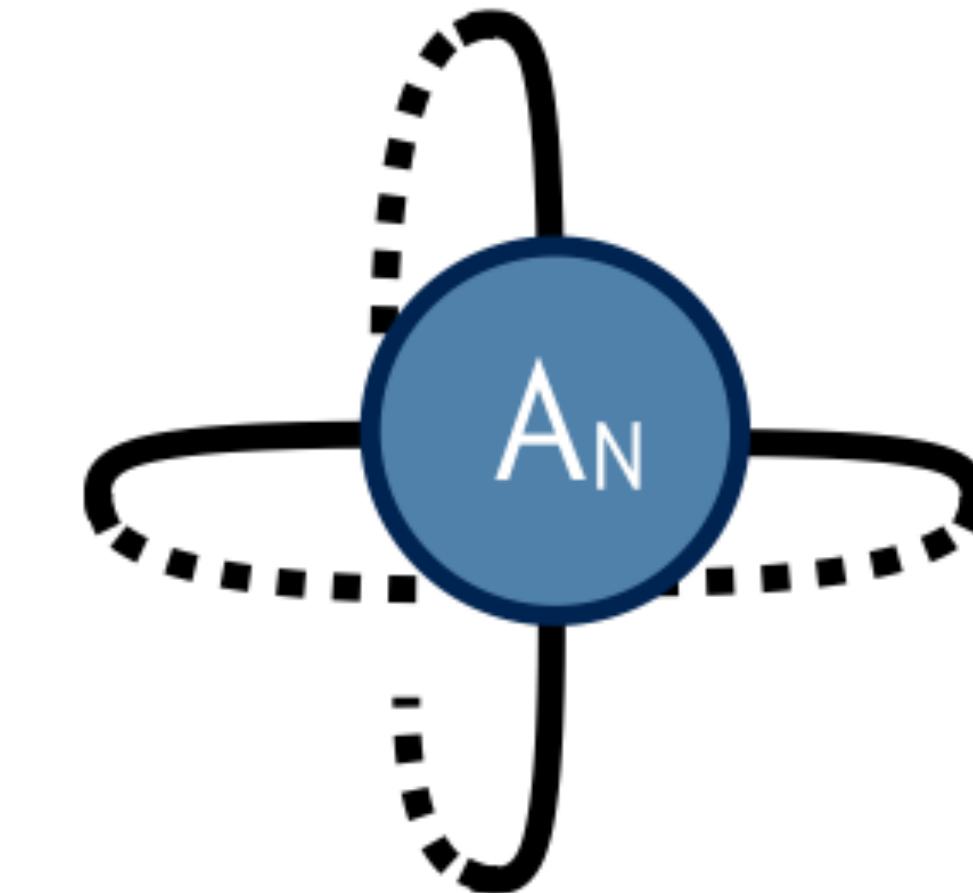
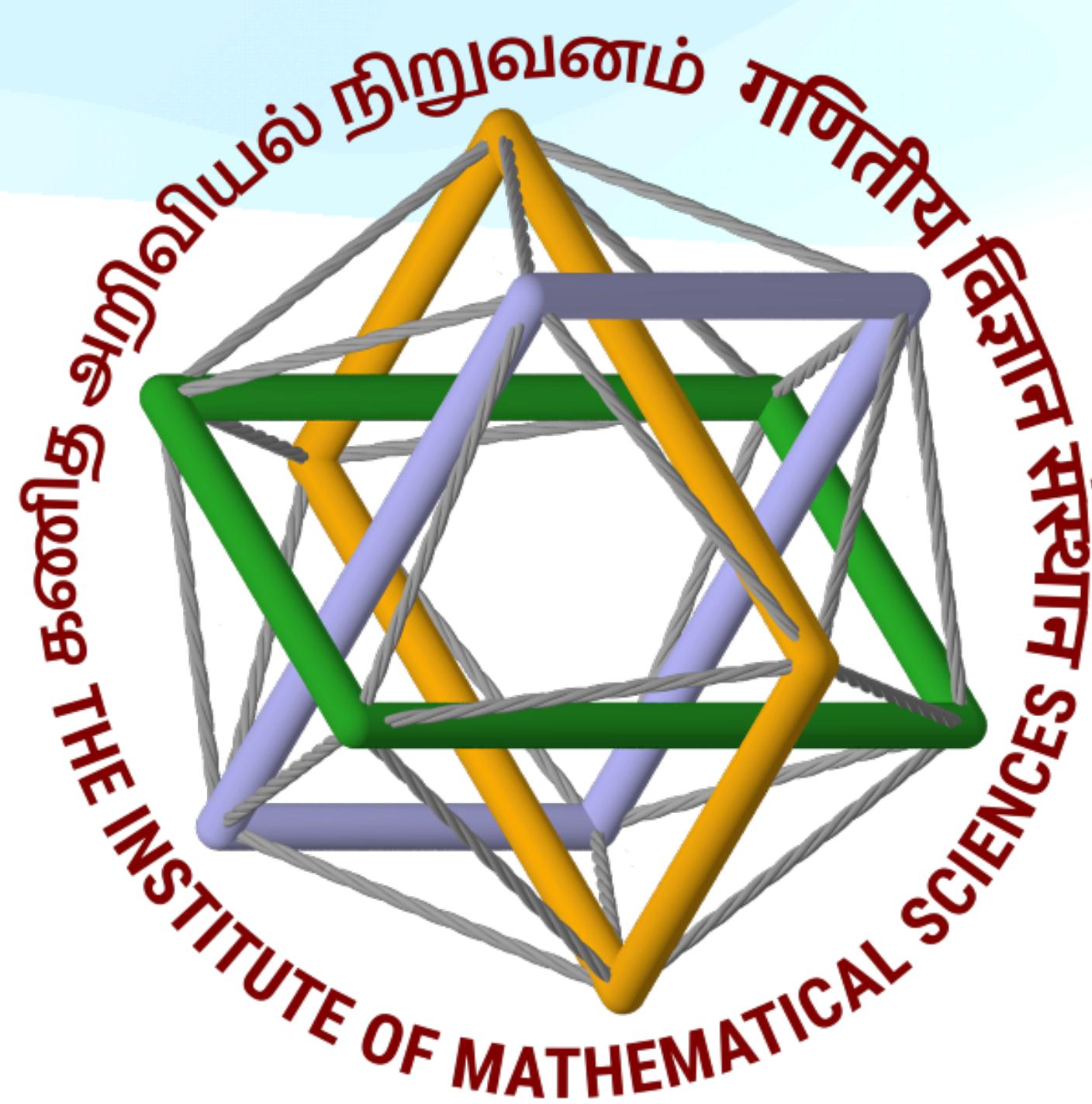
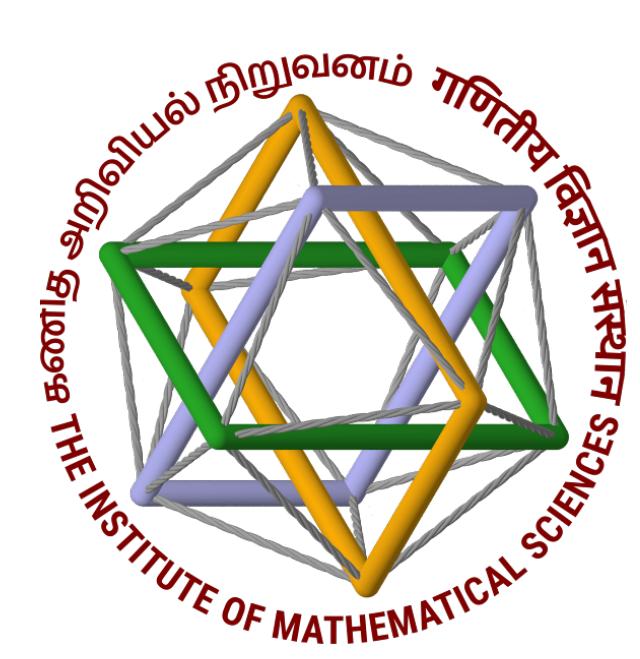


# JC Talk

## Lattice Gauge Theories And Tensors



<https://navdeep-dhindsa.github.io/>  
navdeep.s.dhindsa@gmail.com



arXiv:2312.16167v1 [hep-lat] 26 Dec 2023

# Tensor network representation of non-abelian gauge theory coupled to reduced staggered fermions

Muhammad Asaduzzaman<sup>\*</sup> and Yannick Meurice<sup>†</sup>

*Department of Physics and Astronomy,*

*The University of Iowa, Iowa City, Iowa 52242, USA*

Simon Catterall<sup>‡</sup> and Goksu Can Toga<sup>§</sup>

*Department of Physics, Syracuse University, Syracuse, NY 13244, USA*

Ryo Sakai<sup>¶</sup>

*Jij Inc., Bunkyo-ku, Tokyo 113-0031, Japan and*

*Department of Physics, Syracuse University, Syracuse, NY 13244, USA*

(Dated: December 27, 2023)

## Abstract

We show how to construct a tensor network representation of the path integral for reduced staggered fermions coupled to a non-abelian gauge field in two dimensions. The resulting formulation is both memory and computation efficient because reduced staggered fermions can be represented in terms of a minimal number of tensor indices while the gauge sector can be approximated using Gaussian quadrature with a truncation. Numerical results obtained using the Grassmann TRG algorithm are shown for the case of  $SU(2)$  lattice gauge theory and compared to Monte Carlo results.

