

# Notes of Mathematics

Masato Nakata\*

\*Department of Science, Kyoto University

Since Aug 27, 2017

## Contents

<b>1</b>	<b>MANIFOLDS</b>	<b>1</b>
1.1	Manifolds on Euclidean Spaces . . . . .	1
1.2	Manifolds . . . . .	12
<b>2</b>	<b>P-ADIC NUMBERS</b>	<b>18</b>
2.1	Foundations . . . . .	18
<b>3</b>	<b>LIE ALGEBRAS</b>	<b>23</b>
3.1	Foundations . . . . .	23
3.2	Solvable and Nilpotent Lie algebras . . . . .	26
<b>4</b>	<b>CATEGORIES</b>	<b>28</b>
4.1	Foundations . . . . .	28
<b>5</b>	<b>FUNCTIONAL ANALYSIS</b>	<b>31</b>
5.1	Hahn-Banach Theorems . . . . .	31
<b>6</b>	<b>REAL ANALYSIS AND PROBABILITY</b>	<b>36</b>
6.1	Set Theory . . . . .	36
<b>7</b>	<b>HOMOLOGICAL ALGEBRA</b>	<b>37</b>
7.1	Chain Complexes . . . . .	37

# 1 MANIFOLDS

[6]

## 1.1 Manifolds on Euclidean Spaces

### *Theorem 1.1.1* Taylor's theorem with remainder

A smooth function  $f$  on an open ball  $U \in \mathcal{O}_n$  can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where  $p \in U$  and  $g_i \in C^\infty(U)$  with  $g_i(p) = (\partial f / \partial x^i)(p)$ .

Adapting this to  $g_i$  repeatedly gives the Taylor's expansion of  $f$ .

### *Definition 1.1.2* Tangent vector as an arrow from a point

The **tangent space**  $T_p(\mathbb{R}^n)$  at  $p \in \mathbb{R}^n$  is the set of arrows from  $p$ .

### *Definition 1.1.3* Directional derivative

The **directional derivative** of a smooth function  $f$  in the direction  $v \in T_p(\mathbb{R}^n)$  at  $p \in \mathbb{R}^n$  is

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with  $c^i(t) = p^i + t v^i$ .

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

### *Definition & Proposition 1.1.4* Derivation at a point

A linear map  $D: C_p^\infty \rightarrow \mathbb{R}$  satisfying the Leibniz rule (i.e.,  $D(fg) = (Df)g(p) + f(p)Dg$  for any  $f, g \in C_p^\infty$ ) is called a **derivation** at  $p$  or a **point-derivation** of  $C_p^\infty$ .

The set of all derivations at  $p$  denoted by  $\mathcal{D}_p(\mathbb{R}^n)$  is a real vector space, and a map  $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$  assigning  $D_v$  to each  $v$  is a linear map.

**Lemma 1.1.5 Point-derivation of a constant is zero**

If  $D$  is a point-derivation of  $C_p^\infty$ , then  $D(c) = 0$  for any constant function  $c$ .

**Theorem 1.1.6 Tangent space is isomorphic to the set of point-derivations**

The linear map  $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$  in [Definition & Proposition 1.1.4] is an isomorphism of vector spaces.

**Definition 1.1.7 Tangent vector as a derivation**

By [Theorem 1.1.6],  $v \in T_p(\mathbb{R}^n)$  is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

**Definition 1.1.8 Vector fields on an open set**

A **vector field** on  $U \in \mathcal{O}_n$  is a map  $X: U \rightarrow T_p(\mathbb{R}^n)$ .  $X = \sum a^i \partial / \partial x^i$  means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

$X$  is said to be  $C^\infty$  if all  $a^i$ 's are  $C^\infty$  on  $U$ . The set of all smooth vector fields on  $U$  is denoted by  $\mathfrak{X}(U)$ .

**Definition & Proposition 1.1.9 Multiplication of a smooth vector field and function**

For  $X \in \mathfrak{X}(U)$  and  $f \in C^\infty(U)$ , define  $fX \in \mathfrak{X}(U)$  and  $Xf \in C^\infty(U)$  as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left. \frac{\partial}{\partial x^i} \right|_p,$$

$$(Xf)(p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p).$$

**Proposition 1.1.10 Leibniz rule for a vector field**

For any  $X \in \mathfrak{X}(U), f, g \in C^\infty(U)$ ,

$$X(fg) = (Xf)g + fXg.$$

**Proposition 1.1.11 Derivations one-to-one-correspond to smooth vector fields**

$\varphi: \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \text{Der}(C^\infty(U))$  is an linear isomorphism.

**Definition 1.1.12 k-tensor on a vector space**

A  $k$ -linear function  $f: V^k \rightarrow \mathbf{R}$  on a vector space  $V$  is called a  **$k$ -tensor** on  $V$ . The vector space of all  $k$ -tensors on  $V$  is denoted by  $L_k(V)$ .  $k$  is called the degree of  $f$ .

**Definition 1.1.13 Permutation action on k-tensors**

For  $f \in L_k(V)$  on a vector space  $V$  and  $\sigma \in \mathfrak{S}_n$ , an action of  $\sigma$  on  $f$  is defined by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

**Definition 1.1.14 Symmetric and alternating k-tensor**

A  $k$ -tensor  $f: V^k \rightarrow \mathbf{R}$  is **symmetric** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = f,$$

and  $f$  is **alternating** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = (\text{sgn } \sigma) f.$$

**Definition 1.1.15 The set of all alternating k-tensors**

An alternating  $k$ -tensor on a vector space  $V$  is also called a  **$k$ -covector** or a **multicovector of degree  $k$**  on  $V$ . The set of all  $k$ -covectors on  $V$  is denoted by  $A_k(V)$  for  $k > 0$ ; for  $k = 0$ ,  $A_0(V) = \mathbf{R}$ .

**Definition & Proposition 1.1.16 Symmetrizing and alternating operators on k-covectors**

For a  $f \in A_k(V)$  on a vector space  $V$ ,

$$Sf = \sum_{\sigma \in \mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma \in \mathfrak{S}_n} (\text{sgn } \sigma) \sigma f$$

is alternating.

**Definition 1.1.17 Tensor product of two multilinear functions**

For  $f \in L_k(V)$ ,  $g \in L_\ell(V)$  on a vector space  $V$ , the **tensor product**  $f \otimes g \in L_{k+\ell}(V)$  is defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) g(v_{k+1}, \dots, v_{k+\ell}).$$

**Example 1.1.18 Bilinear map as a tensor product**

Let  $e_1, \dots, e_n$  be a basis for a vector space  $V$ ,  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ , and  $\langle, \rangle: V \times V \rightarrow \mathbf{R}$  a bilinear map on  $V$ . Then,

$$\langle, \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j,$$

where  $g_{ij} = \langle e_i, e_j \rangle$ .

**Definition 1.1.19 Wedge product of two multilinear functions**

For  $f \in A_k(V)$ ,  $g \in A_\ell(V)$  on a vector space  $V$ , their **wedge product** or **exterior product** is

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g).$$

$f \wedge g$  is alternating.

Explicitly,

$$\begin{aligned} (f \wedge g)(v_1, \dots, v_{k+\ell}) &= \frac{1}{k!\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\ &= \sum_{\substack{(k,\ell)\text{-shuffle} \\ \sigma}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}), \end{aligned}$$

where a  $(k, \ell)$ -shuffle means  $\sigma(1) < \dots < \sigma(k)$  and  $\sigma(k+1) < \dots < \sigma(k+\ell)$ .

**Proposition 1.1.20 Wedge product is anticommutative**

For  $f \in A_k(V)$ ,  $g \in A_\ell(V)$  on a vector space  $V$ ,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

If the degree of  $f$  is odd, then  $f \wedge f = 0$ .

**Lemma 1.1.21 Properties of nesting alternating operators**

For  $f \in L_k(V)$  and  $g \in L_\ell(V)$  on a vector space  $V$ ,

- i)  $A(A(f) \otimes g) = k! A(f \otimes g)$ ,
- ii)  $A(f \otimes A(g)) = \ell! A(f \otimes g)$ .

