7 Plasticity Models

Chapter Outline

7.1 Introduction

One historically common way to represent the mechanical behavior of a thermoplastic material in a finite element analysis is to use a metal plasticity model. This type of material model is typically easy to use, and is available in all commercial finite element programs. The main problem with this class of material models is that they were developed for metals and can be inaccurate if used incorrectly for polymers. The purpose of this chapter is to list the most commonly used plasticity models and to highlight both their usefulness and their limitations.

The theory of metal plasticity is well developed and described in numerous textbooks [1–5]. This chapter gives a brief introduction to the theory and how it applies to different types of polymer materials.

Mechanics of Solid Polymers.
DOI: http://dx.doi.org/10.1016/B978-0-323-31150-2.00007-8

7.2 J_2-Plasticity with Isotropic Hardening

J_2-plasticity with isotropic hardening is perhaps the most commonly used type of plasticity theory, and is often simply called *plasticity theory*. In Abaqus [6], this model is created using the following commands:

```
*Elastic
[E], [nu]
*Plastic
[sigY0], 0
[sigY1], [epsY1]
...
*Rate Dependent, type=power law
[D], [n]
```

In this example, some of the needed values are simply listed as variables within square brackets. Also, the yield stress was made dependent on the plastic strain rate (which is optional).

In ANSYS the model can be selected using the following commands: Also in this example some of the needed values are

```
! Isotropic hardening elastic-plastic model
! with Mises isotropic yield and rate dependence
MP, EX, [matid],[E]
MP, NUXY, [matid],[nu]

TB, MISO, [matid], 1, [N]
TBPT,,[strain1],[stress1]
...

TB, RATE, [matid],1, 2, PERZYNA
TBDATA, 1, [m]
TBDATA, 2, [gamma]
```

simply listed as variables within square brackets, and the rate-dependence is optional.

The theory and implementation of this model has been extensively covered [3, 6], here a very simplified implementation will be presented. The theory will be presented for one-dimensional loading in order to demonstrate how the theory works, and also to illustrate strength and limitations to the model.

Start by dividing the total strain into elastic (ε^{e}) and plastic (ε^{p}) contributions: $\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{p}}$. Only the elastic strain contribute to the stress: $\sigma = E(\varepsilon - \varepsilon^{\mathrm{p}})$.

The yield condition can be written as a scalar function:

$$f(\sigma) = |\sigma| - \sigma_{\mathrm{y}}(\alpha) \leq 0, \tag{7.1}$$

where $\sigma_{\mathrm{y}}(\alpha)$ is the yield stress at the accumulated plastic strain magnitude α, which is given by $\dot{\alpha} = |\dot{\varepsilon}^{\mathrm{p}}|$.

If $f(\sigma) < 0$, then the stress magnitude is less than the current yield stress and plastic flow is not active. If $f(\sigma) = 0$, then the stress magnitude is equal to the current yield stress and plastic flow can occur if the loading continues. Note that the yield function $f(\sigma)$ cannot be positive since the stress magnitude cannot be greater than the current yield stress.

This material model can be graphically represented using a set of piecewise linear segments as shown in Figure 7.1. This means that the model can be made to fit almost any monotonic stress-strain data in a single loading mode. Note, however, that the model cannot be made to fit the large-strain behavior of elastomers since these materials often have a tangent modulus at large strains that is larger than the initial small-strain modulus.

7.2.1 Cyclic Loading

All isotropic hardening plasticity models that are based on (non-zero) isotropic hardening will exhibit a gradual increase in the yield stress with increasing plastic strain. As illustrated in Figures 7.2 and 7.3, this behavior will cause each stress-strain

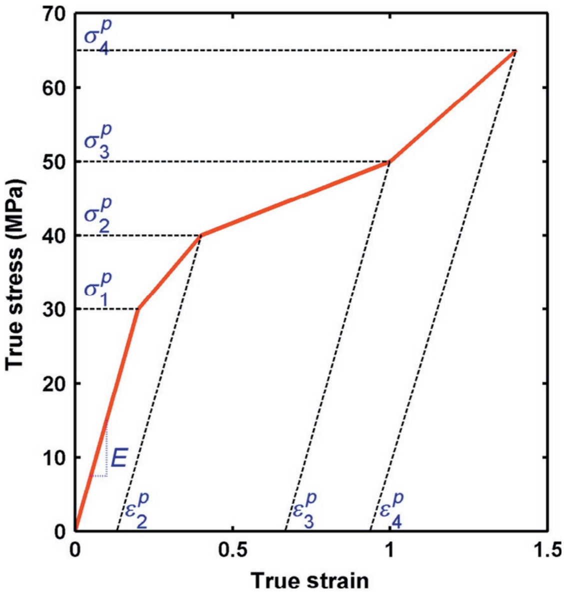

Figure 7.1 Stress-strain representation of the J_2-plasticity model with isotropic hardening. The model is defined by the Young's modulus and the pairs of $(\varepsilon_i^{\mathrm{p}}, \sigma_i^{\mathrm{p}})$.