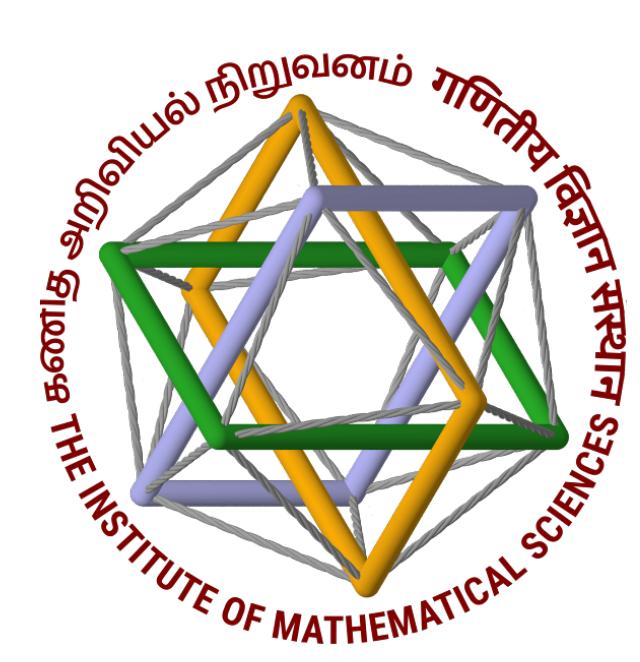


<sup>\*</sup> muhammad-asaduzzaman@uiowa.edu

<sup>†</sup> yannick-meurice@uiowa.edu

<sup>‡</sup> smcatter@syr.edu

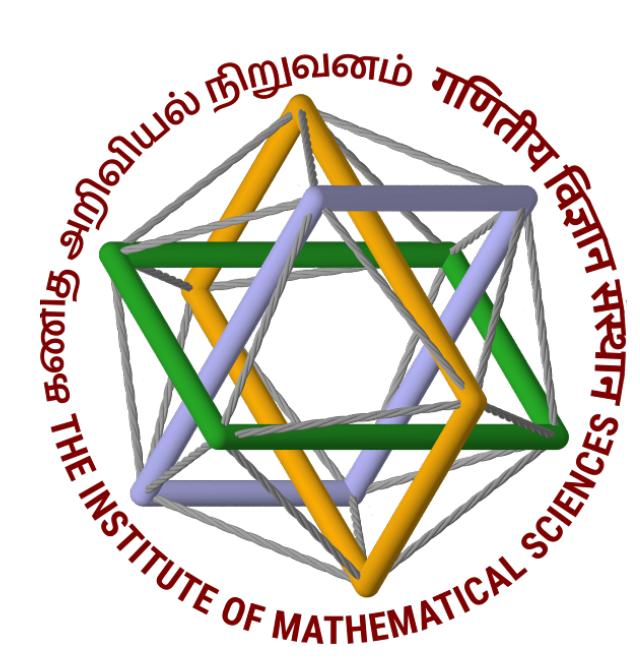
<sup>§</sup> gctoga@syr.edu



## Before we discuss Tensor networks ...

- ◀ Discretizing gauge theories on a spacetime lattice allows for powerful numerical simulations using MCMC methods.
- ◀ Despite the great success, infamous sign problem restricts its usage in certain coupling regimes.

Last JC talk by Sayak



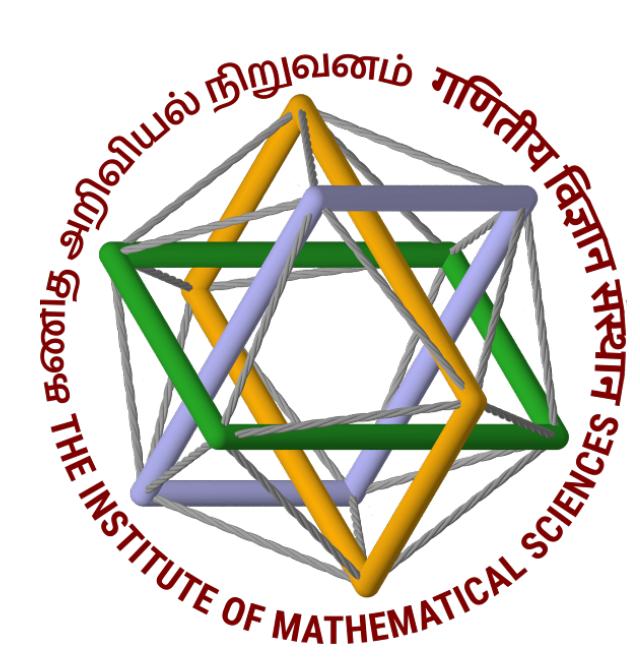
# Sign Problem

Say  $S_1, S_2$  be two action values corresponding to two configurations  $C_1, C_2$

If  $S_1, S_2 \in \mathbb{R}$ , we can comment on which is bigger  $e^{-S_1}, e^{-S_2}$

If  $S_1, S_2 \in \mathbb{C}$ , can we comment on which is bigger ?  $e^{-S_1}, e^{-S_2}$

Which contribute to path integral more ?



# Sign Problem

$$S = \int d\tau \left( -\frac{1}{2} \phi \partial_\tau^2 \phi + \bar{\psi} \partial_\tau \psi + \bar{\psi} W''(\phi) \psi + \frac{1}{2} [W'(\phi)]^2 \right)$$