**Proposition 1.1.22 Associativity of the wedge product**

For  $f \in A_k(V)$ ,  $g \in A_\ell(V)$ ,  $h \in A_m(V)$  on a real vector space  $V$ ,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for  $f_i \in A_{d_i}(V)$  ( $i = 1, \dots, r$ ),

$$f_1 \wedge \dots \wedge f_r = \frac{1}{(d_1)! \dots (d_r)!} A(f_1 \otimes \dots \otimes f_r).$$

**Proposition 1.1.23 Wedge product of covectors is the determinant**

For covectors  $\alpha^1, \dots, \alpha^k$  on a vector space  $V$ ,

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

**Definition 1.1.24 Graded algebra over a field**

An algebra  $A$  over a field  $K$  is said to be **graded** if  $A = \bigoplus_{k=0}^{\infty} A^k$  is a direct sum of vector spaces over  $K$  such that the multiplication sends  $A^k \times A^\ell$  to  $A^{k+\ell}$ .  $A = \bigoplus_{k=0}^{\infty} A^k$  means each nonzero  $a \in A$  is a unique finite sum  $a = a_{i_1} + \dots + a_{i_m}$  with nonzero  $a_{i_j} \in A^{i_j}$ .

$A$  is **anticommutative** or **graded commutative** if  $\forall a \in A^k, b \in A^\ell, ab = (-1)^{k\ell} ba$ .

A **homomorphism** of graded algebras is an algebra homomorphism that preserves the degree.

**Definition & Proposition 1.1.25 Grassmann algebra of multivectors on a vector space**

For a vector space  $V$  of degree  $n < \infty$ , the **exterior algebra** or the **Grassmann algebra** of multivectors on  $V$  is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^n A_k(V)$$

with the wedge product of multivectors as multiplication.

**Lemma 1.1.26 Wedge product of the dual basis applying to a basis**

Let  $e_1, \dots, e_n$  be a basis for a vector space  $V$  and  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ . For  $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k)$  with  $1 \leq i_1 < \dots < i_k \leq n, 1 \leq j_1 < \dots < j_k \leq n$ ,

$$\alpha^I(e_J) = \delta_J^I.$$

**Proposition 1.1.27 Wedge products of the dual basis form a basis for multivectors**

Let  $V$  be a vector space and  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ . Then,  $\alpha^I, I = (i_1 < \dots < i_k)$  form a basis for  $A_k(V)$ .

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

**Definition 1.1.28 Cotangent space to an Euclidean space at a point**

The **cotangent space** to  $R^n$  at  $p$  is  $T_p^*(R^n) = (T_p(R^n))^*$ .

**Definition 1.1.29 Differential 1-form on an open subset of an Euclidean space**

A **covector field** or a **differential 1-form** on  $U \in \mathcal{O}_n$  is  $\omega: U \rightarrow \bigcup_{p \in U} T_p^*(R^n)$  that maps  $U \ni p \mapsto \omega_p \in T_p^*(R^n)$ .

**Definition 1.1.30 Differential of a smooth function**

For  $f \in C^\infty(U)$  on  $U \in \mathcal{O}_n$ , the **differential**  $df$  of  $f$  is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$



In the expression

$$\langle \cdot, \cdot \rangle: T_p(\mathbf{R}^n) \times C_p^\infty(\mathbf{R}^n) \ni (X_p, f) \mapsto \langle X_p, f \rangle = X_p f \in \mathbf{R},$$

a tangent vector is considered as  $\langle X_p, \cdot \rangle$ ; a differential at  $p$  as  $df|_p = (df)_p = \langle \cdot, f \rangle$ .

**Proposition 1.1.31** Differentials of coordinates is the dual basis for the cotangent space

For  $p \in \mathbf{R}^n$ ,  $\{(dx^1)_p, \dots, (dx^n)_p\}$  is the dual basis for  $T_p^*(\mathbf{R}^n)$  to  $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\} \subset T_p(\mathbf{R}^n)$ , where  $x^1, \dots, x^n$  are the standard coordinates on  $\mathbf{R}^n$ .

For any differential 1-form  $\omega$  on  $U \in \mathcal{O}_n$  and  $p \in U$ ,

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some  $a_i(p)$ . In this case,  $\omega$  is written as  $\omega = \sum a_i dx^i$ .

**Definition 1.1.32** Smoothness of a differential 1-form

A differential 1-form  $\omega = \sum a_i dx^i$  on  $U \in \mathcal{O}_n$  is **smooth** if all  $a_i: U \rightarrow \mathbf{R}$  are smooth.

**Proposition 1.1.33** Differentials can be written in terms of partial derivatives

For  $f \in C^\infty(U)$  on  $U \in \mathcal{O}_n$ ,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness of  $f$  implies that of  $df$ .

**Definition 1.1.34** Differential  $k$ -forms on an Euclidean space

A **differential  $k$ -form** or **differential form of degree  $k$**  on  $U \in \mathcal{O}_n$  is  $\omega: U \ni p \mapsto \omega_p \in A_k(T_p(\mathbf{R}^n))$ .

**Definition & Proposition 1.1.35 Basis for differential forms**

Since  $\{dx_p^I \mid I = (1 \leq i_1 < \dots < i_k \leq n)\}$  is a basis for  $A_k(T_p(\mathbf{R}^n))$ , for a differential  $k$ -form  $\omega$  on  $U \in \mathcal{O}_n$  and  $p \in U$ ,

$$\omega_p = \sum a_I(p) dx_p^I, \quad \omega = \sum a_I dx^I.$$

$\omega$  is **smooth** if all  $a_I: U \rightarrow \mathbf{R}$  are smooth. The vector space of  $C^\infty$  differential  $k$ -forms on  $U$  is denoted by  $\Omega^k(U)$ . If  $k = 0$ ,  $\Omega^0(U) = C^\infty(U)$ .

**Definition 1.1.36 Wedge product of differential forms**

For differential  $k$ -form  $\omega$  and  $\ell$ -form  $\tau$  on  $U \in \mathcal{O}_n$ , their **wedge product**  $\omega \wedge \tau$  is a differential  $(k + \ell)$ -form defined by

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

If  $\omega = \sum a_I dx^I$ ,  $\tau = \sum b_J dx^J$ ,

$$\begin{aligned} \omega \wedge \tau &= \sum_{I, J} (a_I b_J) dx^I \wedge dx^J \\ &= \sum_{\text{disjoint } I, J} (a_I b_J) dx^I \wedge dx^J. \end{aligned}$$

For  $\omega \in \Omega^k(U)$ ,  $\tau \in \Omega^\ell(U)$ , the wedge product is a bilinear map

$$\wedge: \Omega^k(U) \times \Omega^\ell(U) \rightarrow \Omega^{k+\ell}(U).$$

In particular, if  $f \in C^\infty(U)$  and  $\omega \in \Omega^k(U)$ , then  $f \wedge \omega = f\omega$ .

**Definition 1.1.37 Graded algebra with smooth differential forms**

For  $U \in \mathcal{O}_n$ , the direct sum  $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$  is an anticommutative graded algebra over  $\mathbf{R}$  with the wedge product as multiplication, which is also a module over  $C^\infty(U)$ .

**Definition 1.1.38 Differential forms as linear maps on a vector field**

For a differential  $k$ -form  $\omega$  on  $U \in \mathcal{O}_n$  and  $X_1, \dots, X_k \in \mathfrak{X}(U)$ , define  $\omega(X_1, \dots, X_k) \in C^\infty(U)$  by

$$(\omega(X_1, \dots, X_k))_p = \omega_p((X_1)_p, \dots, (X_k)_p).$$

The map

$$\mathfrak{X}^k(U) \ni (X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k) \in C^\infty(U)$$

is  $k$ -linear over  $C^\infty(U)$ .

**Definition 1.1.39 Exterior derivatives of differential forms**

For  $k \geq 1$  and  $\omega = \sum a_I dx^I \in \Omega^k(U)$ , the **exterior derivative** of  $\omega$  is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U);$$

for  $k = 0$  and  $f \in C^\infty(U)$ , its exterior derivative is

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

**Definition 1.1.40 Antiderivation of a graded algebra**

An **antiderivation** of a graded algebra  $A = \bigoplus_{k=0}^\infty A^k$  is a linear map  $D: A \rightarrow A$  such that for  $a \in A^k, b \in A^\ell$ ,

$$D(ab) = D(a)b + (-1)^k aD(b).$$

If  $m$  is an integer such that  $D$  sends  $A^k$  to  $A^{k+m}$  for all  $k$ , then  $m$  is called the **degree** of  $D$ .

