Fluids in a Rotating Frame of Reference

Navdeep Brar, Daniel Vass

November 15, 2025

Abstract

In this report we explore the effects of a rotating frame of reference on the dynamics of fluids. First we introduce a general rotation on basis vectors in the inertial reference frame to produce the ficticious forces that arise in a non-inertrial reference frame. A simpler rotating frame of reference is then examined to gain insight on the Coriolis and centrifugal ficticious forces. A dimensional analysis argument is then created to explore the regimes in which the Coriolis force and viscosity force are negligible. Next we focus on a special case of geostrophic flow which is analyzed and a derivation of the Taylor-Proudman Theorem is utilized to show the existence of Taylor Columns. We then attempt to show how geostrophic flow works in the atmosphere where density is not constant . Finally we reintroduce viscosity to describe the flow near a boundary in a rotating frame.

Contents

1 Introduction

In a non-inertial reference frame we obtain fictitious forces. If a solid object is placed in a rotating fluid; under certain conditions, the flow of the rotating fluid is dramatically altered. Rather than disturbing the flow by blocking it; the object produces a creates a column of fluid that acts as a barrier. The column is parallel to the axis of rotation as is centered on the solid object. Any fluid that attempts cross this 'wall' of fluid is instead diverted around the column (see figure 1). The phenomenon are referred to as Taylor columns.
In this report we explore geotrophic balance in an incompressible fluid; first analyzing an ideal fluid and later considering how viscosity affects the fluid. For ideal fluids we obtain a condition that there cannot be any gradients of velocity parallel to the axis of rotation (this is the Taylor-Proudman Theorem). Further analysis is done on geostrophic balance in the atmosphere assuming it is an ideal gas.

2 Generalizing Rotating Frames

First and foremost we will need to introduce a rotating frame of reference. A rotating frame of reference is introduced by applying an orthonormal matrix onto a set of basis vectors used in our reference frame. For a general time dependent orthonormal matrix A its derivative can be written as [4]:

$$\dot{A} = A \times \vec{\Omega}$$

$\vec{\Omega}$ is called the rotation vector and its form will depend on the orthonormal matrix. In general its form is given by satisfying [4]:

$$A_{ki}\dot{A}_{kj} = \epsilon_{ijk}\Omega_k$$

Consider an inertial Cartesian system, the coordinates are given by $\vec{x}'$, in the non- inertial Cartesian system, the coordinates of a point are denoted $\vec{x}$. We also introduce a time evolving origin given by $\vec{C}(t)'$ in the inertial reference frame and $\vec{C}(t)$ in the rotating frame of reference.

The position, velocity and acceleration in the rotating frame of reference is given by [4]:

$$\vec{x} = A(\vec{x}' - \vec{C}') \tag{1}$$

$$\dot{\vec{x}} = A(\dot{\vec{x}}' - \dot{\vec{C}}') - \vec{\Omega} \times \vec{x} \tag{2}$$

$$\ddot{\vec{x}} = A(\ddot{\vec{x}}' - \ddot{\vec{C}}') - \dot{\vec{\Omega}} \times \vec{x} - 2\vec{\Omega} \times \dot{\vec{x}} - \vec{\Omega} \times (\vec{\Omega} \times \vec{x}) \tag{3}$$

Here we can see the Coriolis term $-2\vec{\Omega} \times \vec{v}$ and centrifugal term $-\vec{\Omega} \times (\vec{\Omega} \times \vec{x})$ that arise from the change of basis. There are also the extra terms $-A\ddot{\vec{C}}'$ and $\dot{\vec{\Omega}} \times \vec{x}$. These terms account for the acceleration of the origin and the acceleration of the rotation.

