

# 1 Manifolds

## 1.1 Manifolds on Euclidean Spaces

### 1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function  $f$  on an open ball  $U \in \mathcal{O}_n$  can be written as

$$f(x) = f(p) + \sum (x^i - p^i)g_i(x)$$

where  $p \in U$  and  $g_i \in C^\infty(U)$  with  $g_i(p) = (\partial f / \partial x^i)(p)$ .

Adapting this to  $g_i$  repeatedly gives the Taylor's expansion of  $f$ .

### 1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The *tangent space*  $T_p(\mathbb{R}^n)$  at  $p \in \mathbb{R}^n$  is the set of arrows from  $p$ .

### 1.1.3 Directional derivative [Definition 1.1.3]

The *directional derivative* of a smooth function  $f$  in the direction  $v \in T_p(\mathbb{R}^n)$  at  $p \in \mathbb{R}^n$  is

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with  $c^i(t) = p^i + tv^i$ .

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

### 1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map  $D: C_p^\infty \rightarrow \mathbb{R}$  satisfying the Leibniz rule (i.e.,  $D(fg) = (Df)g(p) + f(p)Dg$  for any  $f, g \in C_p^\infty$ ) is called a *derivation at  $p$*  or a *point-derivation* of  $C_p^\infty$ .

The set of all derivations at  $p$   $\mathcal{D}_p(\mathbb{R}^n)$  is a real vector space, and a map  $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$  assigning  $D_v$  to each  $v$  is a linear map.

### 1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If  $D$  is a point-derivation of  $C_p^\infty$ , then  $D(c) = 0$  for any constant function  $c$ .

### 1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map  $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$  in 1.1.4 is an isomorphism of vector spaces.

### 1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6,  $v \in T_p(\mathbb{R}^n)$  is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

### 1.1.8 Vector fields on an open set [Definition 1.1.8]

A *vector field* on  $U \in \mathcal{O}_n$  is a map  $X: U \rightarrow T_p(\mathbb{R}^n)$ .  $X = \sum a^i \partial / \partial x^i$  means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

$X$  is said to be  $C^\infty$  if all  $a^i$ 's are  $C^\infty$  on  $U$ . The set of all smooth vector fields on  $U$  is denoted by  $\mathfrak{X}(U)$ .

### 1.1.9 Multiplication of a smooth vector field and function [Definition & Proposition ??]

For  $X \in \mathfrak{X}(U)$  and  $f \in C^\infty(U)$ , define  $fX \in \mathfrak{X}(U)$  and  $Xf \in C^\infty(U)$  as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left. \frac{\partial}{\partial x^i} \right|_p,$$

$$(Xf)(p) = X_p f = \sum a^i(p) \left. \frac{\partial f}{\partial x^i} \right|_p.$$

### 1.1.10 Leibniz rule for a vector field [Proposition 1.1.10]

For any  $X \in \mathfrak{X}(U)$ ,  $f, g \in C^\infty(U)$ ,

$$X(fg) = (Xf)g + fXg.$$

### 1.1.11 Derivations one-to-one-correspond to smooth vector fields [Proposition 1.1.11]

$\varphi: \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \text{Der}(C^\infty(U))$  is an linear isomorphism.

### 1.1.12 k-tensor on a vector space [Definition 1.1.12]

A  $k$ -linear function on a vector space  $V$   $f: V^k \rightarrow \mathbb{R}$  is called a *k-tensor* on  $V$ . The vector space of all  $k$ -tensors on  $V$  is denoted by  $L_k(V)$ .  $k$  is called the degree of  $f$ .

### 1.1.13 Permutation action on k-tensors [Definition 1.1.13]

For  $f \in L_k(V)$  on a vector space  $V$  and  $\sigma \in \mathfrak{S}_n$ , an action of  $\sigma$  on  $f$  is defined by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

### 1.1.14 Symmetric and alternating k-tensor [Definition 1.1.14]

A  $k$ -tensor  $f: V^k \rightarrow \mathbb{R}$  is *symmetric* if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = f,$$

and  $f$  is *alternating* if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = (\text{sgn } \sigma) f.$$

### 1.1.15 The set of all alternating k-tensors [Definition 1.1.15]

An alternating  $k$ -tensor on a vector space  $V$  is also called a *k-covector* or a *multicovector of degree k* on  $V$ . The set of all  $k$ -covectors on  $V$  is denoted by  $A_k(V)$  for  $k > 0$ ; for  $k = 0$ ,  $A_0(V) = \mathbb{R}$ .

