

Notes of Mathematics

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1 Manifolds^[4]

1.1 Manifolds on Euclidean Spaces

1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i)g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f .

1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The **tangent space** $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p .

1.1.3 Directional derivative [Definition 1.1.3]

The **directional derivative** of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + tv^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map $D: C_p^\infty \rightarrow \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a **derivation** at p or a **point-derivation** of C_p^∞ .

The set of all derivations at p denoted by $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If D is a point-derivation of C_p^∞ , then $D(c) = 0$ for any constant function c .

1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ in 1.1.4 is an isomorphism of vector spaces.

1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6, $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

1.1.8 Vector fields on an open set [Definition 1.1.8]

A **vector field** on $U \in \mathcal{O}_n$ is a map $X: U \rightarrow T_p(\mathbb{R}^n)$. $X = \sum a^i \partial / \partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

X is said to be C^∞ if all a^i 's are C^∞ on U . The set of all smooth vector fields on U is denoted by $\mathfrak{X}(U)$.

1.1.9 Multiplication of a smooth vector field and function [Definition & Proposition 1.1.9]

For $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, define $fX \in \mathfrak{X}(U)$ and $Xf \in C^\infty(U)$ as follows:

$$\begin{aligned} (fX)_p &= f(p)X_p = \sum (f(p)a^i(p)) \left. \frac{\partial}{\partial x^i} \right|_p, \\ (Xf)(p) &= X_p f = \sum a^i(p) \left. \frac{\partial f}{\partial x^i} \right|_p. \end{aligned}$$

1.1.10 Leibniz rule for a vector field [Proposition 1.1.10]

For any $X \in \mathfrak{X}(U)$, $f, g \in C^\infty(U)$,

$$X(fg) = (Xf)g + fXg.$$

1.1.11 Derivations one-to-one-correspond to smooth vector fields [Proposition 1.1.11]

$\varphi: \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \text{Der}(C^\infty(U))$ is a linear isomorphism.

1.1.12 k-tensor on a vector space [Definition 1.1.12]

A k -linear function on a vector space $V: V^k \rightarrow \mathbb{R}$ is called a **k -tensor** on V . The vector space of all k -tensors on V is denoted by $L_k(V)$. k is called the degree of f .

1.1.13 Permutation action on k-tensors [Definition 1.1.13]

For $f \in L_k(V)$ on a vector space V and $\sigma \in \mathfrak{S}_n$, an action of σ on f is defined by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

1.1.14 Symmetric and alternating k-tensor [Definition 1.1.14]

A k -tensor $f: V^k \rightarrow \mathbb{R}$ is **symmetric** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = f,$$

and f is **alternating** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = (\text{sgn } \sigma) f.$$

1.1.15 The set of all alternating k-tensors [Definition 1.1.15]

An alternating k -tensor on a vector space V is also called a **k -covector** or a **multicovector of degree k** on V . The set of all k -covectors on V is denoted by $A_k(V)$ for $k > 0$; for $k = 0$, $A_0(V) = \mathbb{R}$.

1.1.16 Symmetrizing and alternating operators on k-covectors [Definition & Proposition 1.1.16]

For a $f \in A_k(V)$ on a vector space V ,

$$Sf = \sum_{\sigma \in \mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma \in \mathfrak{S}_n} (\text{sgn } \sigma) \sigma f$$

is alternating.

1.1.17 Tensor product of two multilinear functions [Definition 1.1.17]

For $f \in L_k(V), g \in L_\ell(V)$ on a vector space V , the **tensor product** $f \otimes g \in L_{k+\ell}(V)$ is defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}).$$

1.1.18 Bilinear map as a tensor product [Example 1.1.18]

Let e_1, \dots, e_n be a basis for a vector space V , $\alpha^1, \dots, \alpha^n$ the dual basis in V^* , and $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ a bilinear map on V . Then,

$$\langle \cdot, \cdot \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j,$$

where $g_{ij} = \langle e_i, e_j \rangle$.