2.1 Constant Rotating Frame of Reference

A simpler model is given by having a Cartesian coordinate system rotating at constant angular speed Ω about the z-axis. The origin remains fixed so $\vec{C}(t) = \vec{0}$. If we let A be the rotation matrix we have:

$$A = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we have that for a particle rotating along the z-axis with angular frequency Ω the transformed coordinates $\vec{x}$ in terms of the original inertial coordinates $\vec{x}'$ are given by:

$$\vec{x} = A\vec{x}'$$

$$\frac{d\vec{x}}{dt} = A\frac{d\vec{x}'}{dt} - \vec{\Omega} \times \vec{x}$$

$$\frac{d^2\vec{x}}{dt^2} = A\frac{d^2\vec{x}'}{dt^2} - \vec{\Omega} \times (\vec{\Omega} \times \vec{x}) - 2\vec{\Omega} \times \vec{v}$$

where the rotation vector is given by $\vec{\Omega} = \Omega\hat{z}$. We can now derive Newton's 2nd law in the rotating coordinates:

$$m\frac{d^2\vec{x}}{dt^2} = m\left(A\frac{d^2\vec{x}'}{dt^2} - \vec{\Omega} \times (\vec{\Omega} \times \vec{x}) - 2\vec{\Omega} \times \vec{v}\right)$$

Here we only have the Coriolis term and centrifugal term.

3 Steady Flow in a Rotating Reference Frame

For simplicity we will take the rotation vector to be $\vec{\Omega} = \Omega\hat{z}$, and our rotation takes place on the xy-plane giving $\vec{x} = (x, y, 0)$. The Centrifugal term is a conservative field which gives us:

$$\begin{aligned} \vec{\Omega} \times (\vec{\Omega} \times \vec{x}) &= -\frac{1}{2}\nabla(\vec{\Omega} \times \vec{x})^2 \\ &= (\Omega x^2, \Omega y^2, 0) \end{aligned}$$

We can combine this term with the gravitational field of Earth to get an effective gravity:

$$\vec{g}_{\text{eff}} = (\Omega^2 x, \Omega^2 y, -g_0)$$

Now, consider the momentum equation in a non-rotating reference frame with viscosity:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \vec{g} + \nu \nabla^2 \vec{v}$$

where P is the pressure, $\vec{g}$ is gravity, ν is the viscosity and $\vec{v}$ is the velocity.
In a constant frame we need to incorporate the Coriolis and Centrifugal forces this gives us:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g}_{\text{eff}} + \nu \nabla^2 \vec{v} - 2\vec{\Omega} \times \vec{v}$$

we have no time dependence for velocity so we are left with:

$$(\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g}_{\text{eff}} + \nu \nabla^2 \vec{v} - 2\vec{\Omega} \times \vec{v} \tag{4}$$

This is the momentum equation for a fluid in steady state. The continuity equation under steady state becomes:

$$\nabla \cdot \left(\rho \vec{v}\right) = 0 \tag{5}$$

These two equations (4) and (5) give the Euler Equations in a rotating frame.

3.1 Non-Dimensionalizing the Advective Term and Coriolis Term

The previous equation (4) still has an advective term that gives us a non-linear PDE. Now we analyze under what conditions the advective term and Coriolis term are important. Introducing a length scale L and velocity scale U we have that the advective term has the following form:

$$\left\|(\vec{v} \cdot \nabla)\vec{v}\right\| \approx \frac{U^2}{L}$$

If we have a low velocity or large length-scale the advective term is negligible. The Coriolis term has the form:

$$\left\|2\vec{\Omega} \times \vec{v}\right\| \approx 2\Omega U$$

For large Ω or velocity scale U the Coriolis term is large. Dividing these two equations gives us the *Rossby number*:

$$\boldsymbol{Ro} := \frac{U}{2\Omega L} \tag{6}$$

For large Rossby numbers the advective term dominates while for small Rossby numbers the Coriolis term dominates.

4 Geostrophic Balance

Consider a nearly ideal steady fluid with a small Rossby number. We can use the Euler equations (4) and (5) and ignore the advective term and since it is nearly ideal we can also ignore viscosity. This leaves us with:

$$\frac{1}{\rho}\nabla P = -\vec{g}_{\text{eff}} - 2\vec{\Omega} \times \vec{v} \tag{7}$$

This is called Geostrophic Balance equation. As a sanity check, consider a non-rotating fluid; setting $\vec{v} = \vec{0}$ we get back the hydrostatic balance equation.