**Proposition 1.1.41 Properties of the exterior differentiation**

- i) The exterior differentiation  $d: \Omega^*(U) \rightarrow \Omega^*(U)$  on  $U \in \mathcal{O}_n$  is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

- ii)  $d^2 = 0$ .

- iii) For  $f \in C^\infty(U)$  and  $X \in \mathfrak{X}(U)$ ,  $(df)(X) = Xf$ .

**Proposition 1.1.42 Characterization of the exterior differentiation**

The exterior differentiation  $d: \Omega^*(U) \rightarrow \Omega^*(U)$  on  $U \in \mathcal{O}_n$  is the only antiderivation of  $\Omega^*(U)$ .

**Definition 1.1.43 Closed and exact forms**

A differential  $k$ -form  $\omega$  on  $U \in \mathcal{O}_n$  is said to be **closed** if  $d\omega = 0$ , and said to be **exact** if  $\omega = d\tau$  for some  $(k-1)$ -form  $\tau$  on  $U$ .

Every exact form is closed.

**Definition 1.1.44 Cochain complex and de Rham complex**

A collection of vector spaces  $\{V^k\}_{k=0}^\infty$  with linear maps  $d_k: V^k \rightarrow V^{k+1}$  such that  $d_{k+1} \circ d_k = 0$  is called a **cochain complex** or a **differential complex**.

The **de Rham complex** of  $U \in \mathcal{O}_n$  is a cochain complex

$$0 \rightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

The closed forms are the elements of  $\ker d$ , and the exact forms are the elements of  $\text{im } d$ .

**Proposition 1.1.45 Vector calculus as differential forms**

Under the identifications, for  $U \in \mathcal{O}_3$ ,  $f \in C^\infty(U)$  and  $X = [P \ Q \ R] \in \mathfrak{X}(U)$ ,

$$\text{1-form } Pdx + Qdy + Rdz \longleftrightarrow X,$$

$$\text{2-form } Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \longleftrightarrow X,$$

$$\text{3-form } fdx \wedge dy \wedge dz \longleftrightarrow f,$$

there are correspondences between the exterior derivatives and grad, rot, and div:

$$df \longleftrightarrow \text{grad } f,$$

$$d(Pdx + Qdy + Rdz) \longleftrightarrow \text{rot } X,$$

$$d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \longleftrightarrow \text{div } X.$$

**Definition 1.1.46 k-th de Rham cohomology**

For  $U \in \mathcal{O}_n$ , the  $k$ -th **de Rham cohomology** of  $U$  is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}.$$

**Proposition 1.1.47 Poincaré lemma**

For  $k \geq 1$ , every closed  $k$ -form on  $\mathbb{R}^n$  is exact, i.e.,  $H^k(\mathbb{R}^n)$  vanishes.

## 1.2 Manifolds

**Definition 1.2.1 Locally Euclidean space**

A topological space  $M$  is **locally Euclidean of dimension  $n$**  if  $\forall p \in M, \exists (U, \phi)$ , with a neighborhood  $U$  at  $p$  and a homeomorphism  $\phi: U \rightarrow V \in \mathcal{O}_n$ , called a **chart**, a **coordinate neighborhood** or a **coordinate open set**, and  $\phi$  a **coordinate map** or a **coordinate system** on  $U$ .

A chart  $(U, \phi)$  is said to be **centered** at  $p \in U$  if  $\phi(p) = 0$ .

**Definition 1.2.2 Topological manifold**

A **topological manifold of dimension  $n$**  is a Hausdorff, second countable, locally Euclidean space of dimension  $n$ .

**Definition 1.2.3 Compatible chart**

Two charts  $(U, \phi: U \rightarrow \mathbb{R}^n), (V, \psi: V \rightarrow \mathbb{R}^n)$  of a topological manifold are said to be  $C^\infty$ -**compatible** or simply **compatible** if

$$\phi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \phi(U \cap V), \quad \psi \circ \phi^{-1}: \phi(U \cap V) \rightarrow \psi(U \cap V)$$

called the **transition functions** between charts are  $C^\infty$ . If  $U \cap V = \emptyset$ , they are  $C^\infty$ -compatible.

**Definition 1.2.4 Atlas on a locally Euclidean space**

A  $C^\infty$  **atlas** or simply an **atlas** on a locally Euclidean space  $M$  is a collection  $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$  of pairwise compatible charts that cover  $M$ .

**Definition 1.2.5 Compatibility of a chart with an atlas**

For a locally Euclidean space, a chart  $(V, \psi)$  is compatible with an atlas  $\{(U_\alpha, \phi_\alpha)\}$  if all charts  $(U_\alpha, \phi_\alpha)$  are compatible with  $(V, \psi)$ .

**Lemma 1.2.6 Charts compatible with the same atlas are compatible with each other**

For a locally Euclidean space, charts  $(V, \psi)$ ,  $(W, \sigma)$ , and an atlas  $\{(U_\alpha, \phi_\alpha)\}$  on it, if  $(V, \psi)$  and  $(W, \sigma)$  are both compatible with  $\{(U_\alpha, \phi_\alpha)\}$ , then they are compatible with each other.

**Definition 1.2.7 Maximal Atlas on a locally Euclidean space**

An atlas  $\mathfrak{M}$  on a locally Euclidean space is **maximal** if for another atlas  $\mathfrak{U}$ ,  $\mathfrak{M} \subset \mathfrak{U}$  implies  $\mathfrak{M} = \mathfrak{U}$ .

**Definition 1.2.8 Smooth manifold**

A **smooth** or  $C^\infty$  **manifold** is a topological manifold  $M$  with a maximal atlas called a **differentiable structure** on  $M$ .  $M$  is said to be of dimension  $n$  if all of its connected components are of dimension  $n$ , and then  $M$  is called a  **$n$ -manifold**. A 1-manifold is also called a **curve**, a 2-manifold a **surface**.

**Proposition 1.2.9 A locally Euclidean space with an atlas has a maximal atlas**

In a locally Euclidean space, any atlas is contained in a unique maximal atlas.

**Definition 1.2.10 Conventions of manifold**

- i) A manifold means a smooth manifold.
- ii) The standard coordinates on  $\mathbf{R}^n$  is denoted by  $r^1, \dots, r^n$ .
- iii) For a chart  $(U, \phi)$  of a manifold, let  $x^i = r^i \circ \phi$  the  $i$ -th component of  $\phi$ , and write  $\phi = (x^1, \dots, x^n)$  and  $(U, \phi) = (U, x^1, \dots, x^n)$ .  $x^1, \dots, x^n$  are called **coordinates** or **local coordinates** on  $U$ .
- iv) The notation  $(x^1, \dots, x^n)$  means alternately the local coordinates on  $U$  and a point in  $\mathbf{R}^n$ .
- v) A **chart**  $(U, \phi)$  **about**  $p$  in a manifold  $M$  means a chart in the differentiable structure of  $M$  such that  $p \in U$ .

**Proposition 1.2.11 Product manifold**

For a  $m$ -manifold  $M$  and  $n$ -manifold  $N$ , and atlases  $\{(U_\alpha, \phi_\alpha)\}$  of  $M$  and  $\{(V_{\alpha'}, \psi_{\alpha'})\}$  of  $N$ , the collection

$$\{(U_\alpha \times V_{\alpha'}, \phi_\alpha \times \psi_{\alpha'} : U_\alpha \times V_{\alpha'} \rightarrow \mathbf{R}^m \times \mathbf{R}^n)\}$$

is an atlas on  $M \times N$ , and therefore  $M \times N$  is a manifold of dimension  $m + n$ .

**Definition 1.2.12 Smooth function on a manifold**

For a smooth  $n$ -manifold  $M$ , a function  $f : M \rightarrow \mathbf{R}$  is said to be  $C^\infty$  or **smooth at a point**  $p \in M$  if, for some chart  $(U, \phi)$  about  $p$ ,  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbf{R}^n$  is  $C^\infty$  at  $\phi(p)$ ;  $C^\infty$  **on**  $M$  if it is smooth at every point.