1.1.19 Wedge product of two multilinear functions [Definition 1.1.19]

For $f \in A_k(V)$, $g \in A_\ell(V)$ on a vector space V , their **wedge product** or **exterior product** is

$$f \wedge g = \frac{1}{k! \ell!} A(f \otimes g).$$

$f \wedge g$ is alternating.

Explicitly,

$$\begin{aligned} (f \wedge g)(v_1, \dots, v_{k+\ell}) &= \frac{1}{k! \ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} (\text{sgn } \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\ &= \sum_{\substack{(k, \ell)\text{-shuffle} \\ \sigma}} (\text{sgn } \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}), \end{aligned}$$

where a (k, ℓ) -shuffle means $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+\ell)$.

1.1.20 Wedge product is anticommutative [Proposition 1.1.20]

For $f \in A_k(V)$, $g \in A_\ell(V)$ on a vector space V ,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

If the degree of f is odd, then $f \wedge f = 0$.

1.1.21 Properties of nesting alternating operators [Lemma 1.1.21]

For a k -tensor f and ℓ -tensor g on a vector space V ,

- i) $A(A(f) \otimes g) = k! A(f \otimes g)$,
- ii) $A(f \otimes A(g)) = \ell! A(f \otimes g)$.

1.1.22 Associativity of the wedge product [Proposition 1.1.22]

For $f \in A_k(V)$, $g \in A_\ell(V)$, $h \in A_m(V)$ on a real vector space V ,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for $f_i \in A_{d_i}(V)$ ($i = 1, \dots, r$),

$$f_1 \wedge \dots \wedge f_r = \frac{1}{(d_1)! \dots (d_r)!} A(f_1 \otimes \dots \otimes f_r).$$

1.1.23 Wedge product of covectors is the determinant [Proposition 1.1.23]

For covectors $\alpha^1, \dots, \alpha^k$ on a vector space V ,

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

1.1.24 Graded algebra over a field [Definition 1.1.24]

An algebra \mathbb{A} over a field \mathbb{K} is said to be **graded** if $\mathbb{A} = \bigoplus_{k=0}^{\infty} A^k$ is a direct sum of vector spaces over \mathbb{K} such that the multiplication sends $A^k \times A^l$ to A^{k+l} . $A = \bigoplus_{k=0}^{\infty} A^k$ means each nonzero $a \in \mathbb{A}$ is uniquely a finite sum $a = a_{i_1} + \dots + a_{i_m}$ where nonzero $a_{i_j} \in A^{i_j}$.

\mathbb{A} is **anticommutative** or **graded commutative** if $\forall a \in A^k, b \in A^\ell, ab = (-1)^{k\ell} ba$.

A **homomorphism** of graded algebras is an algebra homomorphism that preserves the degree.

1.1.25 Grassmann algebra of multivectors on a vector space [Definition & Proposition 1.1.25]

For a vector space V of degree $n < \infty$, the **exterior algebra** or the **Grassmann algebra** of multivectors on V is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^n A_k(V)$$

with the wedge product of multivectors as multiplication.

1.1.26 Wedge product of the dual basis applying to a basis [Lemma 1.1.26]

Let e_1, \dots, e_n be a basis for a vector space V and $\alpha^1, \dots, \alpha^n$ the dual basis in V^* . For $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k)$ with $1 \leq i_1 < \dots < i_k \leq n, 1 \leq j_1 < \dots < j_k \leq n$,

$$\alpha^I(e_J) = \delta_J^I.$$

1.1.27 Wedge products of the dual basis form a basis for multivectors [Proposition 1.1.27]

Let V be a vector space and $\alpha^1, \dots, \alpha^n$ the dual basis in V^* . Then, $\alpha^I, I = (i_1 < \dots < i_k)$ form a basis for $A_k(V)$.

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

1.1.28 Cotangent space to an Euclidean space at a point [Definition 1.1.28]

The **cotangent space** to \mathbb{R}^n at p is $T_p^*(\mathbb{R}^n) = (T_p(\mathbb{R}^n))^*$.

