The Three-Body Problem

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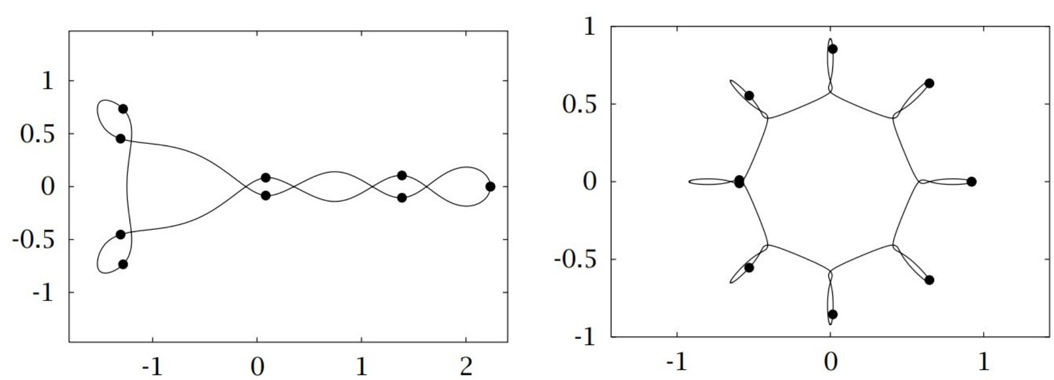
November 2020

Abstract

In this report, we will attempt to solve a simpler version of the three-body problem, commonly referred to as the: Restricted Three Body problem. Two large, finite masses will be in a circular orbit in a plane about their center of mass. A third body of negligible mass will be introduced on the same plane. The objective is to find the equations of motion of the third mass. This will be found using Lagrangian mechanics and an analysis on the chaotic nature of the orbits and possible equilibrium solutions will be discussed. The nature of chaotic orbits and possible equilibrium positions will be of great importance in giving astronomers tools to better understand solar systems and motion of satellites.

1 Introduction

The three-body problem is classical mechanics problem delves into predicting the position and motion of three masses, with initial position and momentum, using Newton's law of motion and universal gravitation. This is a specific case of the n-body problem (with n number of masses).



(a) With nine bodies (b) A floral arrangement with eleven bodies

Figure 1: Choreography with multiple bodies [4]

The two-body problem was initial analyzed by Johannes Kepler in 1609, and solved by Newton, in 1687. In the case for the three-body problem, it was major topic for many mathematicians and physicists from the mids-1700s to the early-1900s. After multiple attempts to solve the problem - notable Einrich Bruns in 1887, Henri Poincaré in mids-1890s - the conclusion was drawn that the three-body problem could not be solved in terms of algebraic formulas and integrals. [9]

However, physicists and mathematicians have continued to find various periodic solutions for the three-body problem, with appropriate initial conditions (Figure 1 and 2).

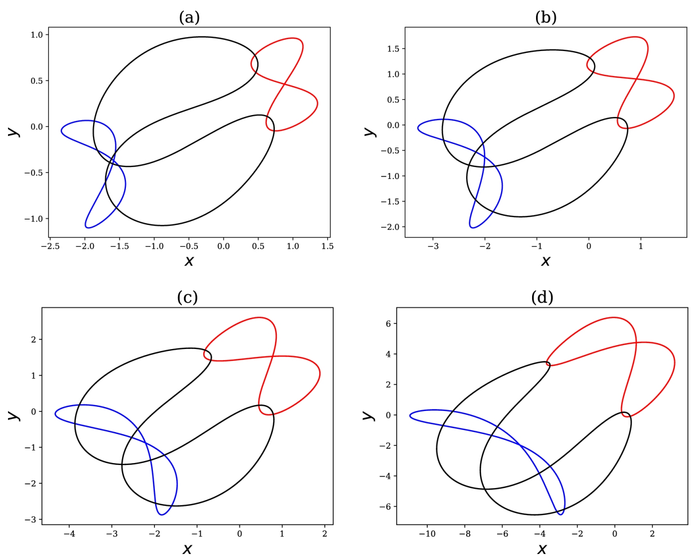


Figure 2: Trajectory of periodic orbits with three unequal masses: (a) $m_1 = 0.975$, $m_2 = 1$, and $m_3 = 0.5764$; (b) $m_1 = 0.95$, $m_2 = 1$, and $m_3 = 0.6454$; (c) $m_1 = 0.925$, $m_2 = 1$, and $m_3 = 0.7169$; (d) $m_1 = 0.9$, $m_2 = 1$, and $m_3 = 0.7569$. Blue line: body-1, red line: body-2, black line: body-3. [3]

