A collection of mathematical formulas involving π

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Abstract

This note presents a collection of mathematical formulas involving the mathematical constant π.

1 Background

The mathematical constant known as $\pi = 3.141592653589793\ldots$ is undeniably the most famous and arguably the most important mathematical constant. Mathematicians since the days of Euclid and Archimedes, up to and including the present day, have analyzed its properties and computed its numerical value.

This is a collection of many of formulas that have been established by mathematicians over the years involving π. While a comprehensive collection is of course not possible, preference is given in this list for formulas that satisfy the following criteria:

- Formulas that give π (or a very simple expression involving π) explicitly, as opposed to implicit relations such as $e^{i\pi} + 1 = 0$.
- Formulas that give π (or a very simple expression involving π) as a straightforward infinite series, infinite product or definite integral.
- Formulas that involve only simple notation, such as summations, integrals, binomial coefficients, square roots, exponentials, logarithms, etc., that would be familiar to anyone who has completed a beginning course in calculus.
- Iterative algorithms for π involving simple expressions of the above types.
- Formulas that are relatively new, discovered within the last 50 years or so.

Included in this listing are several formulas for π that have actually have been used in large calculations of π, both before and since the invention of the computer. These include formulas 2 through 5 prior to the 20th century, and formulas 6, 7, 11, 12, 13, 14, 16, 18, 75 and 76 in the late 20th and early 21st century.

Formulas 13 through 18 have the intriguing property that they permit digits (in certain specific bases) of the constant specified on the left-hand side to be calculated beginning at an arbitrary starting position, without having to calculate any of the digits that came before, by means of a relatively simple algorithm. Formulas 13 and 14 have been used in computations of high-order binary digits of π [9, Sec 3.4–3.6], while formula 16 has been used in computations of high-order binary digits of π^2, and formula 18 has been used in computations of high-order base-3 digits of π^2 [6]. Numerous similar recently-discovered formulas that possess the arbitrary digit-computation property for various mathematical constants are catalogued in [2].

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Many of these formulas are relatively new, in the sense that they were discovered only in the past 50 years or so. The formulas mentioned in the previous paragraph are certainly in this category, having been discovered only since 1996. Many of the formulas from 19 through 52 were not well known until recently. Formulas 68 through 71 are also relatively new, in the sense that they are part of a class of integral formulas that are the subject of current research [3, 4, 5]. Formula 75 was discovered in 1976. Formulas 76 and 77 were discovered in 1984.

2 Credits

- Formula 1 was discovered by Leibniz and Gregory in the 1600s. Formula 2 was attributed to Euler in 1738. Formula 3 was discovered about the same time by Machin [9, pg. 105]. The related arctangent-based formulas 4, 5, 6 and 7 were used by Dase, Ferguson, Kanada and Kanada, respectively [9, pg. 106, 107, 111].
- Formula 8 is due to the Indian mathematician Madhava of Sangamagramma, who lived in the late 1300s and early 1400s [9, pg. 107]. Formula 9 was discovered by Newton in the mid-1600s [9, pg. 106]. Formula 10 was discovered by Wallis at about the same time.
- Formula 11 is due to Ramanujan, and was used by Gosper in 1986 to compute π to over 17 million digits. The similar but more complicated Formula 12 is due to David and Gregory Chudnovsky, and was used by them to compute π to over one billion decimal digits [9, pg. 108].
- Formula 13 is known as the "BBP" formula for π, named for the initials of the co-authors of the 1997 paper where it was first presented [7][9, pg. 119–124]. It was discovered by a computer program running the "PSLQ" algorithm of mathematician-sculptor Helaman Ferguson [14, 8]. Formula 14 is a variant of the BBP formula due to Bellard [9, pg. 124]. Formula 15 was found by Helaman Ferguson and independently by Adamchik and Wagon [16].
- Formula 16 appeared in [7]. Formulas 17 and 18 are due to David Broadhurst [12].
- Some of the summation formulas involving factorials and combinatorial coefficients (i.e., formulas 19 through 51) were found by Ramanujan; others are due to David and Gregory Chudnovsky. The Chudnovskys had these and many other formulas of this general type inscribed on the floor of their research center at Brooklynn Polytechnic University in New York City [13]. Four exceptions are Formula 37, which is due to Ramanujan but appeared in [11, pg. 188], Formulas 46 and 47, which are due to Guillera [15], and 52, which is due to Almkvist and Guillera [1].
- Formulas 53 through 69 have been known for many years; many are from [10, pg. 5, 48, 320–321].
- Formula 70 is an example of numerous formulas of this general type recently discovered by computational methods, typically involving the PSLQ algorithm [14, 8], in studies of Ising theory in mathematical physics [3]. Formulas 71, 72 and 73 are examples of recent discoveries, also by computational methods involving the PSLQ algorithm, in the theory of box integrals [4, 5]. Formula 72, for instance, can be thought of as specifying the average distance from the origin to a point in the unit 3-cube.
- Formula 74, the first of the iterative formulas, is mathematically equivalent to Archimedes' approach involving computing the areas of inscribed and circumscribed regular polygons [9, pg. 104]. Archimedes' polygon approach was used for almost all computations of π in ancient times, including by the fifth century Chinese mathematician Tsu Chung-Chih and, evidently, by the fifth century Indian mathematician Aryabhata.
- Formula 75 is the Brent-Salamin formula, the first quadratically convergent formula. It was discovered independently by Richard Brent and Eugene Salamin in 1976 [9, pg. 109–110]. Formula 76 (a cubically convergent iteration) and 77 (a quartically convergent iteration) are due to Jonathan and Peter Borwein [9, pg. 110].

