Stochastic Analysis of the "Gambler's Ruin" Problem

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Abstract

This paper presents a computational analysis of the "Gambler's Ruin" problem using Monte Carlo simulations in R. We investigate a stochastic process with absorbing barriers in an "unfair game" scenario ($p = 0.48$). The simulation of 1000 trajectories yielded an empirical ruin probability of 98.8%, validating theoretical models with high accuracy. Key findings reveal that while negative drift causes rapid failure, success is a rare and prolonged event. Furthermore, sensitivity analysis identifies a critical phase transition at $p = 0.5$ and strategy optimization demonstrates that aggressive betting significantly minimizes the risk of ruin in sub-fair environments.

1 Introduction

The "Gambler's Ruin" problem is a fundamental concept in stochastic processes, modeling a Random Walk. It illustrates the conflict between short-term variance and long-term expected value. In a realistic casino setting where the house holds a statistical edge ($p < 0.5$), the Law of Large Numbers dictates that a gambler with finite capital will eventually face ruin.

This project bridges theoretical probability with empirical observation using computational simulation. By employing Monte Carlo methods, we analyze 1000 sample paths to understand the dynamics of an unfair game ($p = 0.48$). This study aims to:

- Validate the empirical results against theoretical formulas.
- Visualize the volatility of capital trajectories and the stochastic nature of the stopping time.
- Analyze the distribution of game durations, distinguishing between rapid ruin and endurance.
- Evaluate the impact of betting strategies.

The analysis confirms that in a game with negative drift, the optimal strategy is to minimize the game duration through aggressive betting, rather than relying on the long-term convergence of probabilities.

2 Methodology

2.1 Problem Statement

We consider a gambler entering a casino with an initial capital $h = 50$ monetary units. The gambler plays a roulette-style game where they bet a fixed stake (defaulting to 1 unit) on an outcome with a probability of winning $p = 0.48$. This probability satisfies the condition $p < 0.5$, representing an "unfair game" where the house maintains a statistical edge.

The stochastic process X_t, representing the gambler's capital at time t, stops when one of the following barriers is reached:

- **Ruin:** The capital drops to 0.
- **Success:** The capital reaches the target $H = 100$.

The simulation is performed using the R programming language, employing a Monte Carlo method with $N = 1000$ iterations to estimate the probabilities and analyze the system's behavior.

2.2 Objectives of the Study

The primary goal of this project is to analyze the behavior of the random walk under the constraints defined above.

1. **Simulation and Validation:** implementation of the simulation algorithm and validation of the empirical ruin probability against the theoretical formula derived from difference equations.
2. **Trajectory Visualization:** graphical representation of the sample paths (capital evolution over time) to observe the volatility and the stochastic nature of the stopping time.
3. **Duration Analysis:** investigation of the distribution of the game duration (number of bets until absorption) to determine the expected time of play.
4. **Outcome Comparison:** statistical analysis using boxplots to compare the duration of games resulting in ruin versus those resulting in a win.
5. **Sensitivity Analysis:** examination of the system's sensitivity to the "luck" factor by varying the win probability parameter p in the interval $[0.40, 0.60]$.
6. **Strategy Optimization:** evaluation of the impact of increasing the betting stake (e.g., from 1 to 25 units) on the probability of ruin, testing the hypothesis that aggressive play is optimal in unfair games.

3 Theoretical Background

3.1 Stochastic Processes and Random Walks

In my project, the gambler's capital at any moment t is denoted by X_t. This is a specific type of stochastic process called a **Simple Random Walk**.

- Knowing the current capital X_t does not tell us exactly what X_{t+1} will be.
- At each step, X_t increases by $+1$ or decreases by -1 based on the outcome of a Bernoulli trial.

3.2 The Law of Large Numbers

Since a single game is highly unpredictable, we cannot draw conclusions from just one gambler. However, by simulating $N = 1000$ independent games, the calculated "Simulated Ruin Probability" becomes a stable and accurate estimator of the true mathematical probability.

3.3 The Theoretical Formula for Ruin

For the classic Gambler's Ruin problem with a starting capital h, a target H, and a probability of winning a single round p (where $p \neq 0.5$), the exact probability of going broke is given by the formula:

$$P(\text{Ruin}) = \frac{1 - \left(\frac{q}{p}\right)^h}{1 - \left(\frac{q}{p}\right)^H}, \quad \text{where } q = 1 - p \tag{1}$$

In our specific case ($p = 0.48$, $q = 0.52$), the ratio $\frac{q}{p} > 1$, which means the odds are against the player. This formula is used in the R script to calculate the reference value `prob_teoretica_ruina`, allowing us to verify if our simulation is correct.

