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Linear Systems Coursework Assignment 2022

Contents

1 Question 1

We need to find the solution, $\vec{r}$, to:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = Q\vec{r}, \tag{1}$$

where $Q = HJH^{-1}$, and the three 7×7 matrices $H,\ H^{-1}$ and J are given to be,

$$H = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ H^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and,

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} J_1(-1) & 0 & 0 & 0 \\ 0 & J_1(-1) & 0 & 0 \\ 0 & 0 & J_2(-2) & 0 \\ 0 & 0 & 0 & J_3(0) \end{pmatrix}.$$

Recall that $Q = HJH^{-1}$, so (1) becomes:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = HJH^{-1}\vec{r}.$$

The solution to the system is given to be

$$\vec{r}(t) = e^{Qt}\vec{r_0} = He^{Jt}H^{-1}\vec{r_0},$$

where $\vec{r_0} = [0\ 6\ 0\ 0\ 6\ 0\ 9]^T$. Thus, all we need to do to find the solution is find the matrix exponential of Jt and then multiply the result by $H,\ H^{-1}$ and $\vec{r_0}$.

Now,

$$e^{Jt} = \begin{pmatrix} e^{J_1(-1)t} & 0 & 0 & 0 \\ 0 & e^{J_1(-1)t} & 0 & 0 \\ 0 & 0 & e^{J_2(-2)t} & 0 \\ 0 & 0 & 0 & e^{J_3(0)t} \end{pmatrix},$$

where $e^{J_1(-1)t} = e^{-t}$,

$$e^{J_2(-2)t} = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

and

$$e^{J_3(0)t} = \begin{pmatrix} e^{0t} & te^{0t} & \frac{t^2}{2}e^{0t} \\ 0 & e^{0t} & te^{0t} \\ 0 & 0 & e^{0t} \end{pmatrix} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the matrix exponential is:

$$e^{Jt} = \begin{pmatrix} e^{-t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2t} & te^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We know that $e^{Qt} = He^{Jt}H^{-1}$, so we have to find:

$$e^{Qt} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2t} & te^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times H^{-1}$$

$$= \begin{pmatrix} -e^{-t} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -e^{-t} & 0 & 0 & 0 & 1 & t \\ 0 & 0 & -e^{-2t} & -te^{-2t} & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 0 & e^{-2t} & -1 & -t & -\frac{t^2}{2} \\ 0 & 0 & e^{-2t} & te^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ e^{-t} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times H^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1-e^{-t} \\ t & e^{-t} & 0 & 0 & 0 & 1-e^{-t} & t \\ \frac{t^2}{2} & 0 & 1-te^{-2t} & -te^{-2t} & 1-te^{-2t}-e^{-2t} & t & \frac{t^2}{2} \\ -\frac{t^2}{2} & 0 & e^{-2t}-1 & e^{-2t} & e^{-2t}-1 & -t & -\frac{t^2}{2} \\ 0 & 0 & te^{-2t} & te^{-2t} & e^{-2t}+te^{-2t} & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \end{pmatrix}.$$

Thus, the solution, $\vec{r}(t) = e^{Qt}\vec{r_0}$, to (1) is:

$$\vec{r}(t) = e^{Qt} \times [0\ 6\ 0\ 0\ 6\ 0\ 0\ 9]^T$$

$$= \begin{pmatrix} 9-9e^{-t} \\ 6e^{-t}+9t \\ 6-6te^{-2t}-6e^{-2t}+\frac{9}{2}t^2 \\ 6e^{-2t}-6-\frac{9}{2}t^2 \\ 6e^{-2t}+6te^{-2t} \\ 9t \\ 9e^{-t} \end{pmatrix}.$$

2 Question 2

Consider on $\mathbb{R}^3$ the differential equation

$$\frac{\mathrm{d}\vec{x}}{dt} = A\vec{x}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

2.1 2a) Construct a real matrix P

To find

$$\frac{\mathrm{d}\vec{y}}{dt} = B\vec{y}, \text{ where } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix},$$

we let $\vec{x} = P\vec{y} \iff P^{-1}\vec{x} = \vec{y}$ to transform (2):

$$\begin{aligned} \frac{\mathrm{d}\vec{x}}{dt} = A\vec{x} \quad \longrightarrow \quad P\frac{\mathrm{d}\vec{y}}{dt} &= PBP^{-1}\vec{x} = PB\vec{y}, \\ \Longrightarrow P^{-1}P\frac{\mathrm{d}\vec{y}}{dt} &= P^{-1}PB\vec{y} \\ \Longrightarrow \frac{\mathrm{d}\vec{y}}{dt} &= B\vec{y}. \end{aligned}$$

So we find the matrices P and P^{-1} such that $P^{-1}AP = B \iff A = PBP^{-1}$. We begin by firstly calculating the eigenvalues of the matrix A, then find the corresponding eigenvectors.

To find eigenvalues, we calculate

$$\begin{aligned} \det(A - \lambda\mathbb{I}) = 0 &\Longrightarrow \det\left[\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & -2 \\ 0 & 1 & -\lambda \end{pmatrix}\right] \\ &\Longrightarrow (1-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0 \\ &\Longrightarrow (1-\lambda)(\lambda^2+2) = 0. \end{aligned}$$

We therefore have three eigenvalues: $\lambda_1 = 1,\ \lambda_{2,3} = \pm i\sqrt{2}$.

Now find the associated eigenvectors, starting with $\lambda_1 = 1$:

$$(A - \lambda_1 \mathbb{I})\vec{u}_1 = \vec{0},$$

$$\implies \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

which implies that $u_2 = u_3 = 0$ and $u_1 = \alpha$. Thus,

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we will need to find real matrices P, B and P^{-1}, but we have two complex eigenvalues. Therefore we define the new matrix, C, as:

$$C := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \text{where} \quad \lambda_{j,k} = a \pm ib.$$

Since $a = 0$ and $b = \pm\sqrt{2}$, we let $b = -\sqrt{2}$ to obtain the matrix:

$$C = \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}.$$

This is our Jordan block for the complex eigenvalues and gives:

$$B = \begin{pmatrix} 1 & \mathrm{O} \\ \mathrm{O} & C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Now, we need to find the eigenvectors and then their real form. Begin by finding u_3, then $u_2 = \overline{u_3}$. We find $\vec{u}_3$ first because we use the corresponding eigenvalue in C above:

Therefore, for $\lambda_3 = -i\sqrt{2}$ we have $(A + i\sqrt{2}\mathbb{I})\vec{u}_3$:

$$\begin{pmatrix} 1 + i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{2} & -2 & 0 \\ 0 & 1 & i\sqrt{2} & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 + i\sqrt{2} & 0 & 1 & 0 \\ 0 & 1 & i\sqrt{2} & 0 \\ 0 & i\sqrt{2} & -2 & 0 \end{pmatrix},$$

$R3 = R3 - i\sqrt{2}R2$:

$$\begin{pmatrix} 1+i\sqrt{2} & 0 & 1 & 0 \\ 0 & 1 & i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and we have the two equations:

$$\begin{aligned} (1+i\sqrt{2})u_1 + u_3 &= 0, \\ u_2 + i\sqrt{2}u_3 &= 0. \end{aligned}$$

The first equation gives: $u_3 = -(1+i\sqrt{2})u_1$, so let $u_3 = -(1+i\sqrt{2})$.