4.1 Taylor-Proudman Theorem

Taking the curl of both sides of Geostrophic Balance (7) and using the fact that the effective gravity is a conservative field hence we have $\vec{g}_{\text{eff}} = -\nabla\pi$ for some scalar field π we get [1]

$$2\nabla \times (\vec{\Omega} \times \vec{v}) = \frac{\nabla\rho \times \nabla P}{\rho^2} \tag{8}$$

In the case that we have 2D rotation which gives a rotation vector of the form: $\vec{\Omega} = \Omega\hat{z}$ we can simplify [4].

$$\frac{\partial}{\partial z}\vec{v} = -\frac{\nabla\rho \times \nabla P}{2\Omega\rho^2} - \hat{z}\frac{(\vec{v}\cdot\nabla)\rho}{\rho} \tag{9}$$

Assuming the density is constant (which also gives us that the fluid is incompressible $\nabla \cdot \vec{v} = 0$) the left hand side is zero giving [1]:

$$\begin{aligned}
\vec{0} &= \nabla \times \left(-2\vec{\Omega} \times \vec{v}\right) \\
&= -2(\vec{\Omega}(\nabla\cdot\vec{v}) - (\vec{\Omega}\cdot\nabla)\vec{v}) \\
&= -2(\vec{\Omega}\cdot\nabla)\vec{v}
\end{aligned}$$

For 2D rotation, we had that the rotation vector was given by: $\vec{\Omega} = \Omega\hat{z}$. Plugging this into the above equation we have:

$$\vec{0} = -2\Omega\frac{\partial}{\partial z}\vec{v} \tag{10}$$

Hence the velocity has no explicit z-dependence.

5 Taylor Columns

In the previous section we had shown that a geostrophic balanced incompressible fluid will exhibit no gradient of velocity along the z-axis (10) (parallel to the rotation vector).

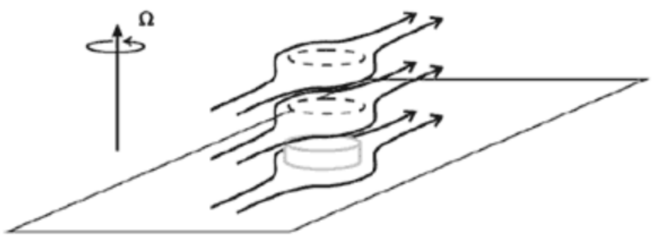

Figure 1: This figure shows a horizontally travelling fluid being diverted around a vertical column centered on the object. This vertical column of undisturbed fluid are called Taylor Columns. Adapted from Taylor Columns by M.Buckley, 2017, [1]

6 Geostrophic Flow in the Atmosphere

The Rossby Number of the atmosphere is negligible [4] making geostrophic balance a good approximation. In the atmosphere we require an equation of state relating the Pressure to the density of temperature:

$$P = P(\rho, T)$$

Now consider the 2D geostrophic balance equation

$$\vec{0} = \frac{1}{\rho}\nabla P + 2\Omega(\hat{z} \times \vec{v}) + \vec{g}_{\text{eff}} \tag{11}$$

From this we can get a velocity defined by the geostrophic flow: [2]

$$\vec{v}_g = \frac{1}{2\rho\Omega}\hat{z} \times \nabla P$$

In component form we get:

$$\vec{v}_g = (u_g, v_g, w_g) = \frac{1}{2\rho\Omega}(-\frac{\partial P}{\partial y}, \frac{\partial P}{\partial x}, 0) \tag{12}$$

From here we can immediately conclude that the magnitude of the geostrophic velocity is proportional to the gradient of the pressure, and its direction is perpendicular to the pressure gradient.

We can convert this to be in terms of pressure coordinates (heights of surfaces of constant pressure) Z [2]:

$$(u_g, v_g, 0) = \left(-\frac{g}{2\Omega}\frac{\partial Z}{\partial y}, \frac{1}{2\Omega}\frac{\partial Z}{\partial x}, 0\right) \tag{13}$$

We will also use the hydrostatic balance equation:

$$\frac{\partial P}{\partial z} = -g\rho$$
$$\implies \frac{\partial Z}{\partial P} = -\frac{1}{g\rho}$$

