$F_X(x) = \int_{-\infty}^{x} \mathbb{1}_{[0,1]}(x)\,dx$ ~~$= \begin{cases} 1 & \text{si } x > 0 \\ 0 & \text{sinon} \end{cases}$~~

$F_Y \; \frac{3}{8}\int_0^1 (1 + xy + y^2)\, \mathbb{1}_{[-1,1]}(y)\, dx$

$= \frac{3}{8}\, \mathbb{1}_{[-1,1]}(y) \left[x + \frac{x^2 y}{2} + x y^2 \right]_0^1 =$

$\frac{3}{8}\left(y^2 + \frac{1}{2}y + 1\right) \mathbb{1}_{[-1,1]}(y)$

$F_Y(y) = \int_{-\infty}^{y} \frac{3}{8}\left(y^2 + \frac{1}{2}y + 1\right) \mathbb{1}_{[-1,1]}(y)\, dy$

$= \begin{cases} 1 & \text{si } y > 1 \\ \frac{3}{8}\left(\frac{y^3}{3} + \frac{1}{4}y^2 + y\right) & \text{si } -1 < y < 1 \\ 0 & \text{si } y < -1 \end{cases}$

3. X est une variable aléatoire constante sur $[0,1]$

4. $EX = \int_{\mathbb{R}} x\, \mathbb{1}_{[0,1]}(x)\, dx = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ V

$Var(Y) = E(Y^2) - E(Y)^2$ X pas Y

$E(Y^2) \overset{\text{transfert}}{=} \int_{\mathbb{R}} y^2 \frac{3}{8}\left(y^2 + \frac{1}{2}y + 1\right) \mathbb{1}_{[-1,1]}(y)\, dy = \frac{3}{8}\int_{-1}^{1} y^4 + \frac{1}{2}y^3 + y^2\, dy$

$= \frac{3}{8}\left[\frac{y^5}{5} + \frac{1}{8}x^4 + \frac{y^3}{3}\right]_{-1}^{1} = \frac{3}{8}\left[\left(\frac{1}{5} + \cancel{\frac{1}{8}} + \frac{1}{3}\right) - \left(\frac{-1}{5} + \cancel{\frac{1}{8}} - \frac{1}{3}\right)\right]$

$= \frac{3}{8}\left(\frac{2}{5} + \frac{2}{3}\right) = \frac{3}{8} \times \frac{16}{15} = \frac{2}{5}$

$E(Y) = \int_{\mathbb{R}} y \frac{3}{8}\left(y^2 + \frac{1}{2}y + 1\right) \mathbb{1}_{[-1,1]}(y)\, dy = \frac{3}{8}\int_{-1}^{1} y^3 + \frac{1}{2}y^2 + y\, dy$

$= \frac{3}{8}\left[\frac{y^4}{4} + \frac{1}{6}y^3 + \frac{y^2}{2}\right]_{-1}^{1} = \frac{3}{8}\left[\left(\frac{1}{4} + \frac{1}{6} + \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{6} + \frac{1}{2}\right)\right]$

$= \frac{3}{8} \times \frac{2}{6} = \frac{1}{8}$ V, $Var(Y) = \frac{2}{5} - \left(\frac{1}{8}\right)^2$

Exo 5

1. $E(X_1) = \lambda$ connu, poisson

$Var(X_1) = \lambda$

$E(S_n) = E(\sum_{i=1}^{n} X_i) \overset{\text{linéarité}}{=} \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} E(X_1) = n\lambda$ ✓

$Var(S_n) = var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + \underbrace{\sum_{i=1}^{n-1}\sum_{j=i}^{n} cov(X_i, X_j)}_{0 \text{ par independance des } X_i}$

$= \sum_{i=1}^{n} Var(X_1) = n\lambda$ ✓

2.

on sait que

$\underbrace{var(X_1+2X_2)}_{} = Var(X_1) + 4Var(X_2) + 2\underbrace{cov(X_1, 2X_2)}_{0}$

$var((X_1+2X_2)+S_4) = var(X_1+2X_2) + var(S_4) + 2\, Cov((X_1+2X_2), S_4)$

donc $cov(X_1+2X_2, S_4) = \dfrac{var((X_1+2X_2)+S_4) - \overset{\lambda}{var(X_1)} - 4\overset{\lambda}{var(X_2)} - \overset{4\lambda}{var(S_4)}}{2}$

$Var((X_1+2X_2)+S_4) = Var(2X_1 + 3X_2 + X_3 + X_4)$

$= 4\underbrace{Var(X_1)}_{\lambda} + 9\underbrace{Var(X_2)}_{\lambda} + \underbrace{Var(X_3)}_{\lambda} + \underbrace{Var(X_4)}_{\lambda} = 15\lambda$

donc $cov(X_1+2X_2, S_4) = \dfrac{15\lambda - \lambda - 4\lambda - 4\lambda}{2} = \dfrac{6\lambda}{2} = 3\lambda$ ✓

3. avec $Var(X_n) < \infty$ donc carré integrable, les X_i i.i.d, et $m = EX_1 = \lambda$, $\bar{X}_n = \frac{S_n}{n}$, on peut appliquer LGN qui dit

$\forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}\{|\bar{X}_n - \lambda| > \varepsilon\} = 0$ i.e, $\frac{S_n}{n}$ converge en

proba vers λ ~~[illegible]~~ ✓