PRP Lecture Scribing
Lecture 24 (19/07/2021)
Convergence of Random Variables

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1 Limit of a sequence of Real Numbers :

Let $x_1, x_2, ...$ be a sequence of real numbers. The limit of the sequence x (if it exists) is:

$$\lim_{n\to\infty} x_n = x$$

An alternate definition can be as follows: Given an $\epsilon > 0$ there exists an integer n_0 such that whenever $n > n_0$, we have

$$|x_n - x| < \epsilon \quad (\epsilon \in \mathbb{R})$$

2 Cauchy Criterion :

A sequence of Real numbers converges to a limit iff

$$|x_n - x_m| \to 0$$

as $n \to 0$ and $m \to 0$. An important difference from the previous definition is that, in the previous definition, we need to know the value of x (the value it converges to), but here, we can conclude if the sequence converges or not, just by looking at the sequence itself. We do not need to guess the value of the limit. Some examples:

$$S_n = \sum_{i=1}^{n} \frac{1}{i} \quad , \quad S_n = (-1)^n$$

do not converge, but $S_n = \sum_{i=1}^{n} \frac{1}{i^2}$ converges

3 Convergence of a sequence of functions :

Let there be a sequence of functions $f_1, f_2, ..., f_i, ...$ which converge to function f. For such a case, we write:

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \in \mathbb{R}$$

This is termed as 'point-wise convergence'. Similar to the case of real numbers, we also have an alternate definition. Given $\epsilon > 0$, there exists an n_ϵ such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n > n_\epsilon \quad \& \quad \forall x \in \mathbb{R}$$

we can use this for Random variables, since random variables are functions from the sample space to the real line.

4 Convergence of RV sequences:

There are four forms of convergence of random variables:

4.1 Almost-Sure Convergence

This is the strongest form of convergence. Let $X_1, X_2, ..$ be a sequence of random variables. Let's say they converge to a random variable X. If

$$P(\omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)) = 1$$

then the sequence is said to be almost-surely convergent. Note that it is possible to have some ω_0 such that $X_n(\omega_0)$ does not tend to $X(\omega_0)$, but for such a case, the $P(\omega_0) = 0$. Almost sure convergence is a form of point wise convergence.

4.2 Convergence in Probability

This is a slightly weaker form of convergence. Let $X_1, X_2, ..$ be a sequence of random variables. Let's say they converge to a random variable X. Given $\epsilon > 0$,

$$\lim_{n \to \infty} P(\omega : |X_n(\omega) - X(\omega)| > \epsilon) = 0$$

is the condition for convergence in probability. Convergence in Probability is also called **Stochastic Convergence**. As we can see, this is not point wise convergence. Rather, it is the convergence of a sequence of probabilities $P_1, P_2, ... \to 0$. Note: Almost-sure convergence implies convergence in probability. The vice-versa is not true.

4.3 Convergence in Mean-Square Sense

This form of convergence is stronger than Convergence in probability. Let $X_1, X_2, ..$ be a sequence of random variables. Let's say they converge to a random variable X. The sequence is said to converge in the mean-square sense if

$$\lim_{n\to\infty} (E[X_n - X])^2 = 0$$

4.4 Convergence in Distribution

It is the weakest form of convergence of random variables. Let $X_1, X_2, ..$ be a sequence of random variables. Let's say they converge to a random variable X. The sequence is said to converge in distribution if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

where F(.) is the cumulative distribution function(cdf). The above equation requires that the cdfs of X_n converge to the cdf of X. Thus, in a strict sense, it is not even the convergence of random variables.

5 Relation between various convergences:

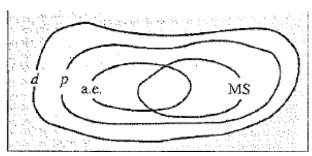

Figure 1: Dependencies

In Figure 1, d represents 'convergence in-distribution', p represents 'convergence in-probability', $a.e$ represents 'almost-sure convergence' and MS represents 'convergence in mean-square sense'. From the Venn-diagram like figure, we can observe that almost-sure convergence and convergence in mean-square sense *imply* convergence in probability and in distribution. Likewise, convergence in probability *implies* convergence in distribution. We can now clearly see why convergence in distribution is the weakest form of convergence (since it is implied by all other forms of convergence).

We will now prove one of the aforementioned dependencies. To show: Let $X_1, X_2, ..$ be a sequence of random variables. If the sequence converges in mean-square sense, then it converges in probability:

Proof: we have seen the Chebyshev Inequality in the previous paper. Using the Chebyshev Inequality for every n, we have

$$P(|X_n - X| > \epsilon) \leq \frac{E((X_n - X)^2)}{\epsilon^2}$$

now apply $\lim_{n\to\infty}$ on both sides

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) \leq \lim_{n\to\infty} \frac{E((X_n - X)^2)}{\epsilon^2}$$

but we know that $\lim_{n\to\infty} E((X_n - X)^2) = 0$ because the sequence in convergent in the mean-square sense. So,

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) \leq 0$$

since probability is non-negative,

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$$

this is the same as

$$\lim_{n\to\infty} P(\omega : |X_n(\omega) - X(\omega)| > \epsilon) = 0$$

This is the condition for convergence in probability. Thus, 'convergence in the mean square sense' $\Rightarrow$ 'convergence in probability'

6 Some pointers to related topics:

6.1 Point-wise Convergence

Definition: Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the sequence (f_n) converges on A_0 to f if, for each $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call f the limit on A_0 of the sequence (f_n). When such a function f exists, we say that the sequence (f_n) is convergent on A_0, or that (f_n) converges 'point-wise' on A_0.

6.2 Uniqueness of Limits

Claim: The limit of a sequence is uniquely determined.
Proof: Suppose that for a sequence $\{x_n\}$, both x' and x'' are the limits. For each $\epsilon > 0$, there exists K' such that $|x_n - x'| < \frac{\epsilon}{2}$ for all $n \geq K'$, and there exists K'' such that $|x_n - x''| < \frac{\epsilon}{2}$ for all $n \geq K''$. Let K be the larger of K' and K''. Apply Triangle inequality for $n \geq K$

$$|x' - x''| = |x' - x_n + x_n - x''|$$

$$|x' - x''| \leq |x' - x_n| + |x_n - x''| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since $\epsilon > 0$ is an arbitrary positive number, we can conclude $x' - x'' = 0$. Thus, there exists only one unique limit to a sequence.

6.3 Some theorems for real line sequences

Here are some theorems learned in Real Analysis which might be useful for dealing with sequences on the real line.

6.3.1 Bolzano Weierstrass Theorem

The Bolzano Weierstrass theorem can be stated as:

'Every bounded sequence has a convergent subsequence.'

Definition of Subsequence: let $\{a_j\}$ be a given sequence. If $0 < j_1 < j_2 < ...$ are positive integers, then the function $k \mapsto a_{j_k}$ is called a subsequence of the given sequence $\{a_j\}$.

6.3.2 Monotone Subsequence Theorem

If $X = \{x_n\}$ is a sequence of real numbers, then there is a subsequence of X that is monotone (i.e it is either only one of non-increasing or non-decreasing).

6.3.3 Theorem:

'A monotone sequence of real numbers converges iff it is bounded.'