

Catalan numbers

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The Catalan numbers (1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, ...), named after [Eugène Charles Catalan](#) (1814–1894), arise in a number of problems in combinatorics. They can be computed using this formula:

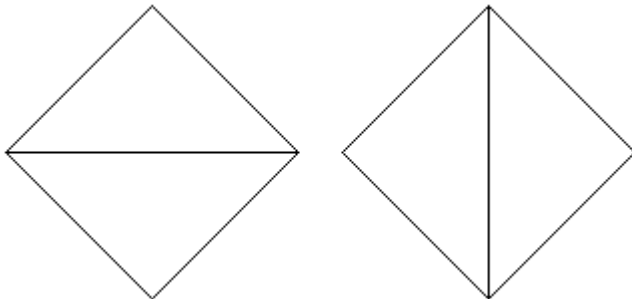
$$\frac{\binom{2n}{n}}{n+1}$$

Among other things, the Catalan numbers describe:

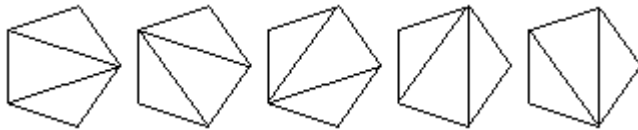
- the number of ways a [polygon](#) with $n+2$ sides can be cut into n triangles
- the number of ways to use n rectangles to tile a [stairstep](#) shape (1, 2, ..., $n-1$, n).
- the number of ways in which [parentheses](#) can be placed in a sequence of numbers to be multiplied, two at a time
- the number of planar binary [trees](#) with $n+1$ leaves
- the number of [paths](#) of length $2n$ through an n -by- n grid that do not rise above the main diagonal

Polygon diagrams:

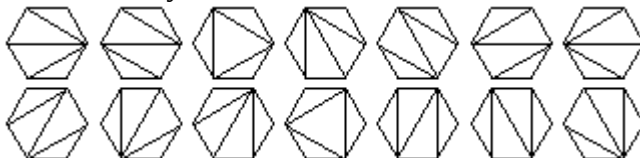
4 sides, 2 ways:



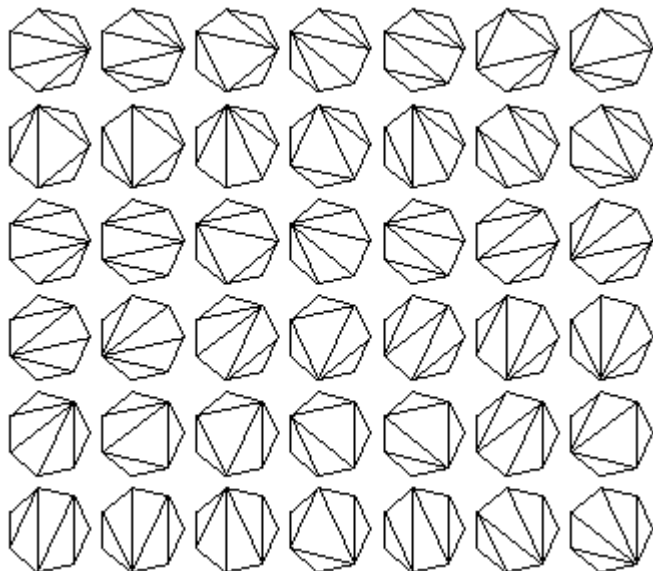
5 sides, 5 ways:



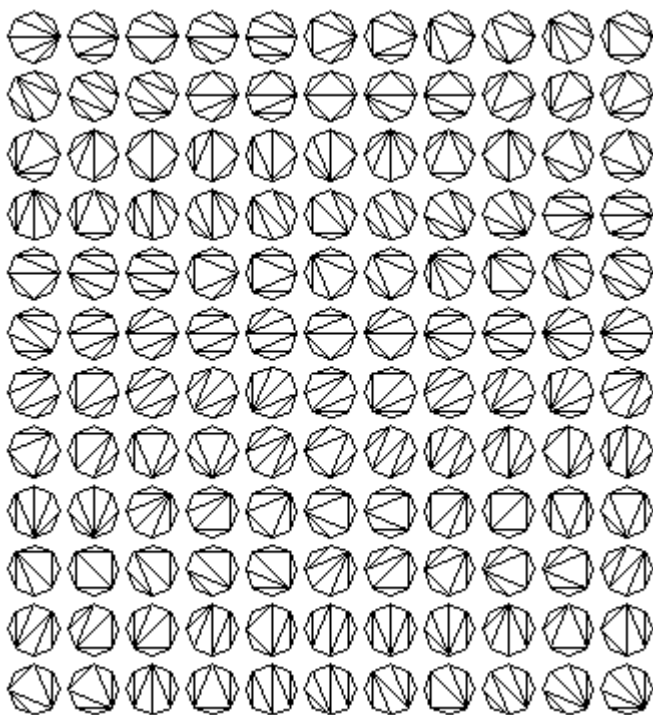
6 sides, 14 ways:



7 sides, 42 ways:



8 sides, 132 ways:



9 sides, 429 ways:

(Hidden in file [catalan9.png](#).)

Step diagrams:

2 rectangles, 2 ways:



The diagrams illustrate the growth of a 3x3 grid from a 1x1 grid. Each diagram shows the current grid in orange and the new cells being added in brown.

- Diagram 1: A single orange square (1x1 grid).
- Diagram 2: The orange square with three brown squares added to its right, top, and bottom (forming a 1x4 grid).
- Diagram 3: The 1x4 grid with two brown squares added to its right, top, and bottom (forming a 2x3 grid).
- Diagram 4: The 2x3 grid with three brown squares added to its right, top, and bottom (forming a 2x5 grid).
- Diagram 5: The 2x5 grid with three brown squares added to its right, top, and bottom (forming a 3x3 grid).

3 numbers:

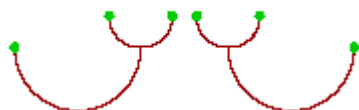
4 numbers:

5 numbers:

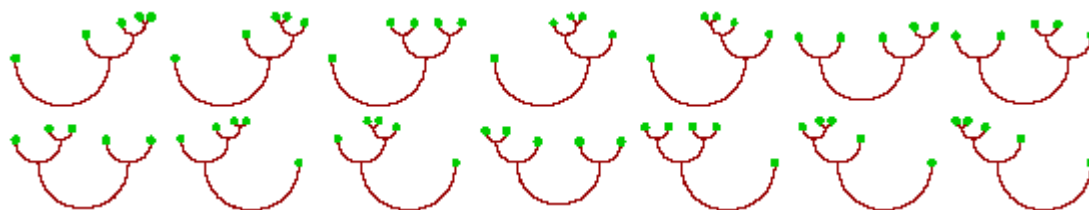
6 numbers:

(1 (2 (3 (4 (5 6))))))	(1 (2 (3 ((4 5) 6))))
(1 (2 ((3 4) (5 6))))	(1 (2 ((3 (4 5)) 6)))
(1 (2 (((3 4) 5) 6)))	(1 ((2 3) (4 (5 6))))
(1 ((2 3) ((4 5) 6)))	(1 ((2 (3 4)) (5 6)))
(1 ((2 (3 (4 5))) 6))	(1 ((2 ((3 4) 5)) 6))
(1 (((2 3) 4) (5 6)))	(1 (((2 3) (4 5)) 6))
(1 (((2 (3 4)) 5) 6))	(1 (((2 3) 4) 5) 6))
((1 2) (3 (4 (5 6))))	((1 2) (3 ((4 5) 6)))
((1 2) ((3 4) (5 6)))	((1 2) ((3 (4 5)) 6))
((1 2) (((3 4) 5) 6))	((1 (2 3)) (4 (5 6)))
((1 (2 3)) ((4 5) 6))	((1 (2 (3 4))) (5 6))
((1 (2 (3 (4 5)))) 6)	((1 (2 ((3 4) 5))) 6)
((1 ((2 3) 4)) (5 6))	((1 ((2 3) (4 5))) 6)
((1 ((2 (3 4)) 5)) 6)	((1 (((2 3) 4) 5)) 6)
((((1 2) 3) (4 (5 6))))	((((1 2) 3) ((4 5) 6)))
((((1 2) (3 4)) (5 6)))	((((1 2) (3 (4 5))) 6))
((((1 2) ((3 4) 5)) 6)	((((1 (2 3)) 4) (5 6)))
((((1 (2 3)) (4 5)) 6)	((((1 (2 (3 4))) 5) 6))
((((1 ((2 3) 4)) 5) 6)	((((1 (2 3) 4) 5) 6))
((((1 (2 3) (4 5)) 6)	((((1 (2) 3) (4 5)) 6))
(((((1 2) 3) (4 5)) 6)	(((((1 2) (3 4)) 5) 6))
(((((1 (2 3)) 4) 5) 6)	(((((1 2) 3) 4) 5) 6))

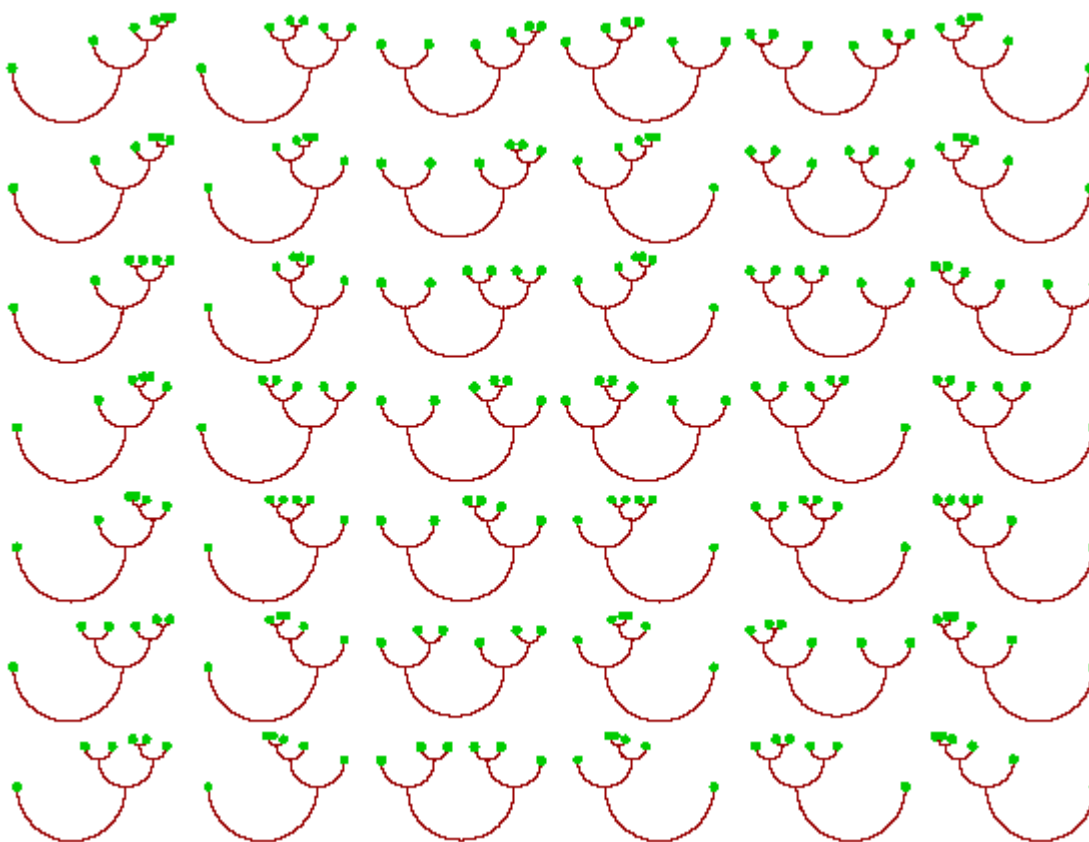
3 nodes:



5 nodes:

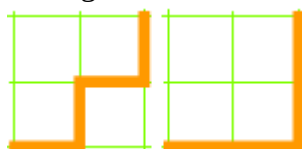


6 nodes:



Path diagrams:

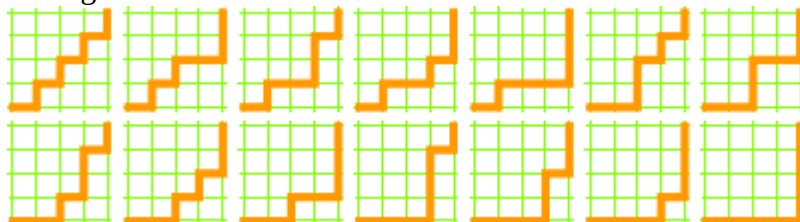
2×2 grid:



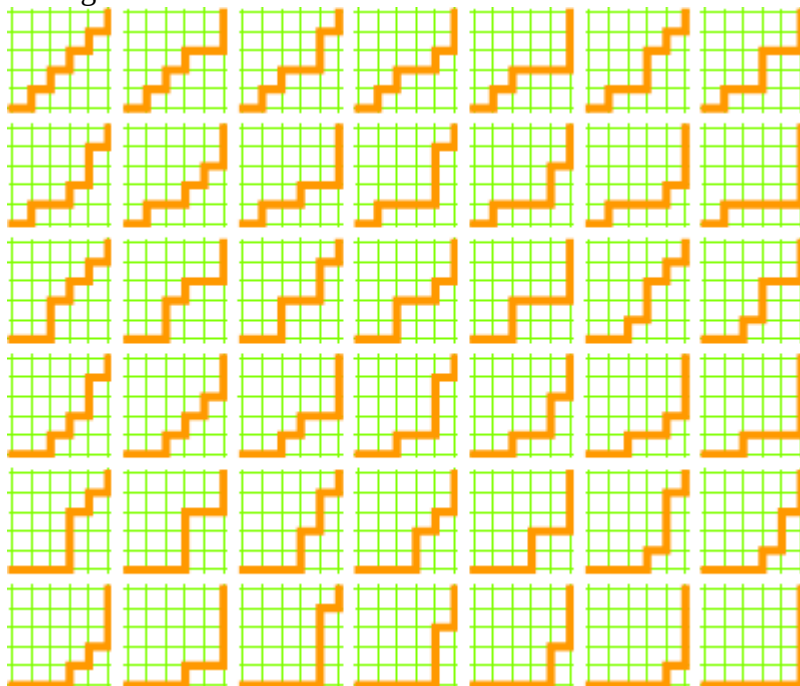
3×3 grid:



4 × 4 grid:



5 × 5 grid:



6 × 6 grid:

(Out of the way in file [catpath6.png](#).)

Originally designed and rendered using [Mathematica](#) 3.0 for the Apple Macintosh. PNG conversions performed with an old version of [ImageMagick](#).

Inspiration and facts (though not figures) by Brian Hayes, "[A Question of Numbers](#)", *American Scientist*, January–February 1996; Steven S. Skiena, *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*, Addison-Wesley, 1990; Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall, 1984; and D. E. Knuth, *Sorting and Searching* (vol. 3 of *The Art of Computer Programming*), Addison-Wesley, 1973. Catalan dates from Florian Cajori, *A History of Mathematics*, The Macmillan Company, 1922.

See also Martin Gardner, *Time Travel and Other Mathematical Bewilderments*, Chapter 20, W. H. Freeman, 1988; and Ilan Vardi, *Computational Recreations in Mathematica*, Chapter 9, Addison-Wesley, 1991.

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