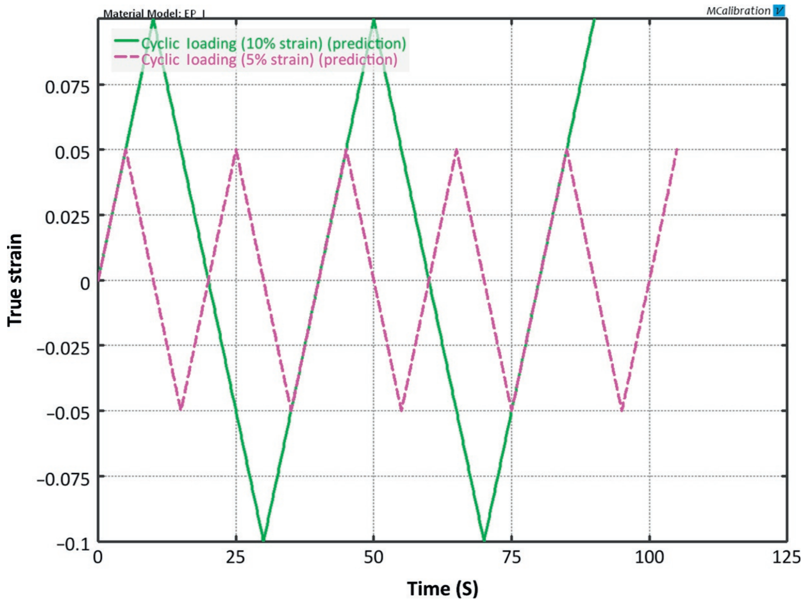

Figure 7.2 In this example two different cyclic strain histories are examined. The first case is using a strain amplitude of 5%, and the second case is using a strain amplitude of 10%.

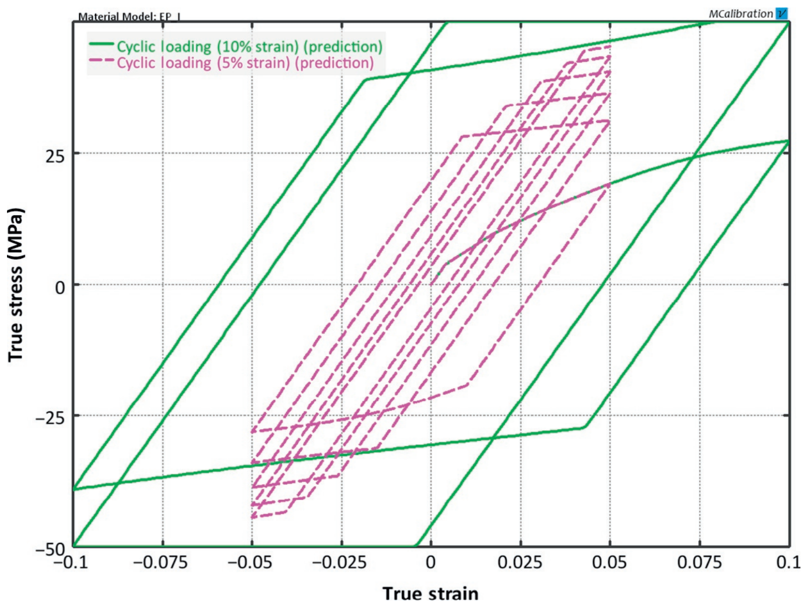

Figure 7.3 Predicted stress-strain response during cyclic loading with a strain amplitude of 5% and 10%.

loop to grow in amplitude during cyclic loading. These figures show the predicted stress-strain response due to two different strain histories. In the first case the strain is cycled between +5% and −5%, and in the second case the strain is cycled between +10% and −10%. The predicted stress-strain response shows that the predicted stress becomes almost bilinear in response, and that the stress amplitude gradually increases during the cyclic loading. These inherent features of the model response are not in agreement with experimental data for either rubbers or thermoplastics for which the stress-strain loops during cyclic loading typically reach a steady-state response, or a response that is close to steady state.

If the goal of the finite element study is to predict the response during cyclic loading then it is typically not appropriate to use a plasticity model based on *isotropic hardening*. In this case it is better to use a plasticity model based on kinematic hardening, or a viscoplastic material model.

