forall \theta_0, \theta_1 \in \Theta.$$

Furthermore, $C(X)$ is called the *uniformly most accurate unbiased* (UMAU) $1-\alpha$ confidence interval for $g(\theta)$ if, for any other $1-\alpha$ unbiased confidence interval $\widetilde{C}(X)$, we have

$$\mathbb{E}_\theta[\text{length}(C(X))] \leq \mathbb{E}_\theta[\text{length}(\widetilde{C}(X))], \quad \forall \theta \in \Theta.$$

Let $C(X)$ be the confidence interval for θ that inverts the family $(\phi_\theta : \theta \in \Theta \subseteq \mathbb{R})$, where $\phi_{\theta_0}(X)$ is a non-randomized level-α test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

(a) Show that $C(X)$ is unbiased if and only if $\phi_\theta(X)$ is unbiased for every θ.

(b) Assuming $C(X)$ has finite expected length, show that $C(X)$ is UMAU if $\phi_\theta(X)$ is UMPU for every θ.

2. Randomized confidence sets

Consider a generic statistical model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ and some parameter of interest $g(\theta) \in \mathbb{R}$.

(a) Suppose that for each value of $z \in \mathbb{R}$, $\phi_z(x)$ is a (possibly randomized) level-α test of $H_0 : g(\theta) = z$. Show that the (non-randomized) set

$$C(X) = \{z : \phi_z(X) < 1\}$$

is a valid $1-\alpha$ confidence set for $g(\theta)$.

(b) Let $U \sim \text{Unif}[0,1]$, independently of the data X. Show that

$$C_r(X, U) = \{z : \phi_z(X) < U\}$$

is a valid $1-\alpha$ confidence set for $g(\theta)$ which is always at least as small as $C(X)$.

(Note: no one uses randomized confidence intervals or randomized tests outside of statistics theory classes; we just use the construction in part (a))

3. Fisher's exact test

Suppose $X_i \sim \text{Binom}(n_i, \pi_i)$ independently for $i = 0, 1$ and $\pi_0, \pi_1 \in (0,1)$. Consider testing $H_0 : \pi_1 \leq \pi_0$ vs. $H_1 : \pi_1 > \pi_0$.

(a) A natural object of inference in this model is the *odds ratio*:

$$\mathrm{OR} = \frac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}.$$

Show that $\mathrm{OR} \le 1$ if and only if H_0 is true.

(b) Find the UMPU level-α test of H_0, giving the cutoffs $c(u)$ in terms of quantiles of a hypergeometric distribution (you do not need to determine the randomization values $\gamma(u)$).

(c) Suppose $n_0 = n_1 = 200$, $X_0 = 90$ and $X_1 = 60$. Give a confidence interval for the odds ratio (you don't need to make it randomized; just use the conservative version).

(Note: This is almost certainly the most important non-Gaussian example of a UMPU test with nuisance parameters, and has been used in countless clinical trials and observational studies. For example, we might give n_1 cardiac disease patients a new drug and give n_0 a placebo, then observe how many patients in each group suffer a heart attack within the next 5 years.)

4. McNemar's test

Suppose we have paired binary data: for $i = 1, \ldots, n$ we observe $(X_i, Y_i) \in \{0,1\}^2$. The pairs are i.i.d. with

$$\mathbb{P}\left[(X_i, Y_i) = (a, b)\right] = \pi_{a,b} \quad a, b \in \{0,1\}.$$

Write $\pi_X = \mathbb{P}(X_i = 1) = \pi_{1,0} + \pi_{1,1}$ and $\pi_Y = \mathbb{P}(Y_i = 1) = \pi_{0,1} + \pi_{1,1}$, and let $N_{a,b} = \sum_{i=1}^n 1\{X_i = a, Y_i = b\}$.

(a) Find the UMPU test of $H_0 : \pi_X \le \pi_Y$ vs. $H_1 : \pi_X > \pi_Y$, giving cutoffs $c(u)$ in terms of quantiles of a binomial distribution (you do not need to determine the randomization values $\gamma(u)$). (Hint: it may help to first reframe the hypothesis in terms of the $\pi_{a,b}$ parameters.)

(b) Suppose $N_{0,0} = N_{1,1} = 1000$, $N_{1,0} = 5$ and $N_{0,1} = 25$. Compute 95% confidence intervals for π_X and π_Y. Then compute a p-value for $H_0 : \pi_X \le \pi_Y$ (you do not need to randomize in either case). Does anything about the respective answers surprise you?

(Note: This test is called McNemar's test; it is very useful for clinical trials with matched pairs of subjects, and also for comparing the performance of different classifiers on a held-out sample.)

5. Permutation tests

(a) Suppose $X_1, \ldots, X_n \in \mathbb{R}$ are independent random variables with $X_i \sim P_i$. Consider testing the null hypothesis $H_0 : P_1 = P_2 = \cdots = P_n$ (i.e., the observations are i.i.d.) against the alternative that there is a systematic trend toward larger values of X_i as i increases. Derive an exact level-α test.

(b) Suppose $(X_1, Y_1), \ldots, (X_n, Y_n) \overset{\text{i.i.d.}}{\sim} P$ where P is an unknown joint distribution on $\mathbb{R}^2$. Consider testing the null hypothesis that X_i and Y_i are independent within each pair (i.e., $P = P_X \times P_Y$, with P_X and P_Y unknown and not necessarily the same) versus the alternative that (X_i, Y_i) are positively correlated within each pair. Derive an exact level α test.

(Note: for the above tests there is some wiggle room in how you choose the test statistic and it will probably not be possible to determine the cutoff explicitly. Just choose a reasonable one and show that your test is level α).