Notes of Mathematics

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Since Aug 27, 2017

Contents

1 MANIFOLDS

[6]

1.1 Manifolds on Euclidean Spaces

Theorem 1.1.1 **Taylor's theorem with remainder**

A smooth function f on an open ball $U \in \mathcal{O}_n$ can be written as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $p \in U$ and $g_i \in C^\infty(U)$ with $g_i(p) = (\partial f / \partial x^i)(p)$.

Adapting this to g_i repeatedly gives the Taylor's expansion of f.

Definition 1.1.2 **Tangent vector as an arrow from a point**

The ***tangent space*** $T_p(\boldsymbol{R}^n)$ at $p \in \boldsymbol{R}^n$ is the set of arrows from p.

Definition 1.1.3 **Directional derivative**

The ***directional derivative*** of a smooth function f in the direction $v \in T_p(\boldsymbol{R}^n)$ at $p \in \boldsymbol{R}^n$ is

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

with $c^i(t) = p^i + t v^i$.

By the chain rule,

$$D_v f = \sum \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

Definition & Proposition 1.1.4 **Derivation at a point**

A linear map $D \colon C_p^\infty \to \boldsymbol{R}$ satisfying the Leibniz rule (i.e., $D(fg) = (Df)g(p) + f(p)Dg$ for any $f, g \in C_p^\infty$) is called a ***derivation*** at p or a ***point-derivation*** of C_p^∞.

The set of all derivations at p denoted by $\mathcal{D}_p(\boldsymbol{R}^n)$ is a real vector space, and a map $\phi\colon T_p(\boldsymbol{R}^n) \to \mathcal{D}_p(\boldsymbol{R}^n)$ assigning D_v to each v is a linear map.

Lemma 1.1.5 **Point-derivation of a constant is zero**

If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

Theorem 1.1.6 **Tangent space is isomorphic to the set of point-derivations**

The linear map $\phi \to T_p(\boldsymbol{R}^n) \to \mathcal{D}_p(\boldsymbol{R}^n)$ in [Definition & Proposition 1.1.4] is an isomorphism of vector spaces.

Definition 1.1.7 **Tangent vector as a derivation**

By [Theorem 1.1.6], $v \in T_p(\boldsymbol{R}^n)$ is identified as

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p \in \mathcal{D}_p(\boldsymbol{R}^n).$$

Definition 1.1.8 **Vector fields on an open set**

A ***vector field*** on $U \in \mathcal{O}_n$ is a map $X\colon U \to T_p(\boldsymbol{R}^n)$. $X = \sum a^i \partial/\partial x^i$ means

$$X(p) = X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{with } a^i(p) \in \boldsymbol{R}$$

X is said to be C^∞ if all a^is are C^∞ on U. The set of all smooth vector fields on U is denoted by $\mathfrak{X}(U)$.

Definition & Proposition 1.1.9 **Multiplication of a smooth vector field and function**

For $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, define $fX \in \mathfrak{X}(U)$ and $Xf \in C^\infty(U)$ as follows:

$$(fX)_p = f(p)X_p = \sum (f(p)a^i(p)) \left.\frac{\partial}{\partial x^i}\right|_p,$$
$$(Xf)(p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p).$$

Proposition 1.1.10 **Leibniz rule for a vector field**

For any $X \in \mathfrak{X}(U)$, $f, g \in C^\infty(U)$,

$$X(fg) = (Xf)g + fXg.$$

Proposition 1.1.11 **Derivations one-to-one-correspond to smooth vector fields**

$\varphi\colon \mathfrak{X}(U) \ni X \mapsto (f \mapsto Xf) \in \mathrm{Der}\,(C^\infty(U))$ is an linear isomorphism.

Definition 1.1.12 **k-tensor on a vector space**

A k-linear function $f\colon V^k \to \boldsymbol{R}$ on a vector space V is called a ***k*-tensor** on V. The vector space of all k-tensors on V is denoted by $L_k(V)$. k is called the degree of f.

Definition 1.1.13 **Permutation action on k-tensors**

For $f \in L_k(V)$ on a vector space V and $\sigma \in \mathfrak{S}_n$, an action of σ on f is defined by

$$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Definition 1.1.14 **Symmetric and alternating k-tensor**

A k-tensor $f\colon V^k \to \boldsymbol{R}$ is ***symmetric*** if

$$\forall \sigma \in \mathfrak{S}_k,\ \sigma f = f,$$

and f is ***alternating*** if

$$\forall\sigma \in \mathfrak{S}_k,\ \sigma f = (\operatorname{sgn}\sigma)f.$$

Definition 1.1.15 **The set of all alternating k-tensors**

An alternating k-tensor on a vector space V is also called a k***-covector*** or a ***multicovector of degree*** k on V. The set of all k-covectors on V is denoted by $A_k(v)$ for $k > 0$; for $k = 0$, $A_0(V) = \boldsymbol{R}$.

Definition & Proposition 1.1.16 **Symmetrizing and alternating operators on k-covectors**

For a $f \in A_k(V)$ on a vector space V,

$$Sf = \sum_{\sigma\in\mathfrak{S}_n} \sigma f$$

is symmetric, and

$$Af = \sum_{\sigma\in\mathfrak{S}_n} (\operatorname{sgn}\sigma)\sigma f$$

is alternating.

Definition 1.1.17 **Tensor product of two multilinear functions**

For $f \in L_k(V), g \in L_\ell(V)$ on a vector space V, the ***tensor product*** $f \otimes g \in L_{k+\ell}(V)$ is defined by

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) = f(v_1, \ldots, v_k)g(v_{k+1}, \ldots, v_{k+\ell}).$$

Example 1.1.18 **Bilinear map as a tensor product**

Let $e_1, \ldots, e_n$ be a basis for a vector space V, $\alpha^1, \ldots, \alpha^n$ the dual basis in V^*, and $\langle\ ,\ \rangle\colon V\times V \to \boldsymbol{R}$ a bilinear map on V. Then,

$$\langle\ ,\ \rangle = \sum g_{ij}\alpha^i \otimes \alpha^j,$$

where $g_{ij} = \langle e_i, e_j \rangle$.

Definition 1.1.19 **Wedge product of two multilinear functions**

For $f \in A_k(V), g \in A_\ell(V)$ on a vector space V, their ***wedge product*** or ***exterior product*** is

$$f \wedge g = \frac{1}{k!\,\ell!} A(f \otimes g).$$

$f \wedge g$ is alternating.

Explicitly,

$$\begin{aligned}(f \wedge g)(v_1, \ldots, v_{k+\ell}) &= \frac{1}{k!\,\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\ &= \sum_{\substack{(k,\ell)\text{-shuffle} \\ \sigma}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}),\end{aligned}$$

where a (k, ℓ)-shuffle means $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+\ell)$.

Proposition 1.1.20 **Wedge product is anticommutative**

For $f \in A_k(V), g \in A_\ell(V)$ on a vector space V,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

If the degree of f is odd, then $f \wedge f = 0$.

Lemma 1.1.21 **Properties of nesting alternating operators**

For $f \in L_k(V)$ and $g \in L_\ell(V)$ on a vector space V,

i) $A(A(f) \otimes g) = k!\, A(f \otimes g)$,

ii) $A(f \otimes A(g)) = \ell!\, A(f \otimes g)$.

