

# Notes of Mathematics

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# 1 Manifolds<sup>[4]</sup>

## 1.1 Manifolds on Euclidean Spaces

### 1.1.1 Taylor's theorem with remainder [Theorem 1.1.1]

A smooth function  $f$  on an open ball  $U \in \mathcal{O}_n$  can be written as

$$f(x) = f(p) + \sum (x^i - p^i)g_i(x)$$

where  $p \in U$  and  $g_i \in C^\infty(U)$  with  $g_i(p) = (\partial f / \partial x^i)(p)$ .

Adapting this to  $g_i$  repeatedly gives the Taylor's expansion of  $f$ .

### 1.1.2 Tangent vector as an arrow from a point [Definition 1.1.2]

The **tangent space**  $T_p(\mathbb{R}^n)$  at  $p \in \mathbb{R}^n$  is the set of arrows from  $p$ .

### 1.1.3 Directional derivative [Definition 1.1.3]

The **directional derivative** of a smooth function  $f$  in the direction  $v \in T_p(\mathbb{R}^n)$  at  $p \in \mathbb{R}^n$  is

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with  $c^i(t) = p^i + tv^i$ .

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

### 1.1.4 Derivation at a point [Definition & Proposition 1.1.4]

A linear map  $D: C_p^\infty \rightarrow \mathbb{R}$  satisfying the Leibniz rule (i.e.,  $D(fg) = (Df)g(p) + f(p)Dg$  for any  $f, g \in C_p^\infty$ ) is called a **derivation** at  $p$  or a **point-derivation** of  $C_p^\infty$ .

The set of all derivations at  $p$  denoted by  $\mathcal{D}_p(\mathbb{R}^n)$  is a real vector space, and a map  $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$  assigning  $D_v$  to each  $v$  is a linear map.

### 1.1.5 Point-derivation of a constant is zero [Lemma 1.1.5]

If  $D$  is a point-derivation of  $C_p^\infty$ , then  $D(c) = 0$  for any constant function  $c$ .

### 1.1.6 Tangent space is isomorphic to the set of point-derivations [Theorem 1.1.6]

The linear map  $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$  in 1.1.4 is an isomorphism of vector spaces.

### 1.1.7 Tangent vector as a derivation [Definition 1.1.7]

By 1.1.6,  $v \in T_p(\mathbb{R}^n)$  is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

### 1.1.8 Vector fields on an open set [Definition 1.1.8]

A **vector field** on  $U \in \mathcal{O}_n$  is a map  $X: U \rightarrow T_p(\mathbb{R}^n)$ .  $X = \sum a^i \partial / \partial x^i$  means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

$X$  is said to be  $C^\infty$  if all  $a^i$ 's are  $C^\infty$  on  $U$ . The set of all smooth vector fields on  $U$  is denoted by  $\mathfrak{X}(U)$ .

### 1.1.9 Multiplication of a smooth vector field and function [Definition & Proposition 1.1.9]

For  $X \in \mathfrak{X}(U)$  and  $f \in C^\infty(U)$ , define  $fX \in \mathfrak{X}(U)$  and  $Xf \in C^\infty(U)$  as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left. \frac{\partial}{\partial x^i} \right|_p,$$

$$(Xf)(p) = X_p f = \sum a^i(p) \left. \frac{\partial f}{\partial x^i} \right|_p.$$

### 1.1.10 Leibniz rule for a vector field [Proposition 1.1.10]

For any  $X \in \mathfrak{X}(U)$ ,  $f, g \in C^\infty(U)$ ,

$$X(fg) = (Xf)g + fXg.$$

### 1.1.11 Derivations one-to-one-correspond to smooth vector fields [Proposition 1.1.11]

$\varphi: \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \text{Der}(C^\infty(U))$  is a linear isomorphism.

### 1.1.12 k-tensor on a vector space [Definition 1.1.12]

A  $k$ -linear function on a vector space  $V: V^k \rightarrow \mathbb{R}$  is called a **k-tensor** on  $V$ . The vector space of all  $k$ -tensors on  $V$  is denoted by  $L_k(V)$ .  $k$  is called the degree of  $f$ .