Then, $u_2 = -i\sqrt{2}(-1-i\sqrt{2}) = i\sqrt{2} + i\sqrt{2}i\sqrt{2} = i\sqrt{2} - 2$. Also then,

$$\begin{aligned} u_1 &= \frac{-u_3}{1+i\sqrt{2}} = -\frac{(-1-i\sqrt{2})}{1+i\sqrt{2}} = -\frac{(-1-i\sqrt{2})(1-i\sqrt{2})}{(1+i\sqrt{2})(1-i\sqrt{2})} \\ &= -\frac{1}{3}(-1+2(-1)) = 1. \end{aligned}$$

Thus, the last two eigenvectors are given as:

$$\vec{u}_3 = \begin{pmatrix} 1 \\ i\sqrt{2}-2 \\ -i\sqrt{2}-1 \end{pmatrix}, \quad \& \quad \vec{u}_2 = \begin{pmatrix} 1 \\ -i\sqrt{2}-2 \\ i\sqrt{2}-1 \end{pmatrix}.$$

To construct P we consider the eigenvector associated with the eigenvalue $\lambda_3 = -i\sqrt{2}$ (since this eigenvalue was used in the construction of C).

$$\begin{aligned} \vec{u}_3 &= \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{2} \\ -\sqrt{2} \end{pmatrix} \\ &= \mathrm{Re}(\vec{u}_3) + i\ \mathrm{Im}(\vec{u}_3). \end{aligned}$$

Define the matrix P as:

$$P = \begin{pmatrix} \vec{u}_1 & \mathrm{Re}(\vec{u}_3) & \mathrm{Im}(\vec{u}_3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & \sqrt{2} \\ 0 & -1 & -\sqrt{2} \end{pmatrix}.$$

To find the inverse of P we take the transpose, find the minors and construct the adjoint matrix:

$$P^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \text{ gives:}$$

$$\begin{aligned}
&|P^T_{11}| = 3\sqrt{2} = \det(P), \quad |P^T_{12}| = -\sqrt{2}, \quad |P^T_{13}| = \sqrt{2}, \\
&|P^T_{21}| = 0, \quad |P^T_{22}| = -\sqrt{2}, \quad |P^T_{23}| = \sqrt{2}, \\
&|P^T_{31}| = 0, \quad |P^T_{32}| = -1, \ \& \ |P^T_{33}| = -2.
\end{aligned}$$

So the matrix of minors is,

$$P_{\min} = \begin{pmatrix} 3\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & -\sqrt{2} & \sqrt{2} \\ 0 & -1 & -2 \end{pmatrix},$$

and multiplying by the checkerboard pattern of pluses and minuses gives:

$$\mathrm{Adj}(P) = \begin{pmatrix} 3\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & -\sqrt{2} & -\sqrt{2} \\ 0 & 1 & -2 \end{pmatrix},$$

Now,

$$P^{-1} = \frac{1}{\det(P)}\mathrm{Adj}(P) = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & -\sqrt{2} & -\sqrt{2} \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{pmatrix},$$

and so,

$$\implies P^{-1} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{3} \end{pmatrix}.$$

Thus we have obtained a real-valued matrix expression for P (and P^{-1}), as required.

2.2 2b) Solve for y and find solution in x

Recall, that we have the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ -\sqrt{2}y_3 \\ \sqrt{2}y_2 \end{pmatrix}.$$

We therefore have the folowing three ODEs:

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_1, \quad \underbrace{\frac{\mathrm{d}y_2}{\mathrm{d}t} = -\sqrt{2}y_3, \quad \frac{\mathrm{d}y_3}{\mathrm{d}t} = \sqrt{2}y_2}_{\text{Coupled equations, solve together.}}$$

Also define some general initial conditions:

$$y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} \quad \text{and} \quad y_3(0) = y_{3,0}. \tag{3}$$

Solving the equation for y_1:

$$\begin{aligned} &\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_1 \implies \frac{\mathrm{d}y_1}{\mathrm{d}t}\frac{1}{y_1} = 1 \implies \int \frac{1}{y_1}\mathrm{d}y_1 = \int \mathrm{d}t \\ &\implies \ln|y_1| = t + c \\ &\implies y_1(t) = Be^t, \quad \text{where} \quad B = y_{1,0}. \end{aligned}$$

Next, solve the coupled ODEs:

$$\frac{\mathrm{d}^2y_2}{\mathrm{d}t^2} = -2y_2, \quad y_2(0) = \frac{\sqrt{2}}{2}y_3'(0) = \tilde{A} \tag{4}$$

$$\frac{\mathrm{d}^2y_3}{\mathrm{d}t^2} = -2y_3, \quad y_3(0) = -\frac{\sqrt{2}}{2}y_2'(0) = -\hat{A} \tag{5}$$

In both cases an exponential ansatz will yield a general solution of the form $\alpha\cos(\sqrt{2}t) + \beta\sin(\sqrt{2}t)$. Consider the ODE for y_2:

$$y_2(t) = a_1\cos(\sqrt{2}t) + a_2\sin(\sqrt{2}t)$$

From (3), and (5) we have two initial conditions and so can find the two constants.

$$\begin{aligned} y_2(0) = a_1 = \tilde{A} &\implies y_2(t) = \tilde{A}\cos(\sqrt{2}) + a_2\sin(\sqrt{2}t), \\ &\implies y_2'(t) = -\sqrt{2}\tilde{A}\sin(\sqrt{2}t) + a_2\sqrt{2}\cos(\sqrt{2}t). \end{aligned}$$

Now the other initial condition reads $y_3(0) = -\hat{A} = -\frac{\sqrt{2}}{2}y_2'(0) \implies y_2'(0) = \frac{2}{\sqrt{2}}\hat{A}$. Now we can find the other constant:

$$\begin{aligned} y_2'(0) &= \frac{2}{\sqrt{2}}\hat{A} = a_2\sqrt{2}, \\ &\implies a_2 = \hat{A}. \end{aligned}$$

Thus,

$$y_2(t) = \tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t).$$

Now, consider the ODE associated with y_3; we obtain again the same general solution:

$$y_3(t) = b_1\cos(\sqrt{2}t) + b_2\sin(\sqrt{2}t).$$

Using the initial conditions stated in (3) and (4) gives:

$$\begin{aligned} y_3(0) = b_1 = -\hat{A} \implies y_3(t) &= -\hat{A}\cos(\sqrt{2}t) + b_2\sin(\sqrt{2}t), \\ \implies y_3'(t) &= \hat{A}\sqrt{2}\sin(\sqrt{2}t) + b_2\sqrt{2}\cos(\sqrt{2}t). \end{aligned}$$

The second initial condition gives $y_3'(0) = \frac{2}{\sqrt{2}}\tilde{A}$:

$$y_3'(0) = \frac{2}{\sqrt{2}}\tilde{A} = b_2\sqrt{2} \implies b_2 = \tilde{A}.$$

Thus,

$$y_3(t) = \tilde{A}\sin(\sqrt{2}t) - \hat{A}\cos(\sqrt{2}t).$$

The solution, in y, is

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} Be^t \\ \tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t) \\ \tilde{A}\sin(\sqrt{2}t) - \hat{A}\cos(\sqrt{2}t). \end{pmatrix}$$

But we want the solution in x thus:

$$\vec{x}(t) = P\vec{y}(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & \sqrt{2} \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ \sqrt{2}y_3 - 2y_2 \\ -y_2 - \sqrt{2}y_3 \end{pmatrix}$$

$$= \begin{pmatrix} Be^t + \tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t) \\ \sqrt{2}(\tilde{A}\sin(\sqrt{2}t) - \hat{A}\cos(\sqrt{2}t)) - 2(\tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t)) \\ -(\tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t)) - \sqrt{2}(\tilde{A}\sin(\sqrt{2}t) - \hat{A}\cos(\sqrt{2}t)) \end{pmatrix},$$

and this simplifies down to:

$$\vec{x}(t) = \begin{pmatrix} Be^t + \tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t) \\ \tilde{A}(\sqrt{2}\sin(\sqrt{2}t) - 2\cos(\sqrt{2}t)) + \hat{A}(-\sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t)) \\ \tilde{A}(-\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t)) + \hat{A}(-\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t)) \end{pmatrix},$$

which is the solution in our original coordinates, as required.

