

Notes of Mathematics

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Contents

1	MANIFOLDS	1
1.1	Manifolds on Euclidean Spaces	1
1.2	Manifolds	12
2	P-ADIC NUMBERS	18
2.1	Foundations	18
3	LIE ALGEBRAS	21
3.1	Foundations	21
3.2	Solvable and Nilpotent Lie algebras	24
4	CATEGORIES	26
4.1	Foundations	26
5	FUNCTIONAL ANALYSIS	29
5.1	Hahn-Banach Theorems	29
6	REAL ANALYSIS AND PROBABILITY	34
6.1	Set Theory	34
7	HOMOLOGICAL ALGEBRA	35
7.1	Chain Complexes	35

1 MANIFOLDS

[6]

1.1 Manifolds on Euclidean Spaces

Theorem 1.1.1 Taylor's theorem with remainder

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f .

Definition 1.1.2 Tangent vector as an arrow from a point

The **tangent space** $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the set of arrows from p .

Definition 1.1.3 Directional derivative

The **directional derivative** of a smooth function f in the direction $v \in T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + t v^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

Definition & Proposition 1.1.4 Derivation at a point

A linear map $D: C_p^\infty \rightarrow \mathbb{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a **derivation** at p or a **point-derivation** of C_p^∞ .

The set of all derivations at p denoted by $\mathcal{D}_p(\mathbb{R}^n)$ is a real vector space, and a map $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ assigning D_v to each v is a linear map.

Lemma 1.1.5 Point-derivation of a constant is zero

If D is a point-derivation of C_p^∞ , then $D(c) = 0$ for any constant function c .

Theorem 1.1.6 Tangent space is isomorphic to the set of point-derivations

The linear map $\phi: T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ in [Definition & Proposition 1.1.4] is an isomorphism of vector spaces.

Definition 1.1.7 Tangent vector as a derivation

By [Theorem 1.1.6], $v \in T_p(\mathbb{R}^n)$ is identified as

$$v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \in \mathcal{D}_p(\mathbb{R}^n).$$

Definition 1.1.8 Vector fields on an open set

A **vector field** on $U \in \mathcal{O}_n$ is a map $X: U \rightarrow T_p(\mathbb{R}^n)$. $X = \sum a^i \partial / \partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

X is said to be C^∞ if all a^i 's are C^∞ on U . The set of all smooth vector fields on U is denoted by $\mathfrak{X}(U)$.

Definition & Proposition 1.1.9 Multiplication of a smooth vector field and function

For $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, define $fX \in \mathfrak{X}(U)$ and $Xf \in C^\infty(U)$ as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left. \frac{\partial}{\partial x^i} \right|_p,$$

$$(Xf)(p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p).$$

Proposition 1.1.10 Leibniz rule for a vector field

For any $X \in \mathfrak{X}(U), f, g \in C^\infty(U)$,

$$X(fg) = (Xf)g + fXg.$$

Proposition 1.1.11 Derivations one-to-one-correspond to smooth vector fields

$\varphi: \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \text{Der}(C^\infty(U))$ is an linear isomorphism.

Definition 1.1.12 k-tensor on a vector space

A k -linear function $f: V^k \rightarrow \mathbf{R}$ on a vector space V is called a **k -tensor** on V . The vector space of all k -tensors on V is denoted by $L_k(V)$. k is called the degree of f .

Definition 1.1.13 Permutation action on k-tensors

For $f \in L_k(V)$ on a vector space V and $\sigma \in \mathfrak{S}_n$, an action of σ on f is defined by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Definition 1.1.14 Symmetric and alternating k-tensor

A k -tensor $f: V^k \rightarrow \mathbf{R}$ is **symmetric** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = f,$$

and f is **alternating** if

$$\forall \sigma \in \mathfrak{S}_k, \sigma f = (\text{sgn } \sigma) f.$$

Definition 1.1.15 The set of all alternating k-tensors

An alternating k -tensor on a vector space V is also called a **k -covector** or a **multicovector of degree k** on V . The set of all k -covectors on V is denoted by $A_k(V)$ for $k > 0$; for $k = 0$, $A_0(V) = \mathbf{R}$.

Definition & Proposition 1.1.16 Symmetrizing and alternating operators on k-covectors

For a $f \in A_k(V)$ on a vector space V ,

$$Sf = \sum_{\sigma \in \mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma \in \mathfrak{S}_n} (\text{sgn } \sigma) \sigma f$$

is alternating.

Definition 1.1.17 Tensor product of two multilinear functions

For $f \in L_k(V)$, $g \in L_\ell(V)$ on a vector space V , the **tensor product** $f \otimes g \in L_{k+\ell}(V)$ is defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) g(v_{k+1}, \dots, v_{k+\ell}).$$

Example 1.1.18 Bilinear map as a tensor product

Let e_1, \dots, e_n be a basis for a vector space V , $\alpha^1, \dots, \alpha^n$ the dual basis in V^* , and $\langle, \rangle: V \times V \rightarrow \mathbf{R}$ a bilinear map on V . Then,

$$\langle, \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j,$$

where $g_{ij} = \langle e_i, e_j \rangle$.

Definition 1.1.19 Wedge product of two multilinear functions

For $f \in A_k(V)$, $g \in A_\ell(V)$ on a vector space V , their **wedge product** or **exterior product** is

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g).$$

$f \wedge g$ is alternating.

Explicitly,

$$\begin{aligned} (f \wedge g)(v_1, \dots, v_{k+\ell}) &= \frac{1}{k!\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\ &= \sum_{\substack{(k,\ell)\text{-shuffle} \\ \sigma}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}), \end{aligned}$$

where a (k, ℓ) -shuffle means $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+\ell)$.

Proposition 1.1.20 Wedge product is anticommutative

For $f \in A_k(V)$, $g \in A_\ell(V)$ on a vector space V ,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

If the degree of f is odd, then $f \wedge f = 0$.

Lemma 1.1.21 Properties of nesting alternating operators

For $f \in L_k(V)$ and $g \in L_\ell(V)$ on a vector space V ,

- i) $A(A(f) \otimes g) = k! A(f \otimes g)$,
- ii) $A(f \otimes A(g)) = \ell! A(f \otimes g)$.

Proposition 1.1.22 Associativity of the wedge product

For $f \in A_k(V)$, $g \in A_\ell(V)$, $h \in A_m(V)$ on a real vector space V ,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for $f_i \in A_{d_i}(V)$ ($i = 1, \dots, r$),

$$f_1 \wedge \dots \wedge f_r = \frac{1}{(d_1)! \dots (d_r)!} A(f_1 \otimes \dots \otimes f_r).$$

Proposition 1.1.23 Wedge product of covectors is the determinant

For covectors $\alpha^1, \dots, \alpha^k$ on a vector space V ,

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

Definition 1.1.24 Graded algebra over a field

An algebra A over a field K is said to be **graded** if $A = \bigoplus_{k=0}^{\infty} A^k$ is a direct sum of vector spaces over K such that the multiplication sends $A^k \times A^\ell$ to $A^{k+\ell}$. $A = \bigoplus_{k=0}^{\infty} A^k$ means each nonzero $a \in A$ is a unique finite sum $a = a_{i_1} + \dots + a_{i_m}$ with nonzero $a_{i_j} \in A^{i_j}$.

A is **anticommutative** or **graded commutative** if $\forall a \in A^k, b \in A^\ell, ab = (-1)^{k\ell} ba$.

A **homomorphism** of graded algebras is an algebra homomorphism that preserves the degree.

Definition & Proposition 1.1.25 Grassmann algebra of multivectors on a vector space

For a vector space V of degree $n < \infty$, the **exterior algebra** or the **Grassmann algebra** of multivectors on V is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^n A_k(V)$$

with the wedge product of multivectors as multiplication.

Lemma 1.1.26 Wedge product of the dual basis applying to a basis

Let e_1, \dots, e_n be a basis for a vector space V and $\alpha^1, \dots, \alpha^n$ the dual basis in V^* . For $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k)$ with $1 \leq i_1 < \dots < i_k \leq n, 1 \leq j_1 < \dots < j_k \leq n$,

$$\alpha^I(e_J) = \delta_J^I.$$

Proposition 1.1.27 Wedge products of the dual basis form a basis for multivectors

Let V be a vector space and $\alpha^1, \dots, \alpha^n$ the dual basis in V^* . Then, $\alpha^I, I = (i_1 < \dots < i_k)$ form a basis for $A_k(V)$.

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

Definition 1.1.28 Cotangent space to an Euclidean space at a point

The **cotangent space** to R^n at p is $T_p^*(R^n) = (T_p(R^n))^*$.

Definition 1.1.29 Differential 1-form on an open subset of an Euclidean space

A **covector field** or a **differential 1-form** on $U \in \mathcal{O}_n$ is $\omega: U \rightarrow \bigcup_{p \in U} T_p^*(R^n)$ that maps $U \ni p \mapsto \omega_p \in T_p^*(R^n)$.

Definition 1.1.30 Differential of a smooth function

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$, the **differential** df of f is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$

In the expression

$$\langle \cdot, \cdot \rangle: T_p(\mathbf{R}^n) \times C_p^\infty(\mathbf{R}^n) \ni (X_p, f) \mapsto \langle X_p, f \rangle = X_p f \in \mathbf{R},$$

a tangent vector is considered as $\langle X_p, \cdot \rangle$; a differential at p as $df|_p = (df)_p = \langle \cdot, f \rangle$.

Proposition 1.1.31 Differentials of coordinates is the dual basis for the cotangent space

For $p \in \mathbf{R}^n$, $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the dual basis for $T_p^*(\mathbf{R}^n)$ to $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\} \subset T_p(\mathbf{R}^n)$, where x^1, \dots, x^n are the standard coordinates on \mathbf{R}^n .

For any differential 1-form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some $a_i(p)$. In this case, ω is written as $\omega = \sum a_i dx^i$.

Definition 1.1.32 Smoothness of a differential 1-form

A differential 1-form $\omega = \sum a_i dx^i$ on $U \in \mathcal{O}_n$ is **smooth** if all $a_i: U \rightarrow \mathbf{R}$ are smooth.

Proposition 1.1.33 Differentials can be written in terms of partial derivatives

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness of f implies that of df .

Definition 1.1.34 Differential k -forms on an Euclidean space

A **differential k -form** or **differential form of degree k** on $U \in \mathcal{O}_n$ is $\omega: U \ni p \mapsto \omega_p \in A_k(T_p(\mathbf{R}^n))$.

Definition & Proposition 1.1.35 Basis for differential forms

Since $\{dx_p^I \mid I = (1 \leq i_1 < \dots < i_k \leq n)\}$ is a basis for $A_k(T_p(\mathbf{R}^n))$, for a differential k -form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_I(p) dx_p^I, \quad \omega = \sum a_I dx^I.$$

ω is **smooth** if all $a_I: U \rightarrow \mathbf{R}$ are smooth. The vector space of C^∞ differential k -forms on U is denoted by $\Omega^k(U)$. If $k=0$, $\Omega^0(U) = C^\infty(U)$.

Definition 1.1.36 Wedge product of differential forms

For differential k -form ω and ℓ -form τ on $U \in \mathcal{O}_n$, their **wedge product** $\omega \wedge \tau$ is a differential $(k+\ell)$ -form defined by

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

If $\omega = \sum a_I dx^I$, $\tau = \sum b_J dx^J$,

$$\begin{aligned} \omega \wedge \tau &= \sum_{I,J} (a_I b_J) dx^I \wedge dx^J \\ &= \sum_{\text{disjoint } I,J} (a_I b_J) dx^I \wedge dx^J. \end{aligned}$$

For $\omega \in \Omega^k(U)$, $\tau \in \Omega^\ell(U)$, the wedge product is a bilinear map

$$\wedge: \Omega^k(U) \times \Omega^\ell(U) \rightarrow \Omega^{k+\ell}(U).$$

In particular, if $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$, then $f \wedge \omega = f\omega$.

Definition 1.1.37 Graded algebra with smooth differential forms

For $U \in \mathcal{O}_n$, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is an anticommutative graded algebra over \mathbf{R} with the wedge product as multiplication, which is also a module over $C^\infty(U)$.

Definition 1.1.38 Differential forms as linear maps on a vector field

For a differential k -form ω on $U \in \mathcal{O}_n$ and $X_1, \dots, X_k \in \mathfrak{X}(U)$, define $\omega(X_1, \dots, X_k) \in C^\infty(U)$ by

$$(\omega(X_1, \dots, X_k))_p = \omega_p((X_1)_p, \dots, (X_k)_p).$$

The map

$$\mathfrak{X}^k(U) \ni (X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k) \in C^\infty(U)$$

is k -linear over $C^\infty(U)$.

Definition 1.1.39 Exterior derivatives of differential forms

For $k \geq 1$ and $\omega = \sum a_I dx^I \in \Omega^k(U)$, the **exterior derivative** of ω is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U);$$

for $k = 0$ and $f \in C^\infty(U)$, its exterior derivative is

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

Definition 1.1.40 Antiderivation of a graded algebra

An **antiderivation** of a graded algebra $A = \bigoplus_{k=0}^\infty A^k$ is a linear map $D: A \rightarrow A$ such that for $a \in A^k, b \in A^\ell$,

$$D(ab) = D(a)b + (-1)^k aD(b).$$

If m is an integer such that D sends A^k to A^{k+m} for all k , then m is called the **degree** of D .

Proposition 1.1.41 Properties of the exterior differentiation

- i) The exterior differentiation $d: \Omega^*(U) \rightarrow \Omega^*(U)$ on $U \in \mathcal{O}_n$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

- ii) $d^2 = 0$.

- iii) For $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(df)(X) = Xf$.

Proposition 1.1.42 Characterization of the exterior differentiation

The exterior differentiation $d: \Omega^*(U) \rightarrow \Omega^*(U)$ on $U \in \mathcal{O}_n$ is the only antiderivation of $\Omega^*(U)$.

Definition 1.1.43 Closed and exact forms

A differential k -form ω on $U \in \mathcal{O}_n$ is said to be **closed** if $d\omega = 0$, and said to be **exact** if $\omega = d\tau$ for some $(k-1)$ -form τ on U .

Every exact form is closed.

Definition 1.1.44 Cochain complex and de Rham complex

A collection of vector spaces $\{V^k\}_{k=0}^\infty$ with linear maps $d_k: V^k \rightarrow V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a **cochain complex** or a **differential complex**.

The **de Rham complex** of $U \in \mathcal{O}_n$ is a cochain complex

$$0 \rightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \dots$$

The closed forms are the elements of $\ker d$, and the exact forms are the elements of $\text{im } d$.

Proposition 1.1.45 Vector calculus as differential forms

Under the identifications, for $U \in \mathcal{O}_3$, $f \in C^\infty(U)$ and $X = [P \ Q \ R] \in \mathfrak{X}(U)$,

$$\text{1-form } Pdx + Qdy + Rdz \longleftrightarrow X,$$

$$\text{2-form } Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \longleftrightarrow X,$$

$$\text{3-form } f dx \wedge dy \wedge dz \longleftrightarrow f,$$

there are correspondences between the exterior derivatives and grad, rot, and div:

$$df \longleftrightarrow \text{grad } f,$$

$$d(Pdx + Qdy + Rdz) \longleftrightarrow \text{rot } X,$$

$$d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \longleftrightarrow \text{div } X.$$

Definition 1.1.46 k-th de Rham cohomology

For $U \in \mathcal{O}_n$, the k -th **de Rham cohomology** of U is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}.$$

Proposition 1.1.47 Poincaré lemma

For $k \geq 1$, every closed k -form on \mathbb{R}^n is exact, i.e., $H^k(\mathbb{R}^n)$ vanishes.

1.2 Manifolds

Definition 1.2.1 Locally Euclidean space

A topological space M is **locally Euclidean of dimension n** if $\forall p \in M, \exists (U, \phi)$, with a neighborhood U at p and a homeomorphism $\phi: U \rightarrow V \in \mathcal{O}_n$, called a **chart**, a **coordinate neighborhood** or a **coordinate open set**, and ϕ a **coordinate map** or a **coordinate system** on U .

A chart (U, ϕ) is said to be **centered** at $p \in U$ if $\phi(p) = 0$.

Definition 1.2.2 Topological manifold

A **topological manifold of dimension n** is a Hausdorff, second countable, locally Euclidean space of dimension n .

Definition 1.2.3 Compatible chart

Two charts $(U, \phi: U \rightarrow \mathbb{R}^n), (V, \psi: V \rightarrow \mathbb{R}^n)$ of a topological manifold are said to be C^∞ -**compatible** or simply **compatible** if

$$\phi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \phi(U \cap V), \quad \psi \circ \phi^{-1}: \phi(U \cap V) \rightarrow \psi(U \cap V)$$

called the **transition functions** between charts are C^∞ . If $U \cap V = \emptyset$, they are C^∞ -compatible.

Definition 1.2.4 Atlas on a locally Euclidean space

A C^∞ **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise compatible charts that cover M .

Definition 1.2.5 Compatibility of a chart with an atlas

For a locally Euclidean space, a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if all charts (U_α, ϕ_α) are compatible with (V, ψ) .

Lemma 1.2.6 Charts compatible with the same atlas are compatible with each other

For a locally Euclidean space, charts (V, ψ) , (W, σ) , and an atlas $\{(U_\alpha, \phi_\alpha)\}$ on it, if (V, ψ) and (W, σ) are both compatible with $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Definition 1.2.7 Maximal Atlas on a locally Euclidean space

An atlas \mathfrak{M} on a locally Euclidean space is **maximal** if for another atlas \mathfrak{U} , $\mathfrak{M} \subset \mathfrak{U}$ implies $\mathfrak{M} = \mathfrak{U}$.

Definition 1.2.8 Smooth manifold

A **smooth** or C^∞ **manifold** is a topological manifold M with a maximal atlas called a **differentiable structure** on M . M is said to be of dimension n if all of its connected components are of dimension n , and then M is called a **n -manifold**. A 1-manifold is also called a **curve**, a 2-manifold a **surface**.

Proposition 1.2.9 A locally Euclidean space with an atlas has a maximal atlas

In a locally Euclidean space, any atlas is contained in a unique maximal atlas.

Definition 1.2.10 Conventions of manifold

- i) A manifold means a smooth manifold.
- ii) The standard coordinates on \mathbf{R}^n is denoted by r^1, \dots, r^n .
- iii) For a chart (U, ϕ) of a manifold, let $x^i = r^i \circ \phi$ the i -th component of ϕ , and write $\phi = (x^1, \dots, x^n)$ and $(U, \phi) = (U, x^1, \dots, x^n)$. x^1, \dots, x^n are called **coordinates** or **local coordinates** on U .
- iv) The notation (x^1, \dots, x^n) means alternately the local coordinates on U and a point in \mathbf{R}^n .
- v) A **chart** (U, ϕ) **about** p in a manifold M means a chart in the differentiable structure of M such that $p \in U$.

Proposition 1.2.11 Product manifold

For a m -manifold M and n -manifold N , and atlases $\{(U_\alpha, \phi_\alpha)\}$ of M and $\{(V_{\alpha'}, \psi_{\alpha'})\}$ of N , the collection

$$\{(U_\alpha \times V_{\alpha'}, \phi_\alpha \times \psi_{\alpha'} : U_\alpha \times V_{\alpha'} \rightarrow \mathbf{R}^m \times \mathbf{R}^n)\}$$

is an atlas on $M \times N$, and therefore $M \times N$ is a manifold of dimension $m + n$.

Definition 1.2.12 Smooth function on a manifold

For a smooth n -manifold M , a function $f : M \rightarrow \mathbf{R}$ is said to be C^∞ or **smooth at a point** $p \in M$ if, for some chart (U, ϕ) about p , $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbf{R}^n$ is C^∞ at $\phi(p)$; C^∞ **on** M if it is smooth at every point.

Proposition 1.2.13 Smoothness of real-valued functions

For a n -manifold M and a function $f : M \rightarrow \mathbf{R}$, the following are equivalent:

- i) f is C^∞ .
- ii) There exists an atlas \mathfrak{U} for M , for any $(U, \phi) \in \mathfrak{U}$, $f \circ \phi^{-1}$ is C^∞ .
- iii) For any chart (U, ϕ) on M , $f \circ \phi^{-1}$ is C^∞ .

Definition 1.2.14 Pullback of a function by a map

For manifolds M, N , the **pullback** of $h: M \rightarrow R$ by $F: N \rightarrow M$ is $F^*h = h \circ F$.

Definition 1.2.15 Smooth map between manifolds

For a m -manifold M and n -manifold N , a continuous map $F: N \rightarrow M$ is C^∞ **at a point** $p \in N$ if, for some charts (U, ϕ) about p and (V, ψ) about $F(p)$, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$; C^∞ if it is C^∞ at every point.

Proposition 1.2.16 Smoothness of maps is independent of charts

Let M be a m -manifold, N a n -manifold, and $F: N \rightarrow M$ be C^∞ at $p \in N$. Then, for any charts (U, ϕ) about p and (V, ψ) about $F(p)$, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Proposition 1.2.17 Smoothness of a map in terms of charts

For a m -manifold M and n -manifold N , and a continuous map $F: N \rightarrow M$, the following are equivalent:

- i) F is C^∞ .
- ii) There exists atlases \mathfrak{U} for N and \mathfrak{V} for M , for any $(U, \phi) \in \mathfrak{U}$ and $(V, \psi) \in \mathfrak{V}$, $\psi \circ F \circ \phi^{-1}$ is C^∞ .
- iii) For any chart (U, ϕ) on N and (V, ψ) on M , $\psi \circ F \circ \phi^{-1}$ is C^∞ .

Proposition 1.2.18 Composite of smooth maps is also smooth

For manifolds M, N, P and C^∞ maps $F: N \rightarrow M$, $G: M \rightarrow P$, $G \circ F: N \rightarrow P$ is also C^∞ .

Definition 1.2.19 Diffeomorphism of manifolds

A **diffeomorphism** of manifolds is a bijective C^∞ map whose inverse is also C^∞ .

Proposition 1.2.20 Coordinate map is a diffeomorphism

A coordinate map $\phi: U \rightarrow \phi(U) \subset \mathbb{R}^n$ for a manifold with a chart (U, ϕ) is a diffeomorphism.

Proposition 1.2.21 Diffeomorphism into an Euclidean space is a coordinate map

For an open subset U of a manifold M with the differentiable structure \mathfrak{U} , if $F: U \rightarrow \mathbb{R}^n$ is a diffeomorphism, then $(U, F) \in \mathfrak{U}$.

Proposition 1.2.22 Smoothness of a vector-valued function

For a continuous map $F: N \rightarrow \mathbb{R}^m$ on a manifold M , the following are equivalent:

- i) F is C^∞ .
- ii) There exists an atlas \mathfrak{U} for M , for any $(U, \phi) \in \mathfrak{U}$, $F \circ \phi^{-1}$ is C^∞ .
- iii) For any chart (U, ϕ) on M , $F \circ \phi^{-1}$ is C^∞ .

Proposition 1.2.23 Vector-valued function is smooth iff its components are all smooth

For a vector-valued function $F = (F^1, \dots, F^m): M \rightarrow \mathbb{R}^m$ on a manifold M , F is C^∞ iff F^1, \dots, F^m are all C^∞ .

Proposition 1.2.24 Smoothness of a map in terms of vector-valued functions

For a continuous map $F: N \rightarrow M$ between a m -manifold M and n -manifold N , the following are equivalent:

- i) F is C^∞ .
- ii) There exists an atlas \mathfrak{U} for M , for any $(U, \phi) \in \mathfrak{U}$, $\phi \circ F$ is C^∞ .

- iii)** For any chart (U, ϕ) on M , $\phi \circ F$ is C^∞ .

Proposition 1.2.25 Smoothness of a map in terms of components

For a continuous map $F: N \rightarrow M$ between a m -manifold M and n -manifold N , the following are equivalent:

- i)** F is C^∞ .
- ii)** There exists an atlas \mathcal{U} for M , for any $(U, \phi^1, \dots, \phi^m) \in \mathcal{U}$, the components $\phi^i \circ F$ of F relative to the chart are all C^∞ .
- iii)** For any chart $(U, \phi^1, \dots, \phi^m)$ on M , the components $\phi^i \circ F$ of F relative to the chart are all C^∞ .

2 P-ADIC NUMBERS

[4]

2.1 Foundations

Definition 2.1.1 Absolute value on a field

An **absolute value** on a field K is a function $|\cdot|: K \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- i) $|x| = 0$ iff $x = 0$
- ii) $\forall x, y \in K, |xy| = |x||y|$
- iii) $\forall x, y \in K, |x + y| \leq |x| + |y|$.

An absolute value that satisfies the condition

- iv) $\forall x, y \in K, |x + y| \leq \max\{|x|, |y|\}$

is said to be **non-archimedean**; otherwise, it is said to be **archimedean**.

Definition 2.1.2 Trivial absolute value

The **trivial absolute value** on a field K is an absolute value on K such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

An absolute value on a finite field must be trivial.

Definition 2.1.3 Valuation on a field

A function $v: A^\times \rightarrow \mathbb{R}$ with an integral domain A is called a **valuation** on A if it satisfies the following conditions:

- i) $\forall x, y \in A^\times, v(xy) = v(x) + v(y)$
- ii) $\forall x, y \in A^\times, v(x + y) \geq \min\{v(x), v(y)\}$

Definition & Proposition 2.1.4 Value group of a valuation

The image of a valuation v on a field is an additive subgroup of R . $\text{im } v$ is called the **value group** of v .

Proposition 2.1.5 Correspondence between valuations and non-archimedean absolute values

Let A be an integral domain and $K = \text{Frac } A$. Let $v: A^\times \rightarrow R$ be a valuation on A and extend v to K by setting $v(a/b) = v(a) - v(b)$, then the function $|\cdot|_v: K \rightarrow R_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on K . Conversely, $-\log |\cdot|$ is a valuation on K for a non-archimedean absolute value $|\cdot|$ on K .

Definition 2.1.6 p-adic valuation

The **p-adic valuation** on \mathbb{Q} with a prime p is a valuation $v_p: \mathbb{Q}^\times \rightarrow R$ defined as follows: for each $n \in \mathbb{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \mathbb{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

Definition 2.1.7 p-adic absolute value

The **p-adic absolute value** $|\cdot|_p: \mathbb{Q} \rightarrow R_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\cdot| = |\cdot|_\infty$.

Definition 2.1.8 Absolute values on a field of rational functions

Here are some absolute values on a field $F(t)$ of rational functions over a field F .

- i) For $f(t) \in F[t]$, $v_\infty(f) = -\deg f$, and for $f(t)/g(t) \in F(t)$, $v_\infty(f/g) = v_\infty(f) - v_\infty(g)$ with $v_\infty(0) = \infty$. Then,

$$|f(t)|_\infty = e^{-v_\infty(f)}.$$

- ii) For an irreducible polynomial $p(t) \in F[t]$, define the $p(t)$ -adic valuation and absolute value.

Lemma 2.1.9 Properties of absolute values on fields

For an absolute value $|\cdot|$ on a field K ,

- i) $|1| = 1$,
- ii) $\forall x \in K, |x^n| = 1 \Rightarrow |x| = 1$,
- iii) $\forall x \in K, |-x| = |x|$,
- iv) If K is finite, then $|\cdot|$ is trivial.

Theorem 2.1.10 Necessary and sufficient conditions of a non-archimedean absolute value

Let K be a field, $|\cdot|$ an absolute value on K . Then,

$$\begin{aligned} |\cdot| \text{ is non-archimedean} &\iff \forall n = 1 + \cdots + 1 \in K, |n| \leq 1 \\ &\iff \sup\{|n| \mid n \in \mathbb{Z}\} = 1. \end{aligned}$$

Furthermore, $\sup\{|n| \mid n \in \mathbb{Z}\} = \infty$ if $|\cdot|$ is archimedean.

3 LIE ALGEBRAS

[5]

3.1 Foundations

Definition 3.1.1 Lie algebra

A vector space \mathfrak{g} over a field K with the Lie bracket satisfying the conditions

- i) Lie bracket is bilinear
- ii) $\forall x \in \mathfrak{g}, [x, x] = 0$
- iii) $\forall x, y, z \in \mathfrak{g}, [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a **Lie algebra** over K .

Definition 3.1.2 General linear Lie algebra

$\mathfrak{gl}_n(\mathbf{R})$ is the Lie algebra $M_n(\mathbf{R})$ with the Lie bracket $[x, y] = xy - yx$.

Definition 3.1.3 Derivation algebra

A linear endomorphism D of an algebra A over \mathbf{R} satisfying $D(xy) = D(x)y + xD(y)$ is called a **derivation** of A . The set of all derivations $\text{Der} A$ with the addition, scalar multiplication, and Lie bracket defined as follows:

- i) $(D + D')(x) = D(x) + D'(x)$
- ii) $(\alpha D)(x) = \alpha D(x)$
- iii) $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the **derivation algebra** of A .

Definition 3.1.4 Lie subalgebra

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra \mathfrak{g} is a **Lie subalgebra** of \mathfrak{g} if $\forall x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$.

For linear subspaces $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ denotes the subspace generated by $[x, y]$ with $x \in \mathfrak{a}, y \in \mathfrak{b}$.

Definition & Proposition 3.1.5 Special linear Lie algebra

$\mathfrak{sl}_n(R) = \{x \in \mathfrak{gl}_n(R) \mid \text{tr } x = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(R)$.

Definition & Proposition 3.1.6 Orthogonal Lie algebra

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(R) \mid {}^t x = -x\}$ is a Lie subalgebra of $\mathfrak{sl}_n(R)$.

Definition & Proposition 3.1.7 Ideal of a Lie algebra

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra \mathfrak{g} is an **ideal** of \mathfrak{g} if $\forall x \in \mathfrak{g}, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$.

For ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ is also an ideal.

Definition 3.1.8 Derived ideal of a Lie algebra

For a Lie algebra \mathfrak{g} , $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of \mathfrak{g} called the **derived ideal** of \mathfrak{g} .

If $\mathfrak{g} = \mathfrak{gl}_n(R)$, $D\mathfrak{g} = \mathfrak{sl}_n(R)$.

Definition & Proposition 3.1.9 Homomorphism of Lie algebras

For Lie algebras $\mathfrak{g}, \mathfrak{h}$, a linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ is called a **homomorphism** if $\forall x, y \in \mathfrak{g}, \varphi([x, y]) = [\varphi(x), \varphi(y)]$. A homomorphism φ is an **isomorphism** if it is bijective. Lie algebras between which there exists an isomorphism are said to be **isomorphic** to each other, written $\mathfrak{g} \cong \mathfrak{h}$.

A composite of homomorphisms is also a homomorphism, and that of isomorphisms is also an isomorphism.

The kernel $\ker \varphi = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$ of a homomorphism φ is an ideal of \mathfrak{g} while the image $\text{im } \varphi = \varphi(\mathfrak{g})$ of φ is a Lie subalgebra of \mathfrak{h} .

Definition 3.1.10 Representation of a Lie algebra on a vector space

For a Lie algebra \mathfrak{g} and a vector space V , a homomorphism $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is called a **representation** of \mathfrak{g} on V .

Definition & Proposition 3.1.11 Adjoint representation of a Lie algebra

For a Lie algebra \mathfrak{g} and $x \in \mathfrak{g}$, define a derivation $\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g}$ by $\text{ad}(x)(y) = [x, y]$. A representation $\text{ad}: \mathfrak{g} \ni x \mapsto \text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is called the **adjoint representation** of \mathfrak{g} . The **center** of \mathfrak{g} is $\mathfrak{z} = \ker(\text{ad})$, which is a commutative ideal. $\text{im}(\text{ad})$ is an ideal of $\text{Der } \mathfrak{g}$. A derivation $\text{ad}(x)$ is called a **inner derivation** of \mathfrak{g} .

Definition 3.1.12 Quotient algebra for Lie algebras

For a Lie algebra \mathfrak{g} and an ideal $\mathfrak{a} \subset \mathfrak{g}$, the **quotient algebra** is

$$\mathfrak{g}/\mathfrak{a} = \{\bar{x} = x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

with canonical operations, where $\bar{x} = \{y \in \mathfrak{g} \mid x \equiv y \pmod{\mathfrak{a}}\} = \{x + a \mid a \in \mathfrak{a}\}$ called the **class** of x . The homomorphism $\varphi: \mathfrak{g} \ni x \mapsto \bar{x} \in \mathfrak{g}/\mathfrak{a}$ is called the **canonical homomorphism**.

Theorem 3.1.13 The first isomorphism theorem for Lie algebras

For Lie algebras $\mathfrak{g}, \mathfrak{h}$ and a homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$,

$$\mathfrak{g}/\ker \varphi \cong \text{im } \varphi.$$

Theorem 3.1.14 The second isomorphism theorem for Lie algebras

For a Lie algebra \mathfrak{g} , an ideal $\mathfrak{a} \subset \mathfrak{g}$, a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the canonical homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$,

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{a}) \cong (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}.$$

3.2 Solvable and Nilpotent Lie algebras

Definition 3.2.1 Solvable Lie algebra

Let \mathfrak{g} be a Lie algebra, and

$$D^0 \mathfrak{g} = \mathfrak{g}, \quad D^k \mathfrak{g} = D(D^{k-1} \mathfrak{g}), \quad k = 1, 2, \dots$$

\mathfrak{g} is said to be **solvable** if $D^r \mathfrak{g} = \{0\}$ for some r called the **length** of \mathfrak{g} .

Example 3.2.2 Lie algebra of triangular matrices is solvable

Let

$$\begin{aligned} \mathfrak{g}_0 &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbf{R}) \mid \xi \text{ is upper triangular}\}, \\ \mathfrak{g}_k &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\mathbf{R}) \mid \xi_{ij} = 0 \text{ for } j - i < k\}. \end{aligned}$$

Then, $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_1$, $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$, $k, \ell = 0, 1, \dots$, and \mathfrak{g}_0 is a solvable Lie algebra of length $\leq n$.

Theorem 3.2.3 Lie subalgebra of a solvable Lie algebra is also solvable

For a solvable Lie algebra \mathfrak{g} , its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also solvable, and if \mathfrak{h} is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also solvable.

Theorem 3.2.4 Lie algebra whose ideal and quotient algebra over it are solvable is solvable

For a Lie algebra \mathfrak{g} and its ideal $\mathfrak{a} \subset \mathfrak{g}$, if \mathfrak{a} and $\mathfrak{g}/\mathfrak{a}$ are both solvable, then \mathfrak{g} is also solvable.

Definition 3.2.5 Nilpotent Lie algebra

Let \mathfrak{g} be a Lie algebra, and

$$C^0 \mathfrak{g} = \mathfrak{g}, \quad C^k \mathfrak{g} = [\mathfrak{g}, C^{k-1} \mathfrak{g}], \quad k = 1, 2, \dots$$

\mathfrak{g} is said to be **nilpotent** if $C^s \mathfrak{g} = \{0\}$ for some s called the **length** of \mathfrak{g} .

Since $D^k \mathfrak{g} \subset C^k \mathfrak{g}$, a nilpotent Lie algebra is solvable.

Example 3.2.6 Lie algebra of strictly triangular matrices is nilpotent

\mathfrak{g}_1 in [Example 3.2.2] is nilpotent while \mathfrak{g}_0 there is not.

Theorem 3.2.7 Lie subalgebra of a nilpotent Lie algebra is also nilpotent

For a nilpotent Lie algebra \mathfrak{g} , its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also nilpotent, and if \mathfrak{h} is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also nilpotent.

Theorem 3.2.8 Center of a nilpotent Lie algebra is nontrivial

For a Lie algebra \mathfrak{g} and its center \mathfrak{z} , $\mathfrak{z} \neq \{0\}$ if \mathfrak{g} is nilpotent while \mathfrak{g} is nilpotent if $\mathfrak{g}/\mathfrak{z}$ is nilpotent.

Lemma 3.2.9 Adjoint representation in a nilpotent Lie algebra is nilpotent

For a nilpotent Lie algebra \mathfrak{g} and $x \in \mathfrak{g}$, $\text{ad } x$ is nilpotent. (i.e., $(\text{ad } x)^s = 0$ for some s).

Theorem 3.2.10 Engel's Theorem

For a Lie algebra \mathfrak{g} , if $\text{ad } x$ is nilpotent for any $x \in \mathfrak{g}$, then \mathfrak{g} is nilpotent.

Theorem 3.2.11 Matrix of a representation of a Lie algebra whose representations are all nilpotent is strictly upper triangular

For a representation $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of a Lie algebra \mathfrak{g} on a vector space V , if $\rho(x)$ is nilpotent for any $x \in \mathfrak{g}$, then a matrix of $\rho(x)$ is strictly upper triangular for some basis for V .

4 CATEGORIES

[1]

4.1 Foundations

Definition 4.1.1 Category

A **category** consists of the followings:

🍃 **Objects** A, B, C, \dots

🍃 **Arrows** f, g, h, \dots with the objects called the domain $\text{dom } f$ and the codomain $\text{cod } f$.

🍃 **Composites** $g \circ f: A \rightarrow C$ for given arrows $f: A \rightarrow B$ and $g: B \rightarrow C$.

🍃 **Identity arrow** 1_A of each object A .

satisfying the following laws:

- i) $\forall \text{arrows } f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D, h \circ (g \circ f) = (h \circ g) \circ f$
- ii) $\forall \text{arrow } f: A \rightarrow B, f \circ 1_A = f = 1_B \circ f$.

Definition 4.1.2 Functor between categories

A **functor** $F: \mathcal{A} \rightarrow \mathcal{B}$ between categories \mathcal{A} and \mathcal{B} is a mapping between objects and between arrows in the following ways:

- i) $F(f: A \rightarrow B) = F(f): F(A) \rightarrow F(B)$,
- ii) $F(1_A) = 1_{F(A)}$,
- iii) $F(g \circ f) = F(g) \circ F(f)$.

Definition 4.1.3 Isomorphism between categories

In a category \mathcal{C} , an arrow $f: A \rightarrow B$ is called an **isomorphism** if

$$\exists g = f^{-1}: B \rightarrow A, g \circ f = 1_A, f \circ g = 1_B.$$

If there is an isomorphism between objects A and B , A is said to be **isomorphic** to B , written $A \cong B$.

Theorem 4.1.4 Category is isomorphic to its Cayley representation

For a category \mathcal{C} with a set of arrows, the Cayley representation $\overline{\mathcal{C}}$ of \mathcal{C} , consisting of

- ▮ object $\overline{C} = \{f \in \mathcal{C} \mid \text{cod } f = C\}$ for an object $C \in \mathcal{C}$,
- ▮ arrow $\overline{g}: \overline{C} \rightarrow \overline{D}$ for an arrow $g: C \rightarrow D$ such that $\overline{g}(f) = g \circ f$,

is isomorphic to \mathcal{C} .

Definition 4.1.5 Product of two categories

The **product** $\mathcal{C} \times \mathcal{D}$ of categories \mathcal{C} and \mathcal{D} consists of

- ▮ object (C, D) for objects $C \in \mathcal{C}$, $D \in \mathcal{D}$,
- ▮ arrow $(f, g): (C, D) \rightarrow (C', D')$ for arrows $f: C \rightarrow C'$, $g: D \rightarrow D'$,

with composition $(f, g) \circ (f', g') = (f \circ f', g \circ g')$ and units $1_{(C, D)} = (1_C, 1_D)$.

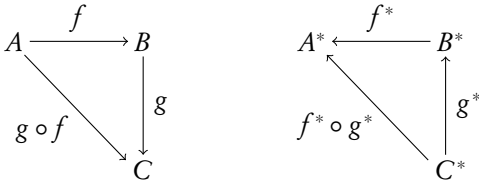
The **projection functors** $\pi_1: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}$ and $\pi_2: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ is defined by $\pi_1(C, D) = C$ and $\pi_1(f, g) = f$, and similarly for π_2 .

Definition 4.1.6 Dual category

For a category \mathcal{C} , its **dual** or **opposite category** \mathcal{C}^{op} consists of

- ▮ object $C^* = C$ for an object $C \in \mathcal{C}$,
- ▮ arrow $f^*: D^* \rightarrow C^*$ for an arrow $f: C \rightarrow D$,

with composition $f^* \circ g^* = (g \circ f)^*$ and units $1_{C^*} = (1_C)^*$.



Definition 4.1.7 Arrow category

For a category \mathcal{C} , its **arrow category** $\mathcal{C}^{\rightarrow}$ consists of

- ▮ object $f: C \rightarrow D$ for an arrow f in \mathcal{C} ,
- ▮ arrow $(g_1, g_2): f \rightarrow f'$, where $f: A \rightarrow B, f': A' \rightarrow B', g_1: A \rightarrow A', g_2: B \rightarrow B'$ in \mathcal{C} , such that $g_2 \circ f = f' \circ g_1$,

with composition $(g_1, g_2) \circ (h_1, h_2) = (g_1 \circ h_1, g_2 \circ h_2)$ and units $1_f = (1_A, 1_B)$.

$$\begin{array}{ccc}
 A & \xrightarrow{g_1} & A' \\
 f \downarrow & & \downarrow f' \\
 B & \xrightarrow{g_2} & B'
 \end{array}$$

There are two functors $\text{dom}, \text{cod}: \mathcal{C}^{\rightarrow} \rightarrow \mathcal{C}$.

Definition 4.1.8 Slice category

For a category \mathcal{C} , its **slice category** \mathcal{C}/C over $C \in \mathcal{C}$ consists of

- ▮ object $f: X \rightarrow C$,
- ▮ arrow $a: X \rightarrow X'$ for arrows $f: X \rightarrow C, f': X' \rightarrow C$ such that $f' \circ a = f$,

with composition and units from those of \mathcal{C} .

$$\begin{array}{ccc}
 X & \xrightarrow{a} & X' \\
 & \searrow f & \swarrow f' \\
 & C &
 \end{array}$$

$U: \mathcal{C}/C \rightarrow \mathcal{C}$ with $U(f: X \rightarrow C) = C$ and $U(a: X \rightarrow X') = a$ is a functor.

5 FUNCTIONAL ANALYSIS

[2]

5.1 Hahn-Banach Theorems

Theorem 5.1.1 Hahn-Banach of analytic form

Let $p: E \rightarrow \mathbf{R}$ be a sublinear function on a vector space E (i.e., $\forall \lambda > 0, x, y \in E, p(\lambda x) = \lambda p(x), p(x+y) \leq p(x) + p(y)$), $G \subset E$ a linear subspace, and $g: G \rightarrow \mathbf{R}$ a linear functional such that $\forall x \in G, g(x) \leq p(x)$. Then, \exists a linear functional $f: E \rightarrow \mathbf{R}$ that extends g and that $\forall x \in E, f(x) \leq p(x)$.

Definition 5.1.2 Norm on the dual space of a normed space

For a normed space E , the **dual norm** on E^* is defined by

$$\|f\|_{E^*} = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} |f(x)| = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} f(x).$$

Definition 5.1.3 Scalar product for the duality

For a vector space E and its dual space E^* , $\langle \cdot, \cdot \rangle: E^* \times E \rightarrow \mathbf{R}$ defined by $\langle f, x \rangle = f(x)$ is called the **scalar product for the duality** E, E^* .

Definition 5.1.4 Strictly convex normed space

A normed space E is said to be **strictly convex** if $\forall t \in (0, 1), x, y \in E$ with $\|x\| = \|y\| = 1$, $\|tx + (1-t)y\| < 1$ except for $x = y$.

Corollary 5.1.5 Hahn-Banach of alternate form

For a continuous linear functional $g: G \rightarrow \mathbf{R}$ on a linear subspace $G \subset E$ of a normed space E , $\exists f \in E^*$ that extends g and that $\|f\|_{E^*} = \|g\|_{G^*}$.

In the case when $G = \mathbf{R}x_0$ and $g(tx_0) = t\|x_0\|^2$ for a given $x_0 \in E$, $\exists f_0 \in E^*$ such that $\|f_0\| = \|x_0\|$ and $\langle f_0, x_0 \rangle = \|x_0\|^2$. If E^* is strictly convex, then f_0 is unique.

Definition 5.1.6 Duality map from a normed space into its dual space

For a normed space E and $x_0 \in E$, define

$$F(x_0) = \{f_0 \in E^* \mid \|f_0\| = \|x_0\|, \langle f_0, x_0 \rangle = \|x_0\|^2\}.$$

The **duality map** from E into E^* is a multivalued map $x_0 \mapsto F(x_0)$.

Corollary 5.1.7 Norm of a vector is the max of its scalar product

For a normed space E and $x \in E$,

$$\|x\| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \max_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle|.$$

Definition 5.1.8 Affine hyperplane of a normed space

For a normed space E and a linear functional $f: E \rightarrow \mathbf{R}$, an affine **hyperplane** is a subset $H = \{x \in E \mid f(x) = \alpha\} \subset E$ with $\alpha \in \mathbf{R}$, written $H = [f = \alpha]$. $f = \alpha$ is called the **equation**.

Proposition 5.1.9 Linear functional is continuous iff its hyperplane is closed

For a linear functional $f: E \rightarrow \mathbf{R}$ on a normed space E and $\alpha \in \mathbf{R}$, $[f = \alpha]$ is closed iff f is continuous.

Definition 5.1.10 Separation by a hyperplane

For two subsets $A, B \subset E$ of a normed space E , the hyperplane $[f = \alpha] \subset E$ **separates** A

and B if

$$\forall x \in A, y \in B, f(x) \leq \alpha \leq f(y);$$

strictly separates if

$$\exists \epsilon > 0, \forall x \in A, y \in B, f(x) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(y).$$

Definition 5.1.11 Convex subset of a normed space

A subset $A \subset E$ of a normed space E is said to be **convex** if

$$\forall t \in [0, 1], x, y \in A, tx + (1-t)y \in A.$$

Theorem 5.1.12 Hahn-Banach of first geometric form

For two disjoint nonempty convex subsets $A, B \subset E$ of a normed space E with one of them open, \exists a closed hyperplane that separates A and B .

Definition & Proposition 5.1.13 Minkowski functional of an open convex set

Let $C \subset E$ be an open convex subset of a normed space E with $0 \in C$, and for $x \in E$

$$p(x) = \inf\{\alpha > 0 \mid \alpha^{-1}x \in C\},$$

called the **gauge** or the **Minkowski functional** of C . Then, p satisfies the following properties:

- i) p is sublinear,
- ii) $\exists M, \forall x \in E, 0 \leq p(x) \leq M\|x\|$,
- iii) $C = \{x \in E \mid p(x) < 1\}$.

Lemma 5.1.14 There exists a hyperplane that separates an open convex and outside point

For a nonempty open convex $C \subset E$ of a normed space E and $x \in E \setminus C$, $\exists f \in E^*$ such that $\forall x \in C, f(x) < f(x_0)$. In particular, the hyperplane $[f = f(x_0)]$ separates $\{x_0\}$ and C .

Theorem 5.1.15 Hahn-Banach of second geometric form

For two disjoint nonempty convex subsets $A, B \subset E$ of a normed space E with A closed and B compact, \exists a closed hyperplane that strictly separates A and B .

Corollary 5.1.16 Some linear functional can vanish on a linear subspace

For a linear subspace $F \subset E$ of a normed space E with $\bar{F} \neq E$, $\exists f \in E^*$ such that

$$\forall x \in F, \langle f, x \rangle = 0, \quad f \neq 0.$$

Definition 5.1.17 Notation of a bidual space

Let E be a normed space, and $J: E \ni x \mapsto Jx \in E^{**}$ a **canonical injection** (i.e., $Jx: f \mapsto \langle f, x \rangle$, or $\langle Jx, f \rangle = \langle f, x \rangle$). Then, J is an **isometry**:

$$\|Jx\|_{E^{**}} = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle Jx, f \rangle| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \|x\|_E.$$

J can be not surjective; if J is surjective, E is said to be **reflexive**.

Definition 5.1.18 Orthogonal complement

For a linear subspace $M \subset E$ of a normed space E and a linear subspace $N \subset E^*$, their **orthogonal complements** are

$$\begin{aligned} M^\perp &= \{f \in E^* \mid \forall x \in M, \langle f, x \rangle = 0\} \subset E^* \\ N^\perp &= \{x \in E \mid \forall f \in N, \langle f, x \rangle = 0\} \subset E, \end{aligned}$$

respectively.

Proposition 5.1.19 Relation between a linear subspace and its orthogonal complement

For a linear subspace $M \subset E$ of a normed space E and a linear subspace $N \subset E^*$,

$$(M^\perp)^\perp = \overline{M}, \quad (N^\perp)^\perp \supset \overline{N}.$$

If E is reflexive, then $(N^\perp)^\perp = \overline{N}$.

6 REAL ANALYSIS AND PROBABILITY

[3]

6.1 Set Theory

7 HOMOLOGICAL ALGEBRA

[7]

7.1 Chain Complexes

Definition 7.1.1 Chain complex of R-modules

A **chain complex** C of R -modules is a family $\{C_n\}_{n \in \mathbb{Z}}$ of R -modules with R -module maps $d_n: C_n \rightarrow C_{n-1}$ such that $d_{n-1} \circ d_n = 0$, called the **differential**. $Z_n = Z_n(C) := \ker d_n$ is the module of n -**cycles** of C , and $B_n = B_n(C) := \operatorname{im} d_{n+1}$ is the module of n -**boundaries** of C . The n -th **homology module** of C is $H_n(C) := Z_n/B_n$.

A **chain complex map** $u: C \rightarrow D$ is a family of R -module homomorphisms $u_n: C_n \rightarrow D_n$ commuting with d (i.e., $u_{n-1}d_n = d_n u_n$).

Definition & Proposition 7.1.2 A category of chain complexes of R-modules

$\mathbf{Ch}(\mathbf{mod}\text{-}R)$ is a category whose objects are chain complexes of (right) R -modules and whose arrows are chain complex maps. An arrow $u: C \rightarrow D$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$ sends boundaries to boundaries and cycles to cycles. $H_n: \mathbf{Ch}(\mathbf{mod}\text{-}R) \rightarrow \mathbf{mod}\text{-}R$ is a functor.

Proposition 7.1.3 Split exact sequence of vector spaces

For a family $\{B_n, H_n\}$ of vector spaces, $\{C_n = B_n \oplus H_n \oplus B_{n-1}\}$ with projection-inclusions $\pi_n: C_n \rightarrow B_{n-1} \subset C_{n-1}$ is a chain complex. Every chain complex of vector spaces is isomorphic to such a complex.

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