1.1.29 Differential 1-form on an open subset of an Euclidean space [Definition 1.1.29]

A **covector field** or a **differential 1-form** on $U \in \mathcal{O}_n$ is $\omega: U \rightarrow \bigcup_{p \in U} T_p^*(\mathbb{R}^n)$ that maps $U \ni p \mapsto \omega_p \in T_p^*(\mathbb{R}^n)$.

1.1.30 Differential of a smooth function [Definition 1.1.30]

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$, the **differential** df of f is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$

In the expression

$$\langle \cdot, \cdot \rangle: T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \ni (X_p, f) \mapsto \langle X_p, f \rangle = X_p f \in \mathbb{R},$$

a tangent vector is considered as $\langle X_p, \cdot \rangle$; a differential at p as $df|_p = (df)_p = \langle \cdot, f \rangle$.

1.1.31 Differentials of coordinates is the dual basis for the cotangent space [Proposition 1.1.31]

For $p \in \mathbb{R}^n$, $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the dual basis for $T_p^*(\mathbb{R}^n)$ to $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\} \subset T_p(\mathbb{R}^n)$, where x^1, \dots, x^n are the standard coordinates on \mathbb{R}^n .

For any differential 1-form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some $a_i(p)$. In this case, ω is written as $\omega = \sum a_i dx^i$.

1.1.32 Smoothness of a differential 1-form [Definition 1.1.32]

A differential 1-form $\omega = \sum a_i dx^i$ on $U \in \mathcal{O}_n$ is **smooth** if all $a_i: U \rightarrow \mathbb{R}$ are smooth.

1.1.33 Differentials can be written in terms of partial derivatives [Proposition 1.1.33]

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness of f implies that of df .

1.1.34 Differential k -forms on an Euclidean space [Definition 1.1.34]

A **differential k -form** or **differential form of degree k** on $U \in \mathcal{O}_n$ is $\omega: U \ni p \mapsto \omega_p \in A_k(T_p(\mathbb{R}^n))$.

1.1.35 Basis for differential forms [Definition & Proposition 1.1.35]

Since $\{dx_p^I \mid I = (1 \leq i_1 < \dots < i_k \leq n)\}$ is a basis for $A_k(T_p(\mathbb{R}^n))$, for a differential k -form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_I(p) dx_p^I, \quad \omega = \sum a_I dx^I.$$

ω is **smooth** if all $a_I: U \rightarrow \mathbb{R}$ are smooth. The vector space of C^∞ differential k -forms on U is denoted by $\Omega^k(U)$. If $k = 0$, $\Omega^0(U) = C^\infty(U)$.

1.1.36 Wedge product of differential forms [Definition 1.1.36]

For differential k -form ω and ℓ -form τ on $U \in \mathcal{O}_n$, their **wedge product** $\omega \wedge \tau$ is a differential $(k + \ell)$ -form defined by

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

If $\omega = \sum a_I dx^I$, $\tau = \sum b_J dx^J$,

$$\begin{aligned} \omega \wedge \tau &= \sum_{I,J} (a_I b_J) dx^I \wedge dx^J \\ &= \sum_{\text{disjoint } I,J} (a_I b_J) dx^I \wedge dx^J. \end{aligned}$$

For $\omega \in \Omega^k(U)$, $\tau \in \Omega^\ell(U)$, the wedge product is a bilinear map

$$\wedge: \Omega^k(U) \times \Omega^\ell(U) \rightarrow \Omega^{k+\ell}(U).$$

In particular, if $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$, then $f \wedge \omega = f\omega$.

1.1.37 Graded algebra with smooth differential forms [Definition 1.1.37]

For $U \in \mathcal{O}_n$, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is an anticommutative graded algebra over \mathbb{R} with the wedge product as multiplication, which is also a module over $C^\infty(U)$.

