

1 Manifolds

1.1 Manifolds on Euclidean Spaces

1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i)g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f .

1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The *tangent space* $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p .

1.1.3 Directional derivative [Definition 1.1.3]

The *directional derivative* of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + tv^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map $D: C_p^\infty \rightarrow \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a *derivation at p* or a *point-derivation* of C_p^∞ .

The set of all derivations at p $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If D is a point-derivation of C_p^∞ , then $D(c) = 0$ for any constant function c .

1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ in 1.1.4 is an isomorphism of vector spaces.

1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6, $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

1.1.8 Vector fields on an open set [Definition 1.1.8]

A *vector field* on $U \in \mathcal{O}_n$ is a map $X: U \rightarrow T_p(\mathbb{R}^n)$. $X = \sum a^i \partial / \partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

2 P-adic Numbers

2.1 Foundations

2.1.1 Absolute value on a field [Definition 2.1.1]

An *absolute value* on a field \mathbb{K} is a function $|\cdot|: \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- i) $|x| = 0$ iff $x = 0$
- ii) $\forall x, y \in \mathbb{K}, |xy| = |x| |y|$
- iii) $\forall x, y \in \mathbb{K}, |x + y| \leq |x| + |y|$.

An absolute value that satisfies the condition

- iv) $\forall x, y \in \mathbb{K}, |x + y| \leq \max\{|x|, |y|\}$

is said to be *non-archimedean*; otherwise, it is said to be *archimedean*.

2.1.2 Trivial absolute value [Definition 2.1.2]

The *trivial absolute value* on a field \mathbb{K} is a absolute value on \mathbb{K} such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

An absolute value on a finite field must be trivial.

2.1.3 Valuation on a field [Definition 2.1.3]

A function $v: \mathbb{A}^\times \rightarrow \mathbb{R}$ with an integral domain \mathbb{A} is called a *valuation* on \mathbb{A} if it satisfies the following conditions:

- i) $\forall x, y \in \mathbb{A}^\times, v(xy) = v(x) + v(y)$
- ii) $\forall x, y \in \mathbb{A}^\times, v(x + y) \geq \min\{v(x), v(y)\}$

2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation v on a field is an additive subgroup of \mathbb{R} . $\text{im } v$ is called the *value group* of v .

2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let \mathbb{A} be an integral domain and $\mathbb{K} = \text{Frac } \mathbb{A}$. Let $v: \mathbb{A}^\times \rightarrow \mathbb{R}$ be a valuation on \mathbb{A} and extend v to \mathbb{K} by setting $v(a/b) = v(a) - v(b)$, then the function $|\cdot|_v: \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on \mathbb{K} . Conversely, $-\log |\cdot|_v$ is a valuation on \mathbb{K} for a non-archimedean absolute value $|\cdot|_v$ on \mathbb{K} .

2.1.6 p -adic valuation [Definition 2.1.6]

The p -adic valuation on \mathbb{Q} with a prime p is a valuation $v_p: \mathbb{Q}^\times \rightarrow \mathbb{R}$ defined as follows: for each $n \in \mathbb{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \mathbb{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

2.1.7 p -adic absolute value [Definition 2.1.7]

The p -adic absolute value $|\cdot|_p: \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\cdot| = |\cdot|_\infty$.

3 Lie Algebra

3.1 Foundations

3.1.1 Lie algebra [Definition 3.1.1]

A vector space \mathfrak{g} over a field \mathbb{K} with the Lie bracket satisfying the conditions

- i) lie bracket is bilinear
- ii) $\forall x \in \mathfrak{g}, [x, x] = 0$
- iii) $\forall x, y, z \in \mathfrak{g}, [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a *Lie algebra* over \mathbb{K} .

3.1.2 General linear Lie algebra [Definition 3.1.2]

$\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra $M_n(\mathbb{R})$ with the Lie bracket $[x, y] = xy - yx$.

3.1.3 Derivation algebra [Definition 3.1.3]

A linear endomorphism D of an algebra \mathbb{A} over \mathbb{R} satisfying $D(xy) = D(x)y + xD(y)$ is called a *derivation* of \mathbb{A} . The set of all derivations $\text{Der } \mathbb{A}$ with the addition, scalar multiplication, and lie bracket defined as follows:

- i) $(D + D')(x) = D(x) + D'(x)$
- ii) $(\alpha D)(x) = \alpha D(x)$
- iii) $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the *derivation algebra* of \mathbb{A} .

3.1.4 Lie subalgebra [Definition 3.1.4]

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra \mathfrak{g} is a *Lie subalgebra* if it satisfies that $\forall x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$.

3.1.5 Special linear Lie algebra [Definition & Proposition 3.1.5]

$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{tr } x = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

3.1.6 Orthogonal Lie algebra [Definition & Proposition 3.1.6]

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^t x = -x\}$ is a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{R})$.

参考文献

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