UC Berkeley
Department of Statistics
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STAT 210A: Introduction to Mathematical Statistics

Problem Set 6

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Note: All measure-theoretic niceties about conditioning on measure-zero sets, almost-sure equality vs. actual equality, "all functions" vs. "all measurable functions," etc. are disregarded for the time being.

1. MLR in location families

Assume $X \sim p_\theta(x) = p_0(x-\theta)$, a location family with p_0 continuous and strictly positive. Show that the family has MLR in x if and only if $\log p_0$ is concave.

2. p-value densities

Suppose $\mathcal{P}$ is a family with monotone likelihood ratio in $T(X)$, and the distribution of $T(X)$ is continuous with common support for all θ. Let ϕ_α denote the UMP level-α test of $H_0 : \theta \leq \theta_0$ vs. $H_0 : \theta > \theta_0$ that rejects when $T(X)$ is large. Let $p(X)$ denote the resulting p-value. Show that $p(X) \sim \text{Unif}[0,1]$ if $\theta = \theta_0$, has non-increasing density on $[0,1]$ if $\theta > \theta_0$, and has non-decreasing density on $[0,1]$ if $\theta < \theta_0$.

3. Uniform UMP test

This problem gives an amusing example of a rare two-sided UMP test. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the uniform distribution on $[0,\theta]$.

(a) Consider the problem of testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Show that any test ϕ for which $\mathbb{E}_{\theta_0}[\phi(X)] = \alpha$, $\mathbb{E}_\theta[\phi(X)] \leq \alpha$ for all $\theta \leq \theta_0$ and $\phi(x) = 1$ when $x_{(n)} = \max\{x_1, \ldots, x_n\} > \theta_0$ is UMP at level α.

(b) Now consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. Show that a unique UMP test exists, and is given by

$$\phi(x) \quad = \quad \begin{cases} 1 & \text{if } x_{(n)} > \theta_0 \text{ or } x_{(n)} < \theta_0 \alpha^{1/n} \\ 0 & \text{otherwise.} \end{cases}$$

4. Poisson testing

Suppose $X \sim \text{Pois}(\theta)$.

(a) Find the UMP level 0.05 test of $H_0 : \theta \leq 1$ vs. $H_1 : \theta > 1$.

(b) Find the UMPU level 0.05 test of $H_0 : \theta = 1$ vs. $H_1 : \theta \neq 1$.

5. Bayesian hypothesis testing

p-values are commonly misinterpreted as representing "the probability that the null hypothesis is true, given the data." But such a quantity could only be computed in a Bayesian framework such as the following.

Suppose that $X \mid \theta \sim N(\theta, 1)$, with $\theta = 0$ under the null hypothesis and having some prior distribution Λ_1 under the alternative hypothesis (assume $\Lambda_1(\{0\}) = 0$). In addition let π_0 denote the *a priori* probability that the null hypothesis is true.

(a) What is the posterior probability $\pi_{\text{post}}(X; \Lambda_1, \pi_0)$ that the null hypothesis is true?

(b) If $X = z_{0.025}$ then the usual two-sided p-value is $p(X) = 0.05$. Assume $\pi_0 = 0.5$ (we are initially agnostic between the null and the alternative), and find the smallest possible value for the posterior null probability; i.e., find

$$\min_{\Lambda_1} \pi_{\text{post}}(X; \Lambda_1, 0.5)$$

and give the minimizing prior Λ_1.

(c) Now restrict $\Lambda_1 = N(0, \tau^2)$ for $\tau > 0$ so the alternative must be symmetric around 0. Again, find

$$\min_{\tau^2 > 0} \pi_{\text{post}}(X; N(0, \tau^2), 0.5),$$

and give the minimizing value of τ^2.

Note that choosing the prior after seeing the data is obviously cheating and not something we would do in practice. The point is that the wrong interpretation of the p-value really is wrong wrong wrong (and would be still more wrong if we didn't optimize the prior), even for a seemingly "objective" prior.