Newton's Law of Cooling and Its Application

Runtime Terror

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GROUP MEMBERS

1. Natnael Alemayehu
2. Mahalinoro Razafimanjato
3. Christopher Cofie Kuzagbe
4. Zubery Ayyub Msemo

1 Introduction

Let's assume that Pierre is in the kitchen. He wants to drink water but there is no water in the fridge and drinking water from the tap is not a solution. So, he decided to boil water instead which means that He has to wait until the water has the perfect temperature to be drinkable. In this situation, we know that as Pierre waits, the temperature of the water will cool down and he will be able to drink it afterwards. This time around, let's assume that Pierre wants to cook his dinner. He is preparing a chicken then places it in the hot oven. Eventually, what is going to happen after several hours is that the chicken will reach the desired temperature and will be cooked. Pierre can enjoy his warm dinner.

These kind of situations are very common in real life and we all have experienced them in different ways. But what we do not know is that it follows a certain pattern which is when an object is at a different temperature than its surroundings, it will gradually cool down or heat up until the temperatures are equal. This leads us to the Newton's Law of Cooling. In the 17^{th} century, Isaac Newton studied the nature of cooling. In his studies, he found out that if there is a less than 10 degree difference between two objects the rate of heat loss is proportional to the temperature difference. Newton applied this principle to estimate the temperature of a red-hot iron ball by observing the time which it took to cool from a red heat to a known temperature, and comparing this with the time taken to cool through a known range at ordinary temperature. According to this law, if the excess of the body above its surrounding is observed at equal intervals of time, the observed values will form a geometrical progression with a common ratio. However, Newton's law was inaccurate at high temperatures. That is why Pierre Dulong and Alexis Petit corrected Newton's law by clarifying the effect of the temperature of the surroundings (slee1111, 2019). Thus:
Newton's Law of Cooling states that the rate at which an object changes temperature is proportional to the difference between its temperature and the temperature of the surrounding.

Considering Pierre's chicken, when it first comes out of the oven, the chicken is much more warmer than the surrounding air, which is at the 'room temperature'. Because the temperature between the cooked chicken and the room temperature is relatively large, the cooling rate is also relatively great. As time passes, the temperature difference decreases and therefore the cooling rate slows. Eventually, if left unattended, the cooling rate decreases to 0 because the the chicken's temperature will have cooled to room temperature. The chicken cannot get any cooler than the air around it.

Based on the situation, we can generate a differential equation to model this phenomenon:

$$\frac{dT}{dt} = k(T - T_a)$$

Where:

- dT/dt = Rate of Change of the object's temperature
- T(t) = Temperature of the object of interest as a function of time
- t = Time
- T_a = Temperature of the environment
- k = Proportionality constant specific to the object of interest

1.1 Experimental Investigation

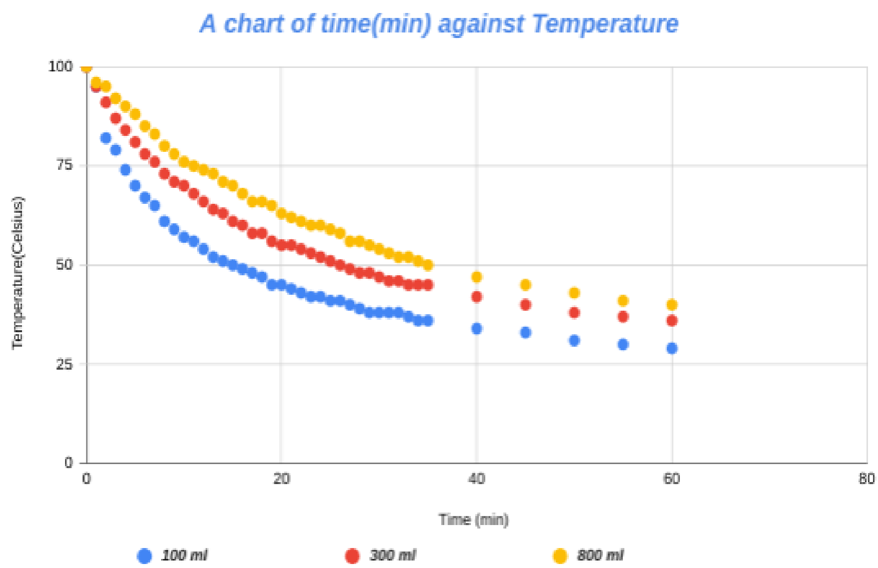
For this exploration, let's test experimentally by measuring the temperature in three beakers of water as they cooled from boiling. The purposes of this experiment are determining if the Newton's Law of Cooling is fitting well with real data and investigating the effect of changing the volume of water being cooled. Using three beakers of water for this experiment. The first hold 100 ml of water, the second 300 ml, and the third 800 ml. Originally, all of the beakers hold water at 100C. Each beaker had its own thermometer and the thermometers are kept in the beakers between measurements so there would be no temperature lag. The temperature of the water in each beaker is measured every minute, always in the same order. The temperature of the surrounding for this investigation is 23C.