Proposition 1.1.22 **Associativity of the wedge product**

For $f \in A_k(V)$, $g \in A_\ell(V)$, $h \in A_m(V)$ on a real vector space V,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Similarly, for $f_i \in A_{d_i}(V)$ $(i = 1, \ldots, r)$,

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{(d_1)! \cdots (d_r)!} A(f_1 \otimes \cdots \otimes f_r).$$

Proposition 1.1.23 **Wedge product of covectors is the determinant**

For covectors $\alpha^1, \ldots, \alpha^k$ on a vector space V,

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = \det(\alpha^i(v_j))_{ij}.$$

Definition 1.1.24 **Graded algebra over a field**

An algebra $\boldsymbol{A}$ over a field $\boldsymbol{K}$ is said to be ***graded*** if $\boldsymbol{A} = \bigoplus_{k=0}^{\infty} A^k$ is a direct sum of vector spaces over $\boldsymbol{K}$ such that the multiplication sends $A^k \times A^\ell$ to $A^{k+\ell}$. $A = \bigoplus_{k=0}^{\infty} A^k$ means each nonzero $a \in \boldsymbol{A}$ is a unique finite sum $a = a_{i_1} + \cdots a_{i_m}$ with nonzero $a_{i_j} \in A^{i_j}$.

$\boldsymbol{A}$ is ***anticommutative*** or ***graded commutative*** if $\forall a \in A^k, b \in A^\ell$, $ab = (-1)^{k\ell} ba$.

A ***homomorphism*** of graded algebras is an algebra homomorphism that preserves the degree.

Definition & Proposition 1.1.25 **Grassmann algebra of multicovectors on a vector space**

For a vector space V of degree $n < \infty$, the ***exterior algebra*** or the ***Grassmann algebra*** of multicovectors on V is the anticommutative graded algebra

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^{n} A_k(V)$$

with the wedge product of multicovectors as multiplication.

Lemma 1.1.26 **Wedge product of the dual basis applying to a basis**

Let $e_1, \ldots, e_n$ be a basis for a vector space V and $\alpha^1, \ldots, \alpha^n$ the dual basis in V^*. For $I = (i_1, \ldots, i_k)$, $J = (j_1, \ldots, j_k)$ with $1 \le i_1 < \cdots < i_k \le n$, $1 \le j_1 < \cdots < j_k \le n$,

$$\alpha^I(e_J) = \delta^I_J.$$

Proposition 1.1.27 **Wedge products of the dual basis form a basis for multicovectors**

Let V be a vector space and $\alpha^1, \ldots, \alpha^n$ the dual basis in V^*. Then, α^I, $I = (i_1 < \cdots < i_k)$ form a basis for $A_k(V)$.

Therefore,

$$\dim A_k(V) = \binom{n}{k},$$

which implies

$$\text{if } k > \dim V, \text{ then } A_k(V) = 0.$$

Definition 1.1.28 **Cotangent space to an Euclidean space at a point**

The ***cotangent space*** to $\boldsymbol{R}^n$ at p is $T_p^*(\boldsymbol{R}^n) = (T_p(\boldsymbol{R}^n))^*$.

Definition 1.1.29 **Differential 1-form on an open subset of an Euclidean space**

A ***covector field*** or a ***differential 1-form*** on $U \in \mathcal{O}_n$ is $\omega\colon U \to \bigcup_{p \in U} T_p^*(\boldsymbol{R}^n)$ that maps $U \ni p \mapsto \omega_p \in T_p^*(\boldsymbol{R}^n)$.

Definition 1.1.30 **Differential of a smooth function**

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$, the ***differential*** df of f is a differential 1-form defined by

$$(df)_p(X_p) = X_p f.$$

In the expression

$$\langle\ ,\ \rangle\colon T_p(\boldsymbol{R}^n) \times C_p^\infty(\boldsymbol{R}^n) \ni (X_p, f) \mapsto \langle X_p, f\rangle = X_p f \in \boldsymbol{R},$$

a tangent vector is considered as $\langle X_p, \cdot\rangle$; a differential at p as $df|_p = (df)_p = \langle \cdot, f\rangle$.

Proposition 1.1.31 **Differentials of coordinates is the dual basis for the cotangent space**

For $p \in \boldsymbol{R}^n$, $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the dual basis for $T_p^*(\boldsymbol{R}^n)$ to $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\} \subset T_p(\boldsymbol{R}^n)$, where $x^1, \dots, x^n$ are the standard coordinates on $\boldsymbol{R}^n$.

For any differential 1-form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some $a_i(p)$. In this case, ω is written as $\omega = \sum a_i dx^i$.

Definition 1.1.32 **Smoothness of a differential 1-form**

A differential 1-form $\omega = \sum a_i dx^i$ on $U \in \mathcal{O}_n$ is ***smooth*** if all $a_i\colon U \to \boldsymbol{R}$ are smooth.

Proposition 1.1.33 **Differentials can be written in terms of partial derivatives**

For $f \in C^\infty(U)$ on $U \in \mathcal{O}_n$,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness of f implies that of df.

Definition 1.1.34 **Differential k-forms on an Euclidean space**

A ***differential k-form*** or ***differential form of degree*** k on $U \in \mathcal{O}_n$ is $\omega\colon U \ni p \mapsto \omega_p \in A_k(T_p(\boldsymbol{R}^n))$.

Definition & Proposition 1.1.35 **Basis for differential forms**

Since $\{dx_p^I \mid I = (1 \leq i_1 < \cdots < i_k \leq n)\}$ is a basis for $A_k(T_p(\boldsymbol{R}^n))$, for a differential k-form ω on $U \in \mathcal{O}_n$ and $p \in U$,

$$\omega_p = \sum a_I(p) dx_p^I, \quad \omega = \sum a_I dx^I.$$

ω is ***smooth*** if all $a_I \colon U \to \boldsymbol{R}$ are smooth. The vector space of C^∞ differential k-forms on U is denoted by $\Omega^k(U)$. If $k = 0$, $\Omega^0(U) = C^\infty(U)$.

Definition 1.1.36 **Wedge product of differential forms**

For differential k-form ω and ℓ-form τ on $U \in \mathcal{O}_n$, their ***wedge product*** $\omega \wedge \tau$ is a differential $(k + \ell)$-form defined by

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p.$$

If $\omega = \sum a_I dx^I$, $\tau = \sum b_J dx^J$,

$$\begin{aligned} \omega \wedge \tau &= \sum_{I,J} (a_I b_J) dx^I \wedge dx^J \\ &= \sum_{\text{disjoint } I,J} (a_I b_J) dx^I \wedge dx^J. \end{aligned}$$

For $\omega \in \Omega^k(U)$, $\tau \in \Omega^\ell(U)$, the wedge product is a bilinear map

$$\wedge \colon \Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U).$$

In particular, if $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$, then $f \wedge \omega = f\omega$.

Definition 1.1.37 **Graded algebra with smooth differential forms**

For $U \in \mathcal{O}_n$, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is an anticommutative graded algebra over $\boldsymbol{R}$ with the wedge product as multiplication, which is also a module over $C^\infty(U)$.

Definition 1.1.38 **Differential forms as linear maps on a vector field**

For a differential k-form ω on $U \in \mathcal{O}_n$ and $X_1, \ldots, X_k \in \mathfrak{X}(U)$, define $\omega(X_1, \ldots, X_k) \in C^\infty(U)$ by

$$(\omega(X_1, \ldots, X_k))_p = \omega_p((X_1)_p, \ldots, (X_k)_p).$$

The map

$$\mathfrak{X}^k(U) \ni (X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k) \in C^\infty(U)$$

is k-linear over $C^\infty(U)$.