3 Formulas for π

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \tag{1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{2n+1}} \tag{2}$$

$$\frac{\pi}{4} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)239^{2n+1}} \tag{3}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)8^{2n+1}} \tag{4}$$

$$\frac{\pi}{4} = 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)20^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)1985^{2n+1}} \tag{5}$$

$$\frac{\pi}{4} = 48\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)49^{2n+1}} + 128\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)57^{2n+1}} - 20\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)239^{2n+1}} + 48\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)110443^{2n+1}} \tag{6}$$

$$\frac{\pi}{4} = 176\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)57^{2n+1}} + 28\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)239^{2n+1}} - 48\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)682^{2n+1}} + 96\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)12943^{2n+1}} \tag{7}$$

$$\pi = \sqrt{12}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot 3^n} \tag{8}$$

$$\pi = \frac{3\sqrt{3}}{4} - 24\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+3)(2n-1)4^{2n+1}} \tag{9}$$

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} \tag{10}$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}} \tag{11}$$

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty} \frac{(-1)^n(6n)!(13591409+545140134n)}{(3n)!(n!)^3 640320^{3n+3/2}} \tag{12}$$

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right) \tag{13}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n+1)} - \frac{1}{64}\sum_{n=0}^{\infty} \frac{(-1)^n}{1024^n}\left(\frac{32}{4n+1} + \frac{8}{4n+2} + \frac{1}{4n+3}\right) \tag{14}$$

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right) \tag{15}$$

$$\pi^2 = \frac{9}{8}\sum_{n=0}^{\infty}\frac{1}{64^n}\left(\frac{16}{(6n+1)^2} - \frac{24}{(6n+2)^2} - \frac{8}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{1}{(6n+5)^2}\right) \tag{16}$$

$$\pi\sqrt{3} = \frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{729^n}\left(\frac{81}{12n+1} - \frac{54}{12n+2} - \frac{9}{12n+4} - \frac{12}{12n+6} - \frac{3}{12n+7} - \frac{2}{12n+8} - \frac{1}{12n+10}\right) \tag{17}$$

$$\begin{aligned}\pi^2 = \frac{2}{27}\sum_{n=0}^{\infty}\frac{1}{729^n}\Bigg(&\frac{243}{(12n+1)^2} - \frac{405}{(12n+2)^2} - \frac{81}{(12n+4)^2} - \frac{27}{(12n+5)^2} - \frac{72}{(12n+6)^2} \\ &-\frac{9}{(12n+7)^2} - \frac{9}{(12n+8)^2} - \frac{5}{(12n+10)^2} + \frac{1}{(12n+11)^2}\Bigg)\end{aligned} \tag{18}$$

$$3\pi + 8 = \sum_{n=0}^{\infty}\frac{12n2^{2n}}{\binom{4n}{2n}} \tag{19}$$

$$\frac{\pi^2}{6} - 2\log^2 2 = \sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{n^2 4^n} \tag{20}$$

$$15\pi + 52 = \sum_{n=0}^{\infty}\frac{(126n^2 - 24n + 8)2^{3n}}{\binom{6n}{3n}} \tag{21}$$

$$105\pi + 304 = \sum_{n=0}^{\infty}\frac{(1920n^3 - 928n^2 + 424n - 16)2^{4n}}{\binom{8n}{4n}} \tag{22}$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}^3(42n+5)}{16^{3n+1}} \tag{23}$$

$$16\pi\sqrt{3} + 81 = \sum_{n=0}^{\infty}\frac{(49n+1)8^n}{3^n\binom{3n}{n}} \tag{24}$$