The temperature is measured every minute for 35 minutes and then every 5 minutes for the remainder of one hour. The following data has been obtained.

Time(min)	100ml Temperature C	300ml Temperature C	800ml Temperature C
0	100	100	100
1	95	95	96
2	82	91	95
3	79	87	92
4	74	84	90
5	70	81	88
6	67	78	85
7	65	76	83
8	61	73	80
9	59	71	78
10	57	70	76
11	56	68	75
12	54	66	74
13	52	64	73
14	51	63	71
15	50	61	70
16	49	60	68
17	48	58	66
18	47	58	66
19	45	56	65
20	45	55	63
21	44	55	62
22	43	54	61
23	42	53	60
24	42	52	60
25	41	51	59
26	41	50	58
27	40	49	56
28	39	48	56
29	38	48	55
30	38	47	54
31	38	46	53
32	38	46	52
33	37	45	52
34	36	45	51
35	36	45	50
40	34	42	47
45	33	40	45
50	31	38	43
55	30	37	41
60	29	36	40

1.1.1 Observation

From this data, it can be observed that the water in the smaller beakers cooled more quickly than the water in the larger beakers. Below is a graph of the data.



The next step is to determine the value of k for each of the beakers of water. We need to integrate our initial function first and then solve for k.

- Newton's Law of cooling integration

$$\frac{dT}{dt} = k(T - T_a) \tag{1}$$

$$\int \frac{1}{T - T_a} dt = \int kdt \tag{2}$$

$$ln|T - T_a| = kt + C \tag{3}$$

$$|T - T_a| = e^{kt+C} = e^{kt}e^C = Ce^{kt} \tag{4}$$

$$T(t) = Ce^{kt} + T_a \tag{5}$$

Where T >= T_a because the temperature is cooling down

$$=> C = T_0 - T_a \tag{6}$$

$$T(t) = (T_0 - T_a)e^{kt} + T_a \tag{7}$$

Where: T_0 is the initial temperature in this situation The first thing we notice is the exponential e. This means that it would imply a continuous rate of cooling and would yield a realistic result, as liquids cool continually.

- Finding k

$$T(t) = (T_0 - T_a)e^{kt} + T_a$$

where:

  - T(t): The temperature of the water in respect of time
  - T_a: The temperature of the surrounding
  - T_0: The initial temperature of the water a t = 0
  - t: The time
  - k: The heat transfer coefficient

