第 1 問

以下の問いに答えよ．

(1) 実変数 x, y の関数 $f(x, y)$ を以下のように定義する．

$$f(x,y) = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}$$

方程式 $f(x, y) = 0$ の解の集合は，xy 平面上の 2 点 (x_1, y_1), (x_2, y_2) を通る直線となることを示せ．ただし，$x_1 \neq x_2$ とする．

(2) 行列式 $\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$ の値を因数分解した形で求めよ．

(3) xy 平面上の 3 点 (x_1, y_1), (x_2, y_2), (x_3, y_3) を通る曲線 $y = a_0 + a_1 x + a_2 x^2$ が唯一存在することを示せ．ただし，a_0, a_1, a_2 は定数，x_1, x_2, x_3 は互いに異なるとする．

(4) (3) の曲線は $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ の形で表せる．ただし，c_1, c_2, c_3 は y_1, y_2, y_3 に依存しないものとする．c_1, c_2, c_3 を求めよ．

(5) xy 平面上の 5 点 $(x_1, y_1), \ldots, (x_5, y_5)$ を通る曲線 $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ を $y = c_1 y_1 + \cdots + c_5 y_5$ の形で表す．ただし，$c_1, \ldots, c_5$ は $y_1, \ldots, y_5$ に依存せず，$x_1, \ldots, x_5$ は互いに異なるとする．c_1 を求めよ．

Problem 1

Answer the following questions.

(1) The function $f(x, y)$ with real variables x, y is defined as follows:

$$f(x,y) = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}.$$

Show that the set of solutions of the equation $f(x, y) = 0$ is a line passing through two points (x_1, y_1), (x_2, y_2) on the xy plane, where $x_1 \neq x_2$.

(2) Find the value of the determinant $\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$ in factored form.

(3) Show that there is a unique curve $y = a_0 + a_1 x + a_2 x^2$ passing through three points (x_1, y_1), (x_2, y_2), (x_3, y_3) on the xy plane, where a_0, a_1, a_2 are constants and x_1, x_2, x_3 are all distinct.

(4) The curve in (3) can be represented in the form $y = c_1 y_1 + c_2 y_2 + c_3 y_3$, where each of c_1, c_2, c_3 does not depend on y_1, y_2, y_3. Find c_1, c_2, c_3.

(5) Let us represent a curve $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ passing through five points $(x_1, y_1), \ldots, (x_5, y_5)$ on the xy plane in the form $y = c_1 y_1 + \cdots + c_5 y_5$, where each of $c_1, \ldots, c_5$ does not depend on $y_1, \ldots, y_5$, and $x_1, \ldots, x_5$ are all distinct. Find c_1.

第 2 問

t を実数の独立変数，$x(t)$ と $y(t)$ を実数値関数として，以下の問いに答えよ．

(1) 常微分方程式

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}+x=\cos(t)$$

の $t\to-\infty$ で有界である解をすべて求めよ．

(2) 常微分方程式

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}+x-y &= \cos(t)\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\frac{\mathrm{d}y}{\mathrm{d}t}+y-x &= 0\end{aligned}$$

の $t\to-\infty$ で有界である解 $x(t)$ と $y(t)$ をすべて求めよ．

(3) 適切な変数変換によって常微分方程式

$$e^{-t}x^2-2\frac{\mathrm{d}x}{\mathrm{d}t}+x=0$$

を線形な常微分方程式に変換し，$x(0)=\frac{1}{2}$ となる解 $x(t)$ を求めよ．

Problem 2

Let t be a real independent variable, and let $x(t)$ and $y(t)$ be real-valued functions. Answer the following questions.

(1) Find all solutions of the following ordinary differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + x = \cos(t),$$

which are bounded when $t \to -\infty$.

(2) Find all solutions $x(t)$ and $y(t)$ of the following ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + x - y &= \cos(t), \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + y - x &= 0, \end{aligned}$$

which are bounded when $t \to -\infty$.

(3) By converting the following ordinary differential equation

$$e^{-t}x^2 - 2\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

to a linear ordinary differential equation with an appropriate change of variable, find the solution $x(t)$ that satisfies $x(0) = \frac{1}{2}$.

第3問

丸石 ○ と四角い石 □ をランダムに左から右に一直線上に一つずつ並べる．$0 < q < 1$ として，丸石を確率 $1-q$，四角い石を確率 q で独立同一分布に従って並べていく．M を正の整数として，四角い石が M 個連続して並べられた直後に並べることを停止する．$M = 4$ の場合の列の例を以下に示す．

列 1　○□□□□

列 2　□○□○○□□□□

停止後の石の数を表す確率変数を L とする．上に示した列の場合，列 1 と列 2 はそれぞれ $L = 5$，$L = 9$ となる．

並べている途中の状態を考える．k を非負整数とし，右端から四角い石が k 個連続している状態を C_k とする．例えば，$M = 4$ の時に以下の列を考える．

列 3　○□□□○○□□

列 4　□○□○○

$M = 4$ の場合を考えているため，列 3 と列 4 はまだ停止していない．列 3 は右端から四角い石が 2 個連続しているので状態 C_2 である．列 4 は右端に四角い石がないので状態 C_0 である．状態 C_k から n 個石を並べたときに初めて停止条件を満たす確率を a_{kn} とする．ここで n は非負整数である．a_{kn} に対して以下のような母関数 $A_k(t)$ を定義する．

$$A_k(t) = \sum_{n=0}^{\infty} t^n a_{kn}$$

この時，以下の問いに答えよ．

(1) $M = 1$ の時，L の平均と分散を求めよ．

(2) $A_k(t)$ が満たす漸化式を求めよ．

(3) $A_k(t)$ を q, M, t, k を用いて表せ．

(4) L の平均を求めよ．

Problem 3

Let us randomly place circle stones ○ and square stones □ one by one in a line from left to right. The circle and square stones are placed with probability $1-q$ and q, respectively, according to the independent and identical distribution, where $0 < q < 1$. The placement stops right after M square stones are placed in a row, where M is a positive integer. We show examples of the lines for $M = 4$ as follows.

line 1 ○□□□□

line 2 □○□○○□□□□

Let L be a random variable which represents the number of the stones after stopping the placement. For the case of the lines shown above, $L = 5$ and $L = 9$ for lines 1 and 2, respectively.

Here, we consider intermediate states during the placement. Let k be a non-negative integer and let C_k be a state of a line where there are k square stones in a row from the right end. For instance, we consider the following lines for $M = 4$.

line 3 ○□□□○○□□

line 4 □○□○○

Since we are considering the case of $M = 4$, lines 3 and 4 are not stopped yet. Line 3 is in state C_2 since there are 2 square stones in a row from the right end. Line 4 is in state C_0 since there is no square stone at the right end. Let a_{kn} be the probability that the stopping condition is met after placing n stones starting from state C_k, where n is a non-negative integer. We define the following generating function $A_k(t)$ for a_{kn}.

$$A_k(t) = \sum_{n=0}^{\infty} t^n a_{kn}$$

Answer the following questions.

(1) Calculate the mean and variance of L for $M = 1$.

(2) Obtain the recurrence relation that $A_k(t)$ satisfies.

(3) Obtain $A_k(t)$ as a function of q, M, t, and k.

(4) Calculate the mean of L.