1.1.38 Differential forms as linear maps on a vector field [Definition 1.1.38]

For a differential k -form ω on $U \in \mathcal{O}_n$ and $X_1, \dots, X_k \in \mathfrak{X}(U)$, define $\omega(X_1, \dots, X_k) \in C^\infty(U)$ by

$$(\omega(X_1, \dots, X_k))_p = \omega_p((X_1)_p, \dots, (X_k)_p).$$

The map

$$\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) \ni (X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k) \in C^\infty(U)$$

is k -linear over $C^\infty(U)$.

1.1.39 Exterior derivatives of differential forms [Definition 1.1.39]

For $k \geq 1$ and $\omega = \sum a_I dx^I \in \Omega^k(U)$, the **exterior derivative** of ω is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U);$$

for $k = 0$ and $f \in C^\infty(U)$, its exterior derivative is

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

1.1.40 Antiderivation of a graded algebra [Definition 1.1.40]

An **antiderivation** of a graded algebra $\mathbb{A} = \bigoplus_{k=0}^\infty A^k$ is a linear map $D: \mathbb{A} \rightarrow \mathbb{A}$ such that for $a \in A^k, b \in A^\ell$,

$$D(ab) = D(a)b + (-1)^k aD(b).$$

If m is an integer such that D sends A^k to A^{k+m} for all k , then m is called the **degree** of D .

1.1.41 Properties of the exterior differentiation [Proposition 1.1.41]

i) The exterior differentiation $d: \Omega^*(U) \rightarrow \Omega^*(U)$ on $U \in \mathcal{O}_n$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

ii) $d^2 = 0$.

iii) For $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(df)(X) = Xf$.

1.1.42 Characterization of the exterior differentiation [Proposition 1.1.42]

The exterior differentiation $d: \Omega^*(U) \rightarrow \Omega^*(U)$ on $U \in \mathcal{O}_n$ is the only antiderivation of $\Omega^*(U)$.

1.1.43 Closed and exact forms [Definition 1.1.43]

A differential k -form ω on $U \in \mathcal{O}_n$ is said to be **closed** if $d\omega = 0$, and said to be **exact** if $\omega = d\tau$ for some $(k-1)$ -form τ on U .

Every exact form is closed.

1.1.44 Cochain complex and de Rham complex [Definition 1.1.44]

A collection of vector spaces $\{V^k\}_{k=0}^\infty$ with linear maps $d_k: V^k \rightarrow V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a **cochain complex** or a **differential complex**.

The **de Rham complex** of $U \in \mathcal{O}_n$ is a cochain complex

$$0 \rightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

The closed forms are the elements of $\ker d$, and the exact forms are the elements of $\operatorname{im} d$.

1.1.45 Vector calculus as differential forms [Proposition 1.1.45]

Under the identifications, for $U \in \mathcal{O}_3$, $f \in C^\infty(U)$ and $X = [P \ Q \ R] \in \mathfrak{X}(U)$,

$$\text{1-form } Pdx + Qdy + Rdz \longleftrightarrow X,$$

$$\text{2-form } Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \longleftrightarrow X,$$

$$\text{3-form } f dx \wedge dy \wedge dz \longleftrightarrow f,$$

there are correspondences between the exterior derivatives and grad, rot, and div:

$$df \longleftrightarrow \operatorname{grad} f,$$

$$d(Pdx + Qdy + Rdz) \longleftrightarrow \operatorname{rot} X,$$

$$d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \longleftrightarrow \operatorname{div} X.$$

1.1.46 k-th de Rham cohomology [Definition 1.1.46]

For $U \in \mathcal{O}_n$, the **k-th de Rham cohomology** of U is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}.$$

1.1.47 Poincaré lemma [Proposition 1.1.47]

For $k \geq 1$, every closed k -form on \mathbb{R}^n is exact, i.e., $H^k(\mathbb{R}^n)$ vanishes.

1.2 Manifolds

1.2.1 Locally Euclidean space [Definition 1.2.1]

A topological space M is **locally Euclidean of dimension n** if $\forall p \in M, \exists (U, \phi)$, where U is a neighborhood of p and a homeomorphism $\phi: U \rightarrow V \in \mathcal{O}_n$, called a **chart**, a **coordinate neighborhood** or a **coordinate open set**, and ϕ a **coordinate map** or a **coordinate system** on U .