$$\frac{T(t) - T_a}{T_0 - T_a} = e^{kt}$$

$$kt = ln(\frac{T(t) - T_a}{T_0 - T_a})$$

$$k = \frac{ln(\frac{T(t)-T_a}{T_0-T_a})}{t}$$

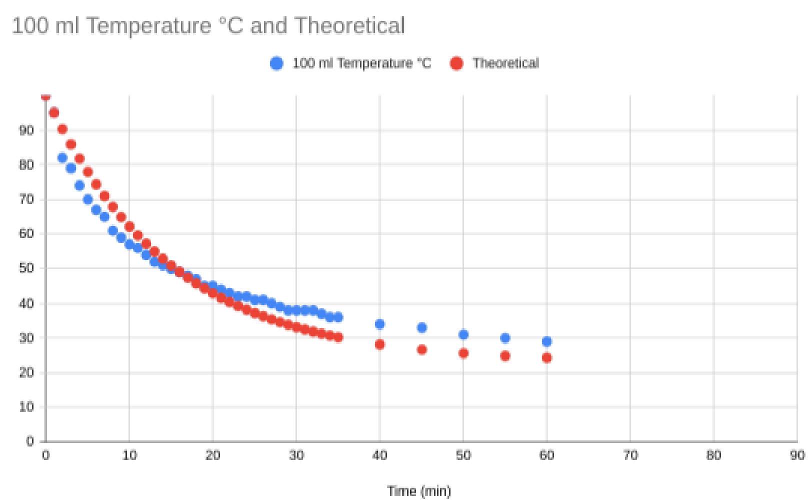
Now, let's find the average k for each beaker and use the result to compute the theoretical values of the experiment, then compare it with the data that we have:

Time(min)	k-value 100ml	k-value 300ml	k-value 800ml
0	100	100	100
1	$k = \frac{ln(\frac{95-100}{100-23})}{1}$	$k = \frac{ln(\frac{95-100}{100-23})}{1}$	$k = \frac{ln(\frac{96-100}{100-23})}{1}$
2	...	...	...
60	$k = \frac{ln(\frac{29-100}{100-23})}{60}$	$k = \frac{ln(\frac{36-100}{100-23})}{60}$	$k = \frac{ln(\frac{40-100}{100-23})}{60}$

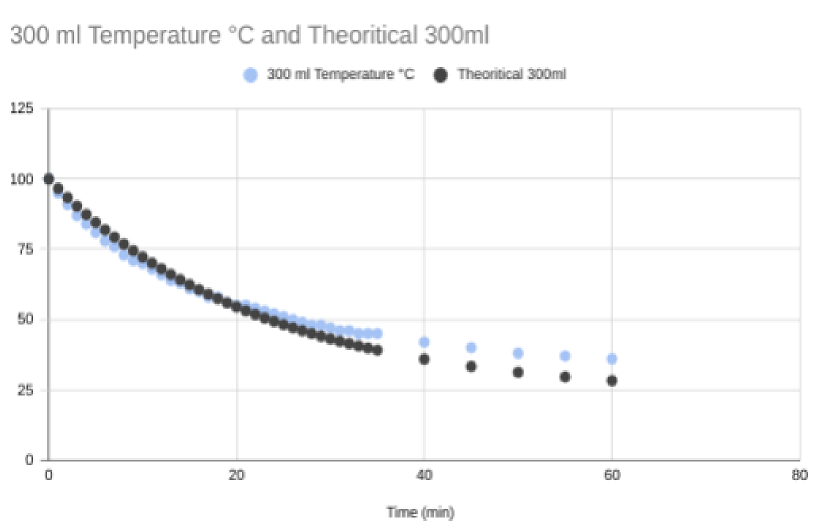
The average k-value for each beaker will be the sum of the k-value in respect of time divided by the maximum time.

The complete dataset and calculation(finding k and calculating the root mean square error) can be found here.
The link to the GitHub Repository can be found here as well.

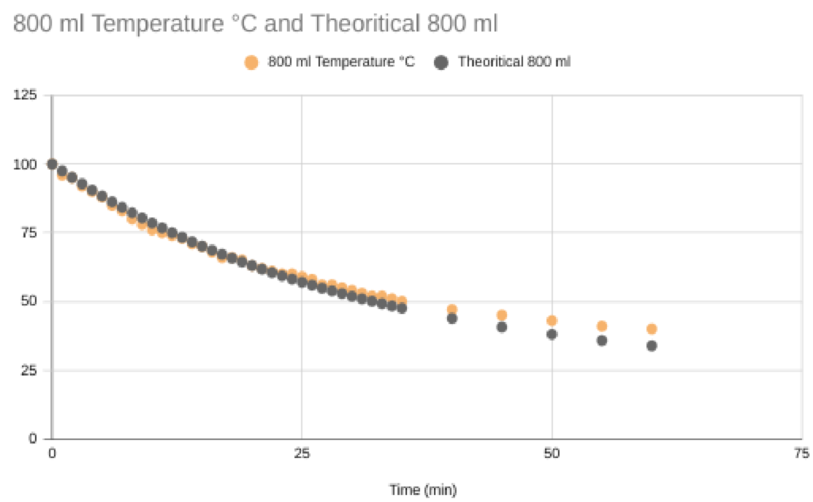
For the 100 ml sample of water, the calculated **k value is -0.0676**. After taking the difference of the observed values and the predicted values, we get the root mean square error which is **4.80**.