4 Results and Discussion

4.1 Simulation and Validation

To validate the stochastic model, we executed the Monte Carlo simulation with the following parameters, reflecting the "unfavorable game" scenario described in the problem statement.

```
Capital Initial: 50 | Tinta: 100 | Probabilitate Castig: 0.48
> cat("Numar total simulari:", nr_simulari, "\n")
Numar total simulari: 1000
> cat("Jucatori ruinati:", jocuri_pierdute, "\n")
Jucatori ruinati: 988
> cat("Jucatori castigatori:", jocuri_castigate, "\n")
Jucatori castigatori: 12
> cat("Probabilitatea Ruinei (SIMULATA): ", round(prob_simulata * 100, 2), "%\n", sep="")
Probabilitatea Ruinei (SIMULATA): 98.8%
> cat("Probabilitatea Ruinei (TEORETICA):", round(prob_teoretica_ruina * 100, 2), "%\n", sep="")
Probabilitatea Ruinei (TEORETICA):98.21%
```

Figure 1: Simulation parameters and the comparison between empirical and theoretical ruin probabilities.

We compared the simulated ruin probability with the theoretical value calculated:

- **Simulated Probability:** 98.80%
- **Theoretical Probability:** 98.21%

The absolute error is approximately 0.59%. This small discrepancy confirms that the Law of Large Numbers holds, as N is sufficiently large, the empirical frequency converges to the theoretical probability.

The result highlights the high risk of the game, the slight disadvantage ($p = 0.48$) leads to ruin in almost 99% of cases.

4.2 Trajectory Visualization

To visualize the stochastic nature of the process, we plotted the capital evolution for the first 5 simulations. Figure 2 illustrates these sample paths, showing how the capital X_t fluctuates over time.

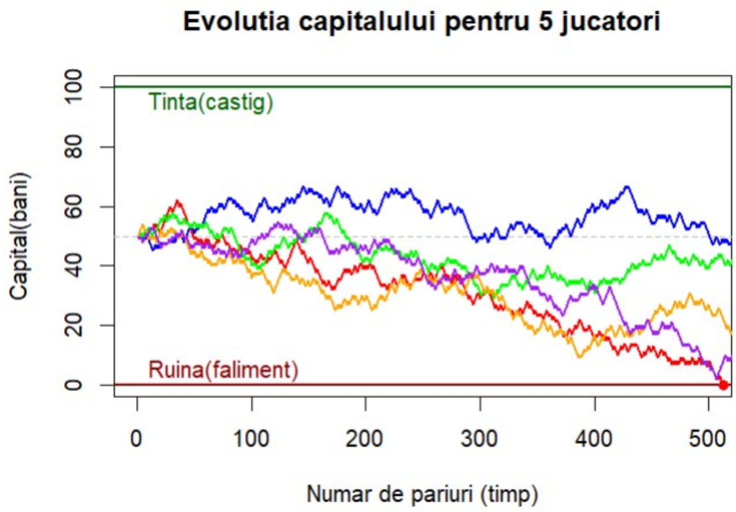

Figure 2: Evolution of capital for 5 distinct gamblers.

The graphical analysis highlights three critical behaviors of the system:

- Although some trajectories (the blue or green lines) may initially rise above the starting capital due to short-term favorable variance, the overall trend is unmistakably downward. This visualizes the mathematical influence of $p = 0.48$, statistical "gravity" eventually pulls the realization towards the ruin barrier.
- The paths are not linear but highly irregular. This demonstrates that while the long-term expectation is negative, the short-term behavior is dominated by noise, making the outcome of the very next step unpredictable.
- A significant contrast exists between trajectories. Some hit the ruin barrier almost immediately (the red line), while others persist for hundreds of iterations. This visually confirms that the duration of the game is a random variable with high variance, while the final destination is often the same (ruin), the "survival time" differs drastically among players.

4.3 General Duration Analysis

The statistical distribution of the stopping time T (the total duration of the game). Figure 3 displays the frequency of game lengths for all 1000 simulations.