Differentiating equation (13) with respect to pressure and using the hydrostatic balance equation we get:

$$u_g = \frac{1}{2\Omega}\frac{\partial}{\partial y}\left[\frac{1}{\rho}\right] \tag{14}$$

$$v_g = -\frac{1}{2\Omega}\frac{\partial}{\partial x}\left[\frac{1}{\rho}\right] \tag{15}$$

Now all we need is an equation of state to determine the geostrophic velocities. A simple choice would be the ideal gas law:

$$P = \rho RT \tag{16}$$

where P is the pressure, ρ is the density, R is the gas constant and T is the temperature. Plugging this into (14) and (15) we get:

$$u_g = \frac{R}{2\Omega P}\frac{\partial T}{\partial y} \tag{17}$$

$$v_g = -\frac{R}{2\Omega P}\frac{\partial T}{\partial x} \tag{18}$$

Here we have the geostrophic velocities in terms of pressure coordinates. Horizontal gradients of temperature produce vertical changes in velocity (here we have changed from z to pressure to act as our vertical coordinate).

7 Ekman Boundary Layer

In the preceding analysis we have made the idealization of neglecting viscosity. Such neglect cannot continue when considering the flow near the boundary, since the no-slip condition must be fulfilled at the boundary friction forces must come into play. Thus we now consider geostrophic balance with the viscosity term reintroduced:

$$\frac{1}{\rho}\nabla p = -\vec{g}_{\text{eff}} - 2\vec{\Omega} \times \vec{v} + \nu\nabla^2\vec{v} \tag{19}$$

As we've done previously we want to compare the relative strength of forces. given a length and velocity scale, L and U, we define the *Ekman number* as the ratio of the viscous force and the Coriolis force [3]:

$$\boldsymbol{Ek} = \frac{\left\|\nu\nabla^2\vec{v}\right\|}{\left\|2\vec{\Omega} \times \vec{v}\right\|} \approx \frac{\nu U/L^2}{2\Omega U} = \frac{\nu}{2\Omega L^2}$$

Recalling that above we defined $\boldsymbol{Ro} = U/2\Omega L$, and that the Reynolds number is $\boldsymbol{Re} = UL/\nu$, we can also express $\boldsymbol{Ek}$ as the following ratio,

$$\boldsymbol{Ek} = \frac{\boldsymbol{Ro}}{\boldsymbol{Re}}$$

In the Ekman layer we have $\boldsymbol{Ek} \sim 1$, this defines a length scale (omitting the factor of $\sqrt{2}$):

$$\frac{\nu}{2\Omega\delta^2} \sim 1 \implies \delta \sim \sqrt{\frac{\nu}{\Omega}}$$

This length scale is a good estimate for the thickness of the layer.

7.1 Ekman Layer with Steady Flow in the Interior

For the purposes of this section we assume once again that the rotation vector is parallel to the z-axis, $\vec{\Omega} = \Omega\hat{z}$. We also assume that far above the Ekman layer we have a thick layer of steady geostrophic flow in the x-direction, that is to say $\vec{v} \to U\hat{x}$ when $z \gg \delta$. Close to the surface of earth we know that the centrifugal force will be negligible [4], so we assume constant gravitational acceleration $\vec{g} = (0, 0, -g_0)$. Finally we assume constant density $\rho = \rho_0$, conservation of mass then takes the form $\nabla \cdot \vec{v} = 0$. With all this in mind the components of (19) become:

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= 2\rho_0\Omega v_y + \nu\rho_0\nabla^2 v_x & \qquad v_x(x,y,0) = 0\\
\frac{\partial p}{\partial y} &= -2\rho_0\Omega v_x + \nu\rho_0\nabla^2 v_y & \qquad v_y(x,y,0) = 0\\
\frac{\partial p}{\partial z} &= -\rho_0 g_0 + \nu\rho_0\nabla^2 v_z & \qquad v_z(x,y,0) = 0
\end{aligned}
$$