Effective Action

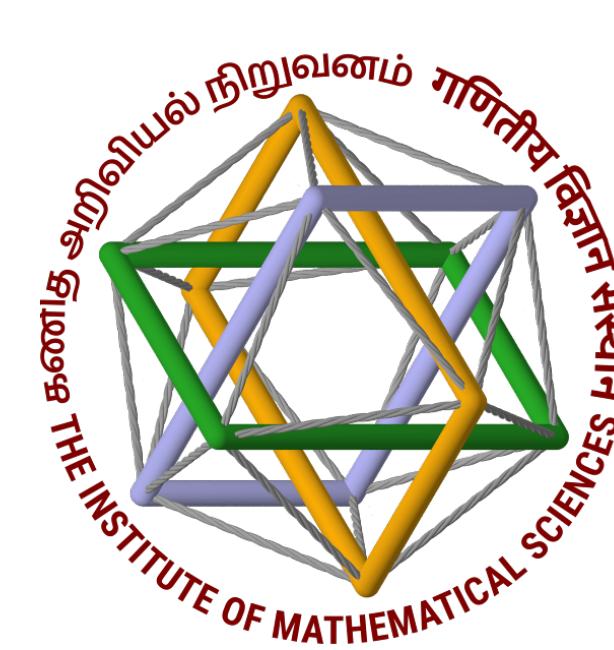
$$S = S_B - \log(\det(M))$$

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_B - S_F}$$

QCD at finite chemical potential

Integrating out fermions

$$\mathcal{Z} = \int \mathcal{D}\phi \det(M) e^{-S_B}$$



# Alternatives

Quantum

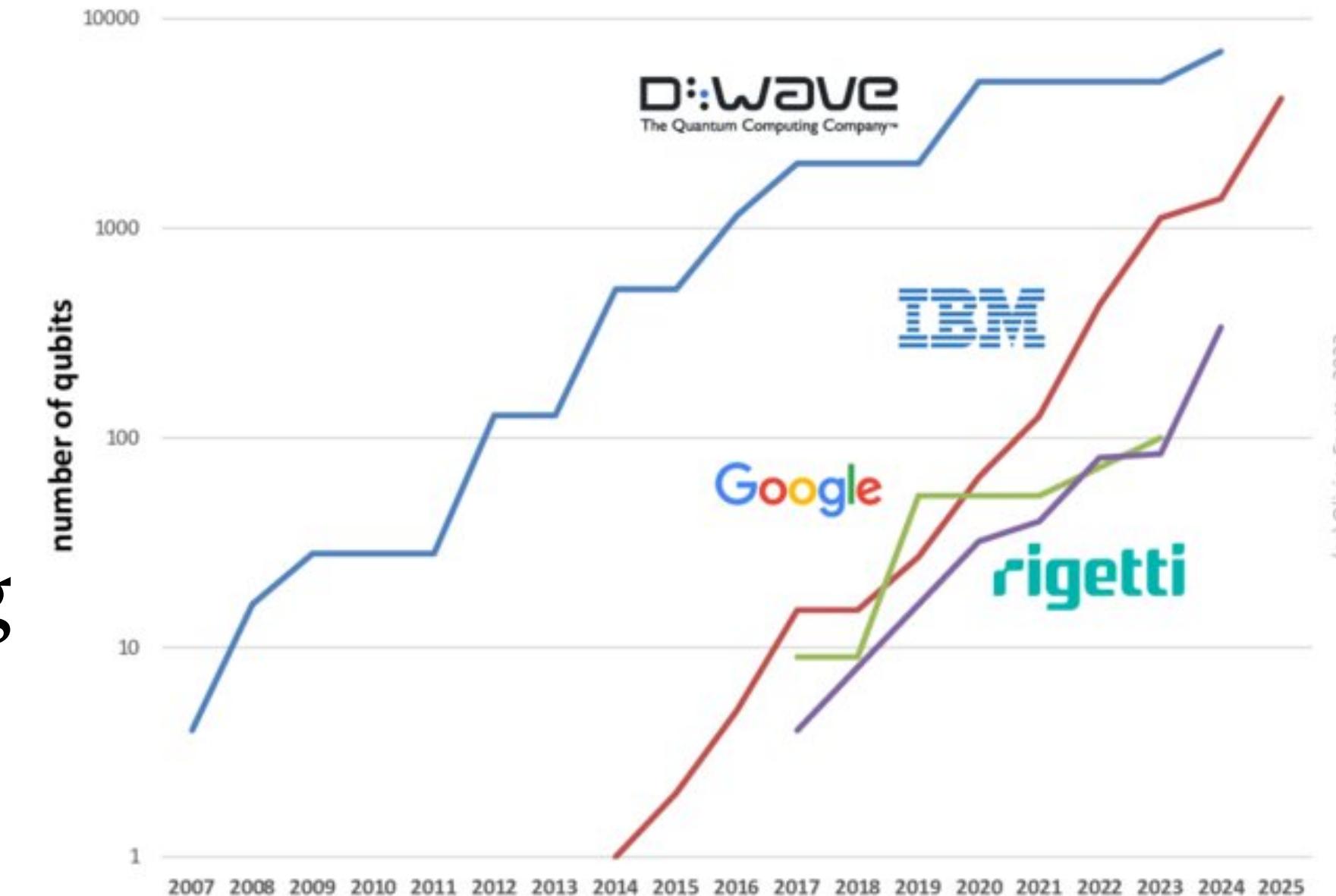
Why Quantum Computation - Real time simulations  
with MC or classical simulation methods challenging

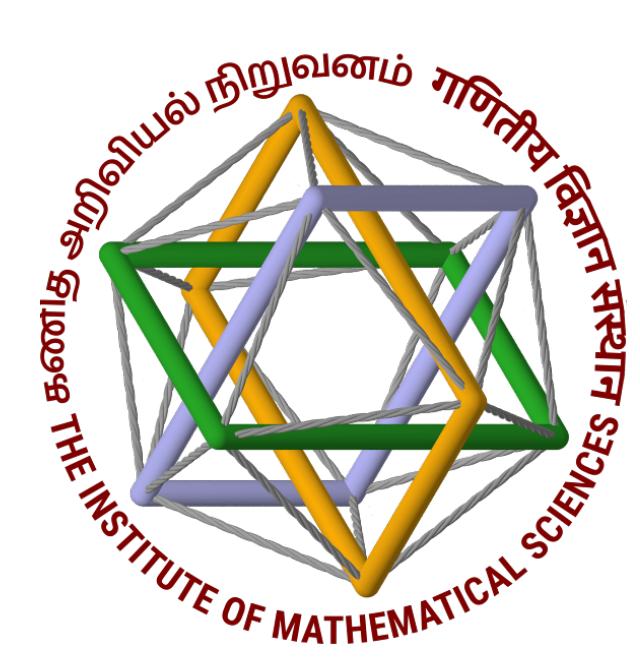
Qubit based approach highly used at the moment to  
study Lattice gauge theories using QC

How many qubits do we need to simulate even a sub parameter space of QCD ?

How long will it take to reach a sufficient number ?

Credit: Olivier Ezratty



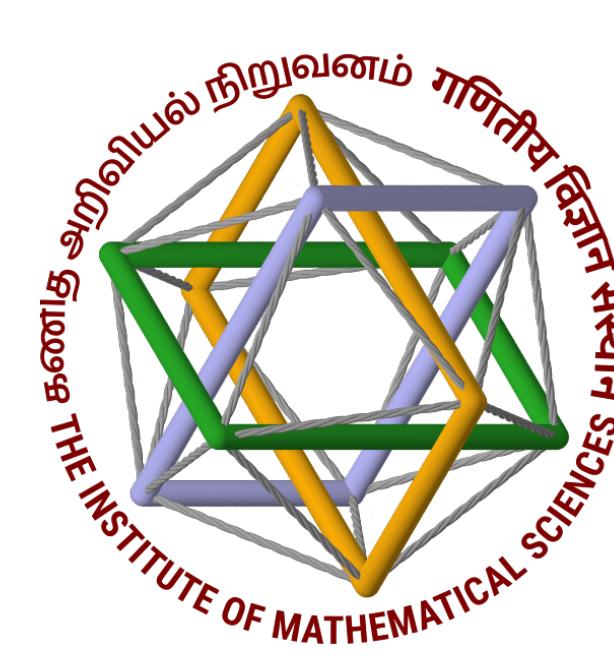


# Alternatives

Classical

- ▶ Phase quenched Monte Carlo
- ▶ Complex Langevin
- ▶ Lefschetz Thimble
- ▶ **Tensor Networks**

Interested in  
Numerical Bootstrap ?

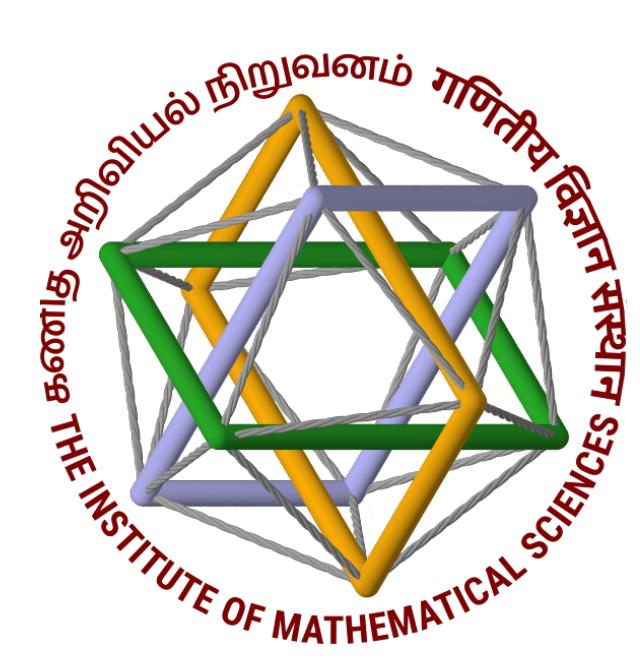


# Tensor Networks

- Tool to represent wave functions of quantum many body systems.
- Partition functions of target models can be visualized as an assembly obtained:
  - by “wiring” together objects carrying multiple “legs”
  - attached to the sites, links or plaquettes of a Euclidean space-time lattice