**Proposition 1.2.13 Smoothness of real-valued functions**

For a  $n$ -manifold  $M$  and a function  $f : M \rightarrow \mathbf{R}$ , the following are equivalent:

- i)  $f$  is  $C^\infty$ .
- ii) There exists an atlas  $\mathfrak{U}$  for  $M$ , for any  $(U, \phi) \in \mathfrak{U}$ ,  $f \circ \phi^{-1}$  is  $C^\infty$ .
- iii) For any chart  $(U, \phi)$  on  $M$ ,  $f \circ \phi^{-1}$  is  $C^\infty$ .

**Definition 1.2.14 Pullback of a function by a map**

For manifolds  $M, N$ , the **pullback** of  $h: M \rightarrow R$  by  $F: N \rightarrow M$  is  $F^*h = h \circ F$ .

**Definition 1.2.15 Smooth map between manifolds**

For a  $m$ -manifold  $M$  and  $n$ -manifold  $N$ , a continuous map  $F: N \rightarrow M$  is  $C^\infty$  **at a point**  $p \in N$  if, for some charts  $(U, \phi)$  about  $p$  and  $(V, \psi)$  about  $F(p)$ ,  $\psi \circ F \circ \phi^{-1}$  is  $C^\infty$  at  $\phi(p)$ ;  $C^\infty$  if it is  $C^\infty$  at every point.

**Proposition 1.2.16 Smoothness of maps is independent of charts**

Let  $M$  be a  $m$ -manifold,  $N$  a  $n$ -manifold, and  $F: N \rightarrow M$  be  $C^\infty$  at  $p \in N$ . Then, for any charts  $(U, \phi)$  about  $p$  and  $(V, \psi)$  about  $F(p)$ ,  $\psi \circ F \circ \phi^{-1}$  is  $C^\infty$  at  $\phi(p)$ .

**Proposition 1.2.17 Smoothness of a map in terms of charts**

For a  $m$ -manifold  $M$  and  $n$ -manifold  $N$ , and a continuous map  $F: N \rightarrow M$ , the following are equivalent:

- i)  $F$  is  $C^\infty$ .
- ii) There exists atlases  $\mathfrak{U}$  for  $N$  and  $\mathfrak{V}$  for  $M$ , for any  $(U, \phi) \in \mathfrak{U}$  and  $(V, \psi) \in \mathfrak{V}$ ,  $\psi \circ F \circ \phi^{-1}$  is  $C^\infty$ .
- iii) For any chart  $(U, \phi)$  on  $N$  and  $(V, \psi)$  on  $M$ ,  $\psi \circ F \circ \phi^{-1}$  is  $C^\infty$ .

**Proposition 1.2.18 Composite of smooth maps is also smooth**

For manifolds  $M, N, P$  and  $C^\infty$  maps  $F: N \rightarrow M$ ,  $G: M \rightarrow P$ ,  $G \circ F: N \rightarrow P$  is also  $C^\infty$ .

**Definition 1.2.19 Diffeomorphism of manifolds**

A **diffeomorphism** of manifolds is a bijective  $C^\infty$  map whose inverse is also  $C^\infty$ .



**Proposition 1.2.20 Coordinate map is a diffeomorphism**

A coordinate map  $\phi: U \rightarrow \phi(U) \subset \mathbb{R}^n$  for a manifold with a chart  $(U, \phi)$  is a diffeomorphism.

**Proposition 1.2.21 Diffeomorphism into an Euclidean space is a coordinate map**

For an open subset  $U$  of a manifold  $M$  with the differentiable structure  $\mathfrak{U}$ , if  $F: U \rightarrow \mathbb{R}^n$  is a diffeomorphism, then  $(U, F) \in \mathfrak{U}$ .

**Proposition 1.2.22 Smoothness of a vector-valued function**

For a continuous map  $F: N \rightarrow \mathbb{R}^m$  on a manifold  $M$ , the following are equivalent:

- i)  $F$  is  $C^\infty$ .
- ii) There exists an atlas  $\mathfrak{U}$  for  $M$ , for any  $(U, \phi) \in \mathfrak{U}$ ,  $F \circ \phi^{-1}$  is  $C^\infty$ .
- iii) For any chart  $(U, \phi)$  on  $M$ ,  $F \circ \phi^{-1}$  is  $C^\infty$ .

**Proposition 1.2.23 Vector-valued function is smooth iff its components are all smooth**

For a vector-valued function  $F = (F^1, \dots, F^m): M \rightarrow \mathbb{R}^m$  on a manifold  $M$ ,  $F$  is  $C^\infty$  iff  $F^1, \dots, F^m$  are all  $C^\infty$ .

**Proposition 1.2.24 Smoothness of a map in terms of vector-valued functions**

For a continuous map  $F: N \rightarrow M$  between a  $m$ -manifold  $M$  and  $n$ -manifold  $N$ , the following are equivalent:

- i)  $F$  is  $C^\infty$ .
- ii) There exists an atlas  $\mathfrak{U}$  for  $M$ , for any  $(U, \phi) \in \mathfrak{U}$ ,  $\phi \circ F$  is  $C^\infty$ .

- iii)** For any chart  $(U, \phi)$  on  $M$ ,  $\phi \circ F$  is  $C^\infty$ .

**Proposition 1.2.25 Smoothness of a map in terms of components**

For a continuous map  $F: N \rightarrow M$  between a  $m$ -manifold  $M$  and  $n$ -manifold  $N$ , the following are equivalent:

- i)**  $F$  is  $C^\infty$ .
- ii)** There exists an atlas  $\mathcal{U}$  for  $M$ , for any  $(U, \phi^1, \dots, \phi^m) \in \mathcal{U}$ , the components  $\phi^i \circ F$  of  $F$  relative to the chart are all  $C^\infty$ .
- iii)** For any chart  $(U, \phi^1, \dots, \phi^m)$  on  $M$ , the components  $\phi^i \circ F$  of  $F$  relative to the chart are all  $C^\infty$ .

## 2 P-ADIC NUMBERS

[4]

### 2.1 Foundations

#### Definition 2.1.1 Absolute value on a field

An **absolute value** on a field  $K$  is a function  $|\cdot|: K \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- i)  $|x| = 0$  iff  $x = 0$
- ii)  $\forall x, y \in K, |xy| = |x||y|$
- iii)  $\forall x, y \in K, |x + y| \leq |x| + |y|$ .

An absolute value that satisfies the condition

- iv)  $\forall x, y \in K, |x + y| \leq \max\{|x|, |y|\}$

is said to be **non-archimedean**; otherwise, it is said to be **archimedean**.

#### Definition 2.1.2 Trivial absolute value

The **trivial absolute value** on a field  $K$  is an absolute value on  $K$  such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

An absolute value on a finite field must be trivial.

#### Definition 2.1.3 Valuation on a field

A function  $v: A^\times \rightarrow \mathbb{R}$  with an integral domain  $A$  is called a **valuation** on  $A$  if it satisfies the following conditions:

- i)  $\forall x, y \in A^\times, v(xy) = v(x) + v(y)$
- ii)  $\forall x, y \in A^\times, v(x + y) \geq \min\{v(x), v(y)\}$

**Definition & Proposition 2.1.4 Value group of a valuation**

The image of a valuation  $v$  on a field is an additive subgroup of  $\mathbf{R}$ .  $\text{im } v$  is called the **value group** of  $v$ .

**Proposition 2.1.5 Correspondence between valuations and non-archimedean absolute values**

Let  $A$  be an integral domain and  $K = \text{Frac } A$ . Let  $v: A^\times \rightarrow \mathbf{R}$  be a valuation on  $A$  and extend  $v$  to  $K$  by setting  $v(a/b) = v(a) - v(b)$ , then the function  $|\cdot|_v: K \rightarrow \mathbf{R}_{\geq 0}$  defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on  $K$ . Conversely,  $-\log |\cdot|$  is a valuation on  $K$  for a non-archimedean absolute value  $|\cdot|$  on  $K$ .