It is important to acknowledge that the three-body problem may not have a closed form solution, approximations for the motions of three masses have have been done with extreme accuracy, long before the age of computers. In 1752, Alexis Clairaut won the St. Petersburg Academy prize for his approximation, which was demonstrated in 1759 when the Halley's comet passed Earth within a month of what his equations had predicted. This also matched his own margin of error. [10]

2 Derivations of the problem

Consider 3 point masses: m_1, m_2, m_3 that have relative positions to some origin O, $\vec{r}_1, \vec{r}_2, \vec{r}_3$. Applying Newton's laws of motion we get:

$$m_1\ddot{\vec{r}_1} = -\frac{Gm_1m_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_1-\vec{r}_2) - \frac{Gm_1m_3}{|\vec{r}_3-\vec{r}_1|^3}(\vec{r}_1-\vec{r}_3) \tag{1}$$

$$m_2\ddot{\vec{r}_2} = -\frac{Gm_2m_1}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_1-\vec{r}_2) - \frac{Gm_2m_3}{|\vec{r}_3-\vec{r}_2|^3}(\vec{r}_2-\vec{r}_3) \tag{2}$$

$$m_3\ddot{\vec{r}_3} = -\frac{Gm_3m_1}{|\vec{r}_1-\vec{r}_3|^3}(\vec{r}_3-\vec{r}_1) - \frac{Gm_2m_3}{|\vec{r}_3-\vec{r}_2|^3}(\vec{r}_3-\vec{r}_1) \tag{3}$$

With the gravitational potential of each masses are

$$U_1 = -\frac{Gm_1m_2}{|\vec{r}_2-\vec{r}_1|} - \frac{Gm_1m_3}{|\vec{r}_3-\vec{r}_1|} \tag{4}$$

$$U_2 = -\frac{Gm_2m_1}{|\vec{r}_1-\vec{r}_2|} - \frac{Gm_1m_3}{|\vec{r}_3-\vec{r}_2|} \tag{5}$$

$$U_3 = -\frac{Gm_3m_1}{|\vec{r}_1-\vec{r}_3|} - \frac{Gm_3m_2}{|\vec{r}_2-\vec{r}_3|} \tag{6}$$

1. **Lagrangian Approach**
   The Lagrangian of the system $\mathcal{L}(r, \dot{r})$, defined to be

   $$\mathcal{L} := T - U \tag{7}$$

   which will satisfy the Lagrange equation:

   $$\frac{\partial \mathcal{L}}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{8}$$

   This gives us a system of 9 nonlinear differential equations - 3 for each masses - which dictates the position of each masses as a function of time. Turns out, there are no closed form solution for this system, and only can be solved numerically.

2. **Hamiltonian Approach**
   The Lagrangian of the system $\mathcal{H}(\vec{r}, \vec{p}, t)$, defined to be

   $$\mathcal{H} := \sum_{n=1}^{3} p_n \dot{r}_n(\vec{r}, \vec{p}) - \mathcal{L}(r, \dot{r}(\vec{r}, \vec{p})) \tag{9}$$

   However, since the gravitational force is conservative and our coordinate system is natural, $\mathcal{H}$ simplifies down to

   $$\mathcal{H} = T + U \tag{10}$$

and will satisfy the Hamiltonian equations

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \tag{11}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \tag{12}$$

This gives us a system of 18 first order differential equations - 3 equations for position and 3 equations for momentum, for each masses. In general, there are no closed form solution for this system either, and only can be solved numerically.

