Notes of Mathematics

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Contents

1 Manifolds [4]

1.1 Manifolds on Euclidean Spaces

1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f.

1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The ***tangent space*** $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p.

1.1.3 Directional derivative [Definition 1.1.3]

The ***directional derivative*** of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + tv^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map $D \colon C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a ***derivation*** at p or a ***point-derivation*** of C_p^∞.

The set of all derivations at p denoted by $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi \colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map $\phi \to T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ in 1.1.4 is an isomorphism of vector spaces.

1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6, $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

1.1.8 Vector fields on an open set [Definition 1.1.8]

A ***vector field*** on $U \in \mathcal{O}_n$ is a map $X\colon U \to T_p(\mathbb{R}^n)$. $X = \sum a^i \partial/\partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

X is said to be C^∞ if all a^is are C^∞ on U. The set of all smooth vector fields on U is denoted by $\mathfrak{X}(U)$.

1.1.9 Multiplication of a smooth vector field and function [Definition & Proposition 1.1.9]

For $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, define $fX \in \mathfrak{X}(U)$ and $Xf \in C^\infty(U)$ as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left. \frac{\partial}{\partial x^i} \right|_p ,$$
$$(Xf)(p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p).$$

1.1.10 Leibniz rule for a vector field [Proposition 1.1.10]

For any $X \in \mathfrak{X}(U)$, $f, g \in C^\infty(U)$,

$$X(fg) = (Xf)g + fXg.$$

1.1.11 Derivations one-to-one-correspond to smooth vector fields [Proposition 1.1.11]

$\varphi\colon \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \mathrm{Der}\,(C^\infty(U))$ is an linear isomorphism.

1.1.12 k-tensor on a vector space [Definition 1.1.12]

A k-linear function on a vector space V $f\colon V^k \to \mathbb{R}$ is called a ***k-tensor*** on V. The vector space of all k-tensors on V is denoted by $L_k(V)$. k is called the degree of f.

1.1.13 Permutation action on k-tensors [Definition 1.1.13]

For $f \in L_k(V)$ on a vector space V and $\sigma \in \mathfrak{S}_n$, an action of σ on f is defined by

$$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

1.1.14 Symmetric and alternating k-tensor [Definition 1.1.14]

A k-tensor $f: V^k \to \mathbb{R}$ is ***symmetric*** if

$$\forall \sigma \in \mathfrak{S}_k,\ \sigma f = f,$$

and f is ***alternating*** if

$$\forall \sigma \in \mathfrak{S}_k,\ \sigma f = (\operatorname{sgn} \sigma) f.$$

1.1.15 The set of all alternating k-tensors [Definition 1.1.15]

An alternating k-tensor on a vector space V is also called a ***k-covector*** or a ***multicovector of degree k*** on V. The set of all k-covectors on V is denoted by $A_k(v)$ for $k > 0$; for $k = 0$, $A_0(V) = \mathbb{R}$.

1.1.16 Symmetrizing and alternating operators on k-covectors [Definition & Proposition 1.1.16]

For a $f \in A_k(V)$ on a vector space V,

$$Sf = \sum_{\sigma \in \mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{sgn} \sigma) \sigma f$$

is alternating.

1.1.17 Tensor product of two multilinear functions [Definition 1.1.17]

For $f \in L_k(V), g \in L_\ell(V)$ on a vector space V, the ***tensor product*** $f \otimes g \in L_{k+\ell}(V)$ is defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) g(v_{k+1}, \dots, v_{k+\ell}).$$

1.1.18 Bilear map as a tensor product [Example 1.1.18]

Let $e_1, \dots, e_n$ be a basis for a vector space V, $\alpha^1, \dots, \alpha^n$ the dual basis in V^*, and $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ a bilinear map on V. Then,

$$\langle\ ,\ \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j,$$

where $g_{ij} = \langle e_i, e_j \rangle$.

1.1.19 Wedge product of two multilinear functions [Definition 1.1.19]

For $f \in A_k(V)$, $g \in A_\ell(V)$ on a vector space V, their ***wedge product*** or ***exterior product*** is

$$f \wedge g = \frac{1}{k!\,\ell!} A(f \otimes g).$$

$f \wedge g$ is alternating.