### 1.1.13 Permutation action on k-tensors [Definition 1.1.13]

For  $f \in L_k(V)$  on a vector space  $V$  and  $\sigma \in \mathfrak{S}_n$ , an action of  $\sigma$  on  $f$  is defined by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

#### 1.1.14 Symmetric and alternating k-tensor [Definition 1.1.14]

A  $k$ -tensor  $f: V^k \rightarrow \mathbb{R}$  is **symmetric** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = f,$$

and  $f$  is **alternating** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = (\text{sgn } \sigma) f.$$

#### 1.1.15 The set of all alternating k-tensors [Definition 1.1.15]

An alternating  $k$ -tensor on a vector space  $V$  is also called a  **$k$ -covector** or a **multicovector of degree  $k$**  on  $V$ . The set of all  $k$ -covectors on  $V$  is denoted by  $A_k(V)$  for  $k > 0$ ; for  $k = 0$ ,  $A_0(V) = \mathbb{R}$ .

#### 1.1.16 Symmetrizing and alternating operators on k-covectors [Definition & Proposition 1.1.16]

For a  $f \in A_k(V)$  on a vector space  $V$ ,

$$Sf = \sum_{\sigma \in \mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma \in \mathfrak{S}_n} (\text{sgn } \sigma) \sigma f$$

is alternating.

#### 1.1.17 Tensor product of two multilinear functions [Definition 1.1.17]

For  $f \in L_k(V), g \in L_\ell(V)$  on a vector space  $V$ , the **tensor product**  $f \otimes g \in L_{k+\ell}(V)$  is defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}).$$

#### 1.1.18 Bilinear map as a tensor product [Example 1.1.18]

Let  $e_1, \dots, e_n$  be a basis for a vector space  $V$ ,  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ , and  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$  a bilinear map on  $V$ . Then,

$$\langle \cdot, \cdot \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j,$$

where  $g_{ij} = \langle e_i, e_j \rangle$ .

### 1.1.19 Wedge product of two multilinear functions [Definition 1.1.19]

For  $f \in A_k(V)$ ,  $g \in A_\ell(V)$  on a vector space  $V$ , their **wedge product** or **exterior product** is

$$f \wedge g = \frac{1}{k! \ell!} A(f \otimes g).$$

$f \wedge g$  is alternating.

Explicitly,

$$\begin{aligned} (f \wedge g)(v_1, \dots, v_{k+\ell}) &= \frac{1}{k! \ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} (\text{sgn } \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\ &= \sum_{\substack{(k, \ell)\text{-shuffle} \\ \sigma}} (\text{sgn } \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}), \end{aligned}$$

where a  $(k, \ell)$ -shuffle means  $\sigma(1) < \dots < \sigma(k)$  and  $\sigma(k+1) < \dots < \sigma(k+\ell)$ .

### 1.1.20 Wedge product is anticommutative [Proposition 1.1.20]

For  $f \in A_k(V)$ ,  $g \in A_\ell(V)$  on a vector space  $V$ ,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

If the degree of  $f$  is odd, then  $f \wedge f = 0$ .

### 1.1.21 Properties of nesting alternating operators [Lemma 1.1.21]

For a  $k$ -tensor  $f$  and  $\ell$ -tensor  $g$  on a vector space  $V$ ,

- i)  $A(A(f) \otimes g) = k! A(f \otimes g)$ ,
- ii)  $A(f \otimes A(g)) = \ell! A(f \otimes g)$ .