A chart (U, ϕ) is said to be **centered** at $p \in U$ if $\phi(p) = 0$.

1.2.2 Topological manifold [Definition 1.2.2]

A **topological manifold of dimension n** is a Hausdorff, second countable, locally Euclidean space of dimension n .

1.2.3 Compatible chart [Definition 1.2.3]

Two charts $(U, \phi: U \rightarrow \mathbb{R}^n)$, $(V, \psi: V \rightarrow \mathbb{R}^n)$ of a topological manifold are said to be C^∞ -**compatible** or simply **compatible** if

$$\phi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \phi(U \cap V), \quad \psi \circ \phi^{-1}: \phi(U \cap V) \rightarrow \psi(U \cap V)$$

called the **transition functions** between charts are C^∞ . If $U \cap V = \emptyset$, they are C^∞ -compatible.

1.2.4 Atlas on a locally Euclidean space [Definition 1.2.4]

A C^∞ **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise compatible charts that cover M .

1.2.5 Compatibility of a chart with an atlas [Definition 1.2.5]

For a locally Euclidean space, a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if all charts (U_α, ϕ_α) are compatible with (V, ψ) .

1.2.6 Charts compatible with the same atlas are compatible with each other [Lemma 1.2.6]

For a locally Euclidean space, charts (V, ψ) , (W, σ) , and an atlas $\{(U_\alpha, \phi_\alpha)\}$ on it, if (V, ψ) and (W, σ) are both compatible with $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

2 P-adic Numbers^[2]

2.1 Foundations

2.1.1 Absolute value on a field [Definition 2.1.1]

An **absolute value** on a field \mathbb{K} is a function $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- i) $|x| = 0$ iff $x = 0$
- ii) $\forall x, y \in \mathbb{K}, |xy| = |x| |y|$
- iii) $\forall x, y \in \mathbb{K}, |x + y| \leq |x| + |y|$.

An absolute value that satisfies the condition

- iv) $\forall x, y \in \mathbb{K}, |x + y| \leq \max\{|x|, |y|\}$

is said to be **non-archimedean**; otherwise, it is said to be **archimedean**.

2.1.2 Trivial absolute value [Definition 2.1.2]

The **trivial absolute value** on a field \mathbb{K} is a absolute value on \mathbb{K} such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

An absolute value on a finite field must be trivial.

2.1.3 Valuation on a field [Definition 2.1.3]

A function $v : \mathbb{A}^\times \rightarrow \mathbb{R}$ with an integral domain \mathbb{A} is called a **valuation** on \mathbb{A} if it satisfies the following conditions:

- i) $\forall x, y \in \mathbb{A}^\times, v(xy) = v(x) + v(y)$
- ii) $\forall x, y \in \mathbb{A}^\times, v(x + y) \geq \min\{v(x), v(y)\}$

2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation v on a field is an additive subgroup of \mathbb{R} . $\text{im } v$ is called the **value group** of v .

2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let \mathbb{A} be an integral domain and $\mathbb{K} = \text{Frac } \mathbb{A}$. Let $v: \mathbb{A}^\times \rightarrow \mathbb{R}$ be a valuation on \mathbb{A} and extend v to \mathbb{K} by setting $v(a/b) = v(a) - v(b)$, then the function $|\cdot|_v: \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on \mathbb{K} . Conversely, $-\log |\cdot|$ is a valuation on \mathbb{K} for a non-archimedean absolute value $|\cdot|$ on \mathbb{K} .

2.1.6 p -adic valuation [Definition 2.1.6]

The **p -adic valuation** on \mathbb{Q} with a prime p is a valuation $v_p: \mathbb{Q}^\times \rightarrow \mathbb{R}$ defined as follows: for each $n \in \mathbb{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \mathbb{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

2.1.7 p -adic absolute value [Definition 2.1.7]

The **p -adic absolute value** $|\cdot|_p: \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\cdot| = |\cdot|_\infty$.