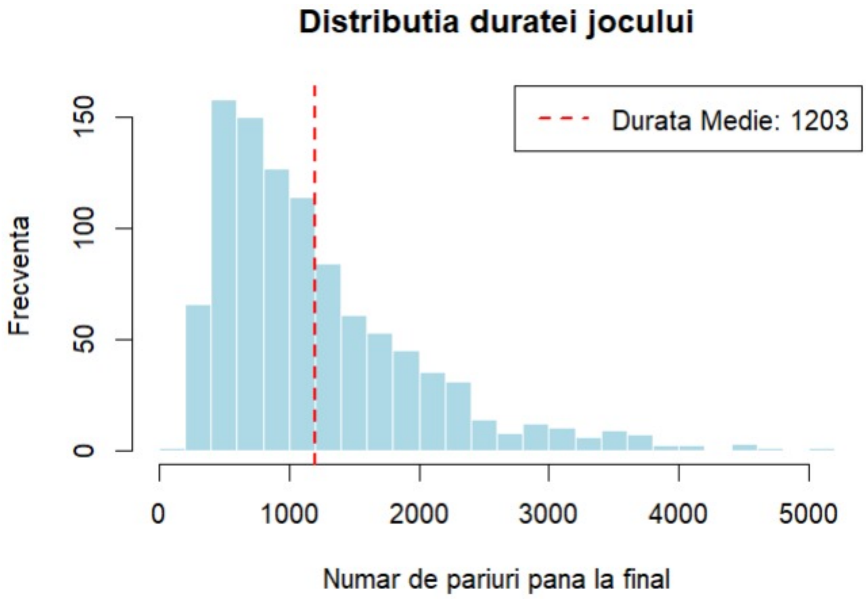

Figure 3: Distribution of game duration.

The histogram reveals a **right-skewed distribution**, which is typical for Random Walk processes.

- The highest frequency bars are concentrated on the left side. This indicates that the most common outcome is a relatively short game, driven by the $p = 0.48$.
- The distribution extends significantly to the right, with some games lasting over 4000 bets.
- The arithmetic mean (red line, ≈ 1203) is "inflated" by those few extremely long games in the distribution's tail. Consequently, a typical player should expect the game to last significantly less than the indicated average.

4.4 Duration Comparison: Ruin vs. Success

Does the outcome of the game influence how long it lasts? Figure 4 compares the duration for gamblers who reached Ruin against those who reached the Target .

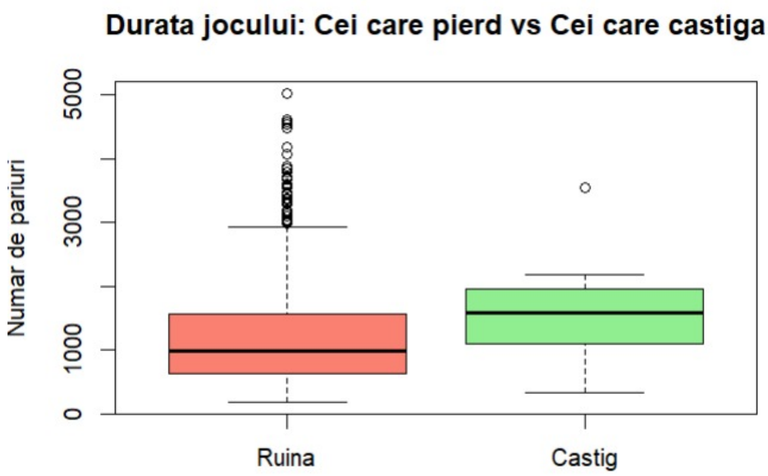

Figure 4: Boxplot comparison of game duration.

- The median duration for the red box is lower than for the green box. Since the game is unfair ($p = 0.48$), the statistical "gravity" naturally pulls the capital towards zero.

- The green box displays a higher median duration. To reach the target of 100, a gambler must overcome the negative drift. This "uphill battle" against the house edge requires a sustained sequence of favorable variance, which typically necessitates a higher number of bets to materialize.

- The numerous outliers (circles) above the red box indicate that while most losers go bankrupt quickly, a subset of gamblers engages in prolonged battles (3000+ bets) before eventually succumbing to the mathematical disadvantage.

4.5 Sensitivity Analysis: The Impact of "Luck" (p)

We have established that with $p = 0.48$, the probability of ruin is near 99%. Figure 5 displays the probability of ruin as a function of the win probability p, varying from 0.40 to 0.60.