3 Restricted Three Body Problem

The simplest form of the three-body problem is called the restricted three-body problem, in which one particle with an infinitesimal mass, is moving in the gravitation field of two massive bodies, of which are orbiting each other. There has been an enormous amount of research papers devoted to this problem, both analytic and numerical developments. Most of the analytic developments has been in the case where all particles are confined to a plane, and the two massive bodies are orbiting in a circular orbit, with respect to the center of mass. The numerical developments allow consideration of the more general problem. [6]

In our case, the following are all of our assumption:

1. **The problem is 2-dimensional**: With m_1 and m_2 holding the majority of the mass of the system, the rotational motion will depend completely on m_1 and m_2. We will use this plane to define our coordinate system. We will also assume the momentum/motion of m_3 will also be on this plane.

2. **Motion of m_1 and m_2 can be described by a two body problem**: Due to the insignificance of m_3, the forces that act on m_1 and m_2, due to m_3 are negligible. This also implies that the center of mass between m_1,m_2 moves with constant velocity. Here we will make the assumption that m_1,m_2 move in a circular orbit about the CM. This will significantly ease our calculations.

3. **The rotational of m_1 and m_2 are circular**: We consider the case when both m_1 and m_2 are orbiting the center of mass in perfect circular orbit, with a constant angular velocity, and no external torque.

Now that we have been able to set up the problem, lets attempt to solve it. From assumption 2, we defined our origin to be the center of mass. Thus we can obtain the position vector $\vec{r_1}, \vec{r_2}$ of mass m_1 and m_2 respectively.

$$\dot{\vec{r_1}} = R \cdot \frac{m_2}{M} \tag{13}$$

$$\dot{\vec{r_2}} = -R \cdot \frac{m_1}{M} \tag{14}$$

and the velocity vector $\vec{r}_1$ and $\vec{r}_2$

$$\dot{\vec{r}}_1 = R \cdot \frac{m_2}{M} \tag{15}$$

$$\dot{\vec{r}}_2 = -R \cdot \frac{m_1}{M} \tag{16}$$

with $M = m_1 + m_2$ and R be the distance between m_1 and m_2. The solution,in spherical coordinates, for this is given from Taylor [8], equation (8.23),(8.24), and (8.26)

$$\dot{\phi} = \frac{l}{\mu r^2} \tag{17}$$

$$\mu \ddot{r} = -\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{l^2}{\mu r^3} \tag{18}$$

with l as the system angular momentum, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$, and U as the potential between the two masses.

4 Equations of Motion for the Smaller Mass

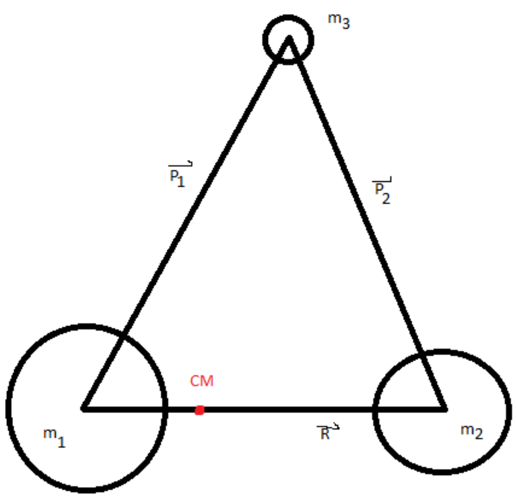


Figure 3: All 3 masses are on a plane and we have the distances of the masses from one another. The positions of m_1,m_2, m_3 relative to the CM (origin) are $(x_1, y_1), (x_2, y_2), (x, y)$ respectively.

Under the assumption that m_1, m_2 are in a circular orbit about their CM, lets try to figure out the equations of motion for m_3. Note that the following derivation is taken from [2]. We will use the Lagrangian approach. For this we will need to find the kinetic and potential energy of m_3. Consider Figure 3 to be our system. The kinetic energy will be given by:

$$T = \frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2) \tag{19}$$

While the potential will be given by:

$$U = -\frac{Gm_1m_3}{p_1^2} - \frac{Gm_2m_3}{p_2^2} \tag{20}$$

Where $p_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2}$ and $p_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2}$. These coordinates are difficult to use because the coordinates m_1 and m_2 are a function of time, since the mass are rotating. However, their path can be parameterized by the form

$$[a\cos(\omega t), a\sin(\omega t)] \tag{21}$$

with a as the distance of the mass (a constant).