2.1.8 Absolute values on a field of rational functions [Definition 2.1.8]

Here are some absolute values on a field $\mathbb{F}(t)$ of rational functions over a field \mathbb{F} .

- i) For $f(t) \in \mathbb{F}[t]$, $v_\infty(f) = -\deg f$, and for $f(t)/g(t) \in \mathbb{F}(t)$, $v_\infty(f/g) = v_\infty(f) - v_\infty(g)$ with $v_\infty(0) = \infty$. Then,

$$|f(t)|_\infty = e^{-v_\infty(f)}.$$

- ii) For an irreducible polynomial $p(t) \in \mathbb{F}[t]$, define the $p(t)$ -adic valuation and absolute value.

2.1.9 Properties of absolute values on fields [Lemma 2.1.9]

For an absolute value $|\cdot|$ on a field \mathbb{K} ,

- i) $|1| = 1$,
- ii) $\forall x \in \mathbb{K}, |x^n| = 1 \Rightarrow |x| = 1$,
- iii) $\forall x \in \mathbb{K}, |-x| = |x|$,
- iv) If \mathbb{K} is finite, then $|\cdot|$ is trivial.

2.1.10 Necessary and sufficient conditions of a non-archimedean absolute value [Theorem 2.1.10]

Let \mathbb{K} be a field, $|\cdot|$ an absolute value on \mathbb{K} . Then,

$$\begin{aligned} |\cdot| \text{ is non-archimedean} &\iff \forall n = 1 + \cdots + 1 \in \mathbb{K}, |n| \leq 1 \\ &\iff \sup\{|n| \mid n \in \mathbb{Z}\} = 1. \end{aligned}$$

Furthermore, $\sup\{|n| \mid n \in \mathbb{Z}\} = \infty$ if $|\cdot|$ is archimedean.

3 Lie Algebra^[3]

3.1 Foundations

3.1.1 Lie algebra [Definition 3.1.1]

A vector space \mathfrak{g} over a field \mathbb{K} with the Lie bracket satisfying the conditions

- i) Lie bracket is bilinear
- ii) $\forall x \in \mathfrak{g}, [x, x] = 0$
- iii) $\forall x, y, z \in \mathfrak{g}, [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a **Lie algebra** over \mathbb{K} .

3.1.2 General linear Lie algebra [Definition 3.1.2]

$\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra $M_n(\mathbb{R})$ with the Lie bracket $[x, y] = xy - yx$.

3.1.3 Derivation algebra [Definition 3.1.3]

A linear endomorphism D of an algebra \mathbb{A} over \mathbb{R} satisfying $D(xy) = D(x)y + xD(y)$ is called a **derivation** of \mathbb{A} . The set of all derivations $\text{Der } \mathbb{A}$ with the addition, scalar multiplication, and lie bracket defined as follows:

- i) $(D + D')(x) = D(x) + D'(x)$
- ii) $(\alpha D)(x) = \alpha D(x)$
- iii) $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the **derivation algebra** of \mathbb{A} .

3.1.4 Lie subalgebra [Definition 3.1.4]

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra \mathfrak{g} is a **Lie subalgebra** of \mathfrak{g} if $\forall x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$.

For linear subspaces $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ denotes the subspace generated by $[x, y]$ with $x \in \mathfrak{a}, y \in \mathfrak{b}$.

3.1.5 Special linear Lie algebra [Definition & Proposition 3.1.5]

$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{tr } x = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

3.1.6 Orthogonal Lie algebra [Definition & Proposition 3.1.6]

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^t x = -x\}$ is a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{R})$.

3.1.7 Ideal of a Lie algebra [Definition & Proposition 3.1.7]

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra \mathfrak{g} is an **ideal** of \mathfrak{g} if $\forall x \in \mathfrak{g}, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$.

For ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ is also an ideal.