2.3 2c) Plotting the trajectories of the solution

We will next plot the solution passing throught the point $\vec{x}(0) = [0,\ 12,\ 9]^T$ over the interval $t \in [0,\ 2]$. Recall that the solution is:

$$\vec{x}(t) = \begin{pmatrix} Be^t + \tilde{A}\cos(\sqrt{2}t) + \hat{A}\sin(\sqrt{2}t) \\ \tilde{A}(\sqrt{2}\sin(\sqrt{2}t) - 2\cos(\sqrt{2}t)) + \hat{A}(-\sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t)) \\ \tilde{A}(-\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t)) + \hat{A}(-\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t)) \end{pmatrix},$$

We can find the three constants using the initial condition as follows:

$$\vec{x}(0) = x_0 = \begin{pmatrix} 0 \\ 12 \\ 9 \end{pmatrix} = \begin{pmatrix} B + \tilde{A} \\ -2\tilde{A} - \sqrt{2}\hat{A} \\ -\tilde{A} + \sqrt{2}\hat{A} \end{pmatrix},$$

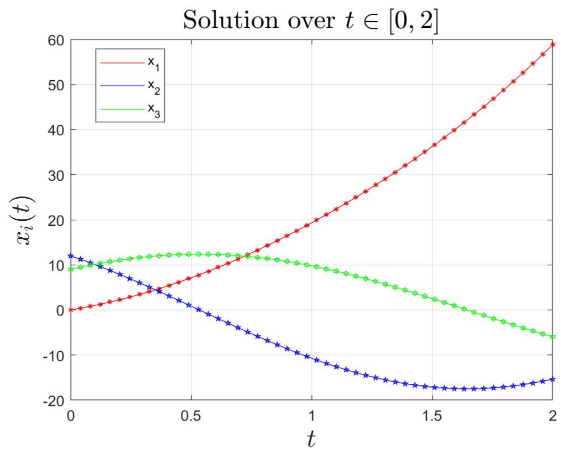
now if we add the 'second row' and the 'third row' in the above equality, we find that:

$$\begin{aligned} 12 + 9 = -3\tilde{A} + 0\hat{A}, &\implies \tilde{A} = -\frac{21}{3} = -7, \\ &\implies \hat{A} = \frac{9 + \tilde{A}}{\sqrt{2}}, \quad \text{(from the third row)} \\ &\implies \hat{A} = \sqrt{2}, \quad \text{and} \quad B = -\tilde{A} = 7. \end{aligned}$$

So the solution is:

$$\vec{x}(t) = \begin{pmatrix} 7e^t - 7\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -9\sqrt{2}\sin(\sqrt{2}t) + 12\cos(\sqrt{2}t) \\ 6\sqrt{2}\sin(\sqrt{2}t) + 9\cos(\sqrt{2}t) \end{pmatrix}.$$

We have the following plot of $\vec{x}(t)$ over the domain $t \in [0,\ 2]$:



(a)

3 Question 3

We need to make a change of variables to express the following as a sum of squares: $S = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$. Firstly, we need to write S in matrix form, say A. We find that

$$A = \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{pmatrix} .* \begin{pmatrix} a & 2a & 2a \\ 2a & a & 2a \\ 2a & 2a & a \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix},$$

$$\text{by comparison to } S\text{, and define } \vec{x} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We need to construct $A = PDP^{-1}$, thus we again find the eigenvalues and eigenvectors of A. Eigenvalues are found by:

$$\det(A - \lambda\mathbb{I}) = 0 \implies \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & -\lambda & 2 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0,$$

$$\begin{aligned} &\implies (1-\lambda)\big\{\lambda(1+\lambda) - 4\big\} + 2\big\{2(1+\lambda)\big\} = 0, \\ &\implies (1-\lambda)(\lambda^2 + \lambda - 4) + 4 + 4\lambda = 0, \\ &\implies \lambda^3 - 9\lambda = 0, \\ &\implies \lambda(\lambda^2 - 9) = 0, \end{aligned}$$

giving distinct eigenvalues $\lambda = 0,\ +3$ and -3. Next, find the corresponding eigenvectors, starting with $\lambda = 0$:

$$\implies (A - \lambda\mathbb{I})\vec{u_1} = A\vec{u_1} = \vec{0}:$$

$$\implies \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} : R2 = R2 + 2R1 \mapsto \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix},$$

$$\equiv \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies u_1 = 2u_2,\ u_3 = 2u_2, \text{ so let } 2u_2 = \alpha, \text{ then,}$$

$$u_1 = u_3 = 2u_2 = \alpha \Longrightarrow \; \vec{u_1} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Next for $\lambda = 3$:

$(A - 3\mathbb{I})\vec{u_2} = \vec{0}$:

$$\Longrightarrow \begin{pmatrix} -2 & -2 & 0 & 0 \\ -2 & -3 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{pmatrix} : R2 = R2 - R1 \mapsto \begin{pmatrix} -2 & -2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{pmatrix},$$

$$\equiv \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\Longrightarrow\; u_1 = -u_2,\; 2u_3 = u_2,\; \text{so let } \; u_2 = \alpha,\; \text{then,}$$

$$u_1 = -\alpha,\; u_3 = \frac{1}{2}\alpha \;\Longrightarrow\; \vec{u_2} = \alpha \begin{pmatrix} -1 \\ 1 \\ \frac{1}{2} \end{pmatrix} \equiv \alpha \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

Lastly for $\lambda = -3$:

$(A + 3\mathbb{I})\vec{u_3} = \vec{0}$:

$$\Longrightarrow \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \equiv \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} : R2 = R2 + R1$$

$$\mapsto \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Longrightarrow\; 2u_1 = u_2,\; u_3 = -u_2,\; \text{so let } \; u_2 = \alpha,\; \text{then,}$$

$$u_1 = \frac{1}{2}\alpha,\; u_3 = -\alpha \;\Longrightarrow\; \vec{u_3} = \alpha \begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \end{pmatrix} \equiv \alpha \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

Next, we need to construct P, where the columns are the normalised eigenvectors.

Recall that for $\lambda \in \{0,\ 3,\ -3\}$ we have:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right\}.$$

The length of each vector above is $\sqrt{3}$, 3 and 3 respectively, and so dividing element-wise gives the orthanormal set of vectors:

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} (\frac{-2}{3}) \\ (\frac{2}{3}) \\ (\frac{1}{3}) \end{pmatrix}, \begin{pmatrix} (\frac{1}{3}) \\ (\frac{2}{3}) \\ (\frac{-2}{3}) \end{pmatrix} \right\},$$

and

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & (\frac{-2}{3}) & (\frac{1}{3}) \\ \frac{1}{\sqrt{3}} & (\frac{2}{3}) & (\frac{2}{3}) \\ \frac{1}{\sqrt{3}} & (\frac{1}{3}) & (\frac{-2}{3}) \end{pmatrix},$$

and where

$$P^{-1} = P^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ (\frac{-2}{3}) & (\frac{2}{3}) & (\frac{1}{3}) \\ (\frac{1}{3}) & (\frac{2}{3}) & (\frac{-2}{3}) \end{pmatrix},$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Make the change of variables, let $\vec{x} = P\vec{y}$. Now we represent the quadratic equation as $S = \vec{x}^{\,T} A\vec{x}$. Note that $\vec{x} = P\vec{y} \implies \vec{y} = P^T\vec{x} \implies \vec{y}^{\,T} = \vec{x}^{\,T}P$, so that

$$\begin{aligned} S &= \vec{x}^{\,T} A\vec{x} = \vec{x}^{\,T} PDP^T\vec{x} = (\vec{x}P^T)^T D\vec{y} \\ &= \vec{y}^{\,T} D\vec{y}, \end{aligned}$$

where

$$\vec{y} = P^T\vec{x} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ (\frac{-2}{3}) & (\frac{2}{3}) & (\frac{1}{3}) \\ (\frac{1}{3}) & (\frac{2}{3}) & (\frac{-2}{3}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \\ \frac{1}{3}(-2x_1 + 2x_2 + x_3) \\ \frac{1}{3}(x_1 + 2x_2 - 2x_3) \end{pmatrix}.$$

Now we calculate $S = \vec{y}^{\,T} D \vec{y}$:

$$S = \vec{y}^{\,T} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \vec{y}^{\,T} \begin{pmatrix} 0 \\ 3y_2 \\ -3y_3 \end{pmatrix},$$

$$\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 0 \\ 3y_2 \\ -3y_3 \end{pmatrix},$$

Thus,

$$\boxed{\implies S = 3y_2^2 - 3y_3^2}$$

is the sum of squares, as required.