7.2.2 Matlab Implementation

The isotropic hardening plasticity model, for the case of uniaxial loading, can be implemented using a short Matlab script. In the implementation shown here, the stress in a given strain increment is obtained from the radial return mapping algorithm which can be written:

1. Initialize variables: $\sigma_0 = 0$, $\alpha_0 = 0$, $i = 1$, $\varepsilon_0^{\mathrm{p}} = 0$. Where σ_0 is the initial stress, α_0 is the initial plastic strain, and $\varepsilon_0^{\mathrm{p}}$ is the initial plastic strain.
2. Take an elastic trial step:
$$\sigma_i^{\text{trial}} = \sigma_{i-1} + E(\varepsilon_i - \varepsilon_{i-1}). \tag{7.2}$$
3. Check if plastic flow is active
$$f = \sigma_i^{\text{trial}} - \sigma_{\mathrm{y}}(\alpha_i).$$
If $f < 0$ then accept the trial step:
$$\sigma_i = \sigma_i^{\text{trial}}.$$

If $f \geq 0$ then let h be the tangent modulus (i.e. hardness) at the current plastic strain magnitude, then perform the following updates:

$$\Delta\gamma = \frac{f}{E+h}, \tag{7.3}$$

$$\sigma_i = \sigma_i^{\text{trial}} - \Delta\gamma\, E \,\text{sign}(\sigma_i^{\text{trial}}), \tag{7.4}$$

$$\alpha_i = \alpha_{i-1} + \Delta\gamma. \tag{7.5}$$

In order to make the theory for the J_2-plasticity model with isotropic hardening more concrete it is useful to study a numerical implementation of the governing equations. A convenient way to do this is to use Matlab, as illustrated below.

Matlab Code "mat_PlasticityJ2Iso.m":

```
function [stress] = mat_PlasticityJ2Iso(time, strain, params)
%mat_PlasticityJ2Iso Elastic-Plastic model with isotropic hardening
%Uniaxial loading
%Params = [E, sigY0, sigY1, eps1, sigY2, eps2, ... sigYn, epsn]

N = length(params);
E = params(1);
sigmaY = [params(2) params(3:2:N)];
epsP = [0 params(4:2:N)];

stress = 0 * strain;
alpha = 0; % plastic strain magnitude
for i = 2 : length(strain)
    stressTrial = stress(i-1) + E * (strain(i) - strain(i-1));
    stressTrialMag = abs(stressTrial);
    sigmaYcurr = interp1(epsP, sigmaY, alpha, 'linear', sigmaY(end));
    fTrial = stressTrialMag - sigmaYcurr;
    if fTrial < 0
        stress(i) = stressTrial; % elastic loading
    else
        h = (interp1(epsP,sigmaY,alpha+1e-4,'linear',sigmaY(end)) …
           - sigmaYcurr) / 1e-4;
        dgamma = fTrial / (E+h); % plastic with radial retun mapping
        stress(i) = stressTrial - dgamma * E * sign(stressTrial);
        alpha = alpha + dgamma;
    end
end
end
```

The code is written in the form of a Matlab function. The function takes as input a vector of time, a vector of strain, and a vector of material parameters. The time vector is not used since the model is strain-rate independent, but kept as an argument to the function in order to make all material model functions presented in this book have the same input arguments.

The material parameters that are required for the Matlab implementation are: $[E, \sigma_y^0, \sigma_y^1, \varepsilon_1^p, \sigma_y^2, \varepsilon_2^p, \ldots]$. That is, the Young's modulus followed by pairs of yield stress and the corresponding plastic strain. Note that the initial plastic strain is automatically set to be zero.

The example Matlab file presented here can be made strain rate dependent by incorporating a yield function that depends on the plastic strain rate. This modification is left as an exercise.

7.2.3 Python Implementation

A slightly more rigorous implementation of isotropic hardening plasticity is presented in Figure 7.4. In this case the Python code includes a search for the plastic strain increment that is consistent