### 1.1.22 Associativity of the wedge product [Proposition 1.1.22]

For  $f \in A_k(V)$ ,  $g \in A_\ell(V)$ ,  $h \in A_m(V)$  on a real vector space  $V$ ,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for  $f_i \in A_{d_i}(V)$  ( $i = 1, \dots, r$ ),

$$f_1 \wedge \dots \wedge f_r = \frac{1}{(d_1)! \dots (d_r)!} A(f_1 \otimes \dots \otimes f_r).$$

### 1.1.23 Wedge product of covectors is the determinant [Proposition 1.1.23]

For covectors  $\alpha^1, \dots, \alpha^k$  on a vector space  $V$ ,

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

### 1.1.24 Graded algebra over a field [Definition 1.1.24]

An algebra  $\mathbb{A}$  over a field  $\mathbb{K}$  is said to be **graded** if  $\mathbb{A} = \bigoplus_{k=0}^{\infty} A^k$  is a direct sum of vector spaces over  $\mathbb{K}$  such that the multiplication sends  $A^k \times A^l$  to  $A^{k+l}$ .  $A = \bigoplus_{k=0}^{\infty} A^k$  means each nonzero  $a \in \mathbb{A}$  is uniquely a finite sum  $a = a_{i_1} + \dots + a_{i_m}$  where nonzero  $a_{i_j} \in A^{i_j}$ .

$\mathbb{A}$  is **anticommutative** or **graded commutative** if  $\forall a \in A^k, b \in A^\ell, ab = (-1)^{k\ell} ba$ .

A **homomorphism** of graded algebras is an algebra homomorphism that preserves the degree.

### 1.1.25 Grassmann algebra of multivectors on a vector space [Definition & Proposition 1.1.25]

For a vector space  $V$  of degree  $n < \infty$ , the **exterior algebra** or the **Grassmann algebra** of multivectors on  $V$  is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^n A_k(V)$$

with the wedge product of multivectors as multiplication.

### 1.1.26 Wedge product of the dual basis applying to a basis [Lemma 1.1.26]

Let  $e_1, \dots, e_n$  be a basis for a vector space  $V$  and  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ . For  $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k)$  with  $1 \leq i_1 < \dots < i_k \leq n, 1 \leq j_1 < \dots < j_k \leq n$ ,

$$\alpha^I(e_J) = \delta_J^I.$$

### 1.1.27 Wedge products of the dual basis form a basis for multivectors [Proposition 1.1.27]

Let  $V$  be a vector space and  $\alpha^1, \dots, \alpha^n$  the dual basis in  $V^*$ . Then,  $\alpha^I, I = (i_1 < \dots < i_k)$  form a basis for  $A_k(V)$ .

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

### 1.1.28 Cotangent space to an Euclidean space at a point [Definition 1.1.28]

The **cotangent space** to  $\mathbb{R}^n$  at  $p$  is  $T_p^*(\mathbb{R}^n) = (T_p(\mathbb{R}^n))^*$ .

### 1.1.29 Differential 1-form on an open subset of an Euclidean space [Definition 1.1.29]

A **covector field** or a **differential 1-form** on  $U \in \mathcal{O}_n$  is  $\omega: U \rightarrow \bigcup_{p \in U} T_p^*(\mathbb{R}^n)$  that maps  $U \ni p \mapsto \omega_p \in T_p^*(\mathbb{R}^n)$ .

### 1.1.30 Differential of a smooth function [Definition 1.1.30]

For  $f \in C^\infty(U)$  on  $U \in \mathcal{O}_n$ , the **differential**  $df$  of  $f$  is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$

In the expression

$$\langle \cdot, \cdot \rangle: T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \ni (X_p, f) \mapsto \langle X_p, f \rangle = X_p f \in \mathbb{R},$$

a tangent vector is considered as  $\langle X_p, \cdot \rangle$ ; a differential at  $p$  as  $df|_p = (df)_p = \langle \cdot, f \rangle$ .

### 1.1.31 Differentials of coordinates is the dual basis for the cotangent space [Proposition 1.1.31]

For  $p \in \mathbb{R}^n$ ,  $\{(dx^1)_p, \dots, (dx^n)_p\}$  is the dual basis for  $T_p^*(\mathbb{R}^n)$  to  $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\} \subset T_p(\mathbb{R}^n)$ , where  $x^1, \dots, x^n$  are the standard coordinates on  $\mathbb{R}^n$ .

For any differential 1-form  $\omega$  on  $U \in \mathcal{O}_n$  and  $p \in U$ ,

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some  $a_i(p)$ . In this case,  $\omega$  is written as  $\omega = \sum a_i dx^i$ .