### 1.1.16 Symmetrizing and alternating operators on k-covectors [Definition & Proposition 1.1.16]

For a  $f \in A_k(V)$  on a vector space  $V$ ,

$$Sf = \sum_{\sigma \in \mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{sgn} \sigma) \sigma f$$

is alternating.

### 1.1.17 Tensor product of two multilinear functions [Definition 1.1.17]

For  $f \in L_k(V), g \in L_l(V)$  on a vector space  $V$ , the *tensor product*  $f \otimes g \in L_{k+l}(V)$  is defined by

$$(f \otimes g)(v_1, \dots, v_{k+l}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+l}).$$

### 1.1.18 Bilinear map as a tensor product [Example 1.1.18]

Let  $e_1, \dots, e_n$  be a basis for a vector space  $V$ ,  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ , and  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$  a bilinear map on  $V$ . Then,

$$\langle \cdot, \cdot \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j,$$

where  $g_{ij} = \langle e_i, e_j \rangle$ .

### 1.1.19 Wedge product of two multilinear functions [Definition 1.1.19]

For  $f \in A_k(V), g \in A_l(V)$  on a vector space  $V$ , their *wedge product* or *exterior product* is

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g).$$

$f \wedge g$  is alternating.

Explicitly,

$$\begin{aligned} (f \wedge g)(v_1, \dots, v_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\ &= \sum_{(k,l)\text{-shuffle } \sigma} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}), \end{aligned}$$

where a  $(k, l)$ -shuffle means  $\sigma(1) < \dots < \sigma(k)$  and  $\sigma(k+1) < \dots < \sigma(k+l)$ .

### 1.1.20 Wedge product is anticommutative [Proposition 1.1.20]

For  $f \in A_k(V), g \in A_l(V)$  on a vector space  $V$ ,

$$f \wedge g = (-1)^{kl} g \wedge f.$$

If the degree of  $f$  is odd, then  $f \wedge f = 0$ .

### 1.1.21 Properties of nesting alternating operators [Lemma 1.1.21]

For a  $k$ -tensor  $f$  and  $l$ -tensor  $g$  on a vector space  $V$ ,

- i)  $A(A(f) \otimes g) = k! A(f \otimes g)$ ,
- ii)  $A(f \otimes A(g)) = l! A(f \otimes g)$ .

### 1.1.22 Associativity of the wedge product [Proposition 1.1.22]

For  $f \in A_k(V), g \in A_l(V), h \in A_m(V)$  on a real vector space  $V$ ,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for  $f_i \in A_{d_i}(V)$  ( $i = 1, \dots, r$ ),

$$f_1 \wedge \dots \wedge f_r = \frac{1}{(d_1)! \dots (d_r)!} A(f_1 \otimes \dots \otimes f_r).$$

### 1.1.23 Wedge product of covectors is the determinant [Proposition 1.1.23]

For covectors  $\alpha^1, \dots, \alpha^k$  on a vector space  $V$ ,

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

### 1.1.24 Graded algebra over a field [Definition 1.1.24]

An algebra  $\mathbb{A}$  over a field  $\mathbb{K}$  is said to be *graded* if  $\mathbb{A} = \bigoplus_{k=0}^{\infty} A^k$  is a direct sum of vector spaces over  $\mathbb{K}$  such that the multiplication sends  $A^k \times A^l$  to  $A^{k+l}$ .  $A = \bigoplus_{k=0}^{\infty} A^k$  means each nonzero  $a \in \mathbb{A}$  is uniquely a finite sum  $a = a_{i_1} + \dots + a_{i_m}$  where nonzero  $a_{i_j} \in A^{i_j}$ .

$\mathbb{A}$  is *anticommutative* or *graded commutative* if  $\forall a \in A^k, b \in A^l, ab = (-1)^{kl}ba$ .

A *homomorphism* of graded algebras is an algebra homomorphism that preserves the degree.

### 1.1.25 Grassmann algebra of multivectors on a vector space [Definition & Proposition 1.1.25]

For a vector space  $V$  of degree  $n < \infty$ , the *exterior algebra* or the *Grassmann algebra* of multivectors on  $V$  is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^n A_k(V)$$

with the wedge product of multivectors as multiplication.