4 Question 4

Find the real canonical form of the following 4×4 matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 6 \end{pmatrix}.$$

As usual, we want to decompose A into the product of three matrices P, D and P^{-1}, such that $A = PDP^{-1}$. Also note that $D = J$ if we use generalised eigenvectors in P. We therefore begine by computing the eigenvalues, then eigenvectors of the matrix A:

$$\det(A - \lambda\mathbb{I}) = 0:$$

$$\implies \begin{vmatrix} -\lambda & 0 & 0 & -8 \\ 1 & -\lambda & 0 & 16 \\ 0 & 1 & -\lambda & -14 \\ 0 & 0 & 1 & 6-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 16 \\ 1 & -\lambda & -14 \\ 0 & 1 & 6-\lambda \end{vmatrix} + 8 \begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix} = 0,$$

$$\implies -\lambda\Big\{ -\lambda \begin{vmatrix} -\lambda & -14 \\ 1 & 6-\lambda \end{vmatrix} + 16 \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} \Big\} + 8 \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} = 0,$$

$$\begin{aligned} &\implies -\lambda\Big\{ -\lambda\big\{ -\lambda(6-\lambda) + 14\big\} + 16 \Big\} + 8\{0\} = 0, \\ &\implies \lambda^4 - 6\lambda^3 + 14\lambda^2 - 16\lambda + 8 = 0, \\ &\implies (\lambda-2)^2(\lambda - 1 - i)(\lambda - 1 + i) = 0, \end{aligned}$$

which gives the following four roots:

$$\begin{aligned} &\lambda = 2, \text{ with algebraic multiplicity} = 2, \text{ and} \\ &\lambda = 1 \pm i. \end{aligned}$$

Because we have a root with AM=2, we might find that the eigenspace corresponding to this root is not two dimensional and so, a diagonal matrix D cannot be found. We would then have to find the Jordan matrix.

Now to find the eigenspace we calculate the following for the case where $\lambda = 2$:

$$(A-2\mathbb{I})\vec{u_1}=\vec{0},$$

$$\implies \begin{pmatrix} -2 & 0 & 0 & -8 & 0 \\ 1 & -2 & 0 & 16 & 0 \\ 0 & 1 & -2 & -14 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 & -4 & 0 \\ 1 & -2 & 0 & 16 & 0 \\ 0 & 1 & -2 & -14 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix},$$

$$\longrightarrow R2 = R2 + R1 \mapsto \begin{pmatrix} -1 & 0 & 0 & -4 & 0 \\ 0 & -2 & 0 & 16 & 0 \\ 0 & 1 & -2 & -14 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix},$$

$$\equiv \begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & -2 & 0 & 16 & 0 \\ 0 & 1 & -2 & -14 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix} \longrightarrow R2 = R2 + 2R3,$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & -4 & -16 & 0 \\ 0 & 1 & -2 & -14 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 1 & -2 & -14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which gives the following equations: $u_1 + 4u_4 = 0$, $u_3 + 4u_4 = 0$ and $-u_2 + 2u_3 + 14u_4 = 0$. We let the free variable be $u_4 = \alpha$, then find that:

$$\begin{aligned} u_1 &= u_3 = -4u_4 = -4\alpha, \\ u_2 &= 2u_3 + 14u_4 = 2(-4\alpha) + 14\alpha = 6\alpha. \end{aligned}$$

Thus, the eigenspace for the eigenvalue 2 is:

$$ES = \left\{ \alpha \begin{pmatrix} -4 \\ 6 \\ -4 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \tag{6}$$

.

Now, GM=1≠AM=1, so we need to find one generalised eigenvector such that $\vec{y} \in \ker(A-2\mathbb{I})^2 = 0$. We know that $\ker(A-2\mathbb{I})^2 = \mathbb{R}^2$, that is, there will be two vectors that form a basis, and we choose one which is linearly independent of the one found in (6). This also means that the matrix J will look like :

$$J = \begin{pmatrix} J_2(2) & & \\ & J_1(1-i) & \\ & & J_1(1+i) \end{pmatrix}.$$

Now we find the kernel of $(A-2\mathbb{I})^2$:

$$(A-2\mathbb{I})^2 = \begin{pmatrix} -2 & 0 & 0 & -8 \\ 1 & -2 & 0 & 16 \\ 0 & 1 & -2 & -14 \\ 0 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 & -8 \\ 1 & -2 & 0 & 16 \\ 0 & 1 & -2 & -14 \\ 0 & 0 & 1 & 4 \end{pmatrix},$$

$$= \begin{pmatrix} 4 & 0 & -8 & 16 \\ -4 & -4 & 16 & 24 \\ 1 & -4 & -10 & -12 \\ 0 & 1 & 2 & 2 \end{pmatrix},$$

so that,

$$\ker(A-2\mathbb{I})^2 = \begin{pmatrix} 1 & -4 & -10 & -12 & 0 \\ 4 & 0 & -8 & -16 & 0 \\ -4 & 4 & 16 & 24 & 0 \\ 0 & 1 & 2 & 2 & 0 \end{pmatrix}$$

$$\longrightarrow \; R2 = R2 - 4R1 \;\text{ and }\; R3 = R3 + 4R1,$$

$$\mapsto \begin{pmatrix} 1 & -4 & -10 & -12 & 0 \\ 0 & 16 & 32 & -32 & 0 \\ 0 & -12 & -24 & -24 & 0 \\ 0 & 1 & 2 & 2 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & -2 & -4 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we have two free variables and the two equations:

$$u_1 = 2u_3 + 4u_4,$$
$$u_2 = -2u_3 - 2u_4.$$

If we set $u_3 = \alpha$ and $u_4 = \beta$, then

$$u_1 = 2\alpha + 4\beta$$
$$u_2 = -2\alpha - 2\beta,$$

and find that the generalised eigenspace is:

$$GES = \left\{ \alpha \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2.$$

Define, say, $\vec{y} := \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix}$, then the two vectors for $\lambda = 2$ are:

$$\{\vec{u_1}, \vec{y}\} = \left\{ \begin{pmatrix} -4 \\ 6 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Next, find the eigenspace for $\lambda = 1 + i \implies \Big(A - (1+i)\mathbb{I}\Big)\vec{u_2} = \vec{0}$, which gives:

$$\implies \begin{pmatrix} -1-i & 0 & 0 & -8 & 0 \\ 1 & -1-i & 0 & 16 & 0 \\ 0 & 1 & -1-i & -14 & 0 \\ 0 & 0 & 1 & 5-i & 0 \end{pmatrix}. \tag{7}$$

However, recall that we want a real matrix J, but we have:

$$J = \begin{pmatrix} J_2(2) & & \\ & J_1(1-i) & \\ & & J_1(1+i) \end{pmatrix}.$$

We had two complex conjugate roots, $\lambda = a \pm ib = 1 \pm i$, and these can be represented as real numbers in a new matrix. Choose $\lambda = 1 + i$ and let

$$C := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then,

$$J = \begin{pmatrix} J_2(2) & \\ & C \end{pmatrix} = \begin{pmatrix} J_2(2) & & \\ & 1 & 1 \\ & -1 & 1 \end{pmatrix}.$$