**Definition 2.1.6 p-adic valuation**

The **p-adic valuation** on  $\mathbf{Q}$  with a prime  $p$  is a valuation  $v_p: \mathbf{Q}^\times \rightarrow \mathbf{R}$  defined as follows: for each  $n \in \mathbf{Z}^\times$ , let  $v_p(n)$  be the greatest integer such that  $p^{v_p(n)} \mid n$ , and for each  $x = a/b \in \mathbf{Q}^\times$ ,  $v_p(x) = v_p(a) - v_p(b)$ .

We often set  $v_p(0) = \infty$ .

**Definition 2.1.7 p-adic absolute value**

The **p-adic absolute value**  $|\cdot|_p: \mathbf{Q} \rightarrow \mathbf{R}_{\geq 0}$  with a prime  $p$  is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as  $|\cdot| = |\cdot|_\infty$ .

**Definition 2.1.8 Absolute values on a field of rational functions**

Here are some absolute values on a field  $F(t)$  of rational functions over a field  $F$ .

i) For  $f(t) \in F[t]$ ,  $v_\infty(f) = -\deg f$ , and for  $f(t)/g(t) \in F(t)$ ,  $v_\infty(f/g) = v_\infty(f) - v_\infty(g)$  with  $v_\infty(0) = \infty$ . Then,

$$|f(t)|_\infty = e^{-v_\infty(f)}.$$

ii) For an irreducible polynomial  $p(t) \in F[t]$ , define the  $p(t)$ -adic valuation and absolute value.

**Lemma 2.1.9 Properties of absolute values on fields**

For an absolute value  $|\cdot|$  on a field  $K$ ,

- i)  $|1| = 1$ ,
- ii)  $\forall x \in K, |x^n| = 1 \Rightarrow |x| = 1$ ,
- iii)  $\forall x \in K, |-x| = |x|$ ,
- iv) If  $K$  is finite, then  $|\cdot|$  is trivial.

**Theorem 2.1.10 Necessary and sufficient conditions of a non-archimedean absolute value**

Let  $K$  be a field,  $|\cdot|$  an absolute value on  $K$ . Then,

$$\begin{aligned} |\cdot| \text{ is non-archimedean} &\iff \forall n = 1 + \dots + 1 \in K, |n| \leq 1 \\ &\iff \sup\{|n| \mid n \in \mathbb{Z}\} = 1. \end{aligned}$$

Furthermore,  $\sup\{|n| \mid n \in \mathbb{Z}\} = \infty$  if  $|\cdot|$  is archimedean.

**Definition 2.1.11 Distance induced from an absolute value on a field**

Given a field  $K$  with an absolute value  $|\cdot|$ , define the distance  $d: K \times K \rightarrow \mathbb{R}$  by  $d(x, y) = |x - y|$ , which makes  $K$  into a metric space.

**Proposition 2.1.12 Field can be a topological field with the induced distance**

For a field  $K$  with the distance  $d: K \times K \rightarrow \mathbb{R}$  induced from an absolute value on  $K$ , addition, multiplication and taking inverse are all continuous: fix points  $x_0, y_0 \in K$ , then

- ☞  $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in K, d(x, x_0), d(y, y_0) < \delta \implies d(x + y, x_0 + y_0) < \epsilon,$
- ☞  $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in K, d(x, x_0), d(y, y_0) < \delta \implies d(x - y, x_0 - y_0) < \epsilon,$
- ☞  $\forall \epsilon > 0, \exists \delta > 0, \forall x \in K, d(x, x_0) < \delta \implies d(1/x, 1/x_0) < \epsilon.$

**Definition & Proposition 2.1.13 Ultrametric space with a non-archimedean absolute value**

For a field  $K$  with the distance  $d$  induced from an absolute value  $|\cdot|$  on  $K$ ,  $|\cdot|$  is non-archimedean iff  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ . This inequality is called the **ultrametric inequality**, a metric which holds the ultrametric inequality an **ultrametric**, and a space with an ultrametric an **ultrametric space**.

**Proposition 2.1.14 Non-archimedean absolute value of a sum is the max of the two**

For a field  $K$  with a non-archimedean absolute value  $|\cdot|$  on  $K$ , and  $x, y \in K$  with  $|x| \neq |y|$ ,

$$|x + y| = \max\{|x|, |y|\}.$$

**Corollary 2.1.15 Triangle on an ultrametric space is isosceles**

In an ultrametric space, all triangles are isosceles.

**Definition 2.1.16 Open ball and closed ball on a field with an absolute value**

In a field  $K$  with an absolute value  $|\cdot|$ , an **open ball** of radius  $r > 0$  and center  $a \in K$  is

$$B(a, r) = \{x \in K \mid |x - a| < r\};$$

a **closed ball** is

$$\overline{B}(a, r) = \{x \in K \mid |x - a| \leq r\}.$$

**Proposition 2.1.17** Properties of an open ball and closed ball with a non-archimedean absolute value

Let  $K$  be a field with a non-archimedean absolute value, and  $a, b \in K$ ,  $r, s > 0$ .

- i)**  $b \in B(a, r) \implies B(a, r) = B(b, r)$ ,
- ii)**  $b \in \overline{B}(a, r) \implies \overline{B}(a, r) = \overline{B}(b, r)$ ,
- iii)**  $B(a, r)$  is clopen,
- iv)**  $\overline{B}(a, r)$  is clopen,
- v)**  $B(a, r) \cap B(b, s) \iff B(a, r) \subset B(b, s) \text{ or } B(a, r) \supset B(b, s)$ .
- vi)**  $\overline{B}(a, r) \cap \overline{B}(b, s) \iff \overline{B}(a, r) \subset \overline{B}(b, s) \text{ or } \overline{B}(a, r) \supset \overline{B}(b, s)$ .

### 3 LIE ALGEBRAS

[5]

#### 3.1 Foundations

##### *Definition 3.1.1* Lie algebra

A vector space  $\mathfrak{g}$  over a field  $K$  with the Lie bracket satisfying the conditions

- i) Lie bracket is bilinear
- ii)  $\forall x \in \mathfrak{g}, [x, x] = 0$
- iii)  $\forall x, y, z \in \mathfrak{g}, [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a **Lie algebra** over  $K$ .

##### *Definition 3.1.2* General linear Lie algebra

$\mathfrak{gl}_n(\mathbf{R})$  is the Lie algebra  $M_n(\mathbf{R})$  with the Lie bracket  $[x, y] = xy - yx$ .

##### *Definition 3.1.3* Derivation algebra

A linear endomorphism  $D$  of an algebra  $A$  over  $\mathbf{R}$  satisfying  $D(xy) = D(x)y + xD(y)$  is called a **derivation** of  $A$ . The set of all derivations  $\text{Der} A$  with the addition, scalar multiplication, and Lie bracket defined as follows:

- i)  $(D + D')(x) = D(x) + D'(x)$
- ii)  $(\alpha D)(x) = \alpha D(x)$
- iii)  $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the **derivation algebra** of  $A$ .

##### *Definition 3.1.4* Lie subalgebra



A linear subspace  $\mathfrak{h} \subset \mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$  is a **Lie subalgebra** of  $\mathfrak{g}$  if  $\forall x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$ .

For linear subspaces  $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ ,  $[\mathfrak{a}, \mathfrak{b}]$  denotes the subspace generated by  $[x, y]$  with  $x \in \mathfrak{a}, y \in \mathfrak{b}$ .

**Definition & Proposition 3.1.5 Special linear Lie algebra**

$\mathfrak{sl}_n(R) = \{x \in \mathfrak{gl}_n(R) \mid \text{tr } x = 0\}$  is a Lie subalgebra of  $\mathfrak{gl}_n(R)$ .

**Definition & Proposition 3.1.6 Orthogonal Lie algebra**

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(R) \mid {}^t x = -x\}$  is a Lie subalgebra of  $\mathfrak{gl}_n(R)$ .

**Definition & Proposition 3.1.7 Ideal of a Lie algebra**

A linear subspace  $\mathfrak{h} \subset \mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$  is an **ideal** of  $\mathfrak{g}$  if  $\forall x \in \mathfrak{g}, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$ .