### 1.1.26 Wedge product of the dual basis applying to a basis [Lemma 1.1.26]

Let  $e_1, \dots, e_n$  be a basis for a vector space  $V$  and  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ . For  $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k)$  with  $1 \leq i_1 < \dots < i_k \leq n, 1 \leq j_1 < \dots < j_k \leq n$ ,

$$\alpha^I(e_J) = \delta_J^I.$$

1.1.27 Wedge products of the dual basis form a basis for multicovectors [Proposition 1.1.27]

Let  $V$  be a vector space and  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ . Then,  $\alpha^I$ ,  $I = (i_1 < \dots < i_k)$  form a basis for  $A_k(V)$ .

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

1.1.28 Cotangent space to an Euclidean space at a point [Definition 1.1.28]

The *cotangent space* to  $\mathbb{R}^n$  at  $p$  is  $T_p^*(\mathbb{R}^n) = (T_p(\mathbb{R}^n))^*$ .

1.1.29 Differential 1-form on an open subset of an Euclidean space [Definition 1.1.29]

A *covector field* or a *differential 1-form* on  $U \in \mathcal{O}_n$  is  $\omega: U \rightarrow \bigcup_{p \in U} T_p^*(\mathbb{R}^n)$  that maps  $U \ni p \mapsto \omega_p \in T_p^*(\mathbb{R}^n)$ .

1.1.30 Differential of a smooth function [Definition 1.1.30]

For  $f \in C^\infty(U)$  on  $U \in \mathcal{O}_n$ , the *differential*  $df$  of  $f$  is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$

In the expression

$$T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \ni (X_p, f) \mapsto \langle X_p, f \rangle = X_p f \in \mathbb{R},$$

a tangent vector is considered as  $\langle X_p, \cdot \rangle$ ; a differential at  $p$  is considered as  $df|_p = (df)_p = \langle \cdot, f \rangle$ .

## 2 P-adic Numbers

### 2.1 Foundations

#### 2.1.1 Absolute value on a field [Definition 2.1.1]

An *absolute value* on a field  $\mathbb{K}$  is a function  $|\cdot|: \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- i)  $|x| = 0$  iff  $x = 0$
- ii)  $\forall x, y \in \mathbb{K}, |xy| = |x| |y|$
- iii)  $\forall x, y \in \mathbb{K}, |x + y| \leq |x| + |y|$ .

An absolute value that satisfies the condition

- iv)  $\forall x, y \in \mathbb{K}, |x + y| \leq \max\{|x|, |y|\}$

is said to be *non-archimedean*; otherwise, it is said to be *archimedean*.

#### 2.1.2 Trivial absolute value [Definition 2.1.2]

The *trivial absolute value* on a field  $\mathbb{K}$  is a absolute value on  $\mathbb{K}$  such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

An absolute value on a finite field must be trivial.

#### 2.1.3 Valuation on a field [Definition 2.1.3]

A function  $v: \mathbb{A}^\times \rightarrow \mathbb{R}$  with an integral domain  $\mathbb{A}$  is called a *valuation* on  $\mathbb{A}$  if it satisfies the following conditions:

- i)  $\forall x, y \in \mathbb{A}^\times, v(xy) = v(x) + v(y)$
- ii)  $\forall x, y \in \mathbb{A}^\times, v(x + y) \geq \min\{v(x), v(y)\}$

#### 2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation  $v$  on a field is an additive subgroup of  $\mathbb{R}$ .  $\text{im } v$  is called the *value group* of  $v$ .

#### 2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let  $\mathbb{A}$  be an integral domain and  $\mathbb{K} = \text{Frac } \mathbb{A}$ . Let  $v: \mathbb{A}^\times \rightarrow \mathbb{R}$  be a valuation on  $\mathbb{A}$  and extend  $v$  to  $\mathbb{K}$  by setting  $v(a/b) = v(a) - v(b)$ , then the function  $|\cdot|_v: \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$  defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on  $\mathbb{K}$ . Conversely,  $-\log |\cdot|_v$  is a valuation on  $\mathbb{K}$  for a non-archimedean absolute value  $|\cdot|_v$  on  $\mathbb{K}$ .