<https://www.tensors.net/> — *Introduction*

<https://tensornetwork.org/trg/> — *TRG (focus of the day)*

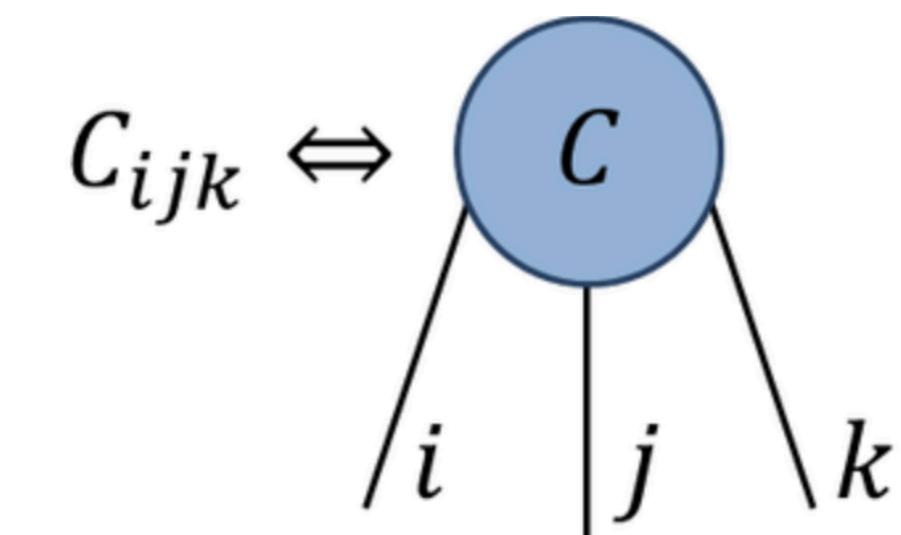
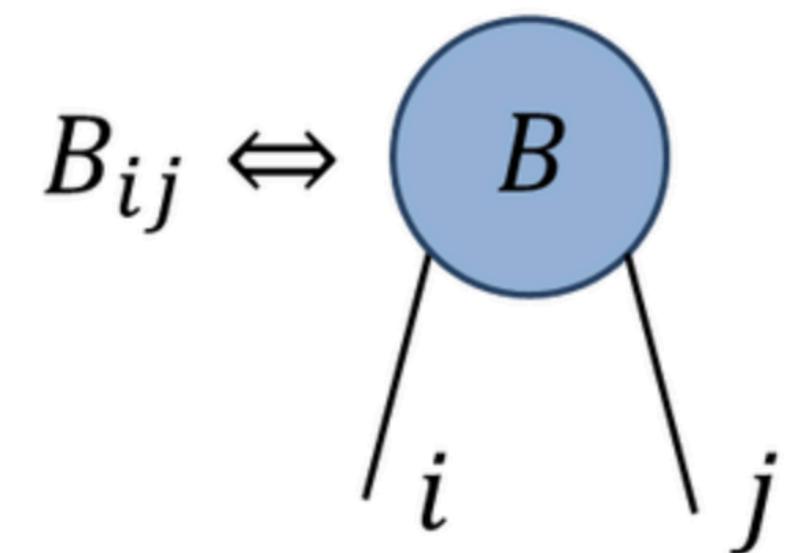
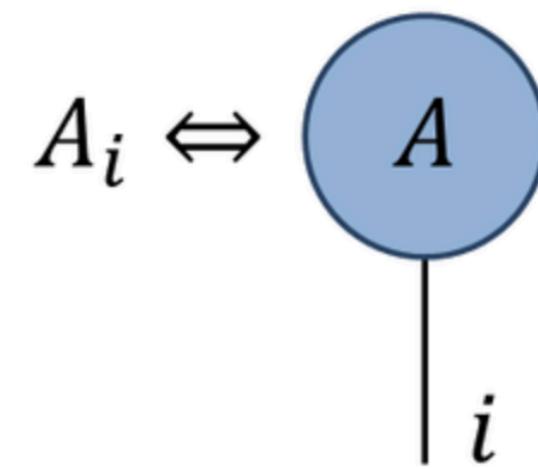


# Tensor Networks

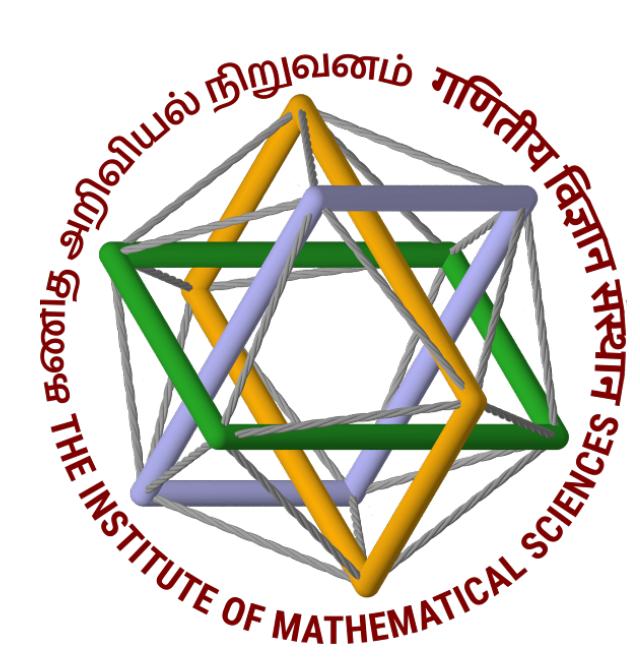
$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$$

$$B = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{\dots\dots\dots}^1 \\ C_{111}^1 & \cdots & C_{1n1}^1 \\ \vdots & \ddots & \vdots \\ C_{m11}^2 & \cdots & C_{mn1}^2 \end{bmatrix}^3$$