Explicitly,

$$\begin{aligned}(f \wedge g)(v_1, \dots, v_{k+\ell}) &= \frac{1}{k!\,\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\ &= \sum_{\substack{(k,\ell)\text{-shuffle} \\ \sigma}} (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}),\end{aligned}$$

where a (k, ℓ)-shuffle means $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+\ell)$.

1.1.20 Wedge product is anticommutative [Proposition 1.1.20]

For $f \in A_k(V)$, $g \in A_\ell(V)$ on a vector space V,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

If the degree of f is odd, then $f \wedge f = 0$.

1.1.21 Properties of nesting alternating operators [Lemma 1.1.21]

For a k-tensor f and ℓ-tensor g on a vector space V,

i) $A(A(f) \otimes g) = k!\, A(f \otimes g)$,
ii) $A(f \otimes A(g)) = \ell!\, A(f \otimes g)$.

1.1.22 Associativity of the wedge product [Proposition 1.1.22]

For $f \in A_k(V)$, $g \in A_\ell(V)$, $h \in A_m(V)$ on a real vector space V,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for $f_i \in A_{d_i}(V)$ $(i = 1, \dots, r)$,

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{(d_1)! \cdots (d_r)!} A(f_1 \otimes \cdots \otimes f_r).$$

1.1.23 Wedge product of covectors is the determinant [Proposition 1.1.23]

For covectors $\alpha^1, \ldots, \alpha^k$ on a vector space V,

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

1.1.24 Graded algebra over a field [Definition 1.1.24]

An algebra $\mathbb{A}$ over a field $\mathbb{K}$ is said to be ***graded*** if $\mathbb{A} = \bigoplus_{k=0}^{\infty} A^k$ is a direct sum of vector spaces over $\mathbb{K}$ such that the multiplication sends $A^k \times A^\ell$ to $A^{k+\ell}$. $A = \bigoplus_{k=0}^{\infty} A^k$ means each nonzero $a \in \mathbb{A}$ is uniquely a finite sum $a = a_{i_1} + \cdots a_{i_m}$ where nonzero $a_{i_j} \in A^{i_j}$.

$\mathbb{A}$ is ***anticommutative*** or ***graded commutative*** if $\forall a \in A^k, b \in A^\ell$, $ab = (-1)^{k\ell} ba$.

A ***homomorphism*** of graded algebras is an algebra homomorphism that preserves the degree.

1.1.25 Grassmann algebra of multicovectors on a vector space [Definition & Proposition 1.1.25]

For a vector space V of degree $n < \infty$, the ***exterior algebra*** or the ***Grassmann algebra*** of multicovectors on V is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^{n} A_k(V)$$

with the wedge product of multicovectors as multiplication.

1.1.26 Wedge product of the dual basis applying to a basis [Lemma 1.1.26]

Let $e_1, \ldots, e_n$ be a basis for a vector space V and $\alpha^1, \ldots, \alpha^n$ the dual basis in V^*. For $I = (i_1, \ldots, i_k)$, $J = (j_1, \ldots, j_k)$ with $1 \le i_1 < \cdots < i_k \le n,\ 1 \le j_1 < \cdots < j_k \le n$,

$$\alpha^I(e_J) = \delta^I_J.$$

1.1.27 Wedge products of the dual basis form a basis for multicovectors [Proposition 1.1.27]

Let V be a vector space and $\alpha^1, \ldots, \alpha^n$ the dual basis in V^*. Then, α^I, $I = (i_1 < \cdots < i_k)$ form a basis for $A_k(V)$.

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

1.1.28 Cotangent space to an Euclidean space at a point [Definition 1.1.28]

The ***cotangent space*** to $\mathbb{R}^n$ at p is $T_p^*(\mathbb{R}^n) = (T_p(\mathbb{R}^n))^*$.

1.1.29 Differential 1-form on an open subset of an Euclidean space [Definition 1.1.29]

A ***covector field*** or a ***differential 1-form*** on $U \in \mathcal{O}_n$ is $\omega \colon U \to \bigcup_{p \in U} T_p^*(\mathbb{R}^n)$ that maps $U \ni p \mapsto \omega_p \in T_p^*(\mathbb{R}^n)$.