We need a coordinate system such that m_1 and m_2 are stationary. Consider first that m_1 and m_2 are moving in a circular orbit about the CM with a constant angular velocity (no external torque). Thus if we switch our frame to one that is rotating with the same angular velocity about the CM then m_1 and m_2 remain stationary. Lets use generalized coordinates q_x, q_y which we defined to be:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

$$x = q_x\cos(\omega t) - q_y\sin(\omega t) \tag{22}$$

$$y = q_x\sin(\omega t) + q_y\cos(\omega t) \tag{23}$$

Using these coordinates we get:

$$\dot{x} = \dot{q}_x\cos(\omega t) - q_x\omega\sin(\omega t) - \dot{q}_y\sin(\omega t) - q_y\omega\cos(\omega t) \tag{24}$$

$$\dot{y} = \dot{q}_x\sin(\omega t) + q_x\omega\cos(\omega t) + \dot{q}_y\cos(\omega t) - q_y\omega\sin(\omega t) \tag{25}$$

Under the transformation, the kinetic energy as a function of q_x and q_y is

$$\begin{aligned} T &= \frac{1}{2}m_3\big(\dot{x}^2 + \dot{y}^2\big) \\ &= \frac{1}{2}m_3\big[\dot{q}_x^2 + \dot{q}_y^2 + 2q_x\omega\dot{q}_y - 2q_y\omega\dot{q}_x + \omega^2\big(q_x^2 + q_y^2\big)\big] \end{aligned} \tag{26}$$

The potential energy will be greatly simplified, since with this coordinate system $(x_1, y_1) \to (u_1, v_2)$ and $(x_2, y_2) \to (u_2, v_2)$ where $(u_1, v_1), (u_2, v_2)$ are both constant vectors. The potential with under this transformation is

$$U = -\frac{Gm_1m_3}{\alpha_1^2} - \frac{Gm_2m_3}{\alpha_2^2} \tag{27}$$

with $\alpha_1 = \sqrt{(q_x - u_1)^2 + (q_y - v_1)^2}$, $\alpha_2 = \sqrt{(q_x - u_2)^2 + (q_y - v_2)^2}$. Now that we have the kinetic energy and potential, we can find the Lagrangian - equation (7)

$$\mathcal{L} = \frac{1}{2}m_3\left[\dot{q}_x^2 + \dot{q}_y^2 + 2q_x\omega\dot{q}_y - 2q_y\omega\dot{q}_x + \omega^2\left(q_x^2 + q_y^2\right)\right] + \frac{Gm_1m_3}{\alpha_1^2} + \frac{Gm_2m_3}{\alpha_2^2} \tag{28}$$

Applying the Lagrange equation - (8) - for both q_x and q_y gives us

$$m_3\ddot{q}_x = -\frac{Gm_1m_3(q_x - u_1)}{\alpha_1^3} - \frac{Gm_2m_3(q_x - u_2)}{\alpha_2^3} + 2m_3\omega\dot{q}_y + m_3\omega^2 q_x$$
$$m_3\ddot{q}_y = -\frac{Gm_1m_3(q_y - v_1)}{\alpha_1^3} - \frac{Gm_2m_3(q_y - v_2)}{\alpha_2^3} - 2m_3\omega\dot{q}_x + m_3\omega^2 q_y$$

We can immediately conclude that m_3 has no effect thus we get:

$$\ddot{q}_x = -\frac{Gm_1(q_x - u_1)}{\alpha_1^3} - \frac{Gm_2(q_x - u_2)}{\alpha_2^3} + 2\omega\dot{q}_y + \omega^2 q_x \tag{29}$$
$$\ddot{q}_y = -\frac{Gm_1(q_y - v_1)}{\alpha_1^3} - \frac{Gm_2(q_y - v_2)}{\alpha_2^3} - 2\omega\dot{q}_x + \omega^2 q_y \tag{30}$$

Thus, we get 2 nonlinear differential equations that describe the motion of mass m_3, in the rotating CM frame. The only constant that we haven't defined yet is frequency of rotation ω. However; since we know both m_1, m_2, and the distance between them R, we can apply Newton's version of Kepler's third law:

$$\left(\frac{2\pi}{\omega}\right)^2 = \frac{4\pi^2}{G(m_1 + m_2)}R^3$$
$$\omega^2 = \frac{G(m_1 + m_2)}{R^3} \tag{31}$$

5 Analysis of the Chaotic Nature of the Non-linear Solutions

A chaotic dynamical system is one in which there is a strong dependence on initial conditions and unstable aperiodic behavior [1].

