1 Manifolds

1.1 Manifolds on Euclidean Spaces

1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f.

1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The *tangent space* $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p.

1.1.3 Directional derivative [Definition 1.1.3]

The *directional derivative* of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + tv^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map $D \colon C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a *derivation at* p or a *point-derivation* of C_p^∞.

The set of all derivations at p $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi \colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map $\phi \to T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ in 1.1.4 is an isomorphism of vector spaces.

1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6, $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

1.1.8 Vector fields on an open set [Definition 1.1.8]

A *vector field* on $U \in \mathcal{O}_n$ is a map $X \colon U \to T_p(\mathbb{R}^n)$. $X = \sum a^i \partial/\partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

2 P-adic Numbers

2.1 Fundations

2.1.1 Absolute value on a field [Definition 2.1.1]

An *absolute value* on a field $\mathbb{K}$ is a function $|\ |\colon \mathbb{K} \to \mathbb{R}_{\geq 0}$ that satisfies:

i) $|x| = 0$ iff $x = 0$.

ii) $\forall x, y \in \mathbb{K},\ |xy| = |x|\,|y|$.

iii) $\forall x, y \in \mathbb{K},\ |x + y| \leq |x| + |y|$.

An absolute value that satisfies the condition

iv) $\forall x, y \in \mathbb{K},\ |x + y| \leq \max\{|x|, |y|\}$

is said to be *non-archimedean*; otherwise, it is said to be *archimedean*.

2.1.2 Trivial absolute value [Definition 2.1.2]

The *trivial absolute value* on a field $\mathbb{K}$ is a absolute value on $\mathbb{K}$ such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

An absolute value on a finite field must be trivial.

2.1.3 Valuation on a field [Definition 2.1.3]

A function $v\colon \mathbb{A}^\times \to \mathbb{R}$ with an integral domain $\mathbb{A}$ is called a *valuation* on $\mathbb{A}$ if it satisfies the following conditions:

i) $\forall x, y \in \mathbb{A}^\times,\ v(xy) = v(x) + v(y)$

ii) $\forall x, y \in \mathbb{A}^\times,\ v(x + y) \geq \min\{v(x), v(y)\}$

2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation v on a field is an additive subgroup of $\mathbb{R}$. $\operatorname{im} v$ is called the *value group* of v.

2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let $\mathbb{A}$ be an integral domain and $\mathbb{K} = \operatorname{Frac}\mathbb{A}$. Let $v\colon \mathbb{A}^\times \to \mathbb{R}$ be a valuation on $\mathbb{A}$ and extend v to $\mathbb{K}$ by setting $v(a/b) = v(a) - v(b)$, then the function $|\ |_v\colon \mathbb{K} \to \mathbb{R}_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on $\mathbb{K}$. Conversely, $-\log|\ |$ is a valuation on $\mathbb{K}$ for a non-archimedean absolute value $|\ |$ on $\mathbb{K}$.

2.1.6 p-adic valuation [Definition 2.1.6]

The *p-adic valuation* on $\mathbb{Q}$ with a prime p is a valuation $v_p \colon \mathbb{Q}^\times \to \mathbb{R}$ defined as follows: for each $n \in \mathbb{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \mathbb{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

2.1.7 p-adic absolute value [Definition 2.1.7]

The *p-adic absolute value* $|\ |_p \colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\ | = |\ |_\infty$.

参考文献

[1] Fernando Q. Gouvêa. *p-adic Numbers - An Introduction, Second Edition.* Springer, 1997.

[2] Loring W. Tu. *An Introduction to Manifolds, Second Edition.* Springer, 2011.