1.1.30 Differential of a smooth function [Definition 1.1.30]

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$, the ***differential*** df of f is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$

In the expression

$$\langle\ ,\ \rangle \colon T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \ni (X_p, f) \mapsto \langle X_p, f \rangle = X_p f \in \mathbb{R},$$

a tangent vector is considered as $\langle X_p, \cdot \rangle$; a differential at p as $df|_p = (df)_p = \langle \cdot, f \rangle$.

1.1.31 Differentials of coordinates is the dual basis for the cotangent space [Proposition 1.1.31]

For $p \in \mathbb{R}^n$, $\{(dx^1)_p, \ldots, (dx^n)_p\}$ is the dual basis for $T_p^*(\mathbb{R}^n)$ to $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\} \subset T_p(\mathbb{R}^n)$, where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$.

For any differential 1-form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some $a_i(p)$. In this case, ω is written as $\omega = \sum a_i dx^i$.

1.1.32 Smoothness of a differential 1-form [Definition 1.1.32]

A differential 1-form $\omega = \sum a_i dx^i$ on $U \in \mathcal{O}_n$ is ***smooth*** if all $a_i \colon U \to \mathbb{R}$ are smooth.

1.1.33 Differentials can be written in terms of partial derivatives [Proposition 1.1.33]

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness of f implies that of df.

1.1.34 Differential k-forms on an Euclidean space [Definition 1.1.34]

A ***differential k-form*** or ***differential form of degree k*** on $U \in \mathcal{O}_n$ is $\omega\colon U \ni p \mapsto \omega_p \in A_k(T_p(\mathbb{R}^n))$.

1.1.35 Basis for differential forms [Definition & Proposition 1.1.35]

Since $\{dx^I_p \mid I = (1 \le i_1 < \cdots < i_k \le n)\}$ is a basis for $A_k(T_p(\mathbb{R}^n)$, for a differential k-form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_I(p)dx^I_p, \quad \omega = \sum a_I dx^I.$$

ω is ***smooth*** if all $a_I\colon U \to \mathbb{R}$ are smooth. The vector space of C^∞ differential k-forms on U is denoted by $\Omega^k(U)$. If $k = 0$, $\Omega^0(U) = C^\infty(U)$.

1.1.36 Wedge product of differential forms [Definition 1.1.36]

For differential k-form ω and ℓ-form τ on $U \in \mathcal{O}_n$, their ***wedge product*** $\omega \wedge \tau$ is a differential $(k+\ell)$-form defined by

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

If $\omega = \sum a_I dx^I$, $\tau = \sum b_J dx^J$,

$$\begin{aligned}\omega \wedge \tau &= \sum_{I,J} (a_I b_J) dx^I \wedge dx^J \\ &= \sum_{\text{disjoint } I,J} (a_I b_J) dx^I \wedge dx^J.\end{aligned}$$

For $\omega \in \Omega^k(U)$, $\tau \in \Omega^\ell(U)$, the wedge product is a bilinear map

$$\wedge\colon \Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U).$$

In particular, if $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$, then $f \wedge \omega = f\omega$.

1.1.37 Graded algebra with smooth differential forms [Definition 1.1.37]

For $U \in \mathcal{O}_n$, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is an anticommutative graded algebra over $\mathbb{R}$ with the wedge product as multiplication, which is also a module over $C^\infty(U)$.

1.1.38 Differential forms as linear maps on a vector field [Definition 1.1.38]

For a differential k-form ω on $U \in \mathcal{O}_n$ and $X_1, \ldots, X_k \in \mathfrak{X}(U)$, define $\omega(X_1, \ldots, X_k) \in C^\infty(U)$ by

$$(\omega(X_1, \ldots, X_k))_p = \omega_p((X_1)_p, \ldots, (X_k)_p).$$

The map

$$\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) \ni (X_1, \ldots, X_k) \mapsto \omega(X_1, \ldots, X_k) \in C^\infty(U)$$

is k-linear over $C^\infty(U)$.

1.1.39 Exterior derivatives of differential forms [Definition 1.1.39]

For $k \geq 1$ and $\omega = \sum a_I dx^I \in \Omega^k(U)$, the ***exterior derivative*** of ω is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U);$$

for $k = 0$ and $f \in C^\infty(U)$, its exterior derivative is

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

1.1.40 Antiderivation of a graded algebra [Definition 1.1.40]

An ***antiderivation*** of a graded algebra $\mathbb{A} = \bigoplus_{k=0}^\infty A^k$ is a linear map $D \colon \mathbb{A} \to \mathbb{A}$ such that for $a \in A^k$, $b \in A^\ell$,

$$D(ab) = D(a)b + (-1)^k a D(b).$$

If m is an integer such that D sends A^k to A^{k+m} for all k, then m is called the ***degree*** of D.