For the 300 ml sample, the calculated **k value is -0.0447**.The root mean square error is **3.71** in this case.



For the 800 ml sample, the calculated value of **k is -0.0327**. We note that the value of k for the 800 ml sample is about half of that for the 100 ml sample. The root mean square error for it is **2.25**.



1.1.2 Conclusion

In all of these cases, the experimental temperature fell more quickly at the beginning of the experiment than that predicted by the theoretical model and more slowly than predicted toward the end. The larger water sample followed the Newton's Law of Cooling model more closely than the smaller samples did. Newton's Law of Cooling accounts primarily for conductive heat exchange and assumes that the only heat lost by the system to the surroundings is that due to the temperature difference. At temperatures near boiling, the rate of evaporation is high. The heat lost through the phase change is greater than the heat lost through convective heat exchange with the environment. The Newton's Law of Cooling did not consider other factors that can be possibly part of how fast the rate of cooling is going down. Despite these complications, we conclude that Newton's Law of Cooling provides a reasonable approximation of the change in temperature for an object cooling in a constant ambient temperature of the beakers.

1.2 Application examples

There are so many ways by which differential equation is being used in real life. These includes, predicting how long it takes for a hot cup of tea to cool down to certain temperature, It is used to control the motion of aeroplane thus, Aircraft Dynamics both first and second order systems, to find the temperature of a soda placed in refrigerator by a certain amount of time,and last but not the least, in cases of crime scenes, Newton's law of cooling can indicate the time of death given probable body temperature at time of death and current body temperature. For this assignment/project, we will be diving deep in to Estimating time of death.

2 Estimating Time of Death

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. When estimating the time of death, one needs to know the temperature of the surroundings and the temperature of the body at two different times in order to make an accurate estimate.

So from the introduction of Newton's law of cooling, we came to the conclusion that, after computing:Newton's Law of cooling integration

$$\frac{dT}{dt} = -k(T - T_a) \tag{8}$$

We came to the conclusion of getting:

$$T(t) = (T_a + Ae^{-kt}) \tag{9}$$

When a person dies, the temperature of their body will gradually decrease from the normal body temperature , to the temperature of the surroundings. The normal body temperature of every human is thirty-seven degrees Celsius.

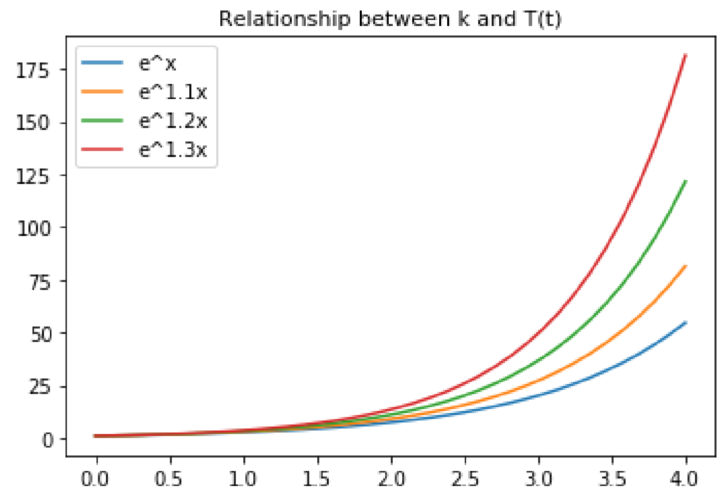
2.1 What K is and What it Represents

As seen in the equation above, K is raised to e as an exponent. Hence a small change in

The constant of proportionality includes, a graphed proportional relationship where x represents the independent variable and y represents the dependent variable, independent variables describe the input values in a relationship, normally represented by the x coordinate in the ordered pairs (x, y), and dependent variables describe the output values in a relationship, normally represented by the y coordinate in the ordered pairs (x, y). The constant of proportionality can never be zero.