For ideals  $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ ,  $[\mathfrak{a}, \mathfrak{b}]$  is also an ideal.

**Definition 3.1.8 Derived ideal of a Lie algebra**

For a Lie algebra  $\mathfrak{g}$ ,  $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$  is an ideal of  $\mathfrak{g}$  called the **derived ideal** of  $\mathfrak{g}$ .

If  $\mathfrak{g} = \mathfrak{gl}_n(R)$ ,  $D\mathfrak{g} = \mathfrak{sl}_n(R)$ .

**Definition & Proposition 3.1.9 Homomorphism of Lie algebras**

For Lie algebras  $\mathfrak{g}, \mathfrak{h}$ , a linear map  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$  is called a **homomorphism** if  $\forall x, y \in \mathfrak{g}, \varphi([x, y]) = [\varphi(x), \varphi(y)]$ . A homomorphism  $\varphi$  is an **isomorphism** if it is bijective. Lie algebras between which there exists an isomorphism are said to be **isomorphic** to each other, written  $\mathfrak{g} \cong \mathfrak{h}$ .

A composite of homomorphisms is also a homomorphism, and that of isomorphisms is also an isomorphism.

The kernel  $\ker \varphi = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$  of a homomorphism  $\varphi$  is an ideal of  $\mathfrak{g}$  while the image  $\text{im } \varphi = \varphi(\mathfrak{g})$  of  $\varphi$  is a Lie subalgebra of  $\mathfrak{h}$ .

**Definition 3.1.10 Representation of a Lie algebra on a vector space**

For a Lie algebra  $\mathfrak{g}$  and a vector space  $V$ , a homomorphism  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is called a **representation** of  $\mathfrak{g}$  on  $V$ .

**Definition & Proposition 3.1.11 Adjoint representation of a Lie algebra**

For a Lie algebra  $\mathfrak{g}$  and  $x \in \mathfrak{g}$ , define a derivation  $\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g}$  by  $\text{ad}(x)(y) = [x, y]$ . A representation  $\text{ad}: \mathfrak{g} \ni x \mapsto \text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$  is called the **adjoint representation** of  $\mathfrak{g}$ . The **center** of  $\mathfrak{g}$  is  $\mathfrak{z} = \ker(\text{ad})$ , which is a commutative ideal.  $\text{im}(\text{ad})$  is an ideal of  $\text{Der } \mathfrak{g}$ . A derivation  $\text{ad}(x)$  is called a **inner derivation** of  $\mathfrak{g}$ .

**Definition 3.1.12 Quotient algebra for Lie algebras**

For a Lie algebra  $\mathfrak{g}$  and an ideal  $\mathfrak{a} \subset \mathfrak{g}$ , the **quotient algebra** is

$$\mathfrak{g}/\mathfrak{a} = \{\bar{x} = x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

with canonical operations, where  $\bar{x} = \{y \in \mathfrak{g} \mid x \equiv y \pmod{\mathfrak{a}}\} = \{x + a \mid a \in \mathfrak{a}\}$  called the **class** of  $x$ . The homomorphism  $\varphi: \mathfrak{g} \ni x \mapsto \bar{x} \in \mathfrak{g}/\mathfrak{a}$  is called the **canonical homomorphism**.

**Theorem 3.1.13 The first isomorphism theorem for Lie algebras**

For Lie algebras  $\mathfrak{g}, \mathfrak{h}$  and a homomorphism  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ ,

$$\mathfrak{g}/\ker \varphi \cong \text{im } \varphi.$$

**Theorem 3.1.14 The second isomorphism theorem for Lie algebras**

For a Lie algebra  $\mathfrak{g}$ , an ideal  $\mathfrak{a} \subset \mathfrak{g}$ , a Lie subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and the canonical homomorphism  $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$ ,

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{a}) \cong (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}.$$

## 3.2 Solvable and Nilpotent Lie algebras

**Definition 3.2.1 Solvable Lie algebra**

Let  $\mathfrak{g}$  be a Lie algebra, and

$$D^0 \mathfrak{g} = \mathfrak{g}, \quad D^k \mathfrak{g} = D(D^{k-1} \mathfrak{g}), \quad k = 1, 2, \dots$$

$\mathfrak{g}$  is said to be **solvable** if  $D^r \mathfrak{g} = \{0\}$  for some  $r$  called the **length** of  $\mathfrak{g}$ .

**Example 3.2.2 Lie algebra of triangular matrices is solvable**

Let

$$\begin{aligned} \mathfrak{g}_0 &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbf{R}) \mid \xi \text{ is upper triangular}\}, \\ \mathfrak{g}_k &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbf{R}) \mid \xi_{ij} = 0 \text{ for } j - i < k\}. \end{aligned}$$

Then,  $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_1$ ,  $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$ ,  $k, \ell = 0, 1, \dots$ , and  $\mathfrak{g}_0$  is a solvable Lie algebra of length  $\leq n$ .

**Theorem 3.2.3 Lie subalgebra of a solvable Lie algebra is also solvable**

For a solvable Lie algebra  $\mathfrak{g}$ , its Lie subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is also solvable, and if  $\mathfrak{h}$  is an ideal,  $\mathfrak{g}/\mathfrak{h}$  is also solvable.

**Theorem 3.2.4 Lie algebra whose ideal and quotient algebra over it are solvable is solvable**

For a Lie algebra  $\mathfrak{g}$  and its ideal  $\mathfrak{a} \subset \mathfrak{g}$ , if  $\mathfrak{a}$  and  $\mathfrak{g}/\mathfrak{a}$  are both solvable, then  $\mathfrak{g}$  is also solvable.

**Definition 3.2.5 Nilpotent Lie algebra**

Let  $\mathfrak{g}$  be a Lie algebra, and

$$C^0 \mathfrak{g} = \mathfrak{g}, \quad C^k \mathfrak{g} = [\mathfrak{g}, C^{k-1} \mathfrak{g}], \quad k = 1, 2, \dots$$

$\mathfrak{g}$  is said to be **nilpotent** if  $C^s \mathfrak{g} = \{0\}$  for some  $s$  called the **length** of  $\mathfrak{g}$ .

Since  $D^k \mathfrak{g} \subset C^k \mathfrak{g}$ , a nilpotent Lie algebra is solvable.

**Example 3.2.6 Lie algebra of strictly triangular matrices is nilpotent**

$\mathfrak{g}_1$  in [Example 3.2.2] is nilpotent while  $\mathfrak{g}_0$  there is not.

**Theorem 3.2.7 Lie subalgebra of a nilpotent Lie algebra is also nilpotent**

For a nilpotent Lie algebra  $\mathfrak{g}$ , its Lie subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is also nilpotent, and if  $\mathfrak{h}$  is an ideal,  $\mathfrak{g}/\mathfrak{h}$  is also nilpotent.

**Theorem 3.2.8 Center of a nilpotent Lie algebra is nontrivial**

For a Lie algebra  $\mathfrak{g}$  and its center  $\mathfrak{z}$ ,  $\mathfrak{z} \neq \{0\}$  if  $\mathfrak{g}$  is nilpotent while  $\mathfrak{g}$  is nilpotent if  $\mathfrak{g}/\mathfrak{z}$  is nilpotent.

**Lemma 3.2.9 Adjoint representation in a nilpotent Lie algebra is nilpotent**

For a nilpotent Lie algebra  $\mathfrak{g}$  and  $x \in \mathfrak{g}$ ,  $\text{ad } x$  is nilpotent. (i.e.,  $(\text{ad } x)^s = 0$  for some  $s$ ).

**Theorem 3.2.10 Engel's Theorem**

For a Lie algebra  $\mathfrak{g}$ , if  $\text{ad } x$  is nilpotent for any  $x \in \mathfrak{g}$ , then  $\mathfrak{g}$  is nilpotent.

**Theorem 3.2.11 Matrix of a representation of a Lie algebra whose representations are all nilpotent is strictly upper triangular**

For a representation  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  of a Lie algebra  $\mathfrak{g}$  on a vector space  $V$ , if  $\rho(x)$  is nilpotent for any  $x \in \mathfrak{g}$ , then a matrix of  $\rho(x)$  is strictly upper triangular for some basis for  $V$ .