Now, continue with (7) to find two vectors associate with C:

$$R2 = R2 + \frac{1}{1+i}R1 \mapsto \begin{pmatrix} -1-i & 0 & 0 & -8 & 0 \\ 0 & -1-i & 0 & 12+4i & 0 \\ 0 & 1 & -1-i & -14 & 0 \\ 0 & 0 & 1 & 5-i & 0 \end{pmatrix},$$

$$R3 = R3 + \frac{1}{1+i}R2 \mapsto \begin{pmatrix} -1-i & 0 & 0 & -8 & 0 \\ 0 & -1-i & 0 & 12+4i & 0 \\ 0 & 0 & -1-i & -6-4i & 0 \\ 0 & 0 & 1 & 5-i & 0 \end{pmatrix},$$

$$R3 \equiv R4 \mapsto \begin{pmatrix} -1-i & 0 & 0 & -8 & 0 \\ 0 & -1-i & 0 & 12+4i & 0 \\ 0 & 0 & 1 & 5-i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $R3 = (-1-i)R4$. We have one free variable, $u_4 = \alpha$, and the three equations:

$$\begin{aligned} -(1+i)u_1 &= 8u_4, \\ -(1+i)u_2 &= -(12+4i)u_4, \\ u_3 &= -(5-i)u_4, \end{aligned}$$

so let $u_4 = \alpha$, then

$$\begin{aligned} u_1 &= \frac{8}{-(1+i)}u_4, \quad u_2 = \frac{-(12+4i)}{-(1+i)}u_4, \text{ and} \\ u_3 &= -(5-i)u_4. \end{aligned}$$

Now,

$$\begin{aligned} u_1 &= \frac{-8\alpha}{1+i} = \frac{-8\alpha(1-i)}{(1+i)(1-i)} = \frac{-8\alpha+8i\alpha}{2} = -4\alpha + 4i\alpha, \\ u_2 &= \frac{(12+4i)(1-i)}{2} = \frac{\alpha}{2}(16-8i) = 8\alpha - 4\alpha i, \\ u_3 &= (i-5)\alpha = -5\alpha + i\alpha. \end{aligned}$$

Thus,

$$\vec{u_2} = \begin{pmatrix} -4\alpha + 4i\alpha \\ 8\alpha - 4\alpha i \\ -5\alpha + i\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -4 \\ 8 \\ -5 \\ 1 \end{pmatrix} + \alpha i \begin{pmatrix} 4 \\ -4 \\ 1 \\ 0 \end{pmatrix} = \alpha \operatorname{Re}(\vec{u_2}) + \alpha i \operatorname{Im}(\vec{u_2}),$$

and we choose the two vectors $\operatorname{Re}(\vec{u_2}),\ \operatorname{Im}(\vec{u_2})$.

The two vectors for $\lambda = 1 + i$ are:

$$\{\operatorname{Re}(\vec{u_2}),\ \operatorname{Im}(\vec{u_2})\} = \left\{ \begin{pmatrix} -4 \\ 8 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

So for

$$\begin{pmatrix} J_2(2) & & \\ & 1 & 1 \\ & -1 & 1 \end{pmatrix},$$

we have

$$P = [\ \vec{u_1}\ \ \vec{y}\ \ \operatorname{Re}(\vec{u_2})\ \ \operatorname{Im}(\vec{u_2})\ ] = \begin{pmatrix} -4 & 2 & -4 & 4 \\ 6 & -2 & 8 & -4 \\ -4 & 1 & -5 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Now we find the inverse of P by forming an augmented matrix and solving such that the left side is the identity.

$$(P \mid \mathbb{I}) = \left(\begin{array}{cccc cccc} -4 & 2 & -4 & 4 & 1 & 0 & 0 & 0 \\ 6 & -2 & 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 1 & -5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$\equiv \left(\begin{array}{cccc cccc} -2 & 1 & -2 & 2 & \frac{1}{2} & 0 & 0 & 0 \\ 3 & -1 & 4 & -2 & 0 & \frac{1}{2} & 0 & 0 \\ -4 & 1 & -5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$\equiv \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & -2 & 2 & \frac{1}{2} & 0 & 0 & 0 \\ 3 & -1 & 4 & -2 & 0 & \frac{1}{2} & 0 & 0 \\ -4 & 1 & -5 & 1 & 0 & 0 & 1 & 0 \end{array}\right),$$

and let

$$\begin{aligned} R2 &= R2 + 2R1, \\ R3 &= R3 - 3R1, \\ R4 &= R4 + 4R1, \end{aligned}$$

which gives:

$$\mapsto \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & \frac{1}{2} & 0 & 0 & 2 \\ 0 & -1 & 1 & -2 & 0 & \frac{1}{2} & 0 & -3 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 4 \end{array}\right).$$

Next, let

$$\begin{aligned} R2 &= R2 + R3, \\ R4 &= R4 + R3, \end{aligned}$$

$$\mapsto \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & -1 & 1 & -2 & 0 & \frac{1}{2} & 0 & -3 \\ 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & 1 & 1 \end{array}\right).$$

Next find

$$\begin{aligned} R1 &= R1 - R2, \\ R3 &= R3 - R2, \end{aligned}$$

which gives:

$$\mapsto \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & -1 & 0 & -2 & -\frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & 1 & 1 \end{array}\right),$$

and lastly find

$$R3 = R3 - 2R4:$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & -1 & 0 & 0 & -\frac{1}{2} & -1 & -2 & -4 \\ 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & 1 & 1 \end{pmatrix},$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & -1 & -1 \end{pmatrix}.$$

And so, after rearranging the rows to be the identity, we have $(\mathbb{I} \mid P^{-1})$ where P^{-1} is:

$$P^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & -\frac{1}{2} & -1 & -1 \end{pmatrix}.$$

Thus, the real canonical form of A is

$$J = P^{-1}AP = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

5 Question 5

For each matrix A, compute the matrix exponential e^{At} for some $t \in \mathbb{R}$:

5.1 a) 3×3 Matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Now, to find e^{At} we need to find the similarity transformation of A. Recall that $A = PDP^{-1}$. It follows that $e^{At} = Pe^{Jt}P^{-1}$ where J is the diagonal matrix of eigenvalues or it is the Jordan normal form of A. To determine whether we have a diagonal or Jordan matrix, we need to find the eigenvalues and corresponding eigenvectors. If the geometric multiplicty is equal to the algebraic multiplicity, then we will have a diagonal matrix; and if it less than, then we will have a Jordan matrix. We, therefore, begin by finding the eigenvalues:

$$\det(A - \lambda\mathbb{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(2-\lambda) = 0,$$

$\implies \lambda = 1, 2$ with algebraic multiplicity (AM) being one and two respectively.
Now find the eigenvectors:
Case $\lambda = 1$:

$$(A - \lambda\mathbb{I})\vec{u_1} = (A - \mathbb{I})\vec{u_1} = \vec{0}$$

$$\implies \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow R2 \mapsto R2 + R3 : \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right).$$

Thus we have the equations $u_2 = -u_3$ & $u_1 = -u_3$, so let $u_3 = \alpha$ ($\alpha \in \mathbb{R}$) then $u_1 = u_2 = -\alpha$ and the vector is therefore:

$$\vec{u_1} = \alpha \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\vec{u_1'} :=},$$

which has geometric multiplicity, GM, = AM = 1.