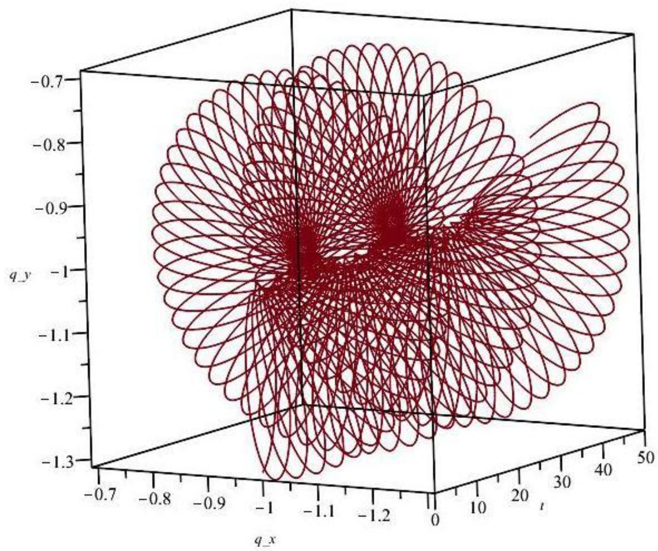


Figure 4: Initial Conditions: $q_x(0) = -1, \dot{q}_x(0) = -0.33, q_y(0) = -1.3, \dot{q}_y(0) = 0.5$. Here we have the red line depicting the trajectory of m_3 from time $t \in [0, 50]$. It is aperiodic and unstable.

Orbits of the third mass can be found by solving equation (29), (30), and (31). Figure 4 and 5 are the solutions, in the case that $m_1 = 1$, $m_2 = 1$, $G = 1$, with the position of the masses m_1 and m_2 at $(1, 1)$ and $(-1, -1)$, respectively.

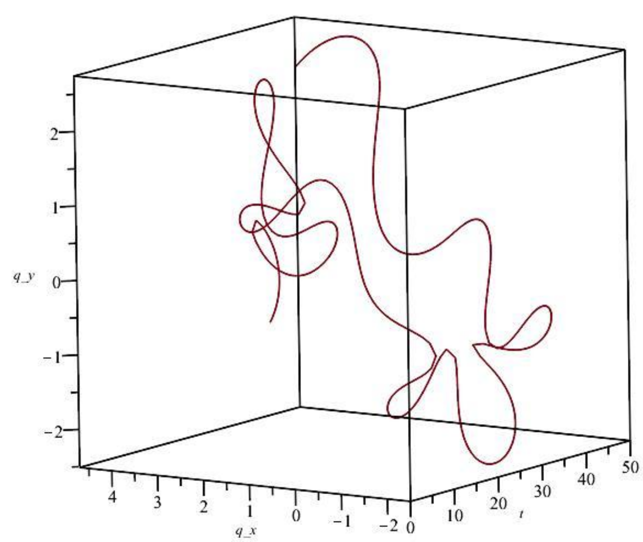


Figure 5: Initial Condition: $q_x(0) = 0.5, \dot{q}_x(0) = -0.33, q_y(0) = -0.3, \dot{q}_y(0) = 0.5$. Here we have the red line depicting the trajectory of m_3 from time $t \in [0, 15]$.

The solutions obtained through Maple show an important aspect of the three body problem which is its chaotic nature. There is no closed closed form solution and more interestingly; there is a strong dependence on initial conditions. A slight change in the initial position or momentum produces a completely different orbit. Observe the Figure 4 and 5. There is also an unstable, aperiodic behavior of the orbits. The orbits do not repeat in a regular fashion.

