1 Manifolds

1.1 Manifolds on Euclidean Spaces

1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f.

1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The *tangent space* $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p.

1.1.3 Directional derivative [Definition 1.1.3]

The *directional derivative* of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + tv^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map $D \colon C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a *derivation at* p or a *point-derivation* of C_p^∞.

The set of all derivations at p $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi \colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map $\phi \to T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ in 1.1.4 is an isomorphism of vector spaces.

1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6, $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

1.1.8 Vector fields on an open set [Definition 1.1.8]

A *vector field* on $U \in \mathcal{O}_n$ is a map $X\colon U \to T_p(\mathbb{R}^n)$. $X = \sum a^i \partial/\partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

X is said to be C^∞ if all a^is are C^∞ on U. The set of all smooth vector fields on U is denoted by $\mathfrak{X}(U)$.

1.1.9 Multiplication of a smooth vector field and function [Definition & Proposition ??]

For $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, define $fX \in \mathfrak{X}(U)$ and $Xf \in C^\infty(U)$ as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left.\frac{\partial}{\partial x^i}\right|_p,$$
$$(Xf)(p) = X_p f = \sum a^i(p)\frac{\partial f}{\partial x^i}(p).$$

1.1.10 Leibniz rule for a vector field [Proposition 1.1.10]

For any $X \in \mathfrak{X}(U), f, g \in C^\infty(U)$,

$$X(fg) = (Xf)g + fXg.$$

1.1.11 Derivations one-to-one-correspond to smooth vector fields [Proposition 1.1.11]

$\varphi\colon \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \mathrm{Der}\,(C^\infty(U))$ is an linear isomorphism.

1.1.12 k-tensor on a vector space [Definition 1.1.12]

A k-linear function on a vector space V $f\colon V^k \to \mathbb{R}$ is called a *k-tensor* on V. The vector space of all k-tensors on V is denoted by $L_k(V)$. k is called the degree of f.

1.1.13 Symmetric and alternating k-tensor [Definition 1.1.13]

A k-tensor $f\colon V^k \to \mathbb{R}$ is *symmetric* if

$$\forall \sigma \in \mathfrak{S}_k,\ f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = f(v_1, \ldots, v_k),$$

and f is *alternating* if

$$\forall \sigma \in \imath S_k,\ f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn}\sigma) f(v_1, \ldots, v_k).$$

1.1.14 The set of all alternating k-tensors [Definition 1.1.14]

An alternating k-tensor on a vector space V is also called a *k-covector* or a *multicovector of degree k* on V. The set of all k-covectors on V is denoted by $A_k(v)$ for $k > 0$; for $k = 0$, $A_0(V) = \mathbb{R}$.

2 P-adic Numbers

2.1 Fundations

2.1.1 Absolute value on a field [Definition 2.1.1]

An *absolute value* on a field $\mathbb{K}$ is a function $|\ |\colon \mathbb{K} \to \mathbb{R}_{\geq 0}$ that satisfies:

i) $|x| = 0$ iff $x = 0$
ii) $\forall x, y \in \mathbb{K},\ |xy| = |x|\,|y|$
iii) $\forall x, y \in \mathbb{K},\ |x + y| \leq |x| + |y|$.

An absolute value that satisfies the condition

iv) $\forall x, y \in \mathbb{K},\ |x + y| \leq \max\{|x|, |y|\}$

is said to be *non-archimedean*; otherwise, it is said to be *archimedean*.

2.1.2 Trivial absolute value [Definition 2.1.2]

The *trivial absolute value* on a field $\mathbb{K}$ is a absolute value on $\mathbb{K}$ such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

An absolute value on a finite field must be trivial.

2.1.3 Valuation on a field [Definition 2.1.3]

A function $v\colon \mathbb{A}^{\times} \to \mathbb{R}$ with an integral domain $\mathbb{A}$ is called a *valuation* on $\mathbb{A}$ if it satisfies the following conditions:

i) $\forall x, y \in \mathbb{A}^{\times},\ v(xy) = v(x) + v(y)$
ii) $\forall x, y \in \mathbb{A}^{\times},\ v(x + y) \geq \min\{v(x), v(y)\}$

2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation v on a field is an additive subgroup of $\mathbb{R}$. $\operatorname{im} v$ is called the *value group* of v.