1.1.41 Properties of the exterior differentiation [Proposition 1.1.41]

i) The exterior differentiation $d \colon \Omega^*(U) \to \Omega^*(U)$ on $U \in \mathcal{O}_n$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

ii) $d^2 = 0$.

iii) For $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(df)(X) = Xf$.

1.1.42 Characterization of the exterior differentiation [Proposition 1.1.42]

The exterior differentiation $d \colon \Omega^*(U) \to \Omega^*(U)$ on $U \in \mathcal{O}_n$ is the only antideriavtion of $\Omega^*(U)$.

1.1.43 Closed and exact forms [Definition 1.1.43]

A differential k-form ω on $U \in \mathcal{O}_n$ is said to be ***closed*** if $d\omega = 0$, and said to be ***exact*** if $\omega = d\tau$ for some $(k-1)$-form τ on U.

Every exact form is closed.

1.1.44 Cochain complex and de Rham complex [Definition 1.1.44]

A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k \colon V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a ***cochain complex*** or a ***differential complex***.

The ***de Rham complex*** of $U \in \mathcal{O}_n$ is a cochain complex

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots .$$

The closed forms are the elements of $\ker d$, and the exact forms are the elements of $\operatorname{im} d$.

1.1.45 Vector calculus as differential forms [Proposition 1.1.45]

Under the identifications, for $U \in \mathcal{O}_3$, $f \in C^\infty(U)$ and $X = [P\ Q\ R] \in \mathfrak{X}(U)$,

$$\text{1-form } Pdx + Qdy + Rdz \longleftrightarrow X,$$

$$\text{2-form } Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \longleftrightarrow X,$$

$$\text{3-form } f dx \wedge dy \wedge dz \longleftrightarrow f,$$

there are correspondences between the exterior derivatives and grad, rot, and div:

$$df \longleftrightarrow \operatorname{grad} f,$$

$$d(Pdx + Qdy + Rdz) \longleftrightarrow \operatorname{rot} X,$$

$$d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \longleftrightarrow \operatorname{div} X.$$

1.1.46 k-th de Rham cohomology [Definition 1.1.46]

For $U \in \mathcal{O}_n$, the k-th ***de Rham cohomology*** of U is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}.$$

1.1.47 Poincaré lemma [Proposition 1.1.47]

For $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact, i.e., $H^k(\mathbb{R}^n)$ vanishes.

1.2 Manifolds

1.2.1 Locally Euclidean space [Definition 1.2.1]

A topological space M is ***locally Euclidean of dimension n*** if $\forall p \in M, \exists (U, \phi)$, with a neighborhood U at p and a homeomorphism $\phi \colon U \to V \in \mathcal{O}_n$, called a ***chart***, a ***coordinate neighborhood*** or a ***coordinate open set***, and ϕ a ***coordinate map*** or a ***coordinate system*** on U.

A chart (U, ϕ) is said to be ***centered*** at $p \in U$ if $\phi(p) = 0$.

1.2.2 Topological manifold [Definition 1.2.2]

A ***topological manifold of dimension n*** is a Hausdorff, second countable, locally Euclidean space of dimension n.

1.2.3 Compatible chart [Definition 1.2.3]

Two charts $(U, \phi\colon U \to \mathbb{R}^n)$, $(V, \psi\colon V \to \mathbb{R}^n)$ of a topological manifold are said to be C^∞***-compatible*** or simply ***compatible*** if

$$\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$$

called the ***transition functions*** between charts are C^∞. If $U \cap V = \emptyset$, they are C^∞-compatible.

1.2.4 Atlas on a locally Euclidean space [Definition 1.2.4]

A C^∞ ***atlas*** or simply an ***atlas*** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise compatible charts that cover M.

1.2.5 Compatibility of a chart with an atlas [Definition 1.2.5]

For a locally Euclidean space, a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if all charts (U_α, ϕ_α) are compatible with (V, ψ).