3.1.8 Derived ideal of a Lie algebra [Definition 3.1.8]

For a Lie algebra \mathfrak{g} , $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of \mathfrak{g} called the **derived ideal** of \mathfrak{g} .

If $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$, $D\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$.

3.1.9 Homomorphism of Lie algebras [Definition & Proposition 3.1.9]

For Lie algebras $\mathfrak{g}, \mathfrak{h}$, a linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ is called a **homomorphism** if $\forall x, y \in \mathfrak{g}, \varphi([x, y]) = [\varphi(x), \varphi(y)]$. A homomorphism φ is an **isomorphism** if it is bijective. Lie algebras between which there exists an isomorphism are said to be **isomorphic** to each other, written as $\mathfrak{g} \cong \mathfrak{h}$.

A composite of homomorphisms is also a homomorphism, and that of isomorphisms is also an isomorphism.

The kernel $\ker \varphi = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$ of a homomorphism φ is an ideal of \mathfrak{g} while the image $\text{im } \varphi = \varphi(\mathfrak{g})$ of φ is a Lie subalgebra of \mathfrak{h} .

3.1.10 Representation of a Lie algebra on a vector space [Definition 3.1.10]

For a Lie algebra \mathfrak{g} and a vector space V , a homomorphism $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is called a **representation** of \mathfrak{g} on V .

3.1.11 Adjoint representation of a Lie algebra [Definition & Proposition 3.1.11]

For a Lie algebra \mathfrak{g} and $x \in \mathfrak{g}$, define a derivation $\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g}$ by $\text{ad}(x)(y) = [x, y]$. A representation $\text{ad}: \mathfrak{g} \ni x \mapsto \text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is called the **adjoint representation** of \mathfrak{g} . The **center** of \mathfrak{g} is $\mathfrak{z} = \ker(\text{ad})$, which is a commutative ideal. $\text{im}(\text{ad})$ is an ideal of $\text{Der } \mathfrak{g}$. A derivation $\text{ad}(x)$ is called a **inner derivation** of \mathfrak{g} .

3.1.12 Quotient algebra for Lie algebras [Definition 3.1.12]

For a Lie algebra \mathfrak{g} and an ideal $\mathfrak{a} \subset \mathfrak{g}$, the *quotient algebra* is

$$\mathfrak{g}/\mathfrak{a} = \{\bar{x} = x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

with canonical operations, where $\bar{x} = \{y \in \mathfrak{g} \mid x \equiv y \pmod{\mathfrak{a}}\} = \{x + a \mid a \in \mathfrak{a}\}$ called the *class* of x . The homomorphism $\varphi: \mathfrak{g} \ni x \mapsto \bar{x} \in \mathfrak{g}/\mathfrak{a}$ is called the *canonical homomorphism*.

3.1.13 The first isomorphism theorem for Lie algebras [Theorem 3.1.13]

For Lie algebras $\mathfrak{g}, \mathfrak{h}$ and a homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$,

$$\mathfrak{g}/\ker \varphi \cong \text{im } \varphi.$$

3.1.14 The second isomorphism theorem for Lie algebras [Theorem 3.1.14]

For a Lie algebra \mathfrak{g} , an ideal $\mathfrak{a} \subset \mathfrak{g}$, a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the canonical homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$,

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{a}) \cong (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}.$$

3.2 Solvable and Nilpotent Lie algebra

3.2.1 Solvable Lie algebra [Definition 3.2.1]

Let \mathfrak{g} be a Lie algebra, and

$$D^0 \mathfrak{g} = \mathfrak{g}, \quad D^k \mathfrak{g} = D(D^{k-1} \mathfrak{g}), \quad k = 1, 2, \dots$$

\mathfrak{g} is said to be *solvable* if $D^r \mathfrak{g} = \{0\}$ for some r called the *length* of \mathfrak{g} .