Definition 1.1.39 Exterior derivatives of differential forms

For $k \geq 1$ and $\omega = \sum a_I dx^I \in \Omega^k(U)$, the ***exterior derivative*** of ω is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U);$$

for $k = 0$ and $f \in C^\infty(U)$, its exterior derivative is

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(U).$$

Definition 1.1.40 Antiderivation of a graded algebra

An ***antiderivation*** of a graded algebra $\boldsymbol{A} = \bigoplus_{k=0}^\infty A^k$ is a linear map $D \colon \boldsymbol{A} \to \boldsymbol{A}$ such that for $a \in A^k$, $b \in A^\ell$,

$$D(ab) = D(a)b + (-1)^k aD(b).$$

If m is an integer such that D sends A^k to A^{k+m} for all k, then m is called the ***degree*** of D.

Proposition 1.1.41 Properties of the exterior differentiation

i) The exterior differentiation $d \colon \Omega^*(U) \to \Omega^*(U)$ on $U \in \mathcal{O}_n$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

ii) $d^2 = 0$.

iii) For $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(df)(X) = Xf$.

Proposition 1.1.42 Characterization of the exterior differentiation

The exterior differentiation $d\colon \Omega^*(U) \to \Omega^*(U)$ on $U \in \mathcal{O}_n$ is the only antideriavtion of $\Omega^*(U)$.

Definition 1.1.43 **Closed and exact forms**

A differential k-form ω on $U \in \mathcal{O}_n$ is said to be ***closed*** if $d\omega = 0$, and said to be ***exact*** if $\omega = d\tau$ for some $(k-1)$-form τ on U.

Every exact form is closed.

Definition 1.1.44 **Cochain complex and de Rham complex**

A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k\colon V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a ***cochain complex*** or a ***differential complex***.

The ***de Rham complex*** of $U \in \mathcal{O}_n$ is a cochain complex

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots .$$

The closed forms are the elements of $\ker d$, and the exact forms are the elements of $\operatorname{im} d$.

Proposition 1.1.45 **Vector calculus as differential forms**

Under the identifications, for $U \in \mathcal{O}_3$, $f \in C^\infty(U)$ and $X = [P\ Q\ R] \in \mathfrak{X}(U)$,

$$\text{1-form } P dx + Q dy + R dz \longleftrightarrow X,$$

$$\text{2-form } P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \longleftrightarrow X,$$

$$\text{3-form } f dx \wedge dy \wedge dz \longleftrightarrow f,$$

there are correspondences between the exterior derivatives and grad, rot, and div:

$$df \longleftrightarrow \operatorname{grad} f,$$

$$d(P dx + Q dy + R dz) \longleftrightarrow \operatorname{rot} X,$$

$$d(P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy) \longleftrightarrow \operatorname{div} X.$$

Definition 1.1.46 **k-th de Rham cohomology**

For $U \in \mathcal{O}_n$, the k-th ***de Rham cohomology*** of U is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}.$$

Proposition 1.1.47 **Poincaré lemma**

For $k \geq 1$, every closed k-form on $\boldsymbol{R}^n$ is exact, i.e., $H^k(\boldsymbol{R}^n)$ vanishes.

1.2 Manifolds

Definition 1.2.1 **Locally Euclidean space**

A topological space M is ***locally Euclidean of dimension*** n if $\forall p \in M, \exists (U, \phi)$, with a neighborhood U at p and a homeomorphism $\phi\colon U \to V \in \mathcal{O}_n$, called a ***chart***, a ***coordinate neighborhood*** or a ***coordinate open set***, and ϕ a ***coordinate map*** or a ***coordinate system*** on U.

A chart (U, ϕ) is said to be ***centered*** at $p \in U$ if $\phi(p) = 0$.

Definition 1.2.2 **Topological manifold**

A ***topological manifold of dimension*** n is a Hausdorff, second countable, locally Euclidean space of dimension n.

Definition 1.2.3 **Compatible chart**

Two charts $(U, \phi\colon U \to \boldsymbol{R}^n)$, $(V, \psi\colon V \to \boldsymbol{R}^n)$ of a topological manifold are said to be C^∞***-compatible*** or simply ***compatible*** if

$$\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$$

called the ***transition functions*** between charts are C^∞. If $U \cap V = \emptyset$, they are C^∞-compatible.

Definition 1.2.4 **Atlas on a locally Euclidean space**

A C^∞ ***atlas*** or simply an ***atlas*** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise compatible charts that cover M.

Definition 1.2.5 **Compatibility of a chart with an atlas**

For a locally Euclidean space, a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if all charts (U_α, ϕ_α) are compatible with (V, ψ).

Lemma 1.2.6 **Charts compatible with the same atlas are compatible with each other**

For a locally Euclidean space, charts (V, ψ), (W, σ), and an atlas $\{(U_\alpha, \phi_\alpha)\}$ on it, if (V, ψ) and (W, σ) are both compatible with $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Definition 1.2.7 **Maximal Atlas on a locally Euclidean space**

An atlas $\mathfrak{M}$ on a locally Euclidean space is ***maximal*** if for another atlas $\mathfrak{U}$, $\mathfrak{M} \subset \mathfrak{U}$ implies $\mathfrak{M} = \mathfrak{U}$.

Definition 1.2.8 **Smooth manifold**

A ***smooth*** or C^∞ ***manifold*** is a topological manifold M with a maximal atlas called a ***differentiable structure*** on M. M is said to be of dimension n if all of its connected components are of dimension n, and then M is called a n-***manifold***. A 1-manifold is also called a ***curve***, a 2-manifold a ***surface***.

Proposition 1.2.9 **A locally Euclidean space with an atlas has a maximal atlas**

In a locally Euclidean space, any atlas is contained in a unique maximal atlas.

Definition 1.2.10 **Conventions of manifold**

i) A manifold means a smooth manifold.

ii) The standard coordinates on $\boldsymbol{R}^n$ is denoted by $r^1, \ldots, r^n$.

iii) For a chart (U, ϕ) of a manifold, let $x^i = r^i \circ \phi$ the i-th component of ϕ, and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. $x^1, \ldots, x^n$ are called ***coordinates*** or ***local coordinates*** on U.

iv) The notation $(x^1, \ldots, x^n)$ means alternately the local coordinates on U and a point in $\boldsymbol{R}^n$

v) A ***chart*** (U, ϕ) ***about*** p in a manifold M means a chart in the differentiable structure of M such that $p \in U$.

Proposition 1.2.11 **Product manifold**

For a m-manifold M and n-manifold N, and atlases $\{(U_\alpha, \phi_\alpha)\}$ of M and $\{(V_{\alpha'}, \psi_{\alpha'})\}$ of N, the collection

$$\{(U_\alpha \times V_{\alpha'}, \phi_\alpha \times \psi_{\alpha'} \colon U_\alpha \times V_{\alpha'} \to \boldsymbol{R}^m \times \boldsymbol{R}^n)\}$$

is an atlas on $M \times N$, and therefore $M \times N$ is a manifold of dimension $m + n$.

Definition 1.2.12 **Smooth function on a manifold**

For a smooth n-manifold M, a function $f \colon M \to \boldsymbol{R}$ is said to be C^∞ or ***smooth at a point*** $p \in M$ if, for some chart (U, ϕ) about p, $f \circ \phi^{-1} \colon \phi(U) \to \boldsymbol{R}^n$ is C^∞ at $\phi(p)$; C^∞ ***on*** M if it is smooth at every point.

Proposition 1.2.13 **Smoothness of real-valued functions**

For a n-manifold M and a function $f \colon M \to \boldsymbol{R}$, the following are equivalent:

i) f is C^∞.

ii) There exists an atlas $\mathfrak{U}$ for M, for any $(U, \phi) \in \mathfrak{U}$, $f \circ \phi^{-1}$ is C^∞.

iii) For any chart (U, ϕ) on M, $f \circ \phi^{-1}$ is C^∞.