1.2.6 Charts compatible with the same atlas are compatible with each other [Lemma 1.2.6]

For a locally Euclidean space, charts (V, ψ), (W, σ), and an atlas $\{(U_\alpha, \phi_\alpha)\}$ on it, if (V, ψ) and (W, σ) are both compatible with $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

1.2.7 Maximal Atlas on a locally Euclidean space [Definition 1.2.7]

An atlas $\mathfrak{M}$ on a locally Euclidean space is ***maximal*** if for another atlas $\mathfrak{U}$, $\mathfrak{M} \subset \mathfrak{U}$ implies $\mathfrak{M} = \mathfrak{U}$.

1.2.8 Smooth manifold [Definition 1.2.8]

A ***smooth*** or C^∞ ***manifold*** is a topological manifold M with a maximal atlas called a ***differentiable structure*** on M. M is said to be of dimension n if all of its connected components are of dimension n, and then M is called a ***n-manifold***. A 1-manifold is also called a ***curve***, a 2-manifold a ***surface***.

1.2.9 A locally Euclidean space with an atlas has a maximal atlas [Proposition 1.2.9]

In a locally Euclidean space, any atlas is contained in a unique maximal atlas.

1.2.10 Conventions of manifold [Definition 1.2.10]

i) A manifold means a smooth manifold.

ii) The standard coordinates on $\mathbb{R}^n$ is denoted by $r^1, \ldots, r^n$.

iii) For a chart (U, ϕ) of a manifold, let $x^i = r^i \circ \phi$ the i-th component of ϕ, and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. $x^1, \ldots, x^n$ are called ***coordinates*** or ***local coordinates*** on U.

iv) The notation $(x^1, \ldots, x^n)$ means alternately the local coordinates on U and a point in $\mathbb{R}^n$

v) A ***chart*** (U, ϕ) ***about p*** in a manifold M means a chart in the differentiable structure of M such that $p \in U$.

1.2.11 Product manifold [Proposition 1.2.11]

For a m-manifold M and n-manifold N, and atlases $\{(U_\alpha, \phi_\alpha)\}$ of M and $\{(V_{\alpha'}, \psi_{\alpha'})\}$ of N, the collection

$$\{(U_\alpha \times V_{\alpha'}, \phi_\alpha \times \psi_{\alpha'} \colon U_\alpha \times V_{\alpha'} \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

is an atlas on $M \times N$, and therefore $M \times N$ is a manifold of dimension $m + n$.

2 P-adic Numbers [2]

2.1 Fundations

2.1.1 Absolute value on a field [Definition 2.1.1]

An ***absolute value*** on a field $\mathbb{K}$ is a function $|\ |\colon \mathbb{K} \to \mathbb{R}_{\geq 0}$ that satisfies:

i) $|x| = 0$ iff $x = 0$
ii) $\forall x, y \in \mathbb{K},\ |xy| = |x|\,|y|$
iii) $\forall x, y \in \mathbb{K},\ |x + y| \leq |x| + |y|$.

An absolute value that satisfies the condition

iv) $\forall x, y \in \mathbb{K},\ |x + y| \leq \max\{|x|, |y|\}$

is said to be ***non-archimedean***; otherwise, it is said to be ***archimedean***.

2.1.2 Trivial absolute value [Definition 2.1.2]

The ***trivial absolute value*** on a field $\mathbb{K}$ is a absolute value on $\mathbb{K}$ such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

An absolute value on a finite field must be trivial.

2.1.3 Valuation on a field [Definition 2.1.3]

A function $v\colon \mathbb{A}^\times \to \mathbb{R}$ with an integral domain $\mathbb{A}$ is called a ***valuation*** on $\mathbb{A}$ if it satisfies the following conditions:

i) $\forall x, y \in \mathbb{A}^\times,\ v(xy) = v(x) + v(y)$
ii) $\forall x, y \in \mathbb{A}^\times,\ v(x + y) \geq \min\{v(x), v(y)\}$

2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation v on a field is an additive subgroup of $\mathbb{R}$. $\operatorname{im} v$ is called the ***value group*** of v.