6 Lagrange Points of the System

Lagrange Points are positions in a restricted three-body problem such that if the smallest mass (m_3 for our case) is at one of the points, it has a tendency to stay there. Lagrange noticed these equilibrium positions, when he wrote the prize winning paper "*Essai sur le Problème des Trois Corps*" in 1772. [2] These points are positioned on our rotating plane where the gravitational forces of the other two bodies, and the centripetal forces are balanced.

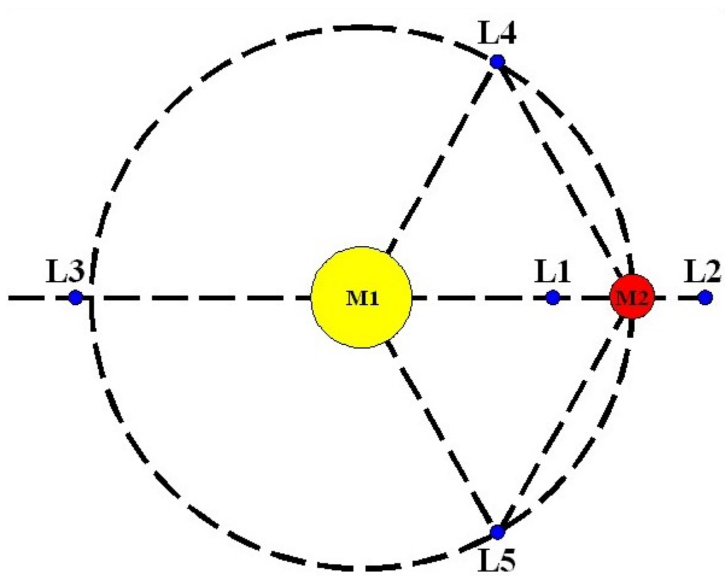


Figure 6: The five Lagrange points of a Earth - Sun system, with $L1$, $L2$, and $L3$ on the line that connects the two masses, and $L4$ and $L5$ form two equilateral triangles, with the two masses at the base. m_2 rotates counter-clockwise. [7]

Given our equation of motion - equation (29) and (30), we can set acceleration and velocity to be zero. This simplifies down to

$$-\frac{Gm_1(q_x - u_1)}{\alpha_1^3} - \frac{Gm_2(q_x - u_2)}{\alpha_2^3} + \omega^2 q_x = 0 \tag{32}$$

$$-\frac{Gm_1(q_y - v_1)}{\alpha_1^3} - \frac{Gm_2(q_y - v_2)}{\alpha_2^3} + \omega^2 q_y = 0 \tag{33}$$

There are 5 solutions for both of the equation above, we will denote them as $L1$ to $L5$. Turns out, only two out of five of these solutions are stable. The unstable solution - $L1, L2$, and $L3$ - lie along the line that connecting the two large masses. The stable points - $L4$ and $L5$ - forms the apex of two equilateral triangles, with the masses at their vertices. $L4$ "leads" the rotation, and $L5$ "follows" the rotation [5]. See figure 6 for more details.

7 Conclusions

As this report showed, solutions for the three-body problem can be solved numerically, with specific sets of assumptions. With the Lagrangian approach, we were able to derive the equation of motion for the insignificant mass, as the two larger bodies rotate around their center of mass, with a constant velocity. It is very important to acknowledge the chaotic nature of the orbit of the smallest mass, and how unstable, and different they are, with just a small deviation of the initial condition. This can be generalized to our solar system, and how it can be considered to be a ten-body problem (nine planets - including Pluto - and the Sun). Therefore, the orbits of each planets are much more unstable than we expected.

Regarding Lagrangian points, it can give us very unique insights onto system. Consider an Earth - Sun - satellite system, Lagragian points give us location that will allow the satellite to use the least amount of fuel. Currently, there are two satellites, located at the two nearest Lagrange points to Earth. $L1$ is home to the Solar and Heliospheric Observatory Satellite, since it has a uninterrupted view of the sun. $L2$ is the home of Planck, future home of the James Webb Space Telescope. It is very ideal for telescopes since it is close enough to Earth for easy communication and the Sun is completely blocked by Earth, which gives $L2$ a very clear view of deep space. [5]

Although we might never find the general solution to the three-body problem, the restricted problem, as we have discussed in this report, has given many unique and practical results.

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