Python Code "mat_PlasticityJ2Iso.py":

```
from pylab import *
import scipy.optimize

def errfunc(dgamma, alpha0, epsP_vec, sigmaY_vec, E, stressTrial):
    sigmaY1 = interp(alpha0+dgamma, epsP_vec, sigmaY_vec)
    return abs(stressTrial - dgamma * E * sign(stressTrial) \
        - sigmaY1)

def plasticity_J2iso(strain, params):
    """Uniaxial loading. [E, sigY0, sigY1, eps1, ...]"""
    N = len(params)
    E = params[0]
    sigmaY_vec = [params[1]]
    sigmaY_vec.extend(params[2::2])
    epsP_vec = [0.0]
    epsP_vec.extend(params[3::2])
    stress = zeros(len(strain))
    alpha0 = 0.0
    for i in range(1, len(strain)):
        stressTrial = stress[i-1] + E * (strain[i] - strain[i-1])
        sigmaY0 = interp(alpha0, epsP_vec, sigmaY_vec)
        fTrial = abs(stressTrial) - sigmaY0
        if fTrial < 0:
            stress[i] = stressTrial
        else:
            sigmaY0 = interp(alpha0, epsP_vec, sigmaY_vec)
            h0 = (interp(alpha0+1.0e-4,epsP_vec,sigmaY_vec) \
                - sigmaY0) / 1.0e-4
            dgamma = fTrial / (E+h0)
            res = scipy.optimize.fmin(errfunc, x0=[dgamma], \
                xtol=1e-9, ftol=1e-9, maxfun=9999, \
                full_output=1, disp=0, \
                args=(alpha0, epsP_vec, sigmaY_vec, E, stressTrial))
            dgamma = res[0][0]
            stress[i] = stressTrial - dgamma * E * sign(stressTrial)
            alpha0 = alpha0 + dgamma
    return stress


strain = linspace(0.0, 0.8, 10)
params = [50.0, 10.0, 15.0, 0.1]

stress = plasticity_J2iso(strain, params)

plot(strain, stress, 'b.-')
```

Figure 7.4 Python implementation of the J_2-plasticity model with isotropic hardening. File name: `mat_PlasticityJ2Iso.py`.

with the yield condition. This can be necessary in cases where the applied strain increment is large so that the hardening response becomes non-linear during the increment. The traditional way to search for the correct plastic strain increment is to use the Newton method, but the example here is simply using a generic search algorithm in order to keep the code clean.

The code in this case also contain the commands to generate and plot the complete stress-strain response during monotonic tension. The resulting figure showing the predicted stress-strain behavior is shown in Figure 7.5.

Also in this case the plasticity model can be made rate-dependent using an equation like the following:

$$\sigma_{\text{yield}} = \sigma_{\text{yield}}^{0} \cdot \left[1 + \left(\frac{\dot{\varepsilon}^{\text{p}}}{D} \right)^{1/n} \right].$$

The implementation of this is left as an exercise.

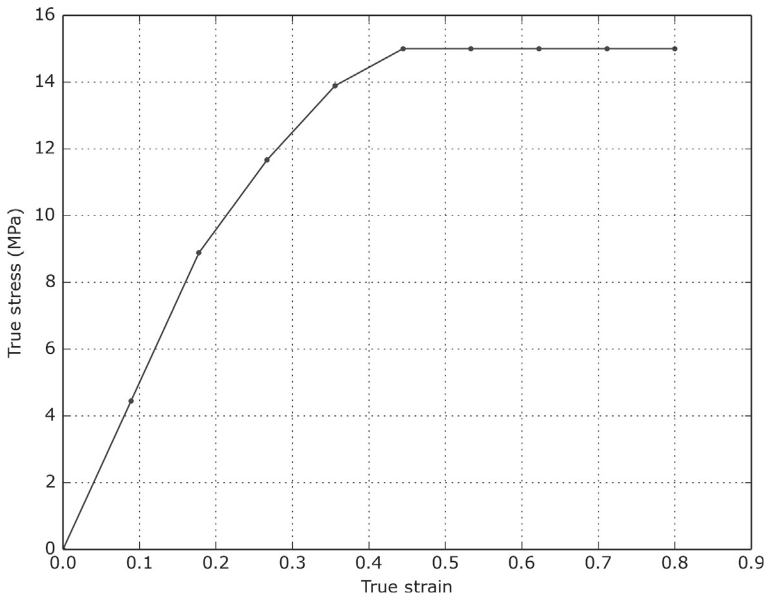

Figure 7.5 Predicted stress-strain response obtained from the file `mat_PlasticityJ2Iso.py`.

7.2.4 Application to Thermoplastics

The applicability of the isotropic hardening plasticity model is in this section examined by direct comparison with experimental data for a ultra-high molecular weight polyethylene (UHMWPE) thermoplastic material. The experimental data in this case was obtained in uniaxial compression at three different engineering strain rates (−0.02/s, −0.05/s, and −0.10/s) to a true strain of about −0.45, followed by unloading to zero stress.