2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let $\mathbb{A}$ be an integral domain and $\mathbb{K} = \operatorname{Frac}\mathbb{A}$. Let $v\colon \mathbb{A}^\times \to \mathbb{R}$ be a valuation on $\mathbb{A}$ and extend v to $\mathbb{K}$ by setting $v(a/b) = v(a) - v(b)$, then the function $|\ |_v\colon \mathbb{K} \to \mathbb{R}_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on $\mathbb{K}$. Conversely, $-\log |\ |$ is a valuation on $\mathbb{K}$ for a non-archimedean absolute value $|\ |$ on $\mathbb{K}$.

2.1.6 p-adic valuation [Definition 2.1.6]

The ***p-adic valuation*** on $\mathbb{Q}$ with a prime p is a valuation $v_p\colon \mathbb{Q}^\times \to \mathbb{R}$ defined as follows: for each $n \in \mathbb{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \mathbb{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

2.1.7 p-adic absolute value [Definition 2.1.7]

The ***p-adic absolute value*** $|\ |_p\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\ | = |\ |_\infty$.

2.1.8 Absolute values on a field of rational functions [Definition 2.1.8]

Here are some absolute values on a field $\mathbb{F}(t)$ of rational functions over a field $\mathbb{F}$.

i) For $f(t) \in \mathbb{F}[t]$, $v_\infty(f) = -\deg f$, and for $f(t)/g(t) \in \mathbb{F}(t)$, $v_\infty(f/g) = v_\infty(f) - v_\infty(g)$ with $v_\infty(0) = \infty$. Then,

$$|f(t)|_\infty = e^{-v_\infty(f)}.$$

ii) For an irreducible polynomial $p(t) \in \mathbb{F}[t]$, define the $p(t)$-adic valuation and absolute value.

2.1.9 Properties of absolute values on fields [Lemma 2.1.9]

For an absolute value $|\ |$ on a field $\mathbb{K}$,

i) $|1| = 1$,

ii) $\forall x \in \mathbb{K},\ |x^n| = 1 \Rightarrow |x| = 1$,

iii) $\forall x \in \mathbb{K},\ |-x| = |x|$,

iv) If $\mathbb{K}$ is finite, then $|\ \ |$ is trivial.

2.1.10 Necessary and sufficient conditions of a non-archimedean absolute value [Theorem 2.1.10]

Let $\mathbb{K}$ be a field, $|\ \ |$ an absolute value on $\mathbb{K}$. Then,

$$\begin{aligned}|\ \ | \text{ is non-archimedean} &\Longleftrightarrow \forall n = 1 + \cdots + 1 \in \mathbb{K},\ |n| \leq 1 \\ &\Longleftrightarrow \sup\{|n| \mid n \in \mathbb{Z}\} = 1.\end{aligned}$$

Furthermore, $\sup\{|n| \mid n \in \mathbb{Z}\} = \infty$ if $|\ \ |$ is archimedean.

3 Lie Algebra [3]

3.1 Fundations

3.1.1 Lie algebra [Definition 3.1.1]

A vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with the Lie bracket satisfying the conditions

i) Lie bracket is bilinear

ii) $\forall x \in \mathfrak{g},\ [x, x] = 0$

iii) $\forall x, y, z \in \mathfrak{g},\ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a ***Lie algebra*** over $\mathbb{K}$.

3.1.2 General linear Lie algebra [Definition 3.1.2]

$\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra $M_n(\mathbb{R})$ with the Lie bracket $[x, y] = xy - yx$.

3.1.3 Derivation algebra [Definition 3.1.3]

A linear endomorphism D of an algebra $\mathbb{A}$ over $\mathbb{R}$ satisfying $D(xy) = D(x)y + xD(y)$ is called a ***derivation*** of $\mathbb{A}$. The set of all derivations $\operatorname{Der} \mathbb{A}$ with the addition, scaler multiplication, and lie bracket defined as follows:

i) $(D + D')(x) = D(x) + D'(x)$

ii) $(\alpha D)(x) = \alpha D(x)$

iii) $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the ***derivation algebra*** of $\mathbb{A}$.

3.1.4 Lie subalgebra [Definition 3.1.4]

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a ***Lie subalgebra*** of $\mathfrak{g}$ if $\forall x, y \in \mathfrak{h},\ [x, y] \in \mathfrak{h}$.

For linear subspaces $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ denotes the subspace generated by $[x, y]$ with $x \in \mathfrak{a}, y \in \mathfrak{b}$.