### 1.1.32 Smoothness of a differential 1-form [Definition 1.1.32]

A differential 1-form  $\omega = \sum a_i dx^i$  on  $U \in \mathcal{O}_n$  is **smooth** if all  $a_i: U \rightarrow \mathbb{R}$  are smooth.

### 1.1.33 Differentials can be written in terms of partial derivatives [Proposition 1.1.33]

For  $f \in C^\infty(U)$  on  $U \in \mathcal{O}_n$ ,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness of  $f$  implies that of  $df$ .

### 1.1.34 Differential $k$ -forms on an Euclidean space [Definition 1.1.34]

A **differential  $k$ -form** or **differential form of degree  $k$**  on  $U \in \mathcal{O}_n$  is  $\omega: U \ni p \mapsto \omega_p \in A_k(T_p(\mathbb{R}^n))$ .

### 1.1.35 Basis for differential forms [Definition & Proposition 1.1.35]

Since  $\{dx_p^I \mid I = (1 \leq i_1 < \dots < i_k \leq n)\}$  is a basis for  $A_k(T_p(\mathbb{R}^n))$ , for a differential  $k$ -form  $\omega$  on  $U \in \mathcal{O}_n$  and  $p \in U$ ,

$$\omega_p = \sum a_I(p) dx_p^I, \quad \omega = \sum a_I dx^I.$$

$\omega$  is **smooth** if all  $a_I: U \rightarrow \mathbb{R}$  are smooth. The vector space of  $C^\infty$  differential  $k$ -forms on  $U$  is denoted by  $\Omega^k(U)$ . If  $k = 0$ ,  $\Omega^0(U) = C^\infty(U)$ .

### 1.1.36 Wedge product of differential forms [Definition 1.1.36]

For differential  $k$ -form  $\omega$  and  $\ell$ -form  $\tau$  on  $U \in \mathcal{O}_n$ , their **wedge product**  $\omega \wedge \tau$  is a differential  $(k + \ell)$ -form defined by

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

If  $\omega = \sum a_I dx^I$ ,  $\tau = \sum b_J dx^J$ ,

$$\begin{aligned} \omega \wedge \tau &= \sum_{I,J} (a_I b_J) dx^I \wedge dx^J \\ &= \sum_{\text{disjoint } I,J} (a_I b_J) dx^I \wedge dx^J. \end{aligned}$$

For  $\omega \in \Omega^k(U)$ ,  $\tau \in \Omega^\ell(U)$ , the wedge product is a bilinear map

$$\wedge: \Omega^k(U) \times \Omega^\ell(U) \rightarrow \Omega^{k+\ell}(U).$$

In particular, if  $f \in C^\infty(U)$  and  $\omega \in \Omega^k(U)$ , then  $f \wedge \omega = f\omega$ .

### 1.1.37 Graded algebra with smooth differential forms [Definition 1.1.37]

For  $U \in \mathcal{O}_n$ , the direct sum  $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$  is an anticommutative graded algebra over  $\mathbb{R}$  with the wedge product as multiplication, which is also a module over  $C^\infty(U)$ .

### 1.1.38 Differential forms as linear maps on a vector field [Definition 1.1.38]

For a differential  $k$ -form  $\omega$  on  $U \in \mathcal{O}_n$  and  $X_1, \dots, X_k \in \mathfrak{X}(U)$ , define  $\omega(X_1, \dots, X_k) \in C^\infty(U)$  by

$$(\omega(X_1, \dots, X_k))_p = \omega_p((X_1)_p, \dots, (X_k)_p).$$



The map

$$\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) \ni (X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k) \in C^\infty(U)$$

is  $k$ -linear over  $C^\infty(U)$ .