### 2.1.6 $p$ -adic valuation [Definition 2.1.6]

The  $p$ -adic valuation on  $\mathbb{Q}$  with a prime  $p$  is a valuation  $v_p: \mathbb{Q}^\times \rightarrow \mathbb{R}$  defined as follows: for each  $n \in \mathbb{Z}^\times$ , let  $v_p(n)$  be the greatest integer such that  $p^{v_p(n)} \mid n$ , and for each  $x = a/b \in \mathbb{Q}^\times$ ,  $v_p(x) = v_p(a) - v_p(b)$ .

We often set  $v_p(0) = \infty$ .

### 2.1.7 $p$ -adic absolute value [Definition 2.1.7]

The  $p$ -adic absolute value  $|\cdot|_p: \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}$  with a prime  $p$  is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as  $|\cdot| = |\cdot|_\infty$ .

## 3 Lie Algebra

### 3.1 Foundations

#### 3.1.1 Lie algebra [Definition 3.1.1]

A vector space  $\mathfrak{g}$  over a field  $\mathbb{K}$  with the Lie bracket satisfying the conditions

- i) lie bracket is bilinear
- ii)  $\forall x \in \mathfrak{g}, [x, x] = 0$
- iii)  $\forall x, y, z \in \mathfrak{g}, [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a *Lie algebra* over  $\mathbb{K}$ .

#### 3.1.2 General linear Lie algebra [Definition 3.1.2]

$\mathfrak{gl}_n(\mathbb{R})$  is the Lie algebra  $M_n(\mathbb{R})$  with the Lie bracket  $[x, y] = xy - yx$ .

#### 3.1.3 Derivation algebra [Definition 3.1.3]

A linear endomorphism  $D$  of an algebra  $\mathbb{A}$  over  $\mathbb{R}$  satisfying  $D(xy) = D(x)y + xD(y)$  is called a *derivation* of  $\mathbb{A}$ . The set of all derivations  $\text{Der } \mathbb{A}$  with the addition, scalar multiplication, and lie bracket defined as follows:

- i)  $(D + D')(x) = D(x) + D'(x)$
- ii)  $(\alpha D)(x) = \alpha D(x)$
- iii)  $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the *derivation algebra* of  $\mathbb{A}$ .

#### 3.1.4 Lie subalgebra [Definition 3.1.4]

A linear subspace  $\mathfrak{h} \subset \mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$  is a *Lie subalgebra* if it satisfies that  $\forall x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$ .

#### 3.1.5 Special linear Lie algebra [Definition & Proposition 3.1.5]

$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{tr } x = 0\}$  is a Lie subalgebra of  $\mathfrak{gl}_n(\mathbb{R})$ .

#### 3.1.6 Orthogonal Lie algebra [Definition & Proposition 3.1.6]

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^t x = -x\}$  is a Lie subalgebra of  $\mathfrak{sl}_n(\mathbb{R})$ .



## 4 Categories

### 4.1.1 Foundations

### 4.1.2 Category [Definition 4.1.2]

A *category* consists of the followings:

- *Objects*  $A, B, C, \dots$
- *Arrows*  $f, g, h, \dots$  with the objects called the domain  $\text{dom}(f)$  and the codomain  $\text{cod}(f)$ .
- *Composites*  $g \circ f: A \rightarrow C$  for given arrows  $f: A \rightarrow B$  and  $g: B \rightarrow C$ .
- *Identity arrow*  $1_A$  of each object  $A$ .

satisfying the following laws:

- i)  $\forall \text{arrows } f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D, h \circ (g \circ f) = (h \circ g) \circ f$
- ii)  $\forall \text{arrow } f: A \rightarrow B, f \circ 1_A = f = 1_B \circ f$ .

### 4.1.3 Functor between categories [Definition 4.1.3]

A *functor*  $F: \mathcal{A} \rightarrow \mathcal{B}$  between categories  $\mathcal{A}$  and  $\mathcal{B}$  is a mapping between objects and between arrows in the following ways:

- i)  $F(f: A \rightarrow B) = F(f): F(A) \rightarrow F(B)$ ,
- ii)  $F(1_A) = 1_{F(A)}$ ,
- iii)  $F(g \circ f) = F(g) \circ F(f)$ .

## 参考文献

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