3.1.5 Special linear Lie algebra [Definition & Proposition 3.1.5]

$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \operatorname{tr} x = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

3.1.6 Orthogonal Lie algebra [Definition & Proposition 3.1.6]

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^t x = -x\}$ is a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{R})$.

3.1.7 Ideal of a Lie algebra [Definition & Proposition 3.1.7]

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is an ***ideal*** of $\mathfrak{g}$ if $\forall x \in \mathfrak{g}, y \in \mathfrak{h},\ [x, y] \in \mathfrak{h}$.

For ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ is also an ideal.

3.1.8 Derived ideal of a Lie algebra [Definition 3.1.8]

For a Lie algebra $\mathfrak{g}$, $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ called the ***derived ideal*** of $\mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$, $D\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$.

3.1.9 Homomorphism of Lie algebras [Definition & Proposition 3.1.9]

For Lie algebras $\mathfrak{g}, \mathfrak{h}$, a linear map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is called a ***homomorphism*** if $\forall x, y \in \mathfrak{g},\ \varphi([x, y]) = [\varphi(x), \varphi(y)]$. A homomorphism φ is an ***isomorphism*** if it is bijective. Lie algebras between which there exists an isomorphism are said to be ***isomorphic*** to each other, written as $\mathfrak{g} \cong \mathfrak{h}$.

A composite of homomorphisms is also a homomorphism, and that of isomorphisms is also an isomorphism.

The kernel $\ker\varphi = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$ of a homomorphism φ is an ideal of $\mathfrak{g}$ while the image $\operatorname{im}\varphi = \varphi(\mathfrak{g})$ of φ is a Lie subalgebra of $\mathfrak{h}$.

3.1.10 Representation of a Lie algebra on a vector space [Definition 3.1.10]

For a Lie algebra $\mathfrak{g}$ and a vector space V, a homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a ***representation*** of $\mathfrak{g}$ on V.

3.1.11 Adjoint representation of a Lie algebra [Definition & Proposition 3.1.11]

For a Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, define a derivation $\operatorname{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ by $\operatorname{ad}(x)(y) = [x, y]$. A representation $\operatorname{ad}\colon \mathfrak{g} \ni x \mapsto \operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is called the ***adjoint representation*** of $\mathfrak{g}$. The ***center*** of $\mathfrak{g}$ is $\mathfrak{z} = \ker(\operatorname{ad})$, which is a commutative ideal. $\operatorname{im}(\operatorname{ad})$ is an ideal of $\operatorname{Der}\mathfrak{g}$. A derivation $\operatorname{ad}(x)$ is called a ***inner derivation*** of $\mathfrak{g}$.

3.1.12 Quotient algebra for Lie algebras [Definition 3.1.12]

For a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{a} \subset \mathfrak{g}$, the ***quotient algebra*** is

$$\mathfrak{g}/\mathfrak{a} = \{\bar{x} = x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

with canonical operations, where $\bar{x} = \{y \in \mathfrak{g} \mid x \equiv y \pmod{\mathfrak{a}}\} = \{x + a \mid a \in \mathfrak{a}\}$ called the ***class*** of x. The homomorphism $\varphi \colon \mathfrak{g} \ni x \mapsto \bar{x} \in \mathfrak{g}/\mathfrak{a}$ is called the ***canonical homomorphism***.

3.1.13 The first isomorphism theorem for Lie algebras [Theorem 3.1.13]

For Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and a homomorphism $\varphi \colon \mathfrak{g} \to \mathfrak{h}$,

$$\mathfrak{g}/\ker\varphi \cong \operatorname{im}\varphi.$$

3.1.14 The second isomorphism theorem for Lie algebras [Theorem 3.1.14]

For a Lie algebra $\mathfrak{g}$, an ideal $\mathfrak{a} \subset \mathfrak{g}$, a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the canonical homomorphism $\varphi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$,

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{a}) \cong (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}.$$

3.2 Solvable and Nilpotent Lie algebra

3.2.1 Solvable Lie algebra [Definition 3.2.1]

Let $\mathfrak{g}$ be a Lie algebra, and

$$D^0\mathfrak{g} = \mathfrak{g}, \quad D^k\mathfrak{g} = D(D^{k-1}\mathfrak{g}), \quad k = 1, 2, \ldots$$

$\mathfrak{g}$ is said to be ***solvable*** if $D^r\mathfrak{g} = \{0\}$ for some r called the ***length*** of $\mathfrak{g}$.