The equations of motion and the BCs do not have an explicit dependence in x or y, thus it is reasonable to assume a solution with only explicit z dependence $\vec{v} = \vec{v}(z)$. This assumption would imply that conservation of mass becomes $\frac{\partial v_z}{\partial z} = 0$, and hence that v_z is constant. Given the boundary condition it must be the case that $v_z = 0$ for all z. This reduces the third equation above to $\frac{\partial p}{\partial z} = -\rho_0 g_0$ and hence that:

$$p = -\rho_0 g_0 z + f(x,y)$$

We know that the pressure in the geostrophic layer is given by [4]:

$$p = -\rho_0 g_0 z - 2\rho_0\Omega U y$$

Since the pressures must agree at the boundary we have that $f(x,y) = -2\rho_0\Omega U y$. Substituting the pressure into the remaining two equations gives us

$$
\begin{aligned}
0 &= 2\rho_0\Omega v_y + \nu\rho_0\frac{\partial^2 v_x}{\partial z^2}\\
-2\rho_0\Omega U &= -2\rho_0\Omega v_x + \nu\rho_0\frac{\partial^2 v_y}{\partial z^2}
\end{aligned}
$$

We can combine these two equations into one fourth order equation:

$$\frac{\partial^4}{\partial z^4}(U - v_x) = -\frac{4\Omega^2}{\nu^2}(U - v_x)$$

The general solution to the above are given by a linear combination of $e^{\lambda z}$, where λ satisfy $\lambda^4 = -4\Omega^2/\nu^2 = -4/\delta^2$. The solutions with $\mathrm{Re}\{\lambda\} > 0$ are inadmissable on physical grounds since the velocity would diverge as $z \to \infty$, thus the General solutions are given by:

$$
\begin{aligned}
U - v_x &= Ae^{-(1+i)z/\delta} + Be^{-(1-i)z/\delta}\\
v_y &= i(Ae^{-(1+i)z/\delta} - Be^{-(1-i)z/\delta})
\end{aligned}
$$

Using the boundary conditions $v_x(0) = v_y(0) = 0$ gives the system of equations,

$$U = A + B$$
$$0 = A - B$$

Which trivially rearrange to get $A = B = U/2$. Hence the final solutions of velocity are:

$$\begin{aligned} v_x &= U\Big(1 - e^{-z/\delta}\cos(z/\delta)\Big) \\ v_y &= Ue^{-z/\delta}\sin(z/\delta) \\ v_z &= 0 \end{aligned}$$

These solutions obey the expected asymptotic behaviour when $z \gg \delta$ of $\vec{v} \to U\hat{x}$.

8 Conclusions

We have shown how a simple, uniformly rotating reference frame produces the fictitious Coriolis and centrifugal force. Assuming a small Rossby number and ideal fluid we obtained the geostrophic balance. This relates the pressure and effective gravity with the Coriolis force. We demonstrated that under the conditions 2D rotation and constant density there is no gradient of velocity parallel to the axis of rotation. This produces the phenomena of Taylor columns.

The atmosphere does not have a constant density; however due to it having a small Rossby number we can ignore the advective term in the momentum equation and analyze the geostrophic balance. Further simplifications of assuming the atmosphere as an ideal gas produced equations that showed horizontal changes in temperature create vertical changes in geostrophic velocity.

Finally we reintroduced viscosity to examine what happens to the flow near a boundary in a rotating frame. The regime where this analysis becomes necessary was characterized by the Ekman number attaining a value on the order of unity. We were able to solve analytically for the flow in the Ekman layer in the case where steady geostrophic flow is achieved far from the boundary.

Further Work

Further work can be done in assuming the Rossby number is intermediate, hence the advective term cannot be neglected. This will produce a non-linear equation which will require numerical solutions to solve. Further analysis will be required in determining the affects of the advective term on 2D rotations. The assumptions that the atmosphere can be considered an ideal gas was a superficial approximation. An interesting extension can be to use a different equation of state; such as the barotropic and polytropic relations to obtain more realistic equation for the geostrophic velocity.

References

[1] Buckley, M. *Taylor Columns*, Course website for Weather and Climate Laboratory at the Massachusetts Institute of Technology, 2004.

[2] Holton, J. R. & Hakim, G. J. (2019). 2.4.1 Geostrophic Approximation and Geostrophic Wind. In An introduction to dynamic meteorology (pp. 42–43). essay, Academic Press.

[3] Kushman-Roisan, B. & Jean-Marie, B. *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic Press, 2011.

[4] Lautrup, B. *Physics of Continuous Matter*, 2nd ed. CRC Press, 2011.