$$162 - 6\pi\sqrt{3} - 18\log 3 = \sum_{n=0}^{\infty}\frac{(-245n+338)8^n}{3^n\binom{3n}{n}} \tag{25}$$

$$\pi = \sum_{n=0}^{\infty}\frac{(50n-6)}{2^n\binom{3n}{n}} \tag{26}$$

$$15\pi + 42 = \sum_{n=1}^{\infty}\frac{(-4)^n(2n)!^2(3n)!(201-952n)}{(6n)!n!} \tag{27}$$

$$15\pi\sqrt{2} + 27 = \sum_{n=0}^{\infty}\frac{8^n(2n)!^2(3n)!(350n-17)}{(6n)!n!} \tag{28}$$

$$40\pi\sqrt{3} + 243 = \sum_{n=1}^{\infty}\frac{(-27)^n(2n)!^2(3n)!(81-1080n)}{(6n)!n!} \tag{29}$$

$$20\pi\sqrt{3} + 89 = \sum_{n=1}^{\infty}\frac{(-1/3)^n(2n)!^2(3n)!(4123-22100n)}{(6n)!n!} \tag{30}$$

$$15\pi + 240\log 2 - 528 = \sum_{n=1}^{\infty} \frac{(-1/2)^n(89012n^3 - 77362n^2 + 482n + 3028)}{\binom{5n}{2n}} \tag{31}$$

$$24516 - 360\pi\sqrt{3} = \sum_{n=1}^{\infty} \frac{9^n(2743n^2 - 130971n - 12724)}{\binom{4n}{n}} \tag{32}$$

$$45\pi + 1164 = \sum_{n=1}^{\infty} \frac{8^n(430n^2 - 6240n - 520)}{\binom{4n}{n}} \tag{33}$$

$$40\pi\sqrt{3} + 1872 = \sum_{n=0}^{\infty} \frac{3^n(7175n^2 - 15215n + 480)}{\binom{4n}{n}} \tag{34}$$

$$288\pi\sqrt{3} - 576\log 2 + 324 = \sum_{n=0}^{\infty} \frac{(9/8)^n(5692 + 6335n - 5415n^2)}{\binom{4n}{n}} \tag{35}$$

$$1008\pi\sqrt{3} - 576\log 2 + 7587 = \sum_{n=0}^{\infty} \frac{(9/8)^n(7517 + 1145n + 18050n^2)}{\binom{4n}{n}} \tag{36}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{42n+5}{4096^n}\binom{2n}{n}^3 \tag{37}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(20n+3)}{4^{4n}(n!)^4 2^{2n+1}} \tag{38}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(260n+23)}{4^{4n}(n!)^4 18^{2n+1}} \tag{39}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(21460n+1123)}{4^{4n}(n!)^4 882^{2n+1}} \tag{40}$$

$$\frac{2}{\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(4n)!(8n+1)}{4^{4n}(n!)^4 3^{2n+1}} \tag{41}$$

$$\frac{1}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(4n)!(10n+1)}{4^{4n}(n!)^4 9^{2n+1}} \tag{42}$$

$$\frac{4}{\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(28n+3)}{4^{4n}(n!)^4 3^n 4^{2n+1}} \tag{43}$$

$$\frac{4}{\pi\sqrt{5}} = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(644n+41)}{4^{4n}(n!)^4 5^n 72^{2n+1}} \tag{44}$$

$$\frac{1}{3\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(4n)!(40n+3)}{4^{4n}(n!)^4 49^{2n+1}} \tag{45}$$

$$\frac{32}{\pi^2} = \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4(120n^2 + 34n + 3)}{2^{16n}} \tag{46}$$

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}^5(820n^2 + 180n + 13)}{2^{20n}} \tag{47}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n}^3 \frac{4n+1}{64^n} \tag{48}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{6n+1}{256^n} \tag{49}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{4096^n} \tag{50}$$

$$\pi + 4 = \sum_{n=0}^{\infty} \frac{2^{n+1}}{\binom{2n}{n}} \tag{51}$$

$$\frac{6}{\pi^2} = 64 \sum_{n=0}^{\infty} \frac{(6n)!(532n^2+126n+9)}{(n!)^6 10^{6n+3}} \tag{52}$$

$$\frac{\pi}{4} = \int_0^1 \frac{\mathrm{d}x}{1+x^2} \tag{53}$$

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \tag{54}$$

$$\frac{\pi-2}{4} = \int_0^1 x \tan^{-1} x \,\mathrm{d}x \tag{55}$$

$$\frac{\pi(\pi-12)}{48} + \frac{\log 2}{2} = \int_0^1 \log x \tan^{-1} x \,\mathrm{d}x \tag{56}$$