#### 1.1.39 Exterior derivatives of differential forms [Definition 1.1.39]

For  $k \geq 1$  and  $\omega = \sum a_I dx^I \in \Omega^k(U)$ , the **exterior derivative** of  $\omega$  is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U);$$

for  $k = 0$  and  $f \in C^\infty(U)$ , its exterior derivative is

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

#### 1.1.40 Antiderivation of a graded algebra [Definition 1.1.40]

An **antiderivation** of a graded algebra  $\mathbb{A} = \bigoplus_{k=0}^\infty A^k$  is a linear map  $D: \mathbb{A} \rightarrow \mathbb{A}$  such that for  $a \in A^k, b \in A^\ell$ ,

$$D(ab) = D(a)b + (-1)^k aD(b).$$

If  $m$  is an integer such that  $D$  sends  $A^k$  to  $A^{k+m}$  for all  $k$ , then  $m$  is called the **degree** of  $D$ .

#### 1.1.41 Properties of the exterior differentiation [Proposition 1.1.41]

- i) The exterior differentiation  $d: \Omega^*(U) \rightarrow \Omega^*(U)$  on  $U \in \mathcal{O}_n$  is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

- ii)  $d^2 = 0$ .

- iii) For  $f \in C^\infty(U)$  and  $X \in \mathfrak{X}(U)$ ,  $(df)(X) = Xf$ .

#### 1.1.42 Characterization of the exterior differentiation [Proposition 1.1.42]

The exterior differentiation  $d: \Omega^*(U) \rightarrow \Omega^*(U)$  on  $U \in \mathcal{O}_n$  is the only antiderivation of  $\Omega^*(U)$ .

## 2 P-adic Numbers<sup>[2]</sup>

### 2.1 Foundations

#### 2.1.1 Absolute value on a field [Definition 2.1.1]

An **absolute value** on a field  $\mathbb{K}$  is a function  $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- i)  $|x| = 0$  iff  $x = 0$
- ii)  $\forall x, y \in \mathbb{K}, |xy| = |x| |y|$
- iii)  $\forall x, y \in \mathbb{K}, |x + y| \leq |x| + |y|$ .

An absolute value that satisfies the condition

- iv)  $\forall x, y \in \mathbb{K}, |x + y| \leq \max\{|x|, |y|\}$

is said to be **non-archimedean**; otherwise, it is said to be **archimedean**.

#### 2.1.2 Trivial absolute value [Definition 2.1.2]

The **trivial absolute value** on a field  $\mathbb{K}$  is a absolute value on  $\mathbb{K}$  such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

An absolute value on a finite field must be trivial.

#### 2.1.3 Valuation on a field [Definition 2.1.3]

A function  $v : \mathbb{A}^\times \rightarrow \mathbb{R}$  with an integral domain  $\mathbb{A}$  is called a **valuation** on  $\mathbb{A}$  if it satisfies the following conditions:

- i)  $\forall x, y \in \mathbb{A}^\times, v(xy) = v(x) + v(y)$
- ii)  $\forall x, y \in \mathbb{A}^\times, v(x + y) \geq \min\{v(x), v(y)\}$

#### 2.1.4 Value group of a valuation [Definition & Proposition 2.1.4]

The image of a valuation  $v$  on a field is an additive subgroup of  $\mathbb{R}$ .  $\text{im } v$  is called the **value group** of  $v$ .

### 2.1.5 Correspondence between valuations and non-archimedean absolute values [Proposition 2.1.5]

Let  $\mathbb{A}$  be an integral domain and  $\mathbb{K} = \text{Frac } \mathbb{A}$ . Let  $v: \mathbb{A}^\times \rightarrow \mathbb{R}$  be a valuation on  $\mathbb{A}$  and extend  $v$  to  $\mathbb{K}$  by setting  $v(a/b) = v(a) - v(b)$ , then the function  $|\cdot|_v: \mathbb{K} \rightarrow \mathbb{R}_{\geq 0}$  defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on  $\mathbb{K}$ . Conversely,  $-\log |\cdot|$  is a valuation on  $\mathbb{K}$  for a non-archimedean absolute value  $|\cdot|$  on  $\mathbb{K}$ .

### 2.1.6 p-adic valuation [Definition 2.1.6]

The **p-adic valuation** on  $\mathbb{Q}$  with a prime  $p$  is a valuation  $v_p: \mathbb{Q}^\times \rightarrow \mathbb{R}$  defined as follows: for each  $n \in \mathbb{Z}^\times$ , let  $v_p(n)$  be the greatest integer such that  $p^{v_p(n)} \mid n$ , and for each  $x = a/b \in \mathbb{Q}^\times$ ,  $v_p(x) = v_p(a) - v_p(b)$ .