The isotropic hardening plasticity model was calibrated to the experimental data in two ways. First the model was calibrated to the monotonic compression to the minimum strain using the intermediate strain rate. The results from that calibration are shown in Figure 7.6. This figure shows that, as expected, the model is very accurate at predicting the monotonic compressive response, but the model prediction of the unloading response is quite inaccurate since it severely underestimates the amount of recovery of the material response during unloading.

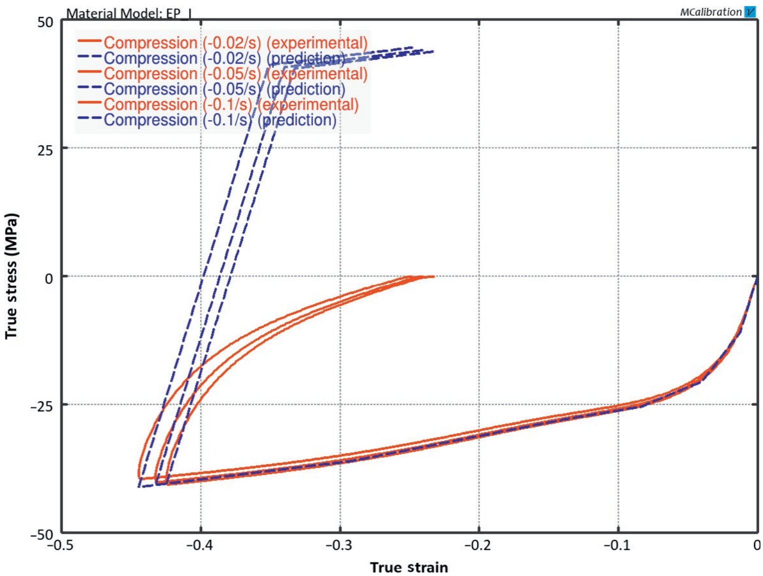

Figure 7.6 Comparison between experimental data for UHMWPE and predictions from the isotropic hardening plasticity model calibrated the monotonic compressive material response.

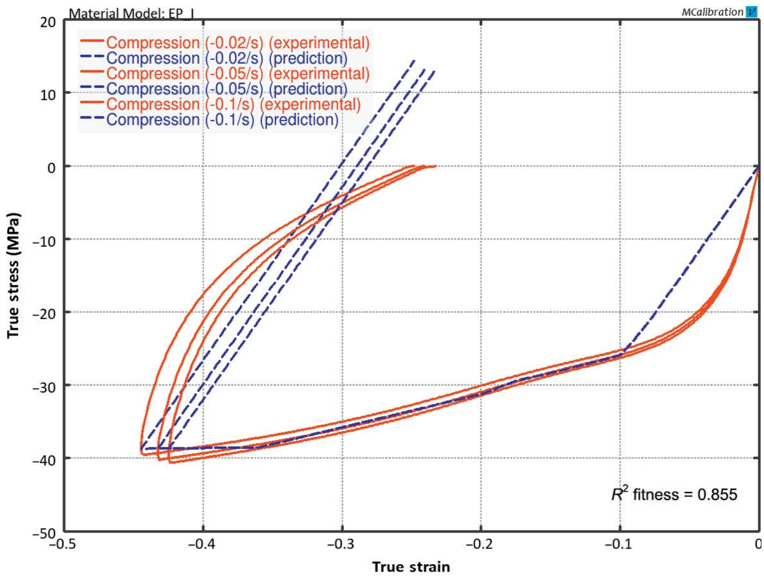

Figure 7.7 Comparison between experimental data for UHMWPE and predictions from the isotropic hardening plasticity model calibrated all the experimental data.

A different way to calibrate the material model is use all compressive data. Figure 7.7 illustrates that when calibrated this way the overall predictions from the model may be better on average, but this calibrated model is likely less useful for most practical applications since the calibrated material model is not accurate at any section of the experimental stress-strain curve.

To accurately predict this material response requires a more advanced viscoplastic material model such as the Three Network (TN) model. This model is discussed in more detail in Section 8.6 of Chapter 8.

7.3 Plasticity with Kinematic Hardening

The stress for a kinematic hardening plasticity model of Chaboche type [7] is given by the following hardening model:

$$\dot{\boldsymbol{\sigma}}_{\text{back}} = \frac{2\alpha}{3\beta}\sigma_{\text{y}}\dot{\varepsilon}^{\text{p}} - \frac{||\dot{\varepsilon}^{\text{p}}||}{\beta}\boldsymbol{\sigma}_{\text{back}}, \tag{7.6}$$

where $\dot{\boldsymbol{\sigma}}_{\text{back}}$ is the time-derivative of the back stress, and $\dot{\varepsilon}^{\text{p}}$ the time-derivative of the plastic strain. The parameter σ_{y} is the initial yield stress, $\alpha\sigma_{\text{y}}$ is the final yield stress, and β the transition strain for the yield evolution.