Case $\lambda = 2$:

$$(A - \lambda\mathbb{I})\vec{u_2} = (A - 2\mathbb{I})\vec{u_2} = \vec{0}$$

$$\implies \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This implies that $u_1 = 0$ and that we have two free variables: let $u_2 = \alpha$ and $u_3 = \beta$. Then

$$\vec{u_2} = \alpha \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v_2} :=} + \beta \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v_3} :=} \quad : \quad \alpha, \beta \in \mathbb{R}.$$

We have that for the second eigenvalue AM = GM = 2. Thus we will have a diagonal matrix $J = D$ such that

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and we have the matrix P given as $P = [\vec{u_1}' \ \vec{v_2} \ \vec{v_3}]$:

$$P = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We need to find P^{-1} such that $PP^{-1} = \mathbb{I}$. We will use the adjoint method, that is, $P^{-1} = \frac{1}{\det(P)}\text{Adj}(P)$. Therefore,

$$\det(P) = -1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1(1) = -1 \neq 0, \text{ (inverse exists).}$$

Take the transpose of P, find the minors and construct the adjoint matrix:

$$P^T = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ gives:}$$

$$
\begin{aligned}
&|P^T_{11}| = 1, \;\; |P^T_{12}| = 0, \;\; |P^T_{13}| = 0,\\
&|P^T_{21}| = -1 \;\; |P^T_{22}| = -1, \;\; |P^T_{23}| = 0,\\
&|P^T_{31}| = -1, \;\; |P^T_{32}| = 0, \; \& \; |P^T_{33}| = -1.
\end{aligned}
$$

So the matrix of minors is,

$$
P_{\min} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix},
$$

and multiplying by the checkerboard pattern of pluses and minuses gives:

$$
\mathrm{Adj}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix},
$$

Now,

$$
P^{-1} = \frac{1}{-1}\mathrm{Adj}(P) = -\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \;\; (= P).
$$

Now, the matrix exponential is:

$$
\begin{aligned}
e^{At} &= e^{PDP^{-1}t} = Pe^{Dt}P^{-1} = P\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} P^{-1} = \begin{pmatrix} -e^t & 0 & 0 \\ -e^t & e^{2t} & 0 \\ e^t & 0 & e^{2t} \end{pmatrix} P^{-1}\\
&= \begin{pmatrix} e^t & 0 & 0 \\ (e^t - e^{2t}) & e^{2t} & 0 \\ (e^{2t} - e^t) & 0 & e^{2t} \end{pmatrix}.
\end{aligned}
$$

5.2 b) 4×4 Matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

We follow the same method as in part a), find the eigenvalues, eigenvectors, construct new matrices (eigendecomposition of A) and find the matrix exponential. Eigenvalues are found as:

$$\det(A - \lambda \mathbb{I}) = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & -1 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda \Bigg\{ -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & -\lambda \\ -1 & 1 \end{vmatrix} \Bigg\}$$

$$\implies \lambda^4 = 0,$$

so we have the eigenvalues zero with AM $= 4$. Next, we find the eingenvector(s) associated with this value:

$$(A - \lambda \mathbb{I})\vec{u} = \vec{0} \implies A\vec{u} = \vec{0}:$$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

giving $u_1 + u_4 = 0$ and $u_3 - u_2 = 0$. We have therefore two free variables: let $u_4 = \alpha$ and $u_3 = \beta$. Then $u_1 = -u_4 = -\alpha$ and $u_2 = u_3 = \beta$ and

$$\vec{u} = \begin{pmatrix} -\alpha \\ \beta \\ \beta \\ \alpha \end{pmatrix} \equiv \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Also note since GM $\neq$ AM, we have that $A = PJP^{-1}$, where

$$J = \begin{pmatrix} J_1(0) & 0 \\ 0 & J_3(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, $(\vec{u}) \in \ker(A) = \mathbb{R}^2$, $\ker(A^2) = \mathbb{R}^3$ and $\ker(A^3) = \mathbb{R}^4$. We have to find three generalised eigenvectors out of four required to construct P (the first vector in P will be $\vec{u}$). Since $\ker(A^3) = \mathbb{R}^4$ we choose a vector in its basis say we take

$$\vec{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and we calculate the following:

$$\begin{aligned} A\vec{v} &= \vec{0}, \\ A\vec{y} &= \vec{v}, \quad \text{and} \\ A\vec{w} &= \vec{y}, \end{aligned}$$

and take the three vectors $[\ \vec{v}\ \ \vec{y}\ \ \vec{w}\ ]$. Thus,

if $\vec{w} \in \ker(A^3)$ then

$$\vec{y} = A\vec{w} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

$$\vec{v} = A\vec{y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix},$$

and

$$A\vec{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} = \vec{0}.$$

So,

$$\vec{w}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \vec{y}=\begin{pmatrix}0\\0\\0\\-1\end{pmatrix},\ \vec{v}=\begin{pmatrix}0\\-1\\-1\\0\end{pmatrix},\ \text{ and }\ \vec{u}=\begin{pmatrix}-1\\1\\1\\1\end{pmatrix},$$

giving us $P=[\ \vec{u}\ \ \vec{v}\ \ \vec{y}\ \ \vec{w}\ ]$:

$$P=\begin{pmatrix}-1&0&0&0\\1&-1&0&1\\1&-1&0&0\\1&0&-1&0\end{pmatrix}$$

To find P^{-1} we solve the augmented matrix such that the left side becomes the identity matrix and the right become the desired inverse. We begin from :

$$(P\mid\mathbb{I})=\begin{pmatrix}-1&0&0&0&1&0&0&0\\1&-1&0&1&0&1&0&0\\1&-1&0&0&0&0&1&0\\1&0&-1&0&0&0&0&1\end{pmatrix}$$

Let

$$\begin{aligned}R2&=R2+R1,\\R3&=R3+R1,\\R4&=R4+R1,\end{aligned}$$

then we have,

$$\begin{pmatrix}-1&0&0&0&1&0&0&0\\0&-1&0&1&1&1&0&0\\0&-1&0&0&1&0&1&0\\0&0&-1&0&1&0&0&1\end{pmatrix}\equiv\begin{pmatrix}1&0&0&0&-1&0&0&0\\0&1&0&-1&-1&-1&0&0\\0&-1&0&0&1&0&1&0\\0&0&-1&0&1&0&0&1\end{pmatrix},$$

let $R2=R2+R3$, then:

$$\begin{pmatrix}1&0&0&0&-1&0&0&0\\0&0&0&-1&0&-1&1&0\\0&-1&0&0&1&0&1&0\\0&0&-1&0&1&0&0&1\end{pmatrix}.$$

Rearranging the columns gives:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \end{array}\right) \equiv \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{array}\right)$$

Thus, P^{-1} is given on the right side of the augmented matrix:

$$P^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Now, since $A = PJP^{-1}$ we have $e^{At} = e^{PJP^{-1}t} = Pe^{Jt}P^{-1}$ where e^{Jt} is given as:

$$e^{Jt} = \begin{pmatrix} e^{J_1(0)t} & 0 \\ 0 & e^{J_3(0)t} \end{pmatrix}, \text{ where } J_1(0) = \begin{pmatrix} 0 \end{pmatrix}, \; J_3(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} e^{J_1(0)t} &= e^{0t} = 1, \text{ and} \\ e^{J_3(0)t} &= e^{(\lambda \mathbb{I}_{3\times 3} + K_3)t}, \text{ where} \end{aligned}$$

$$J_3(0) = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_3} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{K_3}.$$

Thus, since $e^{(\lambda \mathbb{I}_{3\times 3} + K_3)t} = e^{\lambda \mathbb{I}_{3\times 3} t} e^{K_3 t} = e^{0 \mathbb{I}_{3\times 3} t} e^{K_3 t} = e^{K_3 t}$ we have the following, noting that $K^n = 0$ for $n \geq 3$:

$$e^{Kt} = \mathbb{I} + Kt + \frac{1}{2!}K^2t^2 + \frac{1}{3!}K^3t^3 + \ldots \ ,$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ \ldots \ ,$$

$$= \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$e^{J_3(0)t} = e^{K_3t} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}$$

$$e^{Jt} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We therefore have that

$$e^{At} = Pe^{Jt}P^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -t & (1 - \frac{t^2}{2}) \\ 1 & -1 & -t & -\frac{t^2}{2} \\ 1 & 0 & -1 & -t \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 - \frac{t^2}{2} & \frac{t^2}{2} & t \\ t & -\frac{t^2}{2} & (\frac{t^2}{2} + 1) & t \\ 0 & -t & t & 1 \end{pmatrix}.$$

