1 Manifolds

1.1 Manifolds on Euclidean Spaces

1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f.

1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The *tangent space* $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p.

1.1.3 Directional derivative [Definition 1.1.3]

The *directional derivative* of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + tv^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map $D \colon C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a *derivation at* p or a *point-derivation* of C_p^∞.

The set of all derivations at p $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi \colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map $\phi \to T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ in 1.1.4 is an isomorphism of vector spaces.

1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6, $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

1.1.8 Vector fields on an open set [Definition 1.1.8]

A *vector field* on $U \in \mathcal{O}_n$ is a map $X\colon U \to T_p(\mathbb{R}^n)$. $X = \sum a^i \partial/\partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

参考文献

[1] Loring W. Tu. *An Introduction to Manifolds, Second Edition.* Springer, 2011.