The applicability of the kinematic hardening plasticity model was examined by direct comparison with experimental data for a UHMWPE thermoplastic material. The experimental data in this case was obtained in uniaxial compression at three different engineering strain rates (−0.02/s, −0.05/s, and −0.10/s) to a true strain of about −0.45, followed by unloading to zero stress. The kinematic hardening plasticity model was calibrated to the experimental data using the MCalibration software [8]. The results from that calibration are shown in Figures 7.8–7.10.

The first of these figures compare the predictions of the kinematic hardening model with one backstress network to the experimental data. The Abaqus material definition that was used in this figure is shown below.

```
*Material, name=mat
*Elastic
775.664, 0.4
*Plastic, hardening=combined, data type=parameters
16.5909, 86.3193, 0.490949
```

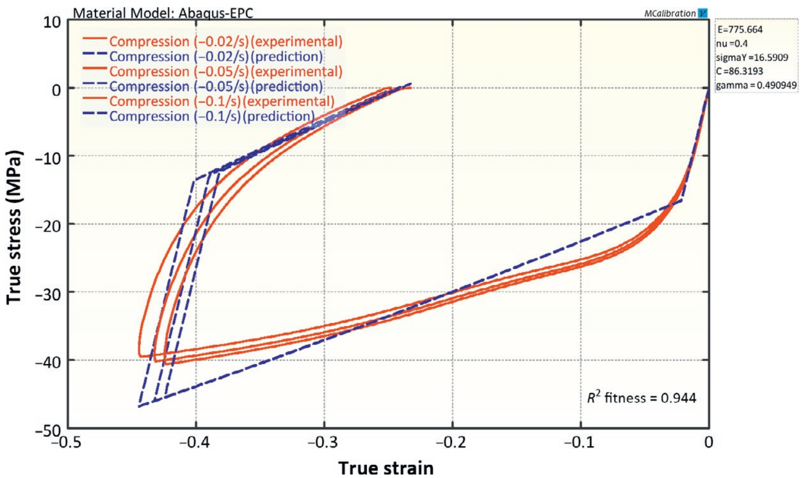

Figure 7.8 Calibration of an elastic-plastic material model with combined kinematic hardening and *one* backstress network.

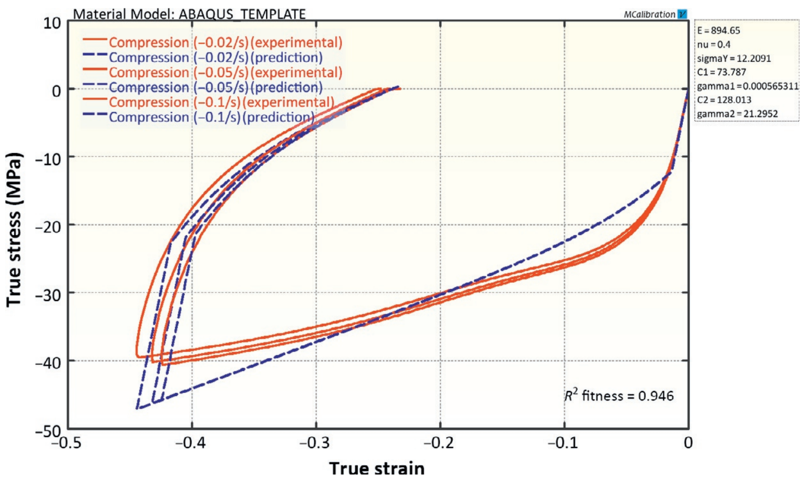

Figure 7.9 Calibration of an elastic-plastic material model with combined kinematic hardening and *two* backstress networks.

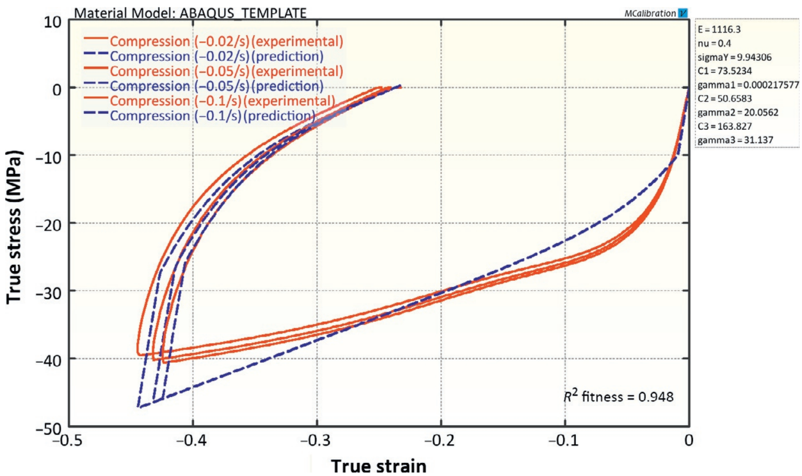

Figure 7.10 Calibration of an elastic-plastic material model with combined kinematic hardening and *three* backstress networks.