2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let $\mathbb{A}$ be an integral domain and $\mathbb{K} = \operatorname{Frac}\mathbb{A}$. Let $v\colon \mathbb{A}^{\times} \to \mathbb{R}$ be a valuation on $\mathbb{A}$ and extend v to $\mathbb{K}$ by setting $v(a/b) = v(a) - v(b)$, then the function $|\ |_v\colon \mathbb{K} \to \mathbb{R}_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on $\mathbb{K}$. Conversely, $-\log|\ |$ is a valuation on $\mathbb{K}$ for a non-archimedean absolute value $|\ |$ on $\mathbb{K}$.

2.1.6 p-adic valuation [Definition 2.1.6]

The *p-adic valuation* on $\mathbb{Q}$ with a prime p is a valuation $v_p\colon \mathbb{Q}^\times \to \mathbb{R}$ defined as follows: for each $n \in \mathbb{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \mathbb{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

2.1.7 p-adic absolute value [Definition 2.1.7]

The *p-adic absolute value* $|\ |_p\colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\ | = |\ |_\infty$.

3 Lie Algebra

3.1 Fundations

3.1.1 Lie algebra [Definition 3.1.1]

A vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with the Lie bracket satisfying the conditions

i) lie bracket is bilinear

ii) $\forall x \in \mathfrak{g},\ [x, x] = 0$

iii) $\forall x, y, z \in \mathfrak{g},\ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a *Lie algebra* over $\mathbb{K}$.

3.1.2 General linear Lie algebra [Definition 3.1.2]

$\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra $M_n(\mathbb{R})$ with the Lie bracket $[x, y] = xy - yx$.

3.1.3 Derivation algebra [Definition 3.1.3]

A linear endomorphism D of an algebra $\mathbb{A}$ over $\mathbb{R}$ satisfying $D(xy) = D(x)y + xD(y)$ is called a *derivation* of $\mathbb{A}$. The set of all derivations $\operatorname{Der}\mathbb{A}$ with the addition, scaler multiplication, and lie bracket defined as follows:

i) $(D + D')(x) = D(x) + D'(x)$

ii) $(\alpha D)(x) = \alpha D(x)$

iii) $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the *derivation algebra* of $\mathbb{A}$.

3.1.4 Lie subalgebra [Definition 3.1.4]

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a *Lie subalgebra* if it satisfies that $\forall x, y \in \mathfrak{h},\ [x, y] \in \mathfrak{h}$.

3.1.5 Special linear Lie algebra [Definition & Proposition 3.1.5]

$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \operatorname{tr} x = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

3.1.6 Orthogonal Lie algebra [Definition & Proposition 3.1.6]

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^t x = -x\}$ is a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{R})$.

4 Categories

4.1.1 Fundations

4.1.2 Category [Definition 4.1.2]

A *category* consists of the followings:

- *Objects* A, B, $C, \dots$
- *Arrows* f, g, $h, \dots$ with the objects called the domain $\mathrm{dom}(f)$ and the codomain $\mathrm{cod}(f)$.
- *Composites* $g \circ f \colon A \to C$ for given arrows $f \colon A \to B$ and $g \colon B \to C$.
- *Identity arrow* 1_A of each object A.

satisfying the following laws:

i) $\forall$arrows $f \colon A \to B, g \colon B \to C, h \colon C \to D,\ h \circ (g \circ f) = (h \circ g) \circ f$

ii) $\forall$arrow $f \colon A \to B,\ f \circ 1_A = f = 1_B \circ f$.

4.1.3 Functor between categories [Definition 4.1.3]

A *functor* $F \colon \mathscr{A} \to \mathscr{B}$ between categories $\mathscr{A}$ and $\mathscr{B}$ is a mapping between objects and between arrows in the following ways:

i) $F(f \colon A \to B) = F(f) \colon F(A) \to F(B)$,

ii) $F(1_A) = 1_{F(A)}$,

iii) $F(g \circ f) = F(g) \circ F(f)$.

参考文献

[1] Steve Awodey. *Category Theory, Second Edition.* Oxford University Press, 2010.

[2] Fernando Q. Gouvêa. *p-adic Numbers - An Introduction, Second Edition.* Springer, 1997.

[3] 佐武一郎. リー環の話. 日本評論社, 1987.

[4] Loring W. Tu. *An Introduction to Manifolds, Second Edition.* Springer, 2011.