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4\,\mathrm{d}x}{1+x^2} \tag{57}$$

$$\frac{\pi}{8} = \int_0^1 \frac{x^2\,\mathrm{d}x}{(1+x^4)\sqrt{1-x^4}} \tag{58}$$

$$\frac{\pi(1+2\log 2)}{8} = \int_0^{\infty} x e^{-x} \sqrt{1-e^{-2x}}\,\mathrm{d}x \tag{59}$$

$$4\pi \log^2 2 + \frac{\pi^3}{3} = \int_0^{\infty} \frac{x^2\,\mathrm{d}x}{\sqrt{e^x-1}} \tag{60}$$

$$\pi \log 2 = \int_0^{\pi/2} \frac{x^2\,\mathrm{d}x}{\sin^2 x} \tag{61}$$

$$\frac{\pi^3}{24} + \frac{\pi \log^2 2}{2} = \int_0^{\pi/2} \log^2(\cos x)\,\mathrm{d}x \tag{62}$$

$$\frac{8\pi^3}{81\sqrt{3}} = \int_0^1 \frac{\log^2 x\,\mathrm{d}x}{x^2+x+1} \tag{63}$$

$$\frac{\pi}{2} - \log 2 = \int_0^1 \frac{\log(1+x^2)\,\mathrm{d}x}{x^2} \tag{64}$$

$$\frac{2\pi \log 3}{\sqrt{3}} = \int_0^1 \frac{\log(1+x^3)\,\mathrm{d}x}{1-x+x^2} \tag{65}$$

$$\frac{\pi^2}{6} = \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} \tag{66}$$

$$\sqrt{\pi} = \Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x}\, \mathrm{d}x \tag{67}$$

$$\frac{\pi^2}{8} = \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{1 - x^2y^2} \tag{68}$$

$$\frac{-\pi}{6} + \log(2+\sqrt{3}) = \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{\sqrt{1+x^2+y^2}} \tag{69}$$

$$5 - \pi^2 - 4\log 2 + 16\log^2 2 = \int_0^1 \int_0^1 \left(\frac{x-1}{x+1}\right)^2 \left(\frac{y-1}{y+1}\right)^2 \left(\frac{xy-1}{xy+1}\right)^2 \mathrm{d}x\, \mathrm{d}y \tag{70}$$

$$-\frac{\pi}{4} + \frac{3\log(2+\sqrt{3})}{2} = \int_0^1 \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z}{\sqrt{x^2+y^2+z^2}} \tag{71}$$

$$-\frac{\pi}{24} + \frac{\sqrt{3}}{4} + \frac{\log(2+\sqrt{3})}{2} = \int_0^1 \int_0^1 \int_0^1 \sqrt{x^2+y^2+z^2}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{72}$$

$$-\frac{\pi}{60} + \frac{2\sqrt{3}}{5} + \frac{7\log(2+\sqrt{3})}{20} = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)^{3/2} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{73}$$

4 Iterations for π

- (The Archimedes iteration). Set $a_0 = 2\sqrt{3}$ and $b_0 = 3$. Iterate

$$a_{n+1} = \frac{2a_nb_n}{a_n+b_n}, \quad b_{n+1} = \sqrt{a_{n+1}b_n}. \tag{74}$$

  Then both a_n and b_n converge to π: each iteration decreases the distance between a_n and b_n (which interval contains π) by a factor of approximately four.
- (The Brent-Salamin iteration). Set $a_0 = 1$, $b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Iterate

$$\begin{aligned} a_k &= \frac{a_{k-1}+b_{k-1}}{2}, \quad b_k = \sqrt{a_{k-1}b_{k-1}}, \\ c_k &= a_k^2 - b_k^2, \quad s_k = s_{k-1} - 2^k c_k, \\ p_k &= \frac{2a_k^2}{s_k}. \end{aligned} \tag{75}$$

  Then p_k converges quadratically to π: each iteration approximately *doubles* the number of correct digits.
- (The Borwein cubic iteration). Set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1+2(1-s_k^3)^{1/3}}, \quad s_{k+1} = \frac{r_{k+1}-1}{2}, \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k(r_{k+1}^2 - 1). \end{aligned} \tag{76}$$

  Then $1/a_k$ converges cubically to π: each iteration approximately *triples* the number of correct digits.
- (The Borwein quartic iteration). Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned} \tag{77}$$

  Then $1/a_k$ converges quartically to π: each iteration approximately *quadruples* the number of correct digits.

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