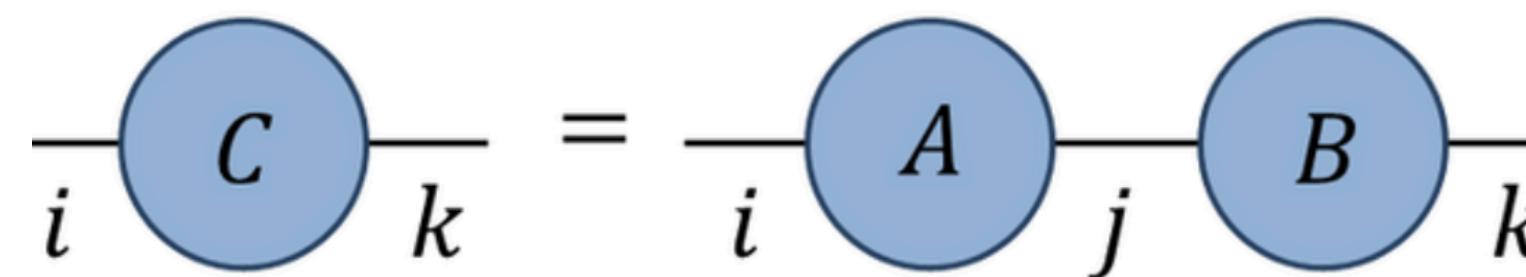


Tensor representation



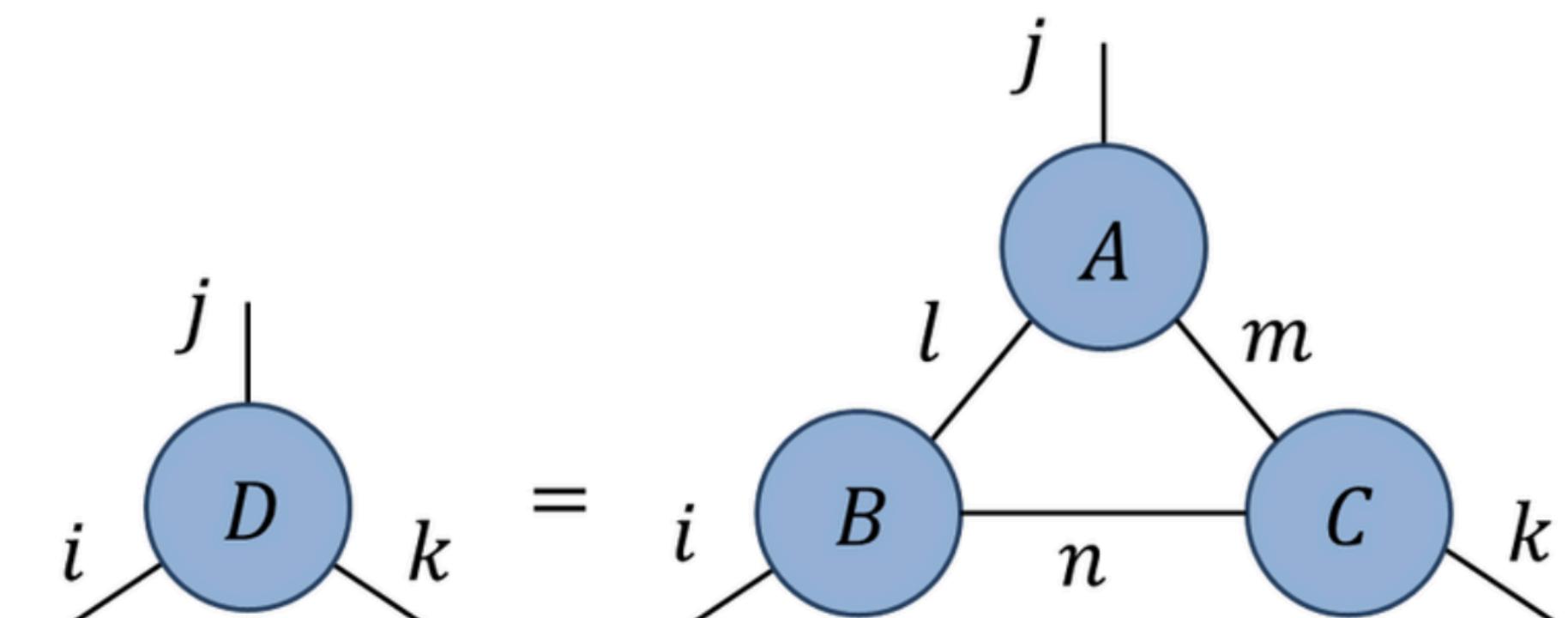
# Tensor Networks

## Contraction of Tensors



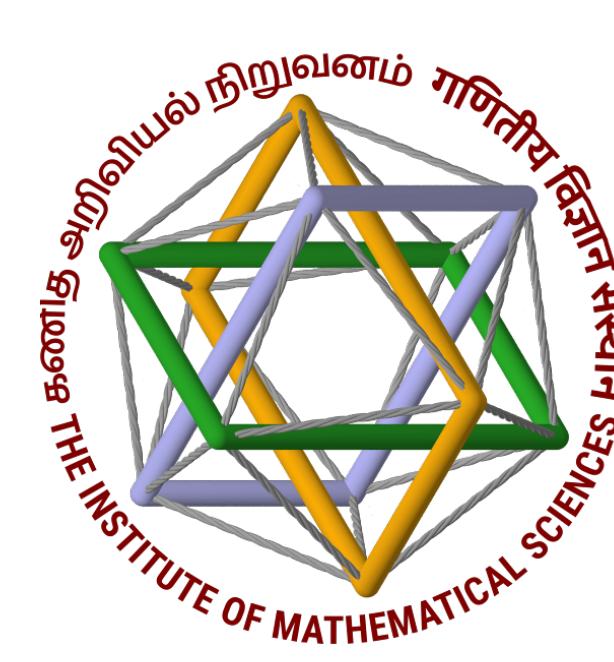
$\Updownarrow$

$$C_{ik} = \sum_i A_{ij} B_{jk}$$



$\Updownarrow$

$$D_{ijk} = \sum_{lmn} A_{ljm} B_{iln} C_{nmk}$$



# Tensor Networks

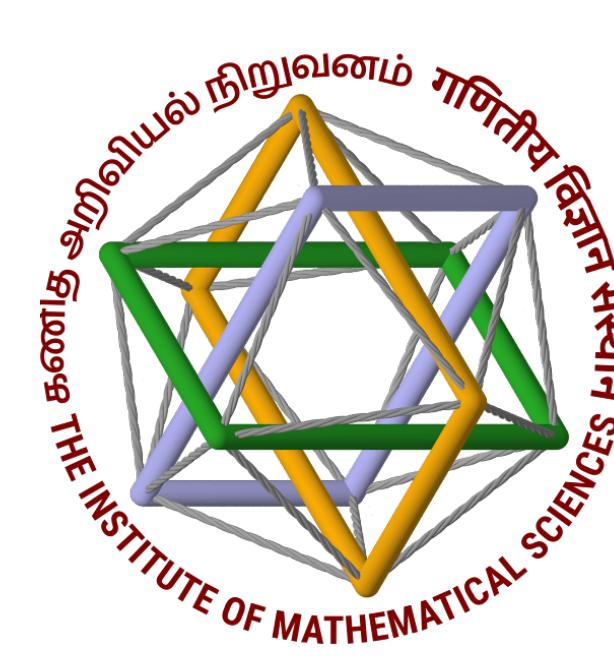
For 1+1 dimensional models various tensor network approaches have been used

**MPS**

**DMRG**

**TRG**

Scalability .... ?

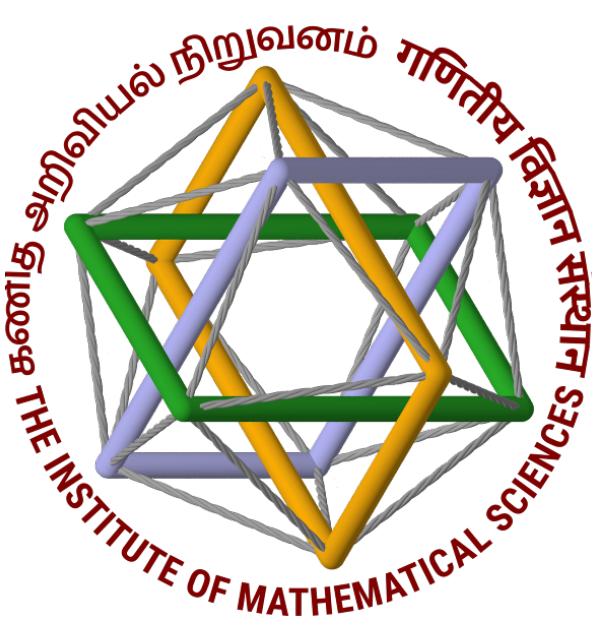


# TRG

Renormalization Group: Systematic investigation of changes of physical system as viewed at different scales.

In simpler terms: Replacing the elementary degrees of freedom by new averaged variables at larger scales.

Performing the same procedure to compute partition function by recursive blocking of tensors — TRG

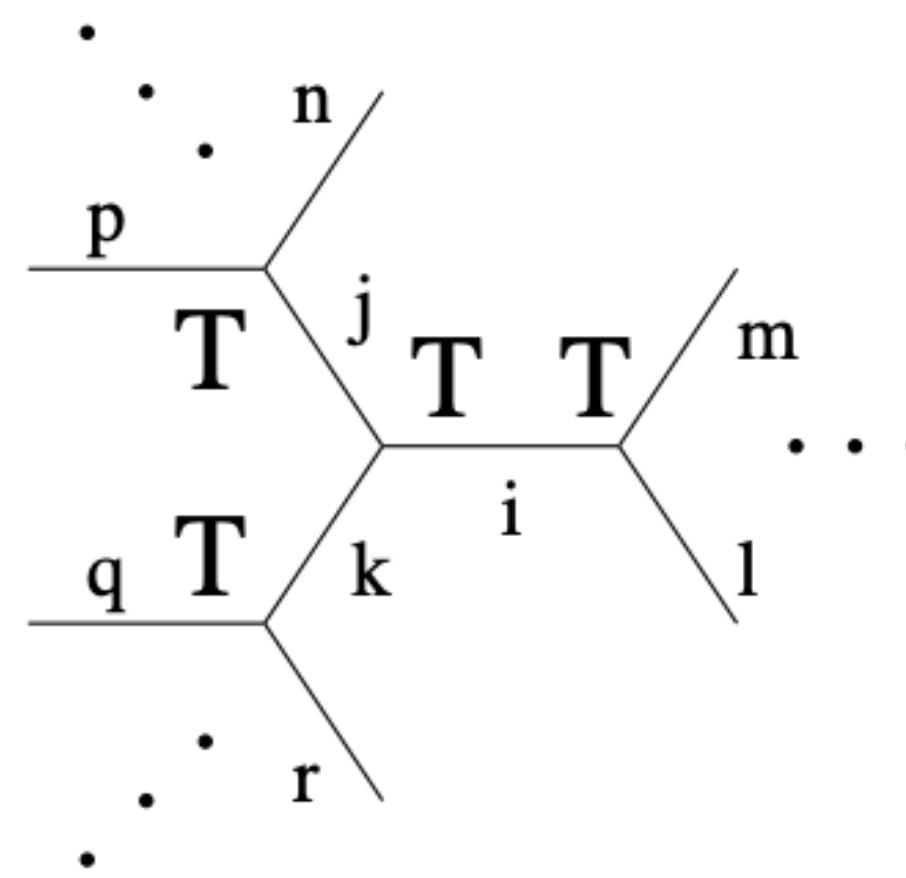


# TRG

arXiv:cond-mat/0611687 [cond-mat.stat-mech]

*Levin, Nave*

## How to write partition function using tensor network algorithms ?

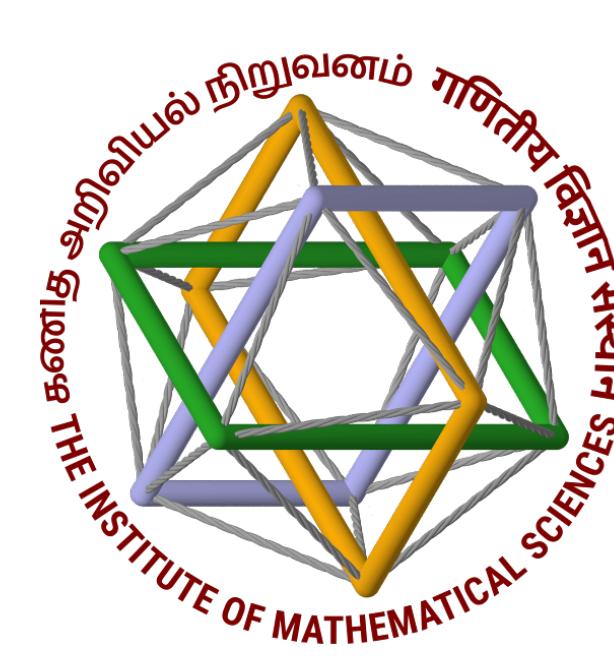


Honeycomb Lattice with

$i, j, k = 1, 2, \dots, D$

Then weight of the configuration  
is given by:

$$e^{-S(i,j,k,\dots)} = T_{ijk}T_{ilm}T_{jnp}T_{kqr}\dots$$



## TRG

How to write partition function using tensor network algorithms ?