2.1.1 How Different Values of k Affect the Solution Graphically

As seen in the above equation the value of k is raised to e. Thus a small change in K can affect the equation significantly. Below is a figure demonstrating how small changes in k affect T(t) assuming other variables remain constant.



2.2 Application of Newton's law of cooling in forensic investigations:

One of the interesting applications of Newton's law of cooling is in finding the time of death of an object / living being. This is especially used in crime scene investigation.

Let's look at the following crime scene:

A person was found murdered in his home, The police arrived on the scene at 10:56 pm. The temperature of the body at that time was 31C and one hour later it was 30C. The temperature of the room in which the body was found was 22C. When was the murder committed?

It is a fact that when a person dies the temperature of his/her body will gradually decrease from 37C (the normal body temperature) to the surrounding temperature if the surrounding temperature is lower than the persons temperature.

Using this information we can distinguish the variables so as to be able to use them in Newton's law of cooling equation as follows:

1. Surrounding temperature $T_a = 22C$
2. $t = 0$ at $T(0) = 31C$
3. $t = 1$ hour and $T(1) = 30C$

We want to find out at what value of t T(t) was 37C?

$$T(t) = (T_a + Ae^{-kt})$$
$$31 = 22 + Ae^{-k(0)} => A = 9$$

Using the above answer we get:

$$T = 22 + 9e^{-kt}$$

and we substitute t=1

$$30 = 22 + 9e^{-k(1)} \qquad 8 = 9e^{-k} \qquad -ln(8/9) = k$$
$$k = 0.11778$$
$$T(t) = 22 + 9e^{At} \qquad 37 = 22 + 9e^{0.11778(t)} \qquad 15 = 9e^{0.11778(t)}$$
$$t = (ln(15/9))/0.11778$$
$$t = -4.337 hours$$

The negative sign tells us that the murder was committed 4.337 hours (4 hours and 20 minutes) before 10:56 pm (our starting time for measurement). Thus the murder happened at 6:36 pm.

2.3 Limitations of Newton's law of cooling in forensic investigations

- The last equation, $t = (ln(15/9))/0.11778$, suggests that there is a time constant in determining the time of death of a person. This constant is given by $1/k = 1/0.11778$ 8.5 This constant however is biased as it can be influenced by many factors, including the size (volume V), the shape (surface area A), body composition (density , and specific heat c), and the heat transfer coefficient h. A more roundly shaped person (large V/ A) has a larger time constant, implying that it is more difficult to lose heat than a skinnier person (small V/A). The amount of fat a person carries varies due to the amount of insulation the body has. If a body is more insulated due to a thicker fat layer, the body temperature will decay at a slower rate. Further improvement in extracting time of death takes into consideration the body size, thermal resistance due to clothes, thermal variations of the environment, and the possibility that an exponential function alone is not accurate to describe the temperature decay.

- Another case where this approach has limitation when the ambient temperature fluctuates. This is due to the fact that Newton's law of cooling is that this law holds true only if the temperature of the surroundings remains constant throughout the cooling if the body. In the case that the environment changes its temperature Newton's law of cooling is not applicable. For instance if the body is found in a forest, it is highly likely the surrounding temperature increases or decreases depending the time of the day.

- Lastly, this approach has limitations as we only took temperature readings only two times. Had the temperature readings occurred as many times as in the above example of water in the beaker, the k value even though not it won't be perfect it would have been more accurate.
- Using the above limitations a murder can commit a murder where the surrounding temperature fluctuates and can increase thermal resistance of the dead body by putting on it clothes or other material, in which both cases lead to inaccurate estimation of time of death.

3 Group Member Roles

- Natnael: Researching on the murder investigation and contributing to the LaTeX document.
- Mahalinoro: Researching on the Newton's Law of Cooling and contributing to the LaTeX document.
- Chris: Researching on the application of Newton's Law of Cooling and contributing to the LaTeX document.
- Zubery: Working on the technology side with python.

4 Bibliography

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