As is shown in Figure 7.9, adding a second backstress network slightly improves the model predictions. The Abaqus material definition that was used in this figure is shown below.

```
*Material, name=mat
*Elastic
894.65, 0.4
*Plastic, hardening=combined, data type=parameters, number backstresses=2
12.2091, 73.787, 0.000565311, 128.013, 21.2952
```

As is shown in Figure 7.10, adding a third backstress network slightly further improves the model predictions. The Abaqus material definition that was used in this figure is shown below.

```
*Material, name=mat
*Elastic
1116.3, 0.4
*Plastic, hardening=combined, data type=parameters, number backstresses=3
9.94306, 73.5234, 0.000217577, 50.6583, 20.0562, 163.827, 31.137
```

One of the main limitations of this material model is that the tangent modulus is monotonically decreasing, making it impossible to capture the large strain tensile response of many ductile thermoplastics that are known to have a stress-strain response that stiffens at large strains.

7.4 Johnson-Cook Plasticity

The Johnson-Cook model is a plasticity model that is based on Mises plasticity with closed-form analytical equations specifying the hardening behavior and the strain-rate dependence of the yield stress. In this model the yield stress is given by the following equation:

$$\sigma_{\text{yield}} = \left[A + B(\varepsilon^{\text{p}})^n\right]\left[1 + C\ln\left(\frac{\dot{\varepsilon}^{\text{p}}}{\dot{\varepsilon}_0}\right)\right](1 - \hat{\theta}^m), \tag{7.7}$$

where $[A, B, n, C, \dot{\varepsilon}_0, m]$ are material parameters that need to be determined from experimental data. In this equation ε^{p} is the effective plastic strain, and $\dot{\varepsilon}^{\text{p}}$ is the time derivative of the effective plastic strain.

Since this material model is based on isotropic hardening, it cannot accurately predict the unloading response of many thermoplastics. Figure 7.11 shows the predicted stress-strain response when calibrated to cyclic uniaxial compression data for UHMWPE. The figure shows that the predicted response under monotonic loading is in good agreement with the experimental behavior, but that the predicted unloading behavior significantly overpredicts the residual deformation after unloading. The calibrated material parameters that were used to generate the model predictions are listed in Abaqus inp-file format below.

```
*Material, name=mat
*Elastic
733.019, 0.4
*Plastic, hardening=Johnson Cook
16.1146, 39.2996, 0.577683, 1, 2000, 1000
*Rate Dependent, type=Johnson Cook
0.0289912, 0.0073417
```

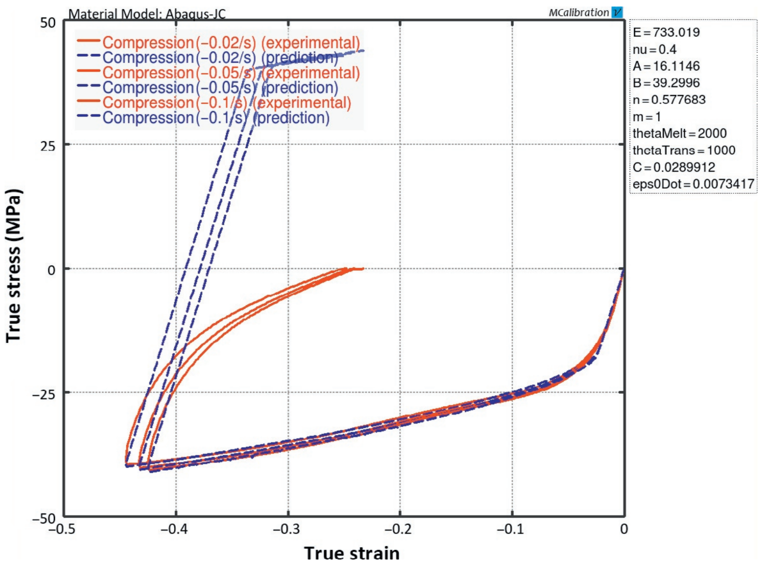

Figure 7.11 Comparison between experimental data for UHMWPE and predictions from a Johnson-Cook plastic material.