We often set  $v_p(0) = \infty$ .

### 2.1.7 p-adic absolute value [Definition 2.1.7]

The **p-adic absolute value**  $|\cdot|_p: \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0}$  with a prime  $p$  is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as  $|\cdot| = |\cdot|_\infty$ .

## 3 Lie Algebra<sup>[3]</sup>

### 3.1 Foundations

#### 3.1.1 Lie algebra [Definition 3.1.1]

A vector space  $\mathfrak{g}$  over a field  $\mathbb{K}$  with the Lie bracket satisfying the conditions

- i) Lie bracket is bilinear
- ii)  $\forall x \in \mathfrak{g}, [x, x] = 0$
- iii)  $\forall x, y, z \in \mathfrak{g}, [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a **Lie algebra** over  $\mathbb{K}$ .

#### 3.1.2 General linear Lie algebra [Definition 3.1.2]

$\mathfrak{gl}_n(\mathbb{R})$  is the Lie algebra  $M_n(\mathbb{R})$  with the Lie bracket  $[x, y] = xy - yx$ .

#### 3.1.3 Derivation algebra [Definition 3.1.3]

A linear endomorphism  $D$  of an algebra  $\mathbb{A}$  over  $\mathbb{R}$  satisfying  $D(xy) = D(x)y + xD(y)$  is called a **derivation** of  $\mathbb{A}$ . The set of all derivations  $\text{Der } \mathbb{A}$  with the addition, scalar multiplication, and lie bracket defined as follows:

- i)  $(D + D')(x) = D(x) + D'(x)$
- ii)  $(\alpha D)(x) = \alpha D(x)$
- iii)  $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the **derivation algebra** of  $\mathbb{A}$ .

#### 3.1.4 Lie subalgebra [Definition 3.1.4]

A linear subspace  $\mathfrak{h} \subset \mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$  is a **Lie subalgebra** of  $\mathfrak{g}$  if  $\forall x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$ .

For linear subspaces  $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ ,  $[\mathfrak{a}, \mathfrak{b}]$  denotes the subspace generated by  $[x, y]$  with  $x \in \mathfrak{a}, y \in \mathfrak{b}$ .

#### 3.1.5 Special linear Lie algebra [Definition & Proposition 3.1.5]

$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{tr } x = 0\}$  is a Lie subalgebra of  $\mathfrak{gl}_n(\mathbb{R})$ .

### 3.1.6 Orthogonal Lie algebra [Definition & Proposition 3.1.6]

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^t x = -x\}$  is a Lie subalgebra of  $\mathfrak{sl}_n(\mathbb{R})$ .

### 3.1.7 Ideal of a Lie algebra [Definition & Proposition 3.1.7]

A linear subspace  $\mathfrak{h} \subset \mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$  is an **ideal** of  $\mathfrak{g}$  if  $\forall x \in \mathfrak{g}, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$ .

For ideals  $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ ,  $[\mathfrak{a}, \mathfrak{b}]$  is also an ideal.

### 3.1.8 Derived ideal of a Lie algebra [Definition 3.1.8]

For a Lie algebra  $\mathfrak{g}$ ,  $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$  is an ideal of  $\mathfrak{g}$  called the **derived ideal** of  $\mathfrak{g}$ .

If  $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ ,  $D\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ .

### 3.1.9 Homomorphism of Lie algebras [Definition & Proposition 3.1.9]

For Lie algebras  $\mathfrak{g}, \mathfrak{h}$ , a linear map  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$  is called a **homomorphism** if  $\forall x, y \in \mathfrak{g}, \varphi([x, y]) = [\varphi(x), \varphi(y)]$ . A homomorphism  $\varphi$  is an **isomorphism** if it is bijective. Lie algebras between which there exists an isomorphism are said to be **isomorphic** to each other, written as  $\mathfrak{g} \cong \mathfrak{h}$ .

A composite of homomorphisms is also a homomorphism, and that of isomorphisms is also an isomorphism.

The kernel  $\ker \varphi = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$  of a homomorphism  $\varphi$  is an ideal of  $\mathfrak{g}$  while the image  $\text{im } \varphi = \varphi(\mathfrak{g})$  of  $\varphi$  is a Lie subalgebra of  $\mathfrak{h}$ .

### 3.1.10 Representation of a Lie algebra on a vector space [Definition 3.1.10]

For a Lie algebra  $\mathfrak{g}$  and a vector space  $V$ , a homomorphism  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is called a **representation** of  $\mathfrak{g}$  on  $V$ .

### 3.1.11 Adjoint representation of a Lie algebra [Definition & Proposition 3.1.11]

For a Lie algebra  $\mathfrak{g}$  and  $x \in \mathfrak{g}$ , define a derivation  $\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g}$  by  $\text{ad}(x)(y) = [x, y]$ . A representation  $\text{ad}: \mathfrak{g} \ni x \mapsto \text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$  is called the **adjoint representation** of  $\mathfrak{g}$ . The **center** of  $\mathfrak{g}$  is  $\mathfrak{z} = \ker(\text{ad})$ , which is a commutative ideal.  $\text{im}(\text{ad})$  is an ideal of  $\text{Der } \mathfrak{g}$ . A derivation  $\text{ad}(x)$  is called a **inner derivation** of  $\mathfrak{g}$ .

### 3.1.12 Quotient algebra for Lie algebras [Definition 3.1.12]

For a Lie algebra  $\mathfrak{g}$  and an ideal  $\mathfrak{a} \subset \mathfrak{g}$ , the *quotient algebra* is

$$\mathfrak{g}/\mathfrak{a} = \{\bar{x} = x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

with canonical operations, where  $\bar{x} = \{y \in \mathfrak{g} \mid x \equiv y \pmod{\mathfrak{a}}\} = \{x + a \mid a \in \mathfrak{a}\}$  called the *class* of  $x$ . The homomorphism  $\varphi: \mathfrak{g} \ni x \mapsto \bar{x} \in \mathfrak{g}/\mathfrak{a}$  is called the *canonical homomorphism*.

### 3.1.13 The first isomorphism theorem for Lie algebras [Theorem 3.1.13]

For Lie algebras  $\mathfrak{g}, \mathfrak{h}$  and a homomorphism  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ ,

$$\mathfrak{g}/\ker \varphi \cong \text{im } \varphi.$$

### 3.1.14 The second isomorphism theorem for Lie algebras [Theorem 3.1.14]

For a Lie algebra  $\mathfrak{g}$ , an ideal  $\mathfrak{a} \subset \mathfrak{g}$ , a Lie subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and the canonical homomorphism  $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$ ,

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{a}) \cong (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}.$$

## 3.2 Solvable and Nilpotent Lie algebra

### 3.2.1 Solvable Lie algebra [Definition 3.2.1]

Let  $\mathfrak{g}$  be a Lie algebra, and

$$D^0 \mathfrak{g} = \mathfrak{g}, \quad D^k \mathfrak{g} = D(D^{k-1} \mathfrak{g}), \quad k = 1, 2, \dots$$

$\mathfrak{g}$  is said to be *solvable* if  $D^r \mathfrak{g} = \{0\}$  for some  $r$  called the *length* of  $\mathfrak{g}$ .

### 3.2.2 Lie algebra of triangular matrices is solvable [Example 3.2.2]

Let

$$\begin{aligned} \mathfrak{g}_0 &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbb{R}) \mid \xi \text{ is upper triangular}\}, \\ \mathfrak{g}_k &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbb{R}) \mid \xi_{ij} = 0 \text{ for } j - i < k\}. \end{aligned}$$

Then,  $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_1$ ,  $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$ ,  $k, \ell = 0, 1, \dots$ , and  $\mathfrak{g}_0$  is a solvable Lie algebra of length  $\leq n$ .

### 3.2.3 Lie subalgebra of a solvable Lie algebra is also solvable [Theorem 3.2.3]

For a solvable Lie algebra  $\mathfrak{g}$ , its Lie subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is also solvable, and if  $\mathfrak{h}$  is an ideal,  $\mathfrak{g}/\mathfrak{h}$  is also solvable.

**3.2.4 Lie algebra whose ideal and quotient algebra over it are solvable is solvable [Theorem 3.2.4]**

For a Lie algebra  $\mathfrak{g}$  and its ideal  $\mathfrak{a} \subset \mathfrak{g}$ , if  $\mathfrak{a}$  and  $\mathfrak{g}/\mathfrak{a}$  are both solvable, then  $\mathfrak{g}$  is also solvable.

**3.2.5 Nilpotent Lie algebra [Definition 3.2.5]**

Let  $\mathfrak{g}$  be a Lie algebra, and

$$C^0 \mathfrak{g} = \mathfrak{g}, \quad C^k \mathfrak{g} = [\mathfrak{g}, C^{k-1} \mathfrak{g}], \quad k = 1, 2, \dots$$

$\mathfrak{g}$  is said to be **nilpotent** if  $C^s \mathfrak{g} = \{0\}$  for some  $s$  called the **length** of  $\mathfrak{g}$ .

Since  $D^k \mathfrak{g} \subset C^k \mathfrak{g}$ , a nilpotent Lie algebra is solvable.

**3.2.6 Lie algebra of strictly triangular matrices is nilpotent [Example 3.2.6]**

$\mathfrak{g}_1$  in 3.2.2 is nilpotent while  $\mathfrak{g}_0$  there is not.

**3.2.7 Lie subalgebra of a nilpotent Lie algebra is also nilpotent [Theorem 3.2.7]**

For a nilpotent Lie algebra  $\mathfrak{g}$ , its Lie subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is also nilpotent, and if  $\mathfrak{h}$  is an ideal,  $\mathfrak{g}/\mathfrak{h}$  is also nilpotent.

**3.2.8 Center of a nilpotent Lie algebra has a nonzero vector [Theorem 3.2.8]**

For a Lie algebra  $\mathfrak{g}$  and its center  $\mathfrak{z}$ ,  $\mathfrak{z} \neq \{0\}$  if  $\mathfrak{g}$  is nilpotent while  $\mathfrak{g}$  is nilpotent if  $\mathfrak{g}/\mathfrak{z}$  is nilpotent.

## 4 Categories<sup>[1]</sup>

### 4.1 Foundations

#### 4.1.1 Category [Definition 4.1.1]

A *category* consists of the followings:

- **Objects**  $A, B, C, \dots$
- **Arrows**  $f, g, h, \dots$  with the objects called the domain  $\text{dom}(f)$  and the codomain  $\text{cod}(f)$ .
- **Composites**  $g \circ f: A \rightarrow C$  for given arrows  $f: A \rightarrow B$  and  $g: B \rightarrow C$ .
- **Identity arrow**  $1_A$  of each object  $A$ .

satisfying the following laws:

- i)  $\forall \text{arrows } f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D, h \circ (g \circ f) = (h \circ g) \circ f$
- ii)  $\forall \text{arrow } f: A \rightarrow B, f \circ 1_A = f = 1_B \circ f.$

#### 4.1.2 Functor between categories [Definition 4.1.2]

A **functor**  $F: \mathcal{A} \rightarrow \mathcal{B}$  between categories  $\mathcal{A}$  and  $\mathcal{B}$  is a mapping between objects and between arrows in the following ways:

- i)  $F(f: A \rightarrow B) = F(f): F(A) \rightarrow F(B),$
- ii)  $F(1_A) = 1_{F(A)},$
- iii)  $F(g \circ f) = F(g) \circ F(f).$

#### 4.1.3 Isomorphism between categories [Definition 4.1.3]

In a category  $\mathcal{C}$ , an arrow  $f: A \rightarrow B$  is called an **isomorphism** if

$$\exists g = f^{-1}: B \rightarrow A, g \circ f = 1_A, f \circ g = 1_B.$$

If there is an isomorphism between objects  $A$  and  $B$ ,  $A$  is said to be **isomorphic** to  $B$ , written  $A \cong B$ .



## References

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