6 Question 6

A coupled spring-mass system consisting of three masses attached to each other by four springs is modelled by the equations:

$$M\frac{\mathrm{d}^2\vec{x}(t)}{\mathrm{d}t^2} = K\vec{x}(t), \tag{8}$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \; M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

$$\text{and } \; K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

6.1 a) Solving for the displacement

We will solve for the displacement, $\vec{x}(t)$, where $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ k_1 = 2,\ k_2 = 1,\ k_3 = 1$ and $k_4 = 2$. This gives us the matrices:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \& \; K = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}.$$

To solve (8) we need to manipulate the expression so that we have:

$$M\frac{\mathrm{d}^2\vec{x}(t)}{\mathrm{d}t^2} = K\vec{x}(t) \Longrightarrow \frac{\mathrm{d}^2\vec{x}(t)}{\mathrm{d}t^2} = M^{-1}K\vec{x}(t),$$

but only if $\det(M) \neq 0$. We quickly see, however, that $M = \mathbb{I} \Longrightarrow M^{-1} = \mathbb{I}^{-1} = \mathbb{I}$, and thus the determinant is non-zero (it is one). This then leaves us to solve the second-order system of ordinary differential equations below:

$$\frac{\mathrm{d}^2\vec{x}(t)}{\mathrm{d}t^2} = K\vec{x}(t) \Longrightarrow \frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

As with the previous questions, we aim to diagonalize K such that $K = PDP^{-1}$. We therefore begin by finding the characteristic equation of K and the roots, eigenvalues, of it. We have

$$\det(K - \lambda\mathbb{I}) = \begin{vmatrix} (-3-\lambda) & 1 & 0 \\ 1 & (-2-\lambda) & 1 \\ 0 & 1 & (-3-\lambda) \end{vmatrix} = 0,$$

$$\begin{aligned}
&\Longrightarrow (-3-\lambda)\begin{vmatrix} (-2-\lambda) & 1 \\ 1 & (-3-\lambda) \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & (-3-\lambda) \end{vmatrix} = 0, \\
&\Longrightarrow (-3-\lambda)\big\{(-2-\lambda)(-3-\lambda) - 1\big\} - 1(-3-\lambda) = 0, \\
&\Longrightarrow -(3+\lambda)(\lambda^2 + 5\lambda + 5) + \lambda + 3 = 0, \\
&\Longrightarrow \lambda^3 + 8\lambda^2 + 19\lambda + 12 = 0, \\
&\Longrightarrow (\lambda+4)(\lambda+3)(\lambda+1) = 0,
\end{aligned}$$

and so, the eigenvalues are $\lambda = -1,\ -3$, and -4. Next, find the corresponding eigenvectors. These satisfy $(K - \lambda\mathbb{I})\vec{u_i} = \vec{0}$.

Beginning with $\lambda = -1$:

$(K + \lambda\mathbb{I})\vec{u_1} = (K + \mathbb{I})\vec{u_1} = \vec{0}$

$$\Longrightarrow \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \longrightarrow R1 \mapsto R1 + 2R2 : \begin{pmatrix} 0 & -1 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix},$$

$$R1 \equiv R3 \Longrightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have the equations $u_1 - u_2 + u_3 = 0$ & $u_2 - 2u_3 = 0$, so let $u_2 = \alpha$ ($\alpha \in \mathbb{R}$) then $u_3 = \frac{1}{2}u_2 = \frac{1}{2}\alpha$ and $u_1 = u_2 - u_3 = \alpha - \frac{1}{2}\alpha = \frac{1}{2}\alpha$ and the vector is therefore:

$$\vec{u_1} \equiv \alpha \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{\vec{u_1'} :=}.$$

Next, $\lambda = -3$:

$(K + 3\mathbb{I})\vec{u_2} = \vec{0}$

$$\Longrightarrow \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ 0 & 1 & 0 & & 0 \end{pmatrix} \longrightarrow R1 \equiv R3 : \begin{pmatrix} 1 & 1 & 1 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \end{pmatrix},$$

Thus we have the equations $u_2 = 0$, $u_3 = -u_1$, so let $u_1 = \alpha$ then $u_3 = -u_1 = -\alpha$ and the vector is therefore:

$$\vec{u_2} \equiv \alpha \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{u_2'} :=}.$$

Lastly for $\lambda = -4$:

$(K + 4\mathbb{I})\vec{u_3} = \vec{0}$

$$\Longrightarrow \begin{pmatrix} 1 & 1 & 0 & & 0 \\ 1 & 2 & 1 & & 0 \\ 0 & 1 & 1 & & 0 \end{pmatrix} \longrightarrow R2 = R2 - R1 : \begin{pmatrix} 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & & 0 \\ 0 & 1 & 1 & & 0 \end{pmatrix},$$

$$R2 \equiv R3 : \begin{pmatrix} 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & & 0 \\ 0 & 0 & 0 & & 0 \end{pmatrix}.$$

And, thus we have the equations $u_1 = -u_2$, $u_3 = -u_2$, so let $u_2 = \alpha$ then the vector is:

$$\vec{u_3} \equiv \alpha \underbrace{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}_{\vec{u_3'} :=}.$$

So, for the eigenvalues $\lambda = -1, \ -3, \ -4$ we have the corresponding eigenvectors:

$$\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix} \right\},$$

and therefore we construct the P-matrix as:

$$P = \left[\ \vec{u_1}'\ \ \vec{u_2}'\ \ \vec{u_3}'\ \right] = \begin{pmatrix}1 & 1 & -1\\2 & 0 & 1\\1 & -1 & -1\end{pmatrix},$$

with

$$P^{-1} = \begin{pmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{1}\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\end{pmatrix}, \text{ and } D = \begin{pmatrix}-1 & 0 & 0\\0 & -3 & 0\\0 & 0 & -4\end{pmatrix}$$

such that $K = PDP^{-1}$.

Recall that we wanted to solve the system $\vec{x}\,''(t) = M^{-1}K\vec{x}(t) = K\vec{x}(t)$. Make a change of variables: let $\vec{x} = P\vec{y} \Longrightarrow \vec{x}\,'' = P\vec{y}\,''$ and $\Longrightarrow \vec{y} = P^{-1}\vec{x}$. So,

$$\begin{aligned}\vec{x}\,'' = K\vec{x} &= PDP^{-1}\vec{x},\\ &\Longrightarrow P\vec{y}\,'' = PD\,\underbrace{P^{-1}\vec{x}}_{\vec{y}},\\ &\Longrightarrow P\vec{y}\,'' = PD\vec{y},\\ &\Longrightarrow P^{-1}P\vec{y}\,'' = D\vec{y},\\ &\Longrightarrow \vec{y}\,'' = D\vec{y},\end{aligned}$$

and so the system we thus solve is:

$$\frac{\mathrm{d}^2\vec{y}(t)}{\mathrm{d}t^2} = \begin{pmatrix}-1 & 0 & 0\\0 & -3 & 0\\0 & 0 & -4\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}-y_1\\-3y_2\\-4y_3\end{pmatrix}.$$

This is an uncoupled system and we can solve each of the ODEs separately and directly. Let us also define some general initial conditions:

$$\vec{y_0} = P^{-1}\vec{x_0} := \begin{pmatrix}a\\b\\c\end{pmatrix} \quad \& \quad \vec{y_0}' = P^{-1}\vec{x_0}' := \begin{pmatrix}d\\e\\f\end{pmatrix}, \text{ where } a,b,c,d,e,f\ \in \mathbb{R}.$$