## 4 CATEGORIES

[1]

### 4.1 Foundations

#### Definition 4.1.1 Category

A **category** consists of the followings:

🍃 **Objects**  $A, B, C, \dots$

🍃 **Arrows**  $f, g, h, \dots$  with the objects called the domain  $\text{dom } f$  and the codomain  $\text{cod } f$ .

🍃 **Composites**  $g \circ f: A \rightarrow C$  for given arrows  $f: A \rightarrow B$  and  $g: B \rightarrow C$ .

🍃 **Identity arrow**  $1_A$  of each object  $A$ .

satisfying the following laws:

- i)  $\forall \text{arrows } f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D, h \circ (g \circ f) = (h \circ g) \circ f$
- ii)  $\forall \text{arrow } f: A \rightarrow B, f \circ 1_A = f = 1_B \circ f$ .

#### Definition 4.1.2 Functor between categories

A **functor**  $F: \mathcal{A} \rightarrow \mathcal{B}$  between categories  $\mathcal{A}$  and  $\mathcal{B}$  is a mapping between objects and between arrows in the following ways:

- i)  $F(f: A \rightarrow B) = F(f): F(A) \rightarrow F(B)$ ,
- ii)  $F(1_A) = 1_{F(A)}$ ,
- iii)  $F(g \circ f) = F(g) \circ F(f)$ .

#### Definition 4.1.3 Isomorphism between categories

In a category  $\mathcal{C}$ , an arrow  $f: A \rightarrow B$  is called an **isomorphism** if

$$\exists g = f^{-1}: B \rightarrow A, g \circ f = 1_A, f \circ g = 1_B.$$

If there is an isomorphism between objects  $A$  and  $B$ ,  $A$  is said to be **isomorphic** to  $B$ , written  $A \cong B$ .

**Theorem 4.1.4 Category is isomorphic to its Cayley representation**

For a category  $\mathcal{C}$  with a set of arrows, the Cayley representation  $\overline{\mathcal{C}}$  of  $\mathcal{C}$ , consisting of

- ▮ object  $\overline{C} = \{f \in \mathcal{C} \mid \text{cod } f = C\}$  for an object  $C \in \mathcal{C}$ ,
- ▮ arrow  $\overline{g}: \overline{C} \rightarrow \overline{D}$  for an arrow  $g: C \rightarrow D$  such that  $\overline{g}(f) = g \circ f$ ,

is isomorphic to  $\mathcal{C}$ .

**Definition 4.1.5 Product of two categories**

The **product**  $\mathcal{C} \times \mathcal{D}$  of categories  $\mathcal{C}$  and  $\mathcal{D}$  consists of

- ▮ object  $(C, D)$  for objects  $C \in \mathcal{C}$ ,  $D \in \mathcal{D}$ ,
- ▮ arrow  $(f, g): (C, D) \rightarrow (C', D')$  for arrows  $f: C \rightarrow C'$ ,  $g: D \rightarrow D'$ ,

with composition  $(f, g) \circ (f', g') = (f \circ f', g \circ g')$  and units  $1_{(C, D)} = (1_C, 1_D)$ .

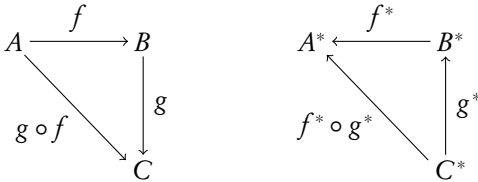
The **projection functors**  $\pi_1: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}$  and  $\pi_2: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$  is defined by  $\pi_1(C, D) = C$  and  $\pi_1(f, g) = f$ , and similarly for  $\pi_2$ .

**Definition 4.1.6 Dual category**

For a category  $\mathcal{C}$ , its **dual** or **opposite category**  $\mathcal{C}^{\text{op}}$  consists of

- ▮ object  $C^* = C$  for an object  $C \in \mathcal{C}$ ,
- ▮ arrow  $f^*: D^* \rightarrow C^*$  for an arrow  $f: C \rightarrow D$ ,

with composition  $f^* \circ g^* = (g \circ f)^*$  and units  $1_{C^*} = (1_C)^*$ .

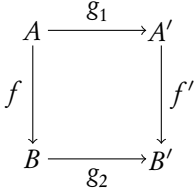


**Definition 4.1.7 Arrow category**

For a category  $\mathcal{C}$ , its **arrow category**  $\mathcal{C}^{\rightarrow}$  consists of

- ▮ object  $f: C \rightarrow D$  for an arrow  $f$  in  $\mathcal{C}$ ,
- ▮ arrow  $(g_1, g_2): f \rightarrow f'$ , where  $f: A \rightarrow B, f': A' \rightarrow B', g_1: A \rightarrow A', g_2: B \rightarrow B'$  in  $\mathcal{C}$ , such that  $g_2 \circ f = f' \circ g_1$ ,

with composition  $(g_1, g_2) \circ (h_1, h_2) = (g_1 \circ h_1, g_2 \circ h_2)$  and units  $1_f = (1_A, 1_B)$ .



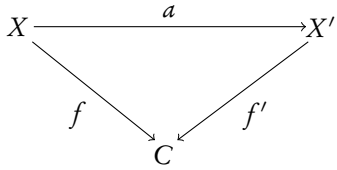
There are two functors  $\text{dom}, \text{cod}: \mathcal{C}^{\rightarrow} \rightarrow \mathcal{C}$ .

**Definition 4.1.8 Slice category**

For a category  $\mathcal{C}$ , its **slice category**  $\mathcal{C}/C$  over  $C \in \mathcal{C}$  consists of

- ▮ object  $f: X \rightarrow C$ ,
- ▮ arrow  $a: X \rightarrow X'$  for arrows  $f: X \rightarrow C, f': X' \rightarrow C$  such that  $f' \circ a = f$ ,

with composition and units from those of  $\mathcal{C}$ .



$U: \mathcal{C}/C \rightarrow \mathcal{C}$  with  $U(f: X \rightarrow C) = C$  and  $U(a: X \rightarrow X') = a$  is a functor.

## 5 FUNCTIONAL ANALYSIS

[2]

### 5.1 Hahn-Banach Theorems

#### *Theorem 5.1.1* Hahn-Banach of analytic form

Let  $p: E \rightarrow \mathbf{R}$  be a sublinear function on a vector space  $E$  (i.e.,  $\forall \lambda > 0, x, y \in E, p(\lambda x) = \lambda p(x), p(x+y) \leq p(x) + p(y)$ ),  $G \subset E$  a linear subspace, and  $g: G \rightarrow \mathbf{R}$  a linear functional such that  $\forall x \in G, g(x) \leq p(x)$ . Then,  $\exists$  a linear functional  $f: E \rightarrow \mathbf{R}$  that extends  $g$  and that  $\forall x \in E, f(x) \leq p(x)$ .

#### *Definition 5.1.2* Norm on the dual space of a normed space

For a normed space  $E$ , the **dual norm** on  $E^*$  is defined by

$$\|f\|_{E^*} = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} |f(x)| = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} f(x).$$

#### *Definition 5.1.3* Scalar product for the duality

For a vector space  $E$  and its dual space  $E^*$ ,  $\langle \cdot, \cdot \rangle: E^* \times E \rightarrow \mathbf{R}$  defined by  $\langle f, x \rangle = f(x)$  is called the **scalar product for the duality**  $E, E^*$ .

#### *Definition 5.1.4* Strictly convex normed space

A normed space  $E$  is said to be **strictly convex** if  $\forall t \in (0, 1), x, y \in E$  with  $\|x\| = \|y\| = 1$ ,  $\|tx + (1-t)y\| < 1$  except for  $x = y$ .

#### *Corollary 5.1.5* Hahn-Banach of alternate form



For a continuous linear functional  $g: G \rightarrow \mathbf{R}$  on a linear subspace  $G \subset E$  of a normed space  $E$ ,  $\exists f \in E^*$  that extends  $g$  and that  $\|f\|_{E^*} = \|g\|_{G^*}$ .

In the case when  $G = \mathbf{R}x_0$  and  $g(tx_0) = t\|x_0\|^2$  for a given  $x_0 \in E$ ,  $\exists f_0 \in E^*$  such that  $\|f_0\| = \|x_0\|$  and  $\langle f_0, x_0 \rangle = \|x_0\|^2$ . If  $E^*$  is strictly convex, then  $f_0$  is unique.