Partition function is obtained by taking  
the product of all the tensors,  
contracting the pairs of indices on each bond.

Then weight of the configuration  
is given by:

$$e^{-S(i,j,k,\dots)} = T_{ijk}T_{ilm}T_{jnp}T_{kqr}\dots$$

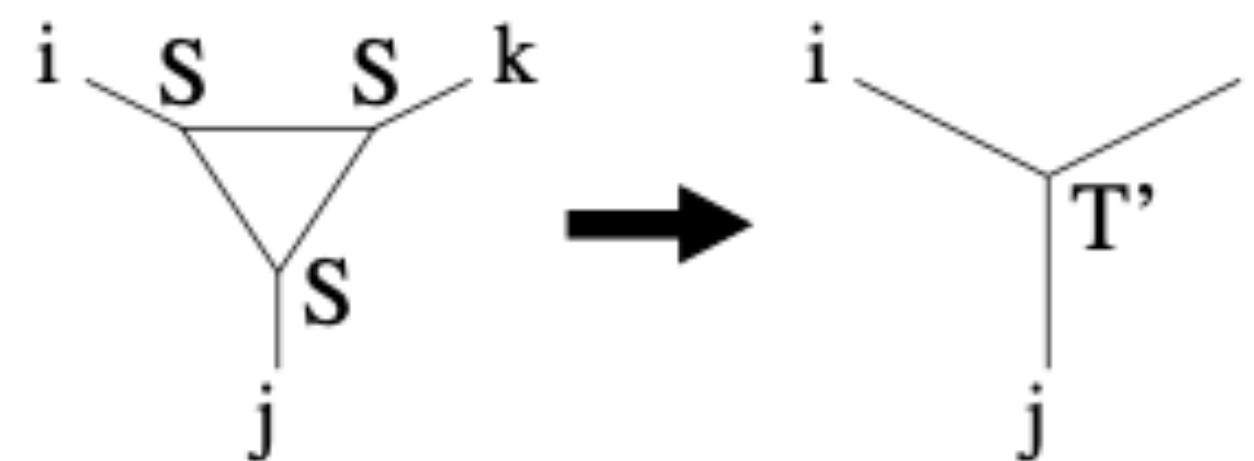
$$Z = \sum_{ijk\dots} e^{-S(i,j,k,\dots)} = \sum_{ijk\dots} T_{ijk}T_{ilm}T_{jnp}T_{kqr}\dots$$

Partition Function

# TRG

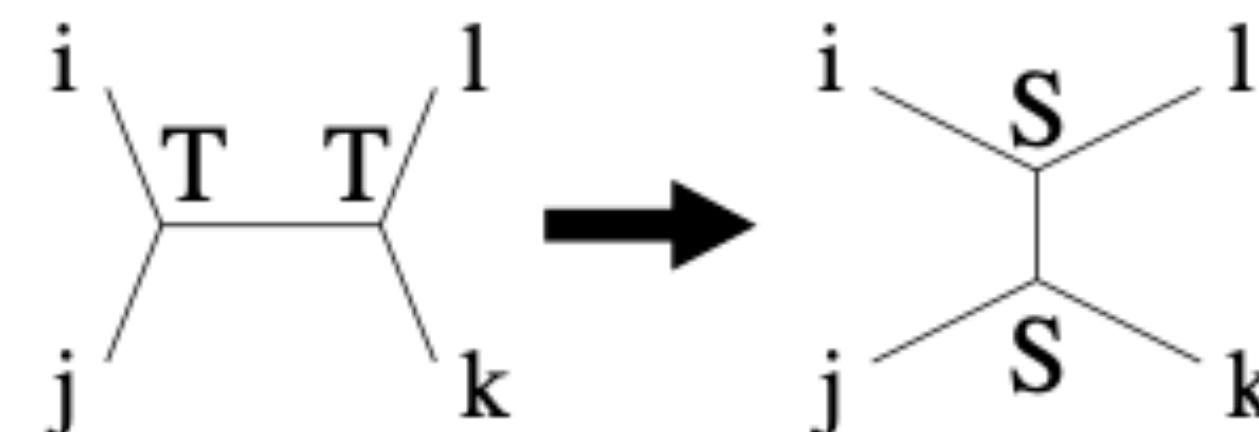
Each iteration consists of two steps

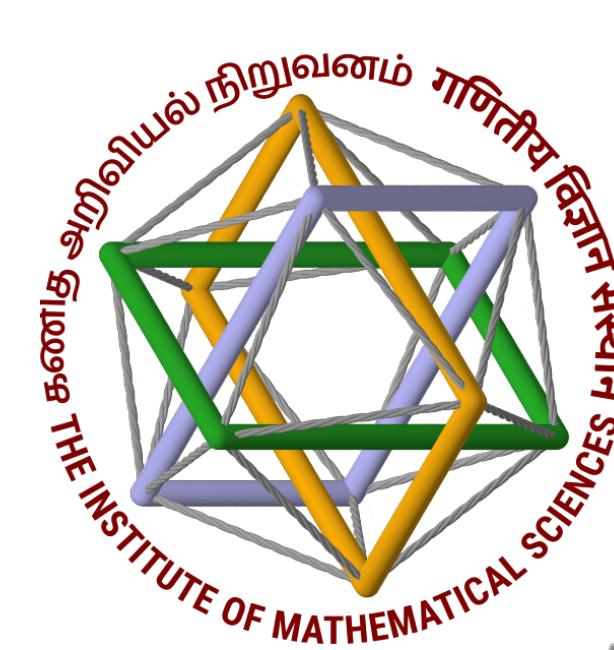
- ▶ Approximate (involving SVD)
- ▶ Exact (Contraction)



$$T'_{ijk} = \sum_{pqr} S_{kpq} S_{jqr} S_{irp}$$

$$\sum_n S_{lin} S_{jkn} \approx \sum_m T_{ijm} T_{klm}$$





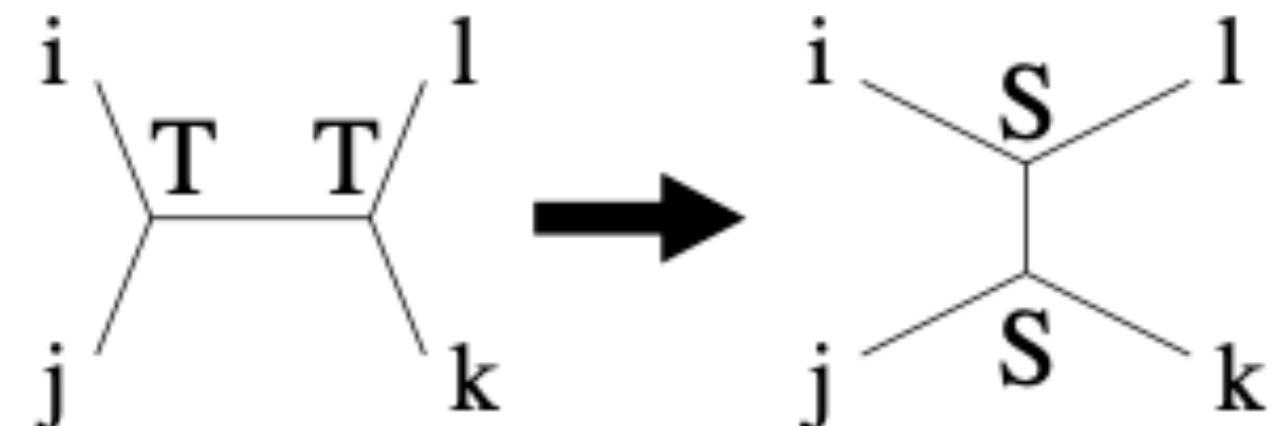
## TRG

- ▶ Approximate (involving SVD)

Why ?