Definition 1.2.14 **Pullback of a function by a map**

For manifolds M, N, the ***pullback*** of $h\colon M \to \boldsymbol{R}$ by $F\colon N \to M$ is $F^*h = h \circ F$.

Definition 1.2.15 **Smooth map between manifolds**

For a m-manifold M and n-manifold N, a continuous map $F\colon N \to M$ is C^∞ ***at a point*** $p \in N$ if, for some chats (U, ϕ) about p and (V, ψ) about $F(p)$, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$; C^∞ if it is C^∞ at every point.

Proposition 1.2.16 **Smoothness of maps is independent of charts**

Let M be a m-manifold, N a n-manifold, and $F\colon N \to M$ be C^∞ at $p \in N$. Then, for any charts (U, ϕ) about p and (V, ψ) about $F(p)$, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Proposition 1.2.17 **Smoothness of a map in terms of charts**

For a m-manifold M and n-manifold N, and a continuous map $F\colon N \to M$, the following are equivalent:

i) F is C^∞.
ii) There exists atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M, for any $(U, \phi) \in \mathfrak{U}$ and $(V, \psi) \in \mathfrak{V}$, $\psi \circ F \circ \phi^{-1}$ is C^∞.
iii) For any chart (U, ϕ) on N and (V, ψ) on M, $\psi \circ F \circ \phi^{-1}$ is C^∞.

Proposition 1.2.18 **Composite of smooth maps is also smooth**

For manifolds M, N, P and C^∞ maps $F\colon N \to M$, $G\colon M \to P$, $G \circ F\colon N \to P$ is also C^∞.

Definition 1.2.19 **Diffeomorphism of manifolds**

A ***diffeomorphism*** of manifolds is a bijective C^∞ map whose inverse is also C^∞.

Proposition 1.2.20 Coordinate map is a diffeomorphism

A coordinate map $\phi\colon U \to \phi(U) \subset \boldsymbol{R}^n$ for a manifold with a chart (U, ϕ) is a diffeomorphism.

Proposition 1.2.21 Diffeomorphism into an Euclidean space is a coordinate map

For an open subset U of a manifold M with the differentiable structure $\mathfrak{U}$, if $F\colon U \to F(U)$ is a diffeomorphism, then $(U, F) \in \mathfrak{U}$.

Proposition 1.2.22 Smoothness of a vector-valued function

For a continuous map $F\colon N \to \boldsymbol{R}^m$ on a manifold M, the following are equivalent:

i) F is C^∞.
ii) There exists an atlas $\mathfrak{U}$ for M, for any $(U, \phi) \in \mathfrak{U}$, $F \circ \phi^{-1}$ is C^∞.
iii) For any chart (U, ϕ) on M, $F \circ \phi^{-1}$ is C^∞.

Proposition 1.2.23 Vector-valued function is smooth iff its components are all smooth

For a vector-valued function $F = (F^1, \ldots, F^m)\colon M \to \boldsymbol{R}^m$ on a manifold M, F is C^∞ iff $F^1, \ldots, F^m$ are all C^∞.

Proposition 1.2.24 Smoothness of a map in terms of vector-valued functions

For a continuous map $F\colon N \to M$ between a m-manifold M and n-manifold N, the following are equivalent:

i) F is C^∞.
ii) There exists an atlas $\mathfrak{U}$ for M, for any $(U, \phi) \in \mathfrak{U}$, $\phi \circ F$ is C^∞.

iii) For any chart (U, ϕ) on M, $\phi \circ F$ is C^∞.

Proposition 1.2.25 **Smoothness of a map in terms of components**

For a continuous map $F\colon N \to M$ between a m-manifold M and n-manifold N, the following are equivalent:

i) F is C^∞.

ii) There exists an atlas $\mathfrak{U}$ for M, for any $(U, \phi^1, \dots, \phi^m) \in \mathfrak{U}$, the components $\phi^i \circ F$ of F relative to the chart are all C^∞.

iii) For any chart $(U, \phi^1, \dots, \phi^m)$ on M, the components $\phi^i \circ F$ of F relative to the chart are al C^∞.

2 P-ADIC NUMBERS

[4]

2.1 Foundations

Definition 2.1.1 Absolute value on a field

An ***absolute value*** on a field $\boldsymbol{K}$ is a function $|\ |\colon \boldsymbol{K} \to \boldsymbol{R}_{\geq 0}$ that satisfies:

i) $|x| = 0$ iff $x = 0$
ii) $\forall x, y \in \boldsymbol{K},\ |xy| = |x|\,|y|$
iii) $\forall x, y \in \boldsymbol{K},\ |x+y| \leq |x| + |y|$.

An absolute value that satisfies the condition

iv) $\forall x, y \in \boldsymbol{K},\ |x+y| \leq \max\{|x|, |y|\}$

is said to be ***non-archimedean***; otherwise, it is said to be ***archimedean***.

Definition 2.1.2 Trivial absolute value

The ***trivial absolute value*** on a field $\boldsymbol{K}$ is a absolute value on $\boldsymbol{K}$ such that

$$|x| = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

An absolute value on a finite field must be trivial.

Definition 2.1.3 Valuation on a field

A function $v\colon \boldsymbol{A}^\times \to \boldsymbol{R}$ with an integral domain $\boldsymbol{A}$ is called a ***valuation*** on $\boldsymbol{A}$ if it satisfies the following conditions:

i) $\forall x, y \in \boldsymbol{A}^\times,\ v(xy) = v(x) + v(y)$
ii) $\forall x, y \in \boldsymbol{A}^\times,\ v(x+y) \geq \min\{v(x), v(y)\}$

Definition & Proposition 2.1.4 **Value group of a valuation**

The image of a valuation v on a field is an additive subgroup of $\boldsymbol{R}$. $\operatorname{im} v$ is called the ***value group*** of v.

Proposition 2.1.5 **Correspondence between valuations and non-archimedean absolute values**

Let $\boldsymbol{A}$ be an integral domain and $\boldsymbol{K} = \operatorname{Frac} \boldsymbol{A}$. Let $v\colon \boldsymbol{A}^\times \to \boldsymbol{R}$ be a valuation on $\boldsymbol{A}$ and extend v to $\boldsymbol{K}$ by setting $v(a/b) = v(a) - v(b)$, then the function $|\ |_v\colon \boldsymbol{K} \to \boldsymbol{R}_{\geq 0}$ defined by

$$|x|_v = \begin{cases} e^{-v(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a non-archimedean absolute value on $\boldsymbol{K}$. Conversely, $-\log |\ |$ is a valuation on $\boldsymbol{K}$ for a non-archimedean absolute value $|\ |$ on $\boldsymbol{K}$.

Definition 2.1.6 **p-adic valuation**

The p***-adic valuation*** on $\boldsymbol{Q}$ with a prime p is a valuation $v_p\colon \boldsymbol{Q}^\times \to \boldsymbol{R}$ defined as follows: for each $n \in \boldsymbol{Z}^\times$, let $v_p(n)$ be the greatest integer such that $p^{v_p(n)} \mid n$, and for each $x = a/b \in \boldsymbol{Q}^\times$, $v_p(x) = v_p(a) - v_p(b)$.

We often set $v_p(0) = \infty$.

Definition 2.1.7 **p-adic absolute value**

The p***-adic absolute value*** $|\ |_p\colon \boldsymbol{Q} \to \boldsymbol{R}_{\geq 0}$ with a prime p is defined as

$$|x|_p = p^{-v_p(x)}, \quad |0| = 0.$$

The usual absolute value is looked as $|\ | = |\ |_\infty$.

Definition 2.1.8 **Absolute values on a field of rational functions**

Here are some absolute values on a field $\boldsymbol{F}(t)$ of rational functions over a field $\boldsymbol{F}$.

i) For $f(t) \in \boldsymbol{F}[t]$, $v_\infty(f) = -\deg f$, and for $f(t)/g(t) \in \boldsymbol{F}(t)$, $v_\infty(f/g) = v_\infty(f) - v_\infty(g)$ with $v_\infty(0) = \infty$. Then,

$$|f(t)|_\infty = e^{-v_\infty(f)}.$$

ii) For an irreducible polynomial $p(t) \in \boldsymbol{F}[t]$, define the $p(t)$-adic valuation and absolute value.

Lemma 2.1.9 **Properties of absolute values on fields**

For an absolute value $|\ |$ on a field $\boldsymbol{K}$,

i) $|1| = 1$,
ii) $\forall x \in \boldsymbol{K},\ |x^n| = 1 \Rightarrow |x| = 1$,
iii) $\forall x \in \boldsymbol{K},\ |-x| = |x|$,
iv) If $\boldsymbol{K}$ is finite, then $|\ |$ is trivial.

Theorem 2.1.10 **Necessary and sufficient conditions of a non-archimedean absolute value**

Let $\boldsymbol{K}$ be a field, $|\ |$ an absolute value on $\boldsymbol{K}$. Then,

$$\begin{aligned} |\ | \text{ is non-archimedean} &\iff \forall n = 1 + \cdots + 1 \in \boldsymbol{K},\ |n| \leq 1 \\ &\iff \sup\{|n| \mid n \in \boldsymbol{Z}\} = 1. \end{aligned}$$

Furthermore, $\sup\{|n| \mid n \in \boldsymbol{Z}\} = \infty$ if $|\ |$ is archimedean.

Definition 2.1.11 **Distance induced from an absolute value on a field**

Given a field $\boldsymbol{K}$ with an absolute value $|\ |$, define the distance $d\colon \boldsymbol{K} \times \boldsymbol{K} \to \boldsymbol{R}$ by $d(x, y) = |x - y|$, which makes $\boldsymbol{K}$ into a metric space.

Proposition 2.1.12 **Field can be a topological field with the induced distance**

For a field $\boldsymbol{K}$ with the distance $d\colon \boldsymbol{K} \times \boldsymbol{K} \to \boldsymbol{R}$ induced from an absolute value on $\boldsymbol{K}$, addition, multiplication and taking inverse are all continuous: fix points $x_0, y_0 \in \boldsymbol{K}$, then

- $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \boldsymbol{K},\ d(x, x_0), d(y, y_0) < \delta \Longrightarrow d(x + y, x_0 + y_0) < \epsilon,$
- $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \boldsymbol{K},\ d(x, x_0), d(y, y_0) < \delta \Longrightarrow d(x - y, x_0 - y_0) < \epsilon,$
- $\forall \epsilon > 0, \exists \delta > 0, \forall x \in \boldsymbol{K},\ d(x, x_0) < \delta \Longrightarrow d(1/x, 1/x_0) < \epsilon.$

Definition & Proposition 2.1.13 **Ultrametric space with a non-archimedean absolute value**

For a field $\boldsymbol{K}$ with the distance d induced from an absolute value $|\ \ |$ on $\boldsymbol{K}$, $|\ \ |$ is non-archimedean iff $d(x, y) \leq \max\{d(x, z), d(z, y)\}$. This inequality is called the ***ultrametric inequality***, a metric which holds the ultrametric inequality an ***ultrametric***, and a space with an ultrametric an ***ultrametric space***.

Proposition 2.1.14 **Non-archimedean absolute value of a sum is the max of the two**

For a field $\boldsymbol{K}$ with a non-archimedean absolute value $|\ \ |$ on $\boldsymbol{K}$, and $x, y \in \boldsymbol{K}$ with $|x| \neq |y|$,

$$|x + y| = \max\{|x|, |y|\}.$$

Corollary 2.1.15 **Triangle on an ultrametric space is isosceles**

In an ultrametric space, all triangles are isosceles.

Definition 2.1.16 **Open ball and closed ball on a field with an absolute value**

In a field $\boldsymbol{K}$ with an absolute value $|\ \ |$, an ***open ball*** of radius $r > 0$ and center $a \in \boldsymbol{K}$ is

$$B(a, r) = \{x \in \boldsymbol{K} \mid |x - a| < r\};$$

a ***closed ball*** is

$$\overline{B}(a, r) = \{x \in \boldsymbol{K} \mid |x - a| \leq r\}.$$

Proposition 2.1.17 **Properties of an open ball and closed ball with a non-archimedean absolute value**

Let $\boldsymbol{K}$ be a field with a non-archimedean absolute value, and $a, b \in \boldsymbol{K},\ r, s > 0$.

i) $b \in B(a, r) \Longrightarrow B(a, r) = B(b, r)$,

ii) $b \in \overline{B}(a, r) \Longrightarrow \overline{B}(a, r) = \overline{B}(b, r)$,

iii) $B(a, r)$ is clopen,

iv) $\overline{B}(a, r)$ is clopen,

v) $B(a, r) \cap B(b, s) \Longleftrightarrow B(a, r) \subset B(b, s)$ or $B(a, r) \supset B(b, s)$.

vi) $\overline{B}(a, r) \cap \overline{B}(b, s) \Longleftrightarrow \overline{B}(a, r) \subset \overline{B}(b, s)$ or $\overline{B}(a, r) \supset \overline{B}(b, s)$.

3 Lie Algebras

[5]

3.1 Foundations

Definition 3.1.1 Lie algebra

A vector space $\mathfrak{g}$ over a field $\boldsymbol{K}$ with the Lie bracket satisfying the conditions

i) Lie bracket is bilinear

ii) $\forall x \in \mathfrak{g},\ [x, x] = 0$

iii) $\forall x, y, z \in \mathfrak{g},\ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

is called a ***Lie algebra*** over $\boldsymbol{K}$.

Definition 3.1.2 General linear Lie algebra

$\mathfrak{gl}_n(\boldsymbol{R})$ is the Lie algebra $M_n(\boldsymbol{R})$ with the Lie bracket $[x, y] = xy - yx$.

Definition 3.1.3 Derivation algebra

A linear endomorphism D of an algebra $\boldsymbol{A}$ over $\boldsymbol{R}$ satisfying $D(xy) = D(x)y + xD(y)$ is called a ***derivation*** of $\boldsymbol{A}$. The set of all derivations $\operatorname{Der}\boldsymbol{A}$ with the addition, scaler multiplication, and lie bracket defined as follows:

i) $(D + D')(x) = D(x) + D'(x)$

ii) $(\alpha D)(x) = \alpha D(x)$

iii) $[D, D'](x) = D(D'(x)) - D'(D(x))$

is a Lie algebra called the ***derivation algebra*** of $\boldsymbol{A}$.

Definition 3.1.4 Lie subalgebra

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a ***Lie subalgebra*** of $\mathfrak{g}$ if $\forall x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

For linear subspaces $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ denotes the subspace generated by $[x, y]$ with $x \in \mathfrak{a}, y \in \mathfrak{b}$.

Definition & Proposition 3.1.5 **Special linear Lie algebra**

$\mathfrak{sl}_n(\boldsymbol{R}) = \{x \in \mathfrak{gl}_n(\boldsymbol{R}) \mid \operatorname{tr} x = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(\boldsymbol{R})$.

Definition & Proposition 3.1.6 **Orthogonal Lie algebra**

$\mathfrak{o}(n) = \{x \in \mathfrak{gl}_n(\boldsymbol{R}) \mid {}^t x = -x\}$ is a Lie subalgebra of $\mathfrak{sl}_n(\boldsymbol{R})$.

Definition & Proposition 3.1.7 **Ideal of a Lie algebra**

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is an ***ideal*** of $\mathfrak{g}$ if $\forall x \in \mathfrak{g}, y \in \mathfrak{h},\ [x, y] \in \mathfrak{h}$.
For ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ is also an ideal.

Definition 3.1.8 **Derived ideal of a Lie algebra**

For a Lie algebra $\mathfrak{g}$, $D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ called the ***derived ideal*** of $\mathfrak{g}$.
If $\mathfrak{g} = \mathfrak{gl}_n(\boldsymbol{R})$, $D\mathfrak{g} = \mathfrak{sl}_n(\boldsymbol{R})$.

Definition & Proposition 3.1.9 **Homomorphism of Lie algebras**

For Lie algebras $\mathfrak{g}, \mathfrak{h}$, a linear map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is called a ***homomorphism*** if $\forall x, y \in \mathfrak{g},\ \varphi([x, y]) = [\varphi(x), \varphi(y)]$. A homomorphism φ is an ***isomorphism*** if it is bijective. Lie algebras between which there exists an isomorphism are said to be ***isomorphic*** to each other, written $\mathfrak{g} \cong \mathfrak{h}$.

A composite of homomorphisms is also a homomorphism, and that of isomorphisms is also an isomorphism.

The kernel $\ker \varphi = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$ of a homomorphism φ is an ideal of $\mathfrak{g}$ while the image $\operatorname{im}\varphi = \varphi(\mathfrak{g})$ of φ is a Lie subalgebra of $\mathfrak{h}$.

Definition 3.1.10 **Representation of a Lie algebra on a vector space**

For a Lie algebra $\mathfrak{g}$ and a vector space V, a homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a ***representation*** of $\mathfrak{g}$ on V.

Definition & Proposition 3.1.11 **Adjoint representation of a Lie algebra**

For a Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, define a derivation $\operatorname{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ by $\operatorname{ad}(x)(y) = [x, y]$. A representation $\operatorname{ad}\colon \mathfrak{g} \ni x \mapsto \operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is called the ***adjoint representation*** of $\mathfrak{g}$. The ***center*** of $\mathfrak{g}$ is $\mathfrak{z} = \ker(\operatorname{ad})$, which is a commutative ideal. $\operatorname{im}(\operatorname{ad})$ is an ideal of $\operatorname{Der}\mathfrak{g}$. A derivation $\operatorname{ad}(x)$ is called a ***inner derivation*** of $\mathfrak{g}$.

Definition 3.1.12 **Quotient algebra for Lie algebras**

For a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{a} \subset \mathfrak{g}$, the ***quotient algebra*** is

$$\mathfrak{g}/\mathfrak{a} = \{\overline{x} = x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

with canonical operations, where $\overline{x} = \{y \in \mathfrak{g} \mid x \equiv y \pmod{\mathfrak{a}}\} = \{x + a \mid a \in \mathfrak{a}\}$ called the ***class*** of x. The homomorphism $\varphi\colon \mathfrak{g} \ni x \mapsto \overline{x} \in \mathfrak{g}/\mathfrak{a}$ is called the ***canonical homomorphism***.

Theorem 3.1.13 **The first isomorphism theorem for Lie algebras**

For Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and a homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$,

$$\mathfrak{g}/\ker\varphi \cong \operatorname{im}\varphi.$$

Theorem 3.1.14 **The second isomorphism theorem for Lie algebras**

For a Lie algebra $\mathfrak{g}$, an ideal $\mathfrak{a} \subset \mathfrak{g}$, a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the canonical homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$,

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{a}) \cong (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}.$$

3.2 Solvable and Nilpotent Lie algebras

Definition 3.2.1 **Solvable Lie algebra**

Let $\mathfrak{g}$ be a Lie algebra, and

$$D^0\mathfrak{g} = \mathfrak{g}, \quad D^k\mathfrak{g} = D(D^{k-1}\mathfrak{g}), \quad k = 1, 2, \dots$$

$\mathfrak{g}$ is said to be ***solvable*** if $D^r\mathfrak{g} = \{0\}$ for some r called the ***length*** of $\mathfrak{g}$.

Example 3.2.2 **Lie algebra of triangular matrices is solvable**

Let

$$\begin{aligned} \mathfrak{g}_0 &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\boldsymbol{R}) \mid \xi \text{ is upper triangular}\}, \\ \mathfrak{g}_k &= \{\xi = (\xi_{ij}) \in \mathfrak{gl}_n(\boldsymbol{R}) \mid \xi_{ij} = 0 \text{ for } j - i < k\}. \end{aligned}$$

Then, $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_1$, $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$, $k, \ell = 0, 1, \dots$, and $\mathfrak{g}_0$ is a solvable Lie algebra of length $\leq n$.

Theorem 3.2.3 **Lie subalgebra of a solvable Lie algebra is also solvable**

For a solvable Lie algebra $\mathfrak{g}$, its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also solvable, and if $\mathfrak{h}$ is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also solvable.

Theorem 3.2.4 **Lie algebra whose ideal and quotient algebra over it are solvable is solvable**

For a Lie algebra $\mathfrak{g}$ and its ideal $\mathfrak{a} \subset \mathfrak{g}$, if $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are both solvable, then $\mathfrak{g}$ is also solvable.

Definition 3.2.5 **Nilpotent Lie algebra**

Let $\mathfrak{g}$ be a Lie algebra, and

$$C^0\mathfrak{g} = \mathfrak{g}, \quad C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}], \quad k = 1, 2, \ldots$$

$\mathfrak{g}$ is said to be ***nilpotent*** if $C^s\mathfrak{g} = \{0\}$ for some s called the ***length*** of $\mathfrak{g}$.

Since $D^k\mathfrak{g} \subset C^k\mathfrak{g}$, a nilpotent Lie algebra is solvable.

Example 3.2.6 **Lie algebra of strictly triangular matrices is nilpotent**

$\mathfrak{g}_1$ in [Example 3.2.2] is nilpotent while $\mathfrak{g}_0$ there is not.

Theorem 3.2.7 **Lie subalgebra of a nilpotent Lie algebra is also nilpotent**

For a nilpotent Lie algebra $\mathfrak{g}$, its Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is also nilpotent, and if $\mathfrak{h}$ is an ideal, $\mathfrak{g}/\mathfrak{h}$ is also nilpotent.

Theorem 3.2.8 **Center of a nilpotent Lie algebra is nontrivial**

For a Lie algebra $\mathfrak{g}$ and its center $\mathfrak{z}$, $\mathfrak{z} \neq \{0\}$ if $\mathfrak{g}$ is nilpotent while $\mathfrak{g}$ is nilpotent if $\mathfrak{g}/\mathfrak{z}$ is nilpotent.

Lemma 3.2.9 **Adjoint representation in a nilpotent Lie algebra is nilpotent**

For a nilpotent Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, $\operatorname{ad} x$ is nilpotent. (i.e., $(\operatorname{ad} x)^s = 0$ for some s).

Theorem 3.2.10 **Engel's Theorem**

For a Lie algebra $\mathfrak{g}$, if $\operatorname{ad} x$ is nilpotent for any $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Theorem 3.2.11 **Matrix of a representation of a Lie algebra whose representations are all nilpotent is strictly upper triangular**

For a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ on a vector space V, if $\rho(x)$ is nilpotent for any $x \in \mathfrak{g}$, then a matrix of $\rho(x)$ is strictly upper triangular for some basis for V.

4 Categories

[1]

4.1 Foundations

Definition 4.1.1 Category

A ***category*** consists of the followings:

- ***Objects*** $A, B, C, \ldots$
- ***Arrows*** $f, g, h, \ldots$ with the objects called the domain $\operatorname{dom} f$ and the codomain $\operatorname{cod} f$.
- ***Composites*** $g \circ f \colon A \to C$ for given arrows $f \colon A \to B$ and $g \colon B \to C$.
- ***Identity arrow*** 1_A of each object A.

satisfying the following laws:

i) $\forall$arrows $f \colon A \to B, g \colon B \to C, h \colon C \to D,\ h \circ (g \circ f) = (h \circ g) \circ f$

ii) $\forall$arrow $f \colon A \to B,\ f \circ 1_A = f = 1_B \circ f$.

Definition 4.1.2 Functor between categories

A ***functor*** $F \colon \mathscr{A} \to \mathscr{B}$ between categories $\mathscr{A}$ and $\mathscr{B}$ is a mapping between objects and between arrows in the following ways:

i) $F(f \colon A \to B) = F(f) \colon F(A) \to F(B)$,

ii) $F(1_A) = 1_{F(A)}$,

iii) $F(g \circ f) = F(g) \circ F(f)$.

Definition 4.1.3 Isomorphism between categories

In a category $\mathscr{C}$, an arrow $f\colon A \to B$ is called an ***isomorphism*** if

$$\exists g = f^{-1}\colon B \to A,\ g \circ f = 1_A,\ f \circ g = 1_B.$$

If there is an isomorphism between objects A and B, A is said to be ***isomorphic*** to B, written $A \cong B$.

Theorem 4.1.4 **Category is isomorphic to its Cayley representation**

For a category $\mathscr{C}$ with a set of arrows, the Cayley representation $\overline{\mathscr{C}}$ of $\mathscr{C}$, consisting of

- object $\overline{C} = \{f \in \mathscr{C} \mid \operatorname{cod} f = C\}$ for an object $C \in \mathscr{C}$,
- arrow $\overline{g}\colon \overline{C} \to \overline{D}$ for an arrow $g\colon C \to D$ such that $\overline{g}(f) = g \circ f$,

is isomorphic to $\mathscr{C}$.

Definition 4.1.5 **Product of two categories**

The ***product*** $\mathscr{C} \times \mathscr{D}$ of categories $\mathscr{C}$ and $\mathscr{D}$ consists of

- object (C, D) for objects $C \in \mathscr{C}$, $D \in \mathscr{D}$,
- arrow $(f, g)\colon (C, D) \to (C', D')$ for arrows $f\colon C \to C'$, $g\colon D \to D'$,

with composition $(f, g) \circ (f', g') = (f \circ f', g \circ g')$ and units $1_{(C,D)} = (1_C, 1_D)$.

The ***projection functors*** $\pi_1\colon \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $\pi_2\colon \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ is defined by $\pi_1(C, D) = C$ and $\pi_1(f, g) = f$, and similarly for π_2.

Definition 4.1.6 **Dual category**

For a category $\mathscr{C}$, its ***dual*** or ***opposite category*** $\mathscr{C}^{\mathrm{op}}$ consists of

- object $C^* = C$ for an object $C \in \mathscr{C}$,
- arrow $f^*\colon D^* \to C^*$ for an arrow $f\colon C \to D$,

with composition $f^* \circ g^* = (g \circ f)^*$ and units $1_{C^*} = (1_C)^*$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A^* & B^* \arrow[l, "f^*"'] \\ & C^* \arrow[u, "g^*"'] \arrow[ul, "f^* \circ g^*"] \end{tikzcd}$$

Definition 4.1.7 Arrow category

For a category $\mathscr{C}$, its ***arrow category*** $\mathscr{C}^{\to}$ consists of

- object $f\colon C \to D$ for an arrow f in $\mathscr{C}$,
- arrow $(g_1, g_2)\colon f \to f'$, where $f\colon A \to B$, $f'\colon A' \to B'$, $g_1\colon A \to A'$, $g_2\colon B \to B'$ in $\mathscr{C}$, such that $g_2 \circ f = f' \circ g_1$,

with composition $(g_1, g_2) \circ (h_1, h_2) = (g_1 \circ h_1, g_2 \circ h_2)$ and units $1_f = (1_A, 1_B)$.

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"'] & B' \end{tikzcd}$$

There are two functors $\mathrm{dom}, \mathrm{cod}\colon \mathscr{C}^{\to} \to \mathscr{C}$.

Definition 4.1.8 Slice category

For a category $\mathscr{C}$, its ***slice category*** $\mathscr{C}/C$ over $C \in \mathscr{C}$ consists of

- object $f\colon X \to C$,
- arrow $a\colon X \to X'$ for arrows $f\colon X \to C$, $f'\colon X' \to C$ such that $f' \circ a = f$,

with composition and units from those of $\mathscr{C}$.

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & C & \end{tikzcd}$$

$U\colon \mathscr{C}/C \to \mathscr{C}$ with $U(f\colon X \to C) = X$ and $U(a\colon X \to X') = a$ is a functor.

5 FUNCTIONAL ANALYSIS

[2]

5.1 Hahn-Banach Theorems

Theorem 5.1.1 **Hahn-Banach of analytic form**

Let $p\colon E \to \boldsymbol{R}$ be a sublinear function on a vector space E (i.e., $\forall \lambda > 0, x, y \in E,\ p(\lambda x) = \lambda p(x), p(x+y) \le p(x) + p(y)$), $G \subset E$ a linear subspace, and $g\colon G \to \boldsymbol{R}$ a linear functional such that $\forall x \in G,\ g(x) \le p(x)$. Then, $\exists$ a linear functional $f\colon E \to \boldsymbol{R}$ that extends g and that $\forall x \in E,\ f(x) \le p(x)$.

Definition 5.1.2 **Norm on the dual space of a normed space**

For a normed space E, the ***dual norm*** on E^* is defined by

$$\|f\|_{E^*} = \sup_{\substack{\|x\| \le 1 \\ x \in E}} |f(x)| = \sup_{\substack{\|x\| \le 1 \\ x \in E}} f(x).$$

Definition 5.1.3 **Scalar product for the duality**

For a vector space E and its dual space E^*, $\langle\ ,\ \rangle\colon E^* \times E \to \boldsymbol{R}$ defined by $\langle f, x \rangle = f(x)$ is called the ***scalar product for the duality*** E, E^*.

Definition 5.1.4 **Strictly convex normed space**

A normed space E is said to be ***strictly convex*** if $\forall t \in (0,1), x, y \in E$ with $\|x\| = \|y\| = 1$, $\|tx + (1-t)y\| < 1$ except for $x = y$.

Corollary 5.1.5 **Hahn-Banach of alternate form**

For a continuous linear functional $g\colon G \to \boldsymbol{R}$ on a linear subspace $G \subset E$ of a normed space E, $\exists f \in E^*$ that extends g and that $\|f\|_{E^*} = \|g\|_{G^*}$.

In the case when $G = \boldsymbol{R}x_0$ and $g(tx_0) = t\|x_0\|^2$ for a given $x_0 \in E$, $\exists f_0 \in E^*$ such that $\|f_0\| = \|x_0\|$ and $\langle f_0, x_0 \rangle = \|x_0\|^2$. If E^* is strictly convex, then f_0 is unique.

Definition 5.1.6 **Duality map from a normed space into its dual space**

For a normed space E and $x_0 \in E$, define

$$F(x_0) = \{f_0 \in E^* \mid \|f_0\| = \|x_0\|,\ \langle f_0, x_0 \rangle = \|x_0\|^2\}.$$

The ***duality map*** from E into E^* is a multivalued map $x_0 \mapsto F(x_0)$.

Corollary 5.1.7 **Norm of a vector is the max of its scalar product**

For a normed space E and $x \in E$,

$$\|x\| = \sup_{\substack{f \in E^* \\ \|f\| \le 1}} |\langle f, x \rangle| = \max_{\substack{f \in E^* \\ \|f\| \le 1}} |\langle f, x \rangle|.$$

Definition 5.1.8 **Affine hyperplane of a normed space**

For a normed space E and a linear functional $f\colon E \to \boldsymbol{R}$, an affine ***hyperplane*** is a subset $H = \{x \in E \mid f(x) = \alpha\} \subset E$ with $\alpha \in \boldsymbol{R}$, written $H = [f = \alpha]$. $f = \alpha$ is called the ***equation***.

Proposition 5.1.9 **Linear functional is continuous iff its hyperplane is closed**

For a linear functional $f\colon E \to \boldsymbol{R}$ on a normed space E and $\alpha \in \boldsymbol{R}$, $[f = \alpha]$ is closed iff f is continuous.

Definition 5.1.10 **Separation by a hyperplane**

For two subsets $A, B \subset E$ of a normed space E, the hyperplane $[f = \alpha] \subset E$ ***separates*** A and B if

$$\forall x \in A, y \in B,\ f(x) \leq \alpha \leq f(y);$$

strictly separates if

$$\exists \epsilon > 0, \forall x \in A, y \in B,\ f(x) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(y).$$

Definition 5.1.11 **Convex subset of a normed space**

A subset $A \subset E$ of a normed space E is said to be ***convex*** if

$$\forall t \in [0, 1], x, y \in A,\ tx + (1 - t)y \in A.$$

Theorem 5.1.12 **Hahn-Banach of first geometric form**

For two disadjoint nonempty convex subsets $A, B \subset E$ of a normed space E with one of them open, $\exists$ a closed hyperplane that separates A and B.

Definition & Proposition 5.1.13 **Minkowski functional of an open convex set**

Let $C \subset E$ be an open convex subset of a normed space E with $0 \in C$, and for $x \in E$

$$p(x) = \inf\{\alpha > 0 \mid \alpha^{-1}x \in C\},$$

called the ***gauge*** or the ***Minkowski functional*** of C. Then, p satisfies the folowing properties:

i) p is sublinear,
ii) $\exists M, \forall x \in E,\ 0 \leq p(x) \leq M \|x\|$,
iii) $C = \{x \in E \mid p(x) < 1\}$.

Lemma 5.1.14 **There exists a hyperplane that separates an open convex and outside point**

For a nonempty open convex $C \subset E$ of a normed space E and $x \in E \setminus C$, $\exists f \in E^*$ such that $\forall x \in C,\ f(x) < f(x_0)$. In particular, the hyperplane $[f = f(x_0)]$ separetes $\{x_0\}$ and C.

Theorem 5.1.15 **Hahn-Banach of second geometric form**

For two disadjoint nonempty convex subsets $A, B \subset E$ of a normed space E with A closed and B compact, $\exists$ a closed hyperplane that strictly separates A and B.

Corollary 5.1.16 **Some linear functional can vanish on a linear subspace**

For a linear subspace $F \subset E$ of a normed space E with $\overline{F} \neq E$, $\exists f \in E^*$ such that

$$\forall x \in F,\ \langle f, x\rangle = 0, \quad f \not\equiv 0.$$

Definition 5.1.17 **Notation of a bidual space**

Let E be a normed space, and $J\colon E \ni x \mapsto Jx \in E^{**}$ a ***canonical injection*** (i.e., $Jx\colon f \mapsto \langle f, x\rangle$, or $\langle Jx, f\rangle = \langle f, x\rangle$). Then, J is an ***isometry***:

$$\|Jx\|_{E^{**}} = \sup_{\substack{f\in E^*\\ \|f\|\le 1}} |\langle Jx, f\rangle| = \sup_{\substack{f\in E^*\\ \|f\|\le 1}} |\langle f, x\rangle| = \|x\|_E\,.$$

J can be not surjective; if J is surjective, E is said to be ***reflexive***.

Definition 5.1.18 **Orthogonal complement**

For a linear subspace $M \subset E$ of a normed space E and a linear subspace $N \subset E^*$, their ***orthogonal complements*** are

$$M^\perp = \{f \in E^* \mid \forall x \in M,\ \langle f, x\rangle = 0\} \subset E^*$$
$$N^\perp = \{x \in E \mid \forall f \in N,\ \langle f, x\rangle = 0\} \subset E,$$

respectively.

Proposition 5.1.19 **Relation between a linear subspace and its orthogonal complement**

For a linear subspace $M \subset E$ of a normed space E and a linear subspace $N \subset E^*$,

$$(M^\perp)^\perp = \overline{M}, \quad (N^\perp)^\perp \supset \overline{N}.$$

If E is reflexive, then $(N^\perp)^\perp = \overline{N}$.

6 Real Analysis and Probability

[3]

6.1 Set Theory

7 Homological Algebra

[7]

7.1 Chain Complexes

Definition 7.1.1 **Chain complex of R-modules**

A ***chain complex*** $C.$ of R-modules is a family $\{C_n\}_{n \in Z}$ of R-modules with R-module maps $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$, called the ***differential***. $Z_n = Z_n(C.) :=$ $\ker d_n$ is the module of n-***cycles*** of $C.$, and $B_n = B_n(C.) := \operatorname{im} d_{n+1}$ is the module of n-***boundaries*** of $C.$. The n-th ***homology module*** of $C.$ is $H_n(C.) := Z_n/B_n$.

A ***chain complex map*** $u : C. \to D.$ is a family of R-module homomorphisms $u_n : C_n \to D_n$ commuting with d (i.e., $u_{n-1}d_n = d_n u_n$).

Definition & Proposition 7.1.2 **A category of chain complexes of R-modules**

Ch(**mod**- R) is a category whose objects are chain complexes of (right) R-modules and whose arrows are chain complex maps. An arrow $u : C. \to D.$ of **Ch**(**mod**- R) sends boundaries to boundaries and cycles to cycles. H_n: **Ch**(**mod**- R) $\to$ **mod**- R is a functor.

Proposition 7.1.3 **Split exact sequence of vector spaces**

For a family $\{B_n, H_n\}$ of vector spaces, $\{C_n = B_n \oplus H_n \oplus B_{n-1}\}$ with projection-inclusions $\pi_n : C_n \to B_{n-1} \subset C_{n-1}$ is a chain complex. Every chain complex of vector spaces is isomorphic to such a complex.

Definition 7.1.4 **Quasi-isomorphism between chain complexes**

A chain map $C. \to D.$ between chain complexes $C., D.$ is called a ***quasi-isomorphism*** if the maps $H_n(C.) \to H_n(D.)$ are all isomorphisms.

Definition 7.1.5 **Cochain complex of R-modules**

A ***cochain complex*** $C^\cdot$ of R-modules is a family $\{C^n\}$ of R-modules with R-module maps $d^n\colon C^n \to C^{n+1}$ such that $d^{n+1}d^n = 0$. $Z^n(C^\cdot) := \ker d^n$ is the module of n-***cocycles***, and $B^n(C^\cdot) := \operatorname{im} d^{n-1}$ is the module of n-***coboundaries***. The n-th ***cohomology module*** of $C^\cdot$ is $H^n(C^\cdot) := Z^n/B^n$.

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