**Definition 5.1.6 Duality map from a normed space into its dual space**

For a normed space  $E$  and  $x_0 \in E$ , define

$$F(x_0) = \{f_0 \in E^* \mid \|f_0\| = \|x_0\|, \langle f_0, x_0 \rangle = \|x_0\|^2\}.$$

The **duality map** from  $E$  into  $E^*$  is a multivalued map  $x_0 \mapsto F(x_0)$ .

**Corollary 5.1.7 Norm of a vector is the max of its scalar product**

For a normed space  $E$  and  $x \in E$ ,

$$\|x\| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \max_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle|.$$

**Definition 5.1.8 Affine hyperplane of a normed space**

For a normed space  $E$  and a linear functional  $f: E \rightarrow \mathbf{R}$ , an affine **hyperplane** is a subset  $H = \{x \in E \mid f(x) = \alpha\} \subset E$  with  $\alpha \in \mathbf{R}$ , written  $H = [f = \alpha]$ .  $f = \alpha$  is called the **equation**.

**Proposition 5.1.9 Linear functional is continuous iff its hyperplane is closed**

For a linear functional  $f: E \rightarrow \mathbf{R}$  on a normed space  $E$  and  $\alpha \in \mathbf{R}$ ,  $[f = \alpha]$  is closed iff  $f$  is continuous.

**Definition 5.1.10 Separation by a hyperplane**

For two subsets  $A, B \subset E$  of a normed space  $E$ , the hyperplane  $[f = \alpha] \subset E$  **separates**  $A$

and  $B$  if

$$\forall x \in A, y \in B, f(x) \leq \alpha \leq f(y);$$

**strictly separates** if

$$\exists \epsilon > 0, \forall x \in A, y \in B, f(x) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(y).$$

**Definition 5.1.11 Convex subset of a normed space**

A subset  $A \subset E$  of a normed space  $E$  is said to be **convex** if

$$\forall t \in [0, 1], x, y \in A, tx + (1-t)y \in A.$$

**Theorem 5.1.12 Hahn-Banach of first geometric form**

For two disjoint nonempty convex subsets  $A, B \subset E$  of a normed space  $E$  with one of them open,  $\exists$  a closed hyperplane that separates  $A$  and  $B$ .

**Definition & Proposition 5.1.13 Minkowski functional of an open convex set**

Let  $C \subset E$  be an open convex subset of a normed space  $E$  with  $0 \in C$ , and for  $x \in E$

$$p(x) = \inf\{\alpha > 0 \mid \alpha^{-1}x \in C\},$$

called the **gauge** or the **Minkowski functional** of  $C$ . Then,  $p$  satisfies the following properties:

- i)  $p$  is sublinear,
- ii)  $\exists M, \forall x \in E, 0 \leq p(x) \leq M\|x\|$ ,
- iii)  $C = \{x \in E \mid p(x) < 1\}$ .

**Lemma 5.1.14 There exists a hyperplane that separates an open convex and outside point**

For a nonempty open convex  $C \subset E$  of a normed space  $E$  and  $x \in E \setminus C$ ,  $\exists f \in E^*$  such that  $\forall x \in C, f(x) < f(x_0)$ . In particular, the hyperplane  $[f = f(x_0)]$  separates  $\{x_0\}$  and  $C$ .

**Theorem 5.1.15 Hahn-Banach of second geometric form**

For two disjoint nonempty convex subsets  $A, B \subset E$  of a normed space  $E$  with  $A$  closed and  $B$  compact,  $\exists$  a closed hyperplane that strictly separates  $A$  and  $B$ .

**Corollary 5.1.16 Some linear functional can vanish on a linear subspace**

For a linear subspace  $F \subset E$  of a normed space  $E$  with  $\overline{F} \neq E$ ,  $\exists f \in E^*$  such that

$$\forall x \in F, \langle f, x \rangle = 0, \quad f \neq 0.$$

**Definition 5.1.17 Notation of a bidual space**

Let  $E$  be a normed space, and  $J: E \ni x \mapsto Jx \in E^{**}$  a **canonical injection** (i.e.,  $Jx: f \mapsto \langle f, x \rangle$ , or  $\langle Jx, f \rangle = \langle f, x \rangle$ ). Then,  $J$  is an **isometry**:

$$\|Jx\|_{E^{**}} = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle Jx, f \rangle| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \|x\|_E.$$

$J$  can be not surjective; if  $J$  is surjective,  $E$  is said to be **reflexive**.

**Definition 5.1.18 Orthogonal complement**

For a linear subspace  $M \subset E$  of a normed space  $E$  and a linear subspace  $N \subset E^*$ , their **orthogonal complements** are

$$\begin{aligned} M^\perp &= \{f \in E^* \mid \forall x \in M, \langle f, x \rangle = 0\} \subset E^* \\ N^\perp &= \{x \in E \mid \forall f \in N, \langle f, x \rangle = 0\} \subset E, \end{aligned}$$

respectively.

**Proposition 5.1.19 Relation between a linear subspace and its orthogonal complement**

For a linear subspace  $M \subset E$  of a normed space  $E$  and a linear subspace  $N \subset E^*$ ,

$$(M^\perp)^\perp = \overline{M}, \quad (N^\perp)^\perp \supset \overline{N}.$$

If  $E$  is reflexive, then  $(N^\perp)^\perp = \overline{N}$ .

## 6 REAL ANALYSIS AND PROBABILITY

[3]

### 6.1 Set Theory

## 7 HOMOLOGICAL ALGEBRA

[7]

### 7.1 Chain Complexes

#### Definition 7.1.1 Chain complex of R-modules

A **chain complex**  $C$  of  $R$ -modules is a family  $\{C_n\}_{n \in \mathbb{Z}}$  of  $R$ -modules with  $R$ -module maps  $d_n: C_n \rightarrow C_{n-1}$  such that  $d_{n-1} \circ d_n = 0$ , called the **differential**.  $Z_n = Z_n(C) := \ker d_n$  is the module of  $n$ -**cycles** of  $C$ , and  $B_n = B_n(C) := \operatorname{im} d_{n+1}$  is the module of  $n$ -**boundaries** of  $C$ . The  $n$ -th **homology module** of  $C$  is  $H_n(C) := Z_n/B_n$ .

A **chain complex map**  $u: C \rightarrow D$  is a family of  $R$ -module homomorphisms  $u_n: C_n \rightarrow D_n$  commuting with  $d$  (i.e.,  $u_{n-1}d_n = d_n u_n$ ).

#### Definition & Proposition 7.1.2 A category of chain complexes of R-modules

$\mathbf{Ch}(\mathbf{mod}\text{-}R)$  is a category whose objects are chain complexes of (right)  $R$ -modules and whose arrows are chain complex maps. An arrow  $u: C \rightarrow D$  of  $\mathbf{Ch}(\mathbf{mod}\text{-}R)$  sends boundaries to boundaries and cycles to cycles.  $H_n: \mathbf{Ch}(\mathbf{mod}\text{-}R) \rightarrow \mathbf{mod}\text{-}R$  is a functor.

#### Proposition 7.1.3 Split exact sequence of vector spaces

For a family  $\{B_n, H_n\}$  of vector spaces,  $\{C_n = B_n \oplus H_n \oplus B_{n-1}\}$  with projection-inclusions  $\pi_n: C_n \rightarrow B_{n-1} \subset C_{n-1}$  is a chain complex. Every chain complex of vector spaces is isomorphic to such a complex.

## References

- [1] Steve Awodey. *Category Theory, Second Edition*. Oxford University Press, 2010.
- [2] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2010.
- [3] R. M. Dudley. *Real Analysis and Probability, Second Edition*. Cambridge University Press, 2010.
- [4] Fernando Q. Gouvêa. *p-adic Numbers - An Introduction, Second Edition*. Springer, 1997.
- [5] 佐武一郎. リー環の話. 日本評論社, 1987.
- [6] Loring W. Tu. *An Introduction to Manifolds, Second Edition*. Springer, 2011.
- [7] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.