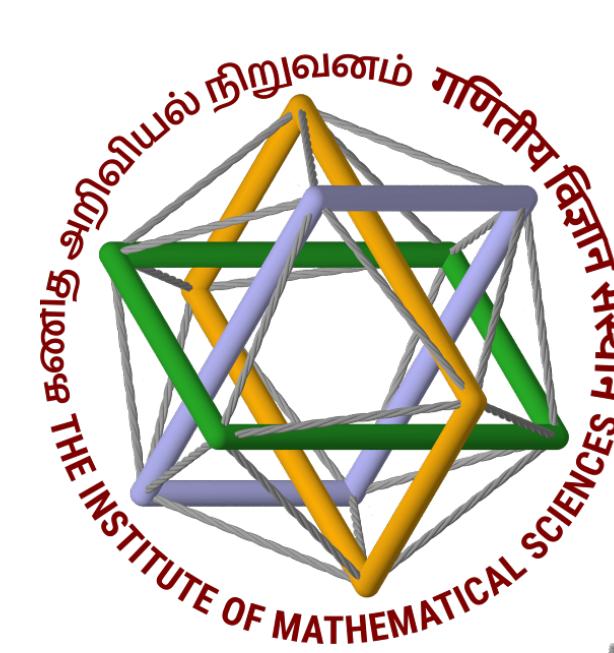
$$D^2 \times D^2 \quad (TT) \rightarrow D^2 \times D \quad (S)$$

$$\sum_n S_{lin} S_{jkn} \approx \sum_m T_{ijm} T_{klm}$$



Find a matrix  $M$  s.t.  $M = S \cdot S^T$

which is not straightforward as we are comparing matrices with rank  $D^2$  and  $D$



## TRG

- ▶ Approximate (involving SVD)

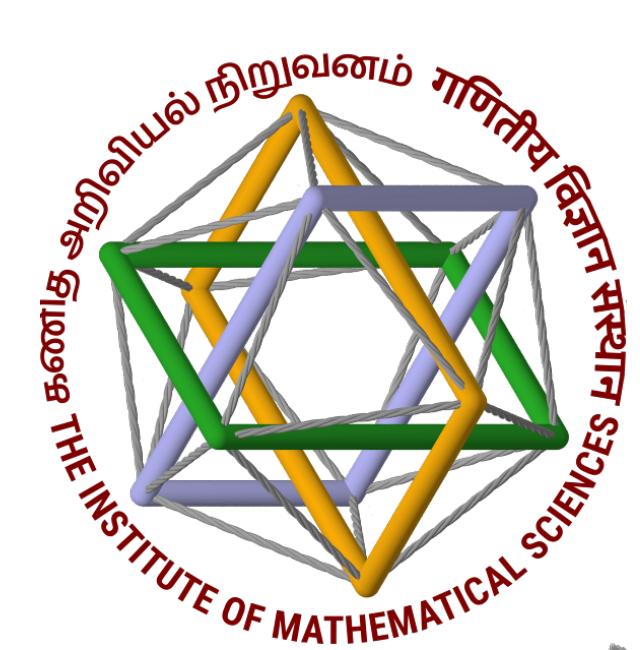
Find a matrix  $M$  s.t.  $M = S \cdot S^T$

which is not straightforward as we are comparing matrices with rank  $D^2$  and  $D$

$$M_{li,jk} = \sum_n s_n U_{li,n} V_{jk,n}^*$$

Truncate  $U$  and  $V$  upto maximum  
 $D$  singular values

$$S_{lin}^A = \sqrt{s_n} \tilde{U}_{li,n}, \quad S_{jkn}^B = \sqrt{s_n} \tilde{V}_{jk,n}^*$$



# TRG

Credit: Vamika

- ▶ Approximate (involving SVD)

