Derivation of the Volume of a Torus

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May 21, 2025

Problem

Let $0 < r < R$. Let T be the solid circular torus ('donut') generated by the circle

$$\left\{(x, y, 0) \in \mathbb{R}^3 : \sqrt{\left(x - \frac{R+r}{2}\right)^2 + y^2} \leq \frac{R-r}{2}\right\}$$

(in the X-Y plane) turning around the vertical axis (Y axis). In this case, R is the distance from the axis of revolution (Y axis) to the farthest part of the tours, and r is the distance from the axis of revolution to the nearest part of the torus. Show that $\mathrm{vol}(T) = \frac{\pi^2}{4}(R+r)(R-r)^2$

Semicircle Definitions

Considering the torus will be generated by rotating a circle with respect to the y axis, and the constants R and r provided in the problem, we can define the equations for the outer semicircle and inner inner circle as:

$$\text{Outer Semicircle:} \quad f_e(y) = \sqrt{-y^2 + \left(\frac{R-r}{2}\right)^2} + \frac{R+r}{2}$$

$$\text{Inner Semicircle:} \quad f_i(y) = -\sqrt{-y^2 + \left(\frac{R-r}{2}\right)^2} + \frac{R+r}{2}$$

Integral for the Volume of the Torus

Given, $\frac{R-r}{r}$ functions as the circle's radius, we can use it's negative as lower bound, and it's positive as uppper bound, to end up with the following integral:

$$V_t = \pi \int_{-\frac{R-r}{2}}^{\frac{R-r}{2}} \left[f_e(y)^2 - f_i(y)^2\right] dy \tag{1}$$

Simplification

We can define the following substitutions to simplify the equation:

$$a = \sqrt{-y^2 + \left(\frac{R-r}{2}\right)^2}$$

$$b = \frac{R+r}{2}$$

And rewrite the functions as follows:

$$\text{Outer Semicircle (simplified):} \quad f_e(y) = a + b$$
$$\text{Inner Semicircle (simplified):} \quad f_i(y) = -a + b$$

Developing the Integral

We can now substitute the functions inside the integral:

$$\begin{aligned}
V_t &= \pi \int \left[(a+b)^2 - (-a+b)^2\right] dy \\
&= \pi \int \left[a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)\right] dy \\
&= \pi \int 4ab \, dy \\
&= 4\pi b \int a \, dy \\
&= 4\pi b \int_{-\frac{R-r}{2}}^{\frac{R-r}{2}} \sqrt{\left(\frac{R-r}{2}\right)^2 - y^2} \, dy
\end{aligned}$$

Trigonometric Substitution

Trigonometric Simplification

To more easily apply trigonometric substitution, we can consider that:

$$\beta = \frac{R-r}{2}$$

Thus:

$$4\pi b \int_{-\beta}^{\beta} \sqrt{\beta^2 - y^2} \, dy$$

Substitution

We can begin by establishing that the square root is the pythagorean addition of b_t, our hypothenuse:

$$b_t = \sqrt{(\beta^2 - y^2)}$$

Which then implies that these are the following trigonometric properties of our pythagorean addition:

$$\sin\theta = \frac{y}{\beta}$$
$$y = \sin(\theta)\beta \Rightarrow dy = \cos(\theta)\beta d\theta$$

and

$$\cos(\theta) = \frac{b_t}{\beta} \Rightarrow b_t = \cos(\theta)\beta$$

With which we can now sutbitute our current integral to:

$$4\pi b\beta^2 \int_{-\beta}^{\beta} \cos^2(\theta) \, d\theta$$

With integration formulas we can now convert our equation to:

$$4\pi b\beta^2 \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_{-\beta}^{\beta}$$

To substitute θ into its equivalence in terms of y, we can refer back to our trigonometric properties and see that:

$$\theta = \arcsin\left(\frac{y}{\beta}\right)$$

Which now allows us to see the transformed integral in terms of y:

$$4\pi b\beta^2 \left[\frac{\arcsin\left(\frac{y}{\beta}\right)}{2} + \frac{\sin(2\arcsin\left(\frac{y}{\beta}\right))}{4}\right]_{-\beta}^{\beta}$$

The limits of the integral can now be evaluated, for simplicity, the trigonometric functions will be evaluated separately for $\pm\beta$ and then replaced onto their respective places within the fully evaluated equation:

$$\begin{aligned} &\text{For } \beta: \\ \arcsin\left(\frac{\beta}{\beta}\right) &= \arcsin 1 = \frac{\pi}{2} \\ \sin\left(2\arcsin\left(\frac{\beta}{\beta}\right)\right) &= \sin(\pi) = 0 \end{aligned}$$

$$\begin{aligned} &\text{For } -\beta: \\ \arcsin\left(\frac{-\beta}{\beta}\right) &= \arcsin -1 = -\frac{\pi}{2} \\ \sin\left(2\arcsin\left(\frac{-\beta}{\beta}\right)\right) &= \sin(-\pi) = 0 \end{aligned}$$

Now we can place them onto the fully evaluated equation:

$$\begin{aligned} 4\pi b\beta^2 \left[\frac{\frac{\pi}{2}}{2} - \frac{-\frac{\pi}{2}}{2}\right] &= 4\pi b\beta^2 \frac{\pi}{2} \\ &= 2\pi^2 b\beta^2 \end{aligned}$$

The equation is now only in terms of b and β, we can recall their definitions and substitute them back to finalize:

$$\begin{aligned} b &= \frac{R+r}{2} \\ \beta &= \frac{R-r}{2} \\ &\Rightarrow 2\pi^2\left(\frac{R+r}{2}\right)\left(\frac{R-r}{2}\right)^2 \\ &= \frac{\pi^2}{4}(R+r)(R-r)^2 \end{aligned}$$

$$\boxed{V_t = \frac{\pi^2}{4}(R+r)(R-r)^2}$$