3.2.2 Lie algebra of triangular matrices is solvable [Example 3.2.2]

Let

$$\begin{aligned} \mathfrak{g}_0 &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbb{R}) \mid \xi \text{ is upper triangular}\}, \\ \mathfrak{g}_k &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbb{R}) \mid \xi_{ij} = 0 \text{ for } j - i < k\}. \end{aligned}$$

Then, $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_1$, $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$, $k, \ell = 0, 1, \dots$, and \mathfrak{g}_0 is a solvable Lie algebra of length $\leq n$.

3.2.3 Lie subalgebra of a solvable Lie algebra is also solvable [Theorem 3.2.3]

For a solvable Lie algebra \mathfrak{g} , its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also solvable, and if \mathfrak{h} is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also solvable.

3.2.4 Lie algebra whose ideal and quotient algebra over it are solvable is solvable [Theorem 3.2.4]

For a Lie algebra \mathfrak{g} and its ideal $\mathfrak{a} \subset \mathfrak{g}$, if \mathfrak{a} and $\mathfrak{g}/\mathfrak{a}$ are both solvable, then \mathfrak{g} is also solvable.

3.2.5 Nilpotent Lie algebra [Definition 3.2.5]

Let \mathfrak{g} be a Lie algebra, and

$$C^0 \mathfrak{g} = \mathfrak{g}, \quad C^k \mathfrak{g} = [\mathfrak{g}, C^{k-1} \mathfrak{g}], \quad k = 1, 2, \dots$$

\mathfrak{g} is said to be **nilpotent** if $C^s \mathfrak{g} = \{0\}$ for some s called the **length** of \mathfrak{g} .

Since $D^k \mathfrak{g} \subset C^k \mathfrak{g}$, a nilpotent Lie algebra is solvable.

3.2.6 Lie algebra of strictly triangular matrices is nilpotent [Example 3.2.6]

\mathfrak{g}_1 in 3.2.2 is nilpotent while \mathfrak{g}_0 there is not.

3.2.7 Lie subalgebra of a nilpotent Lie algebra is also nilpotent [Theorem 3.2.7]

For a nilpotent Lie algebra \mathfrak{g} , its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also nilpotent, and if \mathfrak{h} is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also nilpotent.

3.2.8 Center of a nilpotent Lie algebra has a nonzero vector [Theorem 3.2.8]

For a Lie algebra \mathfrak{g} and its center \mathfrak{z} , $\mathfrak{z} \neq \{0\}$ if \mathfrak{g} is nilpotent while \mathfrak{g} is nilpotent if $\mathfrak{g}/\mathfrak{z}$ is nilpotent.

4 Categories^[1]

4.1 Foundations

4.1.1 Category [Definition 4.1.1]

A *category* consists of the followings:

- **Objects** A, B, C, \dots
- **Arrows** f, g, h, \dots with the objects called the domain $\text{dom}(f)$ and the codomain $\text{cod}(f)$.
- **Composites** $g \circ f: A \rightarrow C$ for given arrows $f: A \rightarrow B$ and $g: B \rightarrow C$.
- **Identity arrow** 1_A of each object A .

satisfying the following laws:

- i) $\forall \text{arrows } f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D, h \circ (g \circ f) = (h \circ g) \circ f$
- ii) $\forall \text{arrow } f: A \rightarrow B, f \circ 1_A = f = 1_B \circ f.$

4.1.2 Functor between categories [Definition 4.1.2]

A **functor** $F: \mathcal{A} \rightarrow \mathcal{B}$ between categories \mathcal{A} and \mathcal{B} is a mapping between objects and between arrows in the following ways:

- i) $F(f: A \rightarrow B) = F(f): F(A) \rightarrow F(B),$
- ii) $F(1_A) = 1_{F(A)},$
- iii) $F(g \circ f) = F(g) \circ F(f).$

4.1.3 Isomorphism between categories [Definition 4.1.3]

In a category \mathcal{C} , an arrow $f: A \rightarrow B$ is called an **isomorphism** if

$$\exists g = f^{-1}: B \rightarrow A, g \circ f = 1_A, f \circ g = 1_B.$$

If there is an isomorphism between objects A and B , A is said to be **isomorphic** to B , written $A \cong B$.

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