7.5 Drucker Prager Plasticity

Drucker Prager plasticity is an isotropic hardening plasticity model specifically developed for frictional materials like granular solids. One interesting feature of this model is that it can have a yield stress that depends on the pressure, and hence can have different yield stresses in tension and compression. This is an attractive feature since many thermoplastics are known to have a pressure dependent yield stress. The model framework can also incorporate strain-rate dependence and progressive failure predictions.

One of the main limitations of the Drucker Prager model is that it is based on isotropic hardening, which means that it cannot accurately predict the unloading response of many thermoplastics. Figure 7.12 shows the predicted stress-strain response when

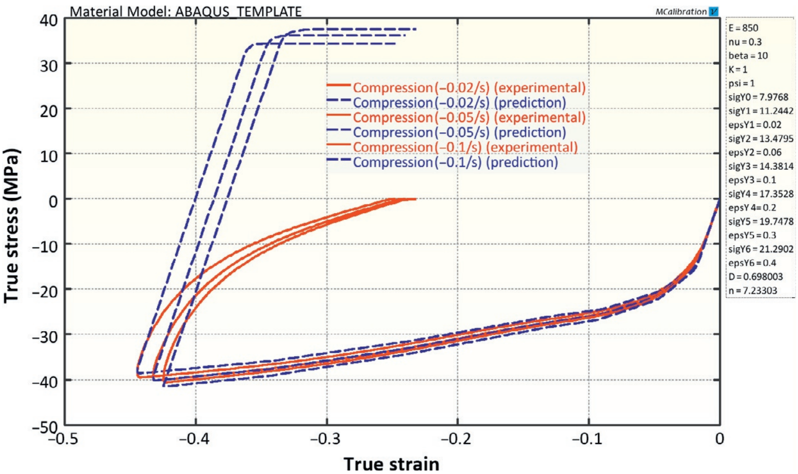

Figure 7.12 Comparison between experimental data for UHMWPE and predictions from a Drucker Prager plastic material.

calibrated to cyclic uniaxial compression data for UHMWPE. The figure shows that the predicted response under monotonic loading is in good agreement with the experimental behavior, but that the predicted unloading behavior significantly overpredicts the residual deformation after unloading. The calibrated material parameters that were used to generate the model predictions are listed in Abaqus inp-file format below.

```
*Material, name=mat
*Elastic
850, 0.3
*Drucker Prager
10, 1, 1
*Drucker Prager Hardening, type=tension
7.9768, 0.0
11.2442, 0.02
13.4795, 0.06
14.3814, 0.1
17.3528, 0.2
19.7478, 0.3
21.2902, 0.4
*Rate Dependent, type=power law
0.698003, 7.23303
```

7.6 Use of Plasticity Models in Polymer Modeling

Plasticity models have a long history of use in finite element simulations, and have also been extensively used to represent the behavior of thermoplastic materials. As was shown in this chapter, however, using a plasticity model to predict the non-linear viscoplastic response of polymers can give poor predictions of the material response, particularly during cyclic loading or during unloading.

There are many different plasticity models that have been developed and that are available in finite element software. In practice, the different models can be divided into two groups: *isotropic hardening* plasticity and *kinematic hardening* plasticity. The isotropic hardening plasticity models are often easier to use and can provide accurate predictions all the way to failure (under monotonic loading). The kinematic hardening models are often based on non-linear equations with material parameters that need to be determined from experimental tests, and hence are often more difficult to calibrate. The kinematic hardening models often have a restriction that the tangent modulus decreases with increasing plastic strain, a restriction that is contradicting experimental behaviors of many polymers.

Plasticity models can still be important for practical use in a finite element setting since they are numerically very efficient. But plasticity models are never the most accurate material modeling framework. The viscoplastic models presented in the next chapter will always be more accurate.

7.7 Exercises

1. Show how to specify an isotropic hardening plasticity model in your favorite finite element program.
2. Why is the isotropic hardening plasticity model so easy to calibrate?
3. What experimental tests would you run in order to calibrate an isotropic hardening plasticity model?
4. What is the main limitation of isotropic hardening plasticity?
5. Describe the difference between the Matlab and the Python implementations of the isotropic hardening plasticity model.

6. What are some of the limitations of the kinematic hardening plasticity model?
7. Create a Matlab or Python implementation of the combined kinematic plasticity model.
8. What experimental tests would you run in order to calibrate a kinematic hardening plasticity model?
9. What are some features of the Johnson-Cook plasticity model that make it interesting?
10. What are some differences between the Johnson-Cook and the Drucker Prager plasticity models?

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