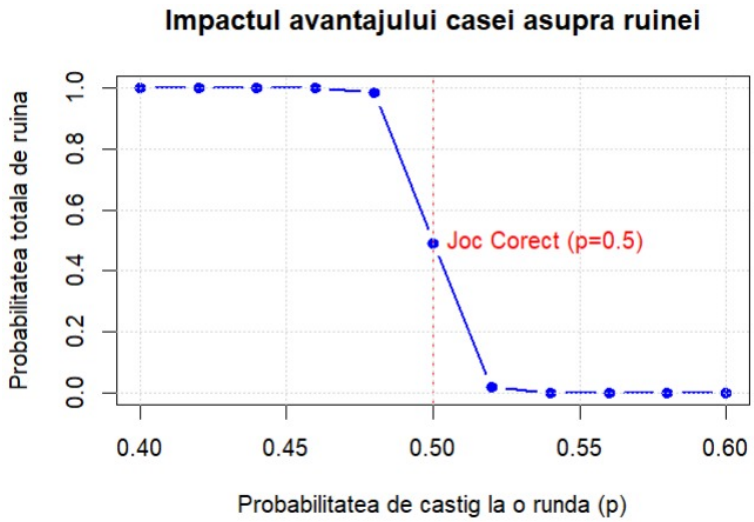

Figure 5: Sensitivity Analysis.

- **The Cliff ($p < 0.5$):** for any value slightly below 0.5 (even 0.48 or 0.49), the probability of ruin stays critically high (near 1.0). This explains why casinos only need a small edge to guarantee profit.

- **The Fair Game ($p = 0.5$):** at exactly 0.5, the probability of ruin is linearly proportional to the starting capital.

- **The Winning Zone ($p > 0.5$):** The steep slope around 0.5 indicates that a very small change in the win probability (from 0.49 to 0.51) fundamentally alters the gambler's fate from certain failure to probable success.

4.6 Strategy Optimization: Prudent vs. Aggressive Play

We investigated whether the player can alter their fate by changing the betting strategy. Figure 6 compares the probability of ruin for different stake sizes (1, 2, 5, 10, and 25 units), assuming the same starting capital ($h = 50$) and target ($H = 100$).

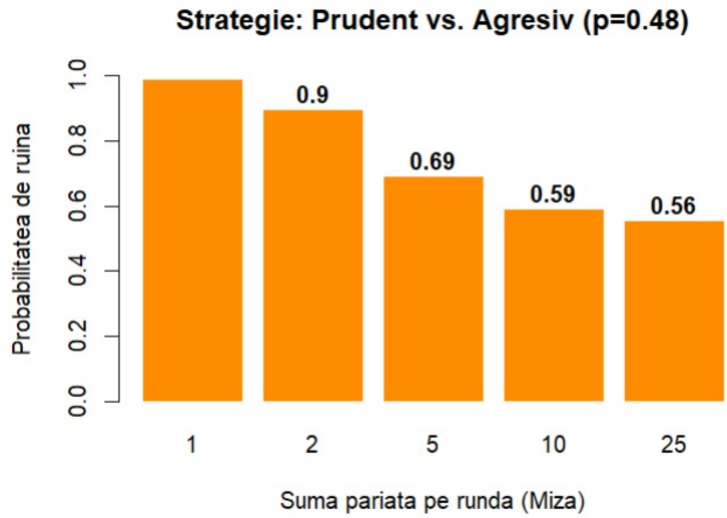

Figure 6: Impact of betting stake size on ruin probability.

- **Prudent Strategy:** when betting small amounts (1 unit), the gambler is forced to play a large number of rounds to reach the target. This allows the Law of Large Numbers to converge, ensuring the casino's mathematical edge ($p = 0.48$) dominates. The ruin probability is nearly 100%.
- **Aggressive Strategy:** by increasing the stake to 25 units, the gambler minimizes the number of rounds played. This reduces the exposure to the negative drift and increases the variance. The player relies on short-term luck to jump over the target barrier before the statistical disadvantage takes effect.
- **Conclusion:** In a sub-fair game ($p < 0.5$), the optimal strategy is to bet as much as possible to shorten the game duration.

5 Conclusion

The computational analysis of the "Gambler's Ruin" confirms the deterministic nature of stochastic processes over long horizons. Our simulation ($N = 1000$) validated the theoretical model with a margin of error of less than 0.6%, demonstrating the robustness of the Law of Large Numbers.

- **Dynamics of ruin:** the negative drift ($p = 0.48$) acts as a "gravitational pull," causing failure to be the path of least resistance and shortest duration.
- **System sensitivity:** the process exhibits a phase transition at $p = 0.5$, where a marginal shift in probability dramatically alters the outcome.
- **Optimal strategy:** in an unfair game, the "Bold Play" strategy is mathematically superior. Increasing the stake minimizes the game duration, thereby reducing the cumulative effect of the house edge.

References

[1] D. Stirzaker, *Elementary Probability*, Cambridge University Press, New York, 2025.