We solve the ODEs individually in order:

$$y_1''(t) = -y_1 \implies m^2 = -1 \implies m = \pm i, \text{ by the auxiliary equation, thus,}$$
$$\implies y_1(t) = A\cos(t) + B\sin(t),$$

and where the constants A, B are given as $A = a,\ B = d$, using the initial conditions, so that

$$y_1(t) = a\cos(t) + d\sin(t).$$

Next,

$$y_2''(t) = -3y_2 \implies m^2 = -3 \implies m = \pm i\sqrt{3}, \quad \text{thus,}$$
$$\implies y_2(t) = C\cos(\sqrt{3}t) + D\sin(\sqrt{3}t),$$

and where the constants C, D are given as $C = b,\ D = e$, by the initial conditions, so that

$$y_2(t) = b\cos(\sqrt{3}t) + e\sin(\sqrt{3}t).$$

Lastly,

$$y_3''(t) = -4y_2 \implies m^2 = -4 \implies m = \pm 2i, \quad \text{thus,}$$
$$\implies y_3(t) = E\cos(2t) + F\sin(2t),$$

and where the constants E, F are given as $E = c,\ F = f$, using, again, the initial conditions, so that we have

$$y_3(t) = c\cos(2t) + f\sin(2t).$$

Which gives us the solution in $\vec{y}$:

$$\vec{y}(t) = \begin{pmatrix} a\cos(t) + d\sin(t) \\ b\cos(\sqrt{3}t) + e\sin(\sqrt{3}t) \\ c\cos(2t) + f\sin(2t) \end{pmatrix}.$$

Now, $\vec{x} = P\vec{y}$ so that the solution for the displacement is:

$$\vec{x}(t) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \vec{y}(t) = \begin{pmatrix} y_1 + y_2 - y_3 \\ 2y_1 + y_3 \\ y_1 - y_2 - y_3 \end{pmatrix},$$

$$= \begin{pmatrix} a\cos(t) + d\sin(t) + b\cos(\sqrt{3}t) + e\sin(\sqrt{3}t) - (c\cos(2t) + f\sin(2t)) \\ 2(a\cos(t) + d\sin(t)) + c\cos(2t) + f\sin(2t) \\ a\cos(t) + d\sin(t) - (b\cos(\sqrt{3}t) + e\sin(\sqrt{3}t)) - (c\cos(2t) + f\sin(2t)) \end{pmatrix}.$$

6.2 b) Plotting the trajectories of the solution

Recall that the solution for the displacement is given as the vector $\vec{x}(t)$:

$$\begin{pmatrix} a\cos(t) + d\sin(t) + b\cos(\sqrt{3}t) + e\sin(\sqrt{3}t) - (c\cos(2t) + f\sin(2t)) \\ 2(a\cos(t) + d\sin(t)) + c\cos(2t) + f\sin(2t) \\ a\cos(t) + d\sin(t) - (b\cos(\sqrt{3}t) + e\sin(\sqrt{3}t)) - (c\cos(2t) + f\sin(2t)) \end{pmatrix},$$

with the two initial conditions:

$$\vec{x_0} = \vec{x}(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \text{and } \vec{x_0}' = \vec{x}\,'(0) = \begin{pmatrix} 6+9 \\ 0 \\ 6+9 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \\ 15 \end{pmatrix}.$$

Applying them in order gives:

$$\vec{x}(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} a+b-c \\ 2a+c \\ a-b-c \end{pmatrix}, \tag{9}$$

$$\vec{x}\,'(0) = \begin{pmatrix} 15 \\ 0 \\ 15 \end{pmatrix} = \begin{pmatrix} d+\sqrt{3}e-2f \\ 2d+2f \\ d-e\sqrt{3}-2f \end{pmatrix}. \tag{10}$$

Solving for (9), we have the three equations:

$$\begin{aligned} a+b-c &= 3 \quad - \; R1, \\ 2a+c &= 3 \quad - \; R2, \\ a-b-c &= 3 \quad - \; R3, \end{aligned}$$

and solving gives:

$$\begin{gathered} R1+R2 \mapsto \;\; 3a+b=6, \\ R2+R3 \mapsto \;\; 3a-b=6, \\ \Longrightarrow \text{let } b = 6-3a, \text{ then } 3a-6+3a=6 \Longrightarrow a=2, \\ \Longrightarrow b = 6-3(2) = 0,\; c = 3-2(a) = 3-4 = -1, \end{gathered}$$

$$\Longrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

Solving for (10), we have that the three equations are:

$$\begin{aligned} d + \sqrt{3}e - 2f = 15 \qquad &- \ R1, \\ 2d + 2f = 0 \qquad &- \ R2, \\ d - e\sqrt{3} - 2f = 15 \qquad &- \ R3, \end{aligned}$$

and solving gives:

$$\begin{aligned} R2:\ 2f = -2d, \\ \implies R1 = d + \sqrt{3}e + 2d = 15,\ \&\ R3 = d - e\sqrt{3} + 2d = 15 \\ R1 + R3 \mapsto 2d + 4d = 30 \implies d = 5,\ \&\ f = -d \implies f = -5, \\ \implies \sqrt{3}e = 15 - d + 2f \implies \frac{1}{\sqrt{3}}(15 - 5 + 2(-5)) = 0, \end{aligned}$$

$$\implies \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}.$$

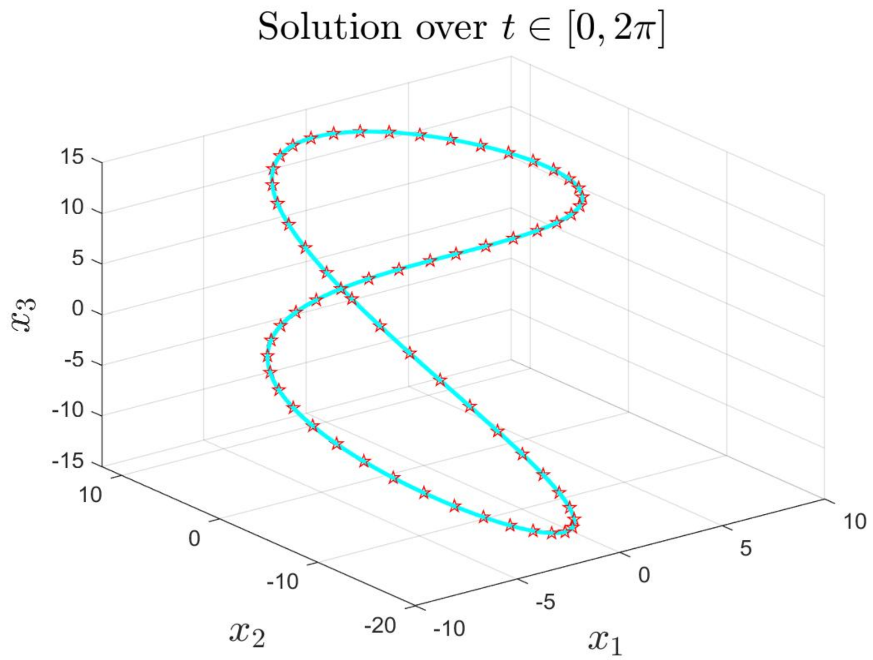
The solution for the displacement $\vec{x}$ is therefore

$$\vec{x}(t) = \begin{pmatrix} 2\cos(t) + 5\sin(t) - (-1\cos(2t) - 5\sin(2t)) \\ 2(2\cos(t) + 5\sin(t)) + (-1\cos(2t) - 5\sin(2t)) \\ 2\cos(t) + 5\sin(t) - (-1\cos(2t) - 5\sin(2t)) \end{pmatrix},$$

$$= \begin{pmatrix} 2\cos(t) + 5\sin(t) + 1\cos(2t) + 5\sin(2t) \\ 4\cos(t) + 10\sin(t)) - 1\cos(2t) - 5\sin(2t) \\ 2\cos(t) + 5\sin(t) + 1\cos(2t) + 5\sin(2t) \end{pmatrix}.$$

We now plot the 3-dimensional trajectory of the particular solution $\vec{x}(t)$:

The solution is plotted for $t \in [0,\ 2\pi]$ and gives the following:



(b)