3.2.2 Lie algebra of triangular matrices is solvable [Example 3.2.2]

Let

$$\begin{aligned}
\mathfrak{g}_0 &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbb{R}) \mid \xi \text{ is upper triangular}\}, \\
\mathfrak{g}_k &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbb{R}) \mid \xi_{ij} = 0 \text{ for } j - i < k\}.
\end{aligned}$$

Then, $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_1$, $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$, $k, \ell = 0, 1, \ldots$, and $\mathfrak{g}_0$ is a solvable Lie algebra of length $\leq n$.

3.2.3 Lie subalgebra of a solvable Lie algebra is also solvable [Theorem 3.2.3]

For a solvable Lie algebra $\mathfrak{g}$, its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also solvable, and if $\mathfrak{h}$ is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also solvable.

3.2.4 Lie algebra whose ideal and quotient algebra over it are solvable is solvable [Theorem 3.2.4]

For a Lie algebra $\mathfrak{g}$ and its ideal $\mathfrak{a} \subset \mathfrak{g}$, if $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are both solvable, then $\mathfrak{g}$ is also solvable.

3.2.5 Nilpotent Lie algebra [Definition 3.2.5]

Let $\mathfrak{g}$ be a Lie algebra, and

$$C^0\mathfrak{g} = \mathfrak{g}, \quad C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}], \quad k = 1, 2, \ldots$$

$\mathfrak{g}$ is said to be ***nilpotent*** if $C^s\mathfrak{g} = \{0\}$ for some s called the ***length*** of $\mathfrak{g}$.

Since $D^k\mathfrak{g} \subset C^k\mathfrak{g}$, a nilpotent Lie algebra is solvable.

3.2.6 Lie algebra of strictly triangular matrices is nilpotent [Example 3.2.6]

$\mathfrak{g}_1$ in 3.2.2 is nilpotent while $\mathfrak{g}_0$ there is not.

3.2.7 Lie subalgebra of a nilpotent Lie algebra is also nilpotent [Theorem 3.2.7]

For a nilpotent Lie algebra $\mathfrak{g}$, its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also nilpotent, and if $\mathfrak{h}$ is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also nilpotent.

3.2.8 Center of a nilpotent Lie algebra has a nonzero vector [Theorem 3.2.8]

For a Lie algebra $\mathfrak{g}$ and its center $\mathfrak{z}$, $\mathfrak{z} \neq \{0\}$ if $\mathfrak{g}$ is nilpotent while $\mathfrak{g}$ is nilpotent if $\mathfrak{g}/\mathfrak{z}$ is nilpotent.

4 Categories[1]

4.1 Fundations

4.1.1 Category [Definition 4.1.1]

A ***category*** consists of the followings:

- ***Objects*** $A, B, C, \ldots$
- ***Arrows*** $f, g, h, \ldots$ with the objects called the domain $\mathrm{dom}(f)$ and the codomain $\mathrm{cod}(f)$.
- ***Composites*** $g \circ f : A \to C$ for given arrows $f : A \to B$ and $g : B \to C$.
- ***Identity arrow*** 1_A of each object A.

satisfying the following laws:

i) $\forall$arrows $f : A \to B, g : B \to C, h : C \to D,\ h \circ (g \circ f) = (h \circ g) \circ f$

ii) $\forall$arrow $f : A \to B,\ f \circ 1_A = f = 1_B \circ f$.

4.1.2 Functor between categories [Definition 4.1.2]

A ***functor*** $F : \mathscr{A} \to \mathscr{B}$ between categories $\mathscr{A}$ and $\mathscr{B}$ is a mapping between objects and between arrows in the following ways:

i) $F(f : A \to B) = F(f) : F(A) \to F(B)$,

ii) $F(1_A) = 1_{F(A)}$,

iii) $F(g \circ f) = F(g) \circ F(f)$.

4.1.3 Isomorphism between categories [Definition 4.1.3]

In a category $\mathscr{C}$, an arrow $f : A \to B$ is called an ***isomorphism*** if

$$\exists g = f^{-1} : B \to A,\ g \circ f = 1_A,\ f \circ g = 1_B.$$

If there is an isomorphism between objects A and B, A is said to be ***isomorphic*** to B, written $A \cong B$.

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