Marienplatz  
Plaza in Munich, Germany  
 $3825 \times 4861$  real matrix



$D_{\text{cut}} = 30$



$D_{\text{cut}} = 100$

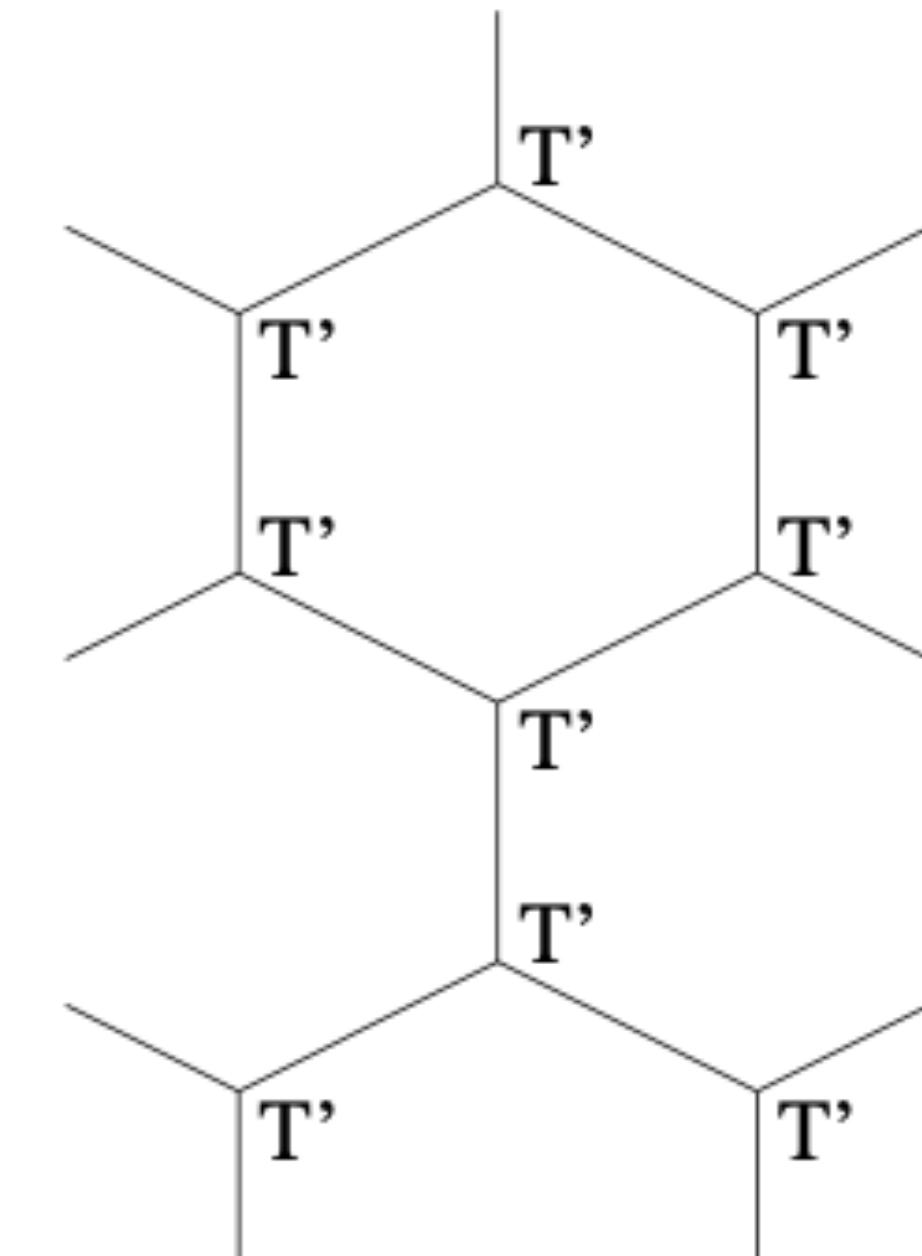
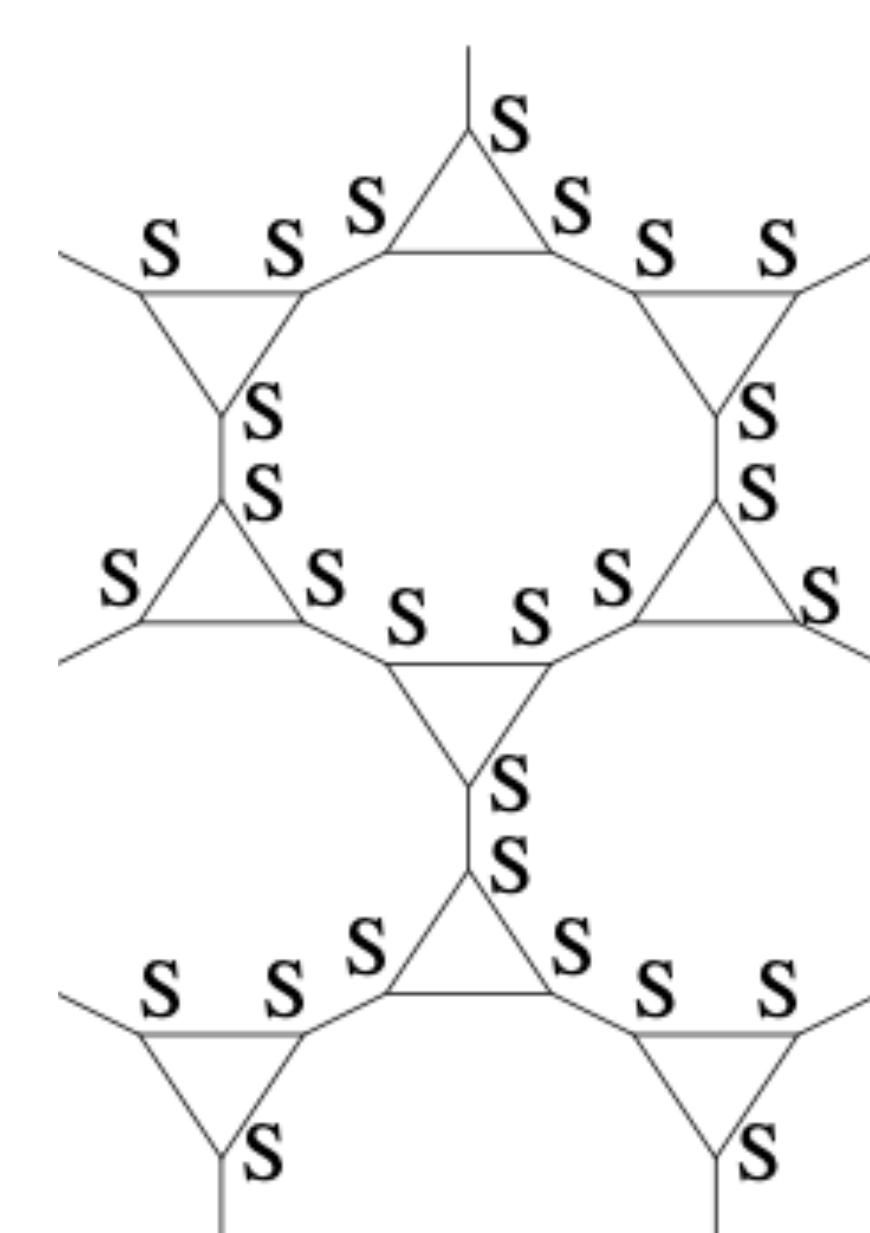
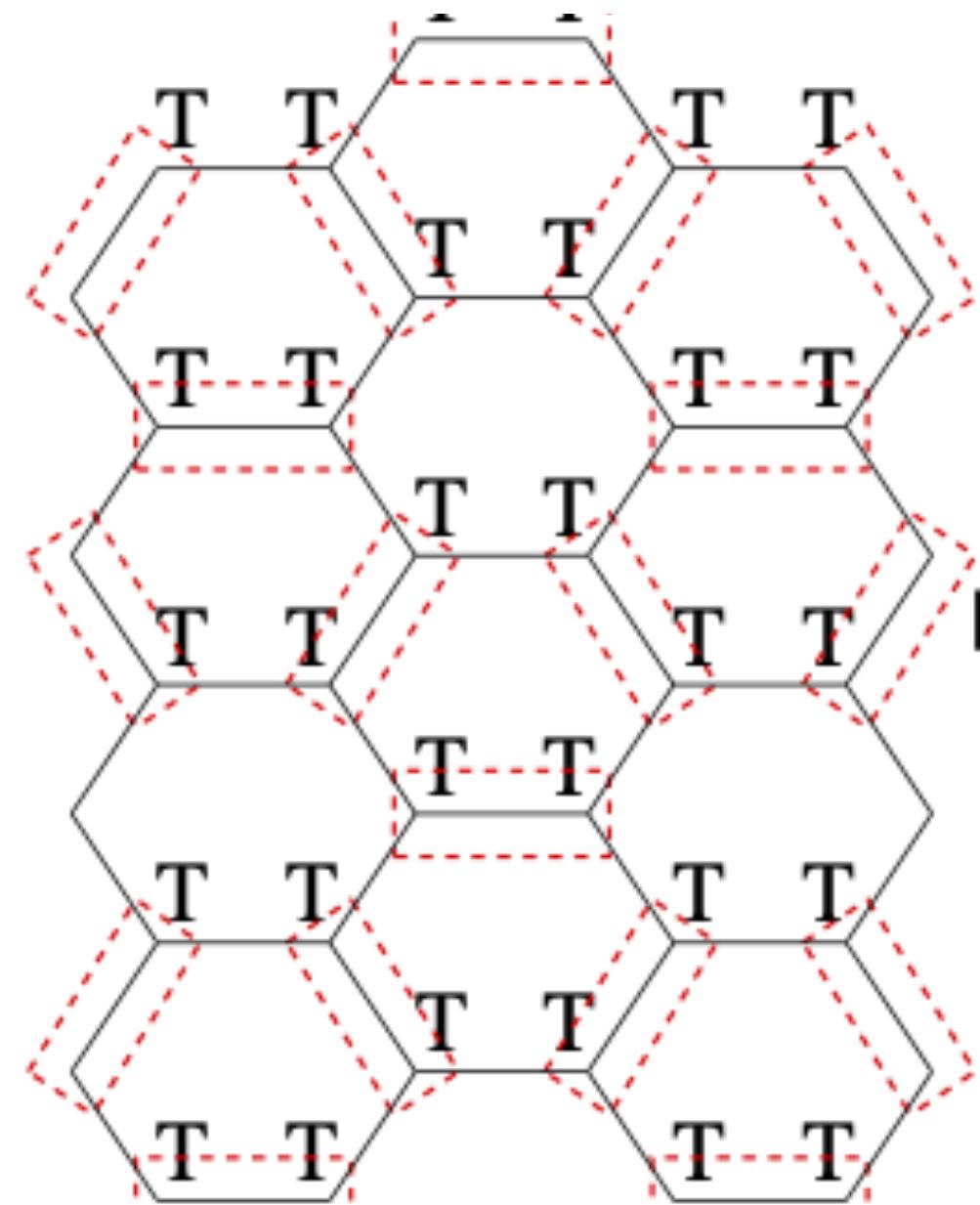


$D_{\text{cut}} = 300$



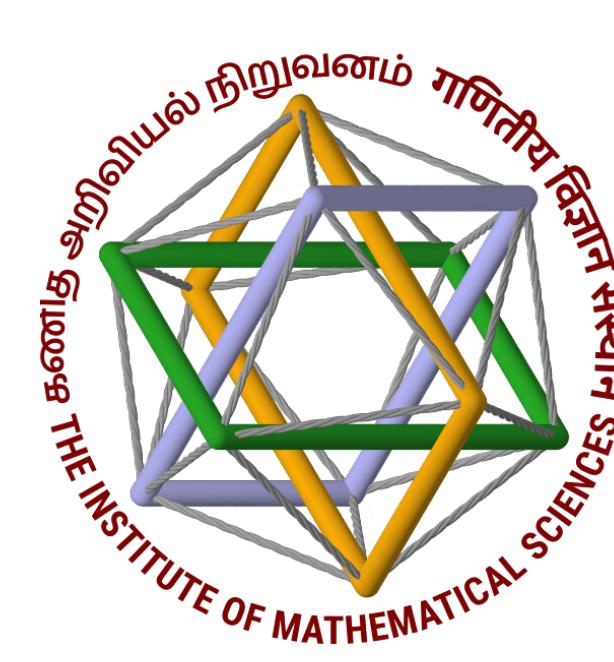
$D_{\text{cut}} = 1000$

# TRG



$$\sum_n S_{lin} S_{jkn} \approx \sum_m T_{ijm} T_{klm}$$

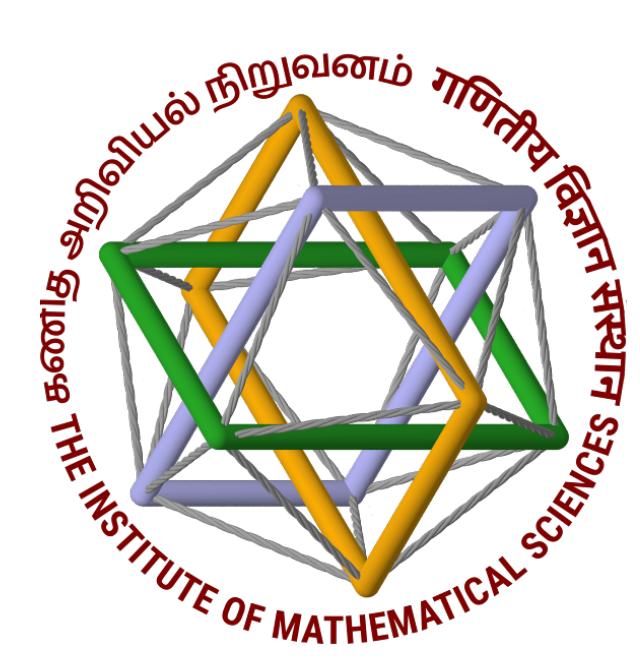
$$T'_{ijk} = \sum_{pqr} S_{kpq} S_{jqr} S_{irp}$$



# TRG

- ▶ Iteratively proceed in same manner
- ▶ In this honeycomb example lattice points decreased by a factor of 3
- ▶ At last we are down to a single tensor
- ