$a+4b = 5a$
$2a+3b = 5b$

Exercise - Chapter III : Eigenvalues and Eigenvectors

1. Find bases for the eigenspace of the following matrix

   (a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

   $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   $(1-\lambda)(3-\lambda)-8=0$

   $\lambda^2-4\lambda-5=0 \to \lambda=5,-1$

   $\lambda=5$, $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$

   $\lambda=-1$, $\vec{v}=\begin{bmatrix}2\\-1\end{bmatrix}$

   eigenspace $=$ span$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\-1\end{bmatrix}\right\}$

   (b) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

   $\begin{pmatrix} -\frac{x}{2} \\ x \\ x \end{pmatrix}$

   null$\left(\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}\right) \longrightarrow \begin{pmatrix} -\frac{x}{2} \\ x \\ x \end{pmatrix}$

   (c) $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

   2 11 22 13 14

2. Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda-1)(\lambda-3)^2(\lambda-4)^3.$$

   (a) What is the size of A? 6×6

   $\det(\lambda I - A) = 0$

   (b) Is A invertible? invertible

   $\lambda = 1,3,3,4,4,4$

   (c) How many eigenspaces does A have? 3

   (d) Find an algebraic multiplicity of each eigenvalue of A. $\lambda=1,3,4$

3. Find a matrix P that diagonalizes

   $P_\lambda = (2-\lambda)(3-\lambda)(3-\lambda)$

   eigen value $= 2,3$

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   $(A-\lambda I) = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$ $\begin{bmatrix}1\\0\\0\end{bmatrix}$

   $(A-\lambda I) = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix}1\\0\\-\frac{1}{2}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$

   and compute $P^{-1}AP$.

   $S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$ $S^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$

   $S^{-1}AS = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & -\frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. Let

   $P^{-1}AP$

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

   $\det(P) = 1 \to$ invertible

   $P^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 5 & 4 & -1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{pmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

characteristic equation

$$(2-\lambda)(3-\lambda)(3-\lambda)=0$$

$$\lambda = 2, 3, 3$$

$$(A-2I) = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \rightarrow \quad x_2 = 0, \; x_3 = 0$$

$$\text{eigenvector} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A-3I = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \quad x_1 + 2x_3 = 0$$

$$x_3 = -\frac{x_1}{2}$$

$\vec{x}$ ที่ทำให้ $(A-3I)\vec{x} = \vec{0}$ เขียนได้ในรูป

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ -\frac{x_1}{2} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{eigenvector} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & -\frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$P^{-1}AP$ = diagonal

(a) Show that P diagonalizes A.

$P^{-1}AP$ = diagonal

$P^{-1}AP$

$$\boxed{\begin{matrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}}$$

(b) Find A^{12}

$$A^{12} = \begin{pmatrix} 1 & -20483 & 4095 \\ 0 & 1 & 0 \\ 0 & -3047 & 4096 \end{pmatrix}$$

$$\underbrace{P^{-1}APP^{-1}AP\ldots P^{-1}AP}_{12} = P^{-1}A^{12}P = (\text{diagonal})^{12} = [a_{ii}^{12}] = \begin{bmatrix} 2^{12} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{12} = P\begin{bmatrix} 2^{12} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}$$

$$= \begin{matrix} 0 & -20480 & 4096 \\ 1 & -4 & -1 \\ 0 & 1 & 0 \end{matrix}$$

5. Let A be 3×3 matrix such that $A^3 = A$. Find all possible eigenvalues of A?

$A\vec{x} = \lambda\vec{x}$ $\quad \therefore \lambda^3 = 1,\ \lambda = 1, \operatorname{cis}(\frac{2\pi}{3}), \operatorname{cis}(\frac{4\pi}{3})$

6. (a) Prove that the characteristic equation of a 2×2 matrix A can be expressed as

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0,$$

where $\text{tr}(A)$ is the trace of A.

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0$

$(a-\lambda)(d-\lambda) - bc = 0$

$ad + \lambda^2 - (a+d)\lambda - bc = 0$

$\lambda^2 - tr(A)\lambda + det(A) = 0$

(b) Show that for every 2×2 matrix A

$$A^2 - \text{tr}(A)A + \det(A)I_2 = \underline{0}_2.$$

$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$= \begin{bmatrix} (a+b)a & (a+b)b \\ (a+b)c & (a+b)d \end{bmatrix} - \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$\therefore A^2 - tr(A)A + det(A)I_2 = 0$

7. Use the result in (6a) to show that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the solutions of the characteristic equation of A are

$$\lambda = \frac{1}{2}[(a+d) \pm \sqrt{(a-d)^2 + 4bc}].$$

$(a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$

$\lambda = \frac{a+d}{2} \pm \frac{\sqrt{(a+d)^2 - 4ad + 4bc}}{2}$

Use this result to show that A has

(a) two distinct real eigenvalues if $(a-d)^2 + 4bc > 0$. ✓

(b) two repeated real eigenvalues if $(a-d)^2 + 4bc = 0$. ✓

(c) complex conjugate eigenvalues if $(a-d)^2 + 4bc < 0$. ✓

8. Let A be $n \times n$ matrix and $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A. Show that

$$\det(A) = \lambda_1 \cdots \lambda_n.$$

9. For every $n \times n$ matrix A. Let
invertible A

$$p_A(\lambda) = \det(\lambda I_n - A).$$

$= det(\lambda I_n - A^{-1})$

$= det(\lambda AA^{-1} - I_nA^{-1}) = det(\lambda A - I_n)\,det(A^{-1})$

$= \frac{(-\lambda)^n}{\det(A)} p_A(\frac{1}{\lambda})$

(a) Show that

$$p_{A^{-1}}(\lambda) = \frac{(-\lambda)^n}{\det A} p_A\left(\frac{1}{\lambda}\right).$$

(b) Use (a) to find $p_{A^{-1}}(\lambda)$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -7 & 2 \\ 5 & 4 & 3 \end{bmatrix}.$$

10. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$(1-\lambda)(5-\lambda)(9-\lambda) + 84 + 96 - 72 - 48 - 105 + 21\lambda = 0$

$-59\lambda + 15\lambda^2 - \lambda^3 + 21\lambda = 0$ $\quad \lambda(\lambda^2 - 15\lambda + 38)$

(a) Find characteristic equation of A and find all eigenvalues of A.

$\lambda = 0$

(b) Find characteristic equation of $A - 2I_3$ and find all eigenvalues of $A - 2I_3$.

11. Find all matrix A such that

$$A^3 = \begin{bmatrix} -51 & 118 & -52 \\ -27 & 62 & -27 \\ -4 & 8 & -3 \end{bmatrix}.$$

$P^{-1}A^3P = D$

$[(a_{ii})^{1/3}]$

$A = PDP^{-1}$

eigen value $= -8, -1, 1$

$D = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 2 & 5 & -1 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$d = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} -3 & 7 & -3 \\ 1 & -2 & 1 \\ -2 & 4 & -1 \end{bmatrix}$

12. Prove that the intersection of any two distinct eigenvalues of a matrix A is $\{\vec{0}\}$.

13. Let A, B and C be $n \times n$ matrices such that A is similar to B and B is similar to C. Prove that A is similar to C.

14. Find the determinant of the $n \times n$ matrix

$$A = \begin{bmatrix} -5 & -2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & -5 & -2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & -5 & -2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & -5 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & -5 & -2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 & -5 \end{bmatrix}.$$

diagonalize

$125 - 30 - 30$

35

$\det(A_n) = (-5)\det(A_{n-1}) + (-2)\det \begin{vmatrix} 3 & -2 & 0 & 0 & \cdots \\ 0 & -5 & -2 & 0 & \cdots \\ \vdots & 3 & -5 & \cdots & \\ 0 & 0 & 0 & \cdots & \\ & 0 & 0 & \cdots & \end{vmatrix}$

$= -5\det(A_{n-1}) + (-2)(3)\det(A_{n-1})$

$$= -11 \det(A_{n-1})$$
$$= (-11)^{n-2} \det(A_{n-2}) = (-11)^{n-2}(31)$$

15. The fibonacci numbers $\{F_n\}$ are defined for non-negative integers n such that

$$F_0 = F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \text{ for } n \geq 0.$$

(a) Find the matrix A such that

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = A \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

for $n \geq 0$.

(b) Show that the matrix A in (15a) is diagonalizable and find matrix P that diagonalize A.

$$(1-\lambda)(-\lambda) - 1 = 0 \qquad \lambda := \psi_1, \psi_2$$
$$\lambda^2 - \lambda - 1 = 0$$

(c) Let λ_1 and λ_2 be eigenvalues of A such that $\lambda_1 > \lambda_2$. Show that F_n can be expressed as

$$F_n = c_1\lambda_1^n + c_2\lambda_2^n$$

for some $c_1, c_2 \in \mathbb{R}$ and find c_1, c_2.

(d) Show that $\lim_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \lambda_1$.

16. Polar bears and seals live in the North Pole. The number of polar bears P and seals S each year is determined by the populations the previous year, according to the formulas

$$P(n) = 1.5P(n-1) + S(n-1), \quad \text{and} \quad S(n) = P(n-1).$$

We know that in year 0, there are 20 polar bears and 10 seals.

(a) What will the populations be in year k?

(b) After a long time, approximately what will the ratio of polar bears to seals be?

17. Consider the sequence of fraction

$$2, \quad 2 + \frac{1}{2}, \quad 2 + \frac{1}{2 + \frac{1}{2}}, \quad 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \quad \ldots,$$

$$\frac{a_n}{b_n} = 2 \qquad \frac{a_{n+1}}{b_{n+1}} = 2 + \frac{b_n}{a_n} = \frac{2a_n + b_n}{a_n}$$

(a.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

(b.) $(1-\lambda)(-\lambda) - 1 = 0$

$\lambda^2 - \lambda - 1 = 0 \rightarrow$ 2 distinct eigenvalues

$\therefore$ diagonalizable

$$A - \lambda_1 I = \begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $(A - \lambda_1 I)\vec{x} = \vec{0}$

$\rightarrow (1-\lambda_1) x_1 - x_2 = 0 \rightarrow (-\lambda_1)(1-\lambda_1) x_1 + \lambda_1 x_2 = 0$

and $x_1 - \lambda_1 x_2 = 0$

$\therefore x_1 = \lambda_1 x_2$

eigenvector $= \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$

Likewise,

another eigenvector $= \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$; $\lambda_1 = \frac{+1 + \sqrt{5}}{2}$, $\lambda_2 = \frac{+1 - \sqrt{5}}{2}$

$\therefore P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$

and diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\therefore D^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \rightarrow P D^k P^{-1} = A^k$

$$A^k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} P^{-1}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{k+1} & \lambda_2^{k+1} \\ \lambda_1^k & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} & \lambda_1 \lambda_2^{k+1} - \lambda_2 \lambda_1^{k+1} \\ \lambda_1^k - \lambda_2^k & \lambda_1 \lambda_2^k - \lambda_2 \lambda_1^k \end{bmatrix}$$

from $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = A \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$

$$= A^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \ F_1 = 1, \ F_0 = 0$$

$$F_{n+1} = \frac{1}{\sqrt{5}} \left(\lambda_1^{n+1} - \lambda_2^{n+1} \right)$$

$$= \frac{\lambda_1^{n+1}}{\sqrt{5}} + \frac{\lambda_2^{n+1}}{-\sqrt{5}}$$

$$c_1 = \sqrt{5}^{-1}, \ c_2 = (-\sqrt{5})^{-1}$$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \lambda_1 = \frac{1+\sqrt{5}}{2}$$

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

This is an example of what is called *partial fraction*. If we denote the n^{th} in this sequence by $\frac{a_n}{b_n}$.

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(a) Find the explicit formulars of a_n and b_n.

(b) Find $\lim_{n\to\infty} \frac{a_n}{b_n}$. $1+\sqrt{2}$

$\lambda^2 - 2\lambda - 1 = 0$

λ_1, λ_2

18. For every $n \times n$ matrix A. Define

diag

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

(a) If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, show that $e^A = \begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix}$.

(b) If $B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, find e^B.

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^5 = A$

$$[a_{11}]_{e^A} = 1 + 0 + \frac{(-1)}{2!} + 0 + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \dots = \cos 1$$

$$[a_{12}]_{e^A} = 0 + 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots = \sin 1$$

$$[a_{21}]_{e^A} = -1 + \frac{1}{3!} - \frac{1}{5!} + \dots = -\sin 1$$

$$[a_{22}]_{e^A} = [a_{11}]_{e^A}$$

$$= \cos 1$$

(b.) $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$$(-\lambda)(2-\lambda)(3-\lambda) + (2-\lambda)(+2)(1) = 0$$

$$6 - 5\lambda + \lambda^2$$

$$-6\lambda + 5\lambda^2 - \lambda^3 + 4 - 2\lambda = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda-2)(\lambda^2 - 3\lambda + 2)$$

$$(\lambda-2)^2(\lambda-1)$$

2. Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

(a) What is the size of A? 6×6 $\qquad$ $\det(\lambda I - A) = 0$

(b) Is A invertible? invertible $\qquad$ $\lambda = 1, 3, 3, 4, 4, 4$

(c) How many eigenspaces does A have? 3

(d) Find an algebraic multiplicity of each eigenvalue of A. $\qquad$ $\lambda = 1, 3, 4$

a. 6×6

b. $\lambda = 0$ is not a solution of $P(\lambda)$

invertible

c. $\lambda = 1, 3, 4$

eigenspace = 3

d. algebraic multiplicity

$\lambda = 1 \rightarrow$ 1

$= 3 \rightarrow$ 2

$= 4 \rightarrow$ 3

11.

characteristic equation of A^3

$$P(\lambda) = (-51-\lambda)(62-\lambda)(-3-\lambda) + 27\times4\times118 + 27\times52\times8 - (-3-\lambda)(-27)118 - 4\times52\times(62-\lambda) - (-51-\lambda)(-27)(8)$$

$P(\lambda) = 0$; $\lambda = +8, -1, 1$

$$A^3 - \lambda_1 I = \begin{bmatrix} -59 & 118 & -52 \\ -27 & 54 & -27 \\ -4 & 8 & -11 \end{bmatrix} \xrightarrow[-7R_3+R_2]{-15R_3+R_1} \begin{bmatrix} 1 & -2 & 113 \\ 1 & -2 & 50 \\ -4 & 8 & -11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 113 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 = 0$, $x_3 = 0$

$$\text{eigen vector} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$A^3-\lambda_2 I = \begin{bmatrix}-50 & 118 & -52\\ -27 & 63 & -27\\ -4 & 8 & -2\end{bmatrix} \xrightarrow[-7R_3+R_2]{-13R_3+R_1} \begin{bmatrix}2 & 14 & -26\\ 1 & 7 & -13\\ -4 & 8 & -2\end{bmatrix} \to \begin{bmatrix}0 & 0 & 0\\ 1 & 7 & -13\\ 0 & 36 & -54\end{bmatrix}$$

$$\left.\begin{aligned} x_1+7x_2-13x_3&=0\\ 2x_2-3x_3&=0\end{aligned}\right\}\ \text{eigenvector} = \begin{bmatrix}5/2\\ 3/2\\ 1\end{bmatrix}$$

$$A^3-\lambda_3 I = \begin{bmatrix}-52 & 118 & -52\\ -27 & 61 & -27\\ -4 & 8 & -4\end{bmatrix} \xrightarrow[-7R_3+R_2]{-13R_3+R_1} \begin{bmatrix}0 & 14 & 0\\ 1 & 5 & 1\\ -4 & 8 & -4\end{bmatrix} \to \begin{bmatrix}0 & 1 & 0\\ 1 & 5 & 1\\ 0 & 28 & 0\end{bmatrix}$$

$$\downarrow$$

$$\text{eigenvector} - \begin{bmatrix}1\\ 0\\ -1\end{bmatrix} \leftarrow x_2=0,\ x_1+x_3=0 \leftarrow \begin{bmatrix}0 & 1 & 0\\ 1 & 5 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$P = \begin{bmatrix}2 & 5 & 1\\ 1 & 3 & 0\\ 0 & 2 & -1\end{bmatrix}$$

Find P^{-1},

$$\left[\begin{array}{ccc|ccc}2 & 5 & 1 & 1 & 0 & 0\\ 1 & 3 & 0 & 0 & 1 & 0\\ 0 & 2 & -1 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}0 & -1 & 1 & 1 & -2 & 0\\ 1 & 3 & 0 & 0 & 1 & 0\\ 0 & 2 & -1 & 0 & 0 & 1\end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc}1 & 3 & 0 & 0 & 1 & 0\\ 0 & 1 & -1 & -1 & 2 & 0\\ 0 & 1 & 0 & 1 & -2 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -3 & 7 & -3\\ 0 & 1 & 0 & 1 & -2 & 1\\ 0 & 0 & 1 & +2 & -4 & +1\end{array}\right]$$

$$A^3 = \begin{bmatrix}-51 & 118 & -52\\ -27 & 62 & -27\\ -4 & 8 & -3\end{bmatrix}$$

$$P^{-1} = \begin{bmatrix}-3 & 7 & -3\\ 1 & -2 & 1\\ +2 & -4 & +1\end{bmatrix}$$

$$D^3 = P^{-1}A^3P$$

$$D^3 = \begin{bmatrix}-24 & 56 & -24\\ -1 & 2 & -1\\ 2 & -4 & 1\end{bmatrix}\begin{bmatrix}2 & 5 & 1\\ 1 & 3 & 0\\ 0 & 2 & -1\end{bmatrix}$$

$$D^3 = \begin{bmatrix}8 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & +1\end{bmatrix}$$

$$D = \begin{bmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & +1\end{bmatrix}$$

$$\therefore \quad A = PDP^{-1}$$

$$= \begin{bmatrix} 2 & 5 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} -3 & 7 & -3 \\ 1 & -2 & 1 \\ -2 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 34 & -16 \\ -9 & 20 & -9 \\ -4 & 8 & -3 \end{bmatrix}$$

$$6a+9b+6c = N$$
$$7a+2b+7c = N$$
$$a-7b+c = 0$$
$$a+c = 7b$$
$$51b = N$$

$$\sqrt{2^{3x+y+2} - 2^{6x} - 2^{2y+2}} + 5 = x+y$$

$$\left(2^{3x} - 2^{y+1}\right)^2$$

$3x = y+1$ $\qquad 4x-1 = 5$ $\qquad \frac{9}{2}-1$

$x+y = 5$ $\qquad 4x = 6$

$x = 3/2 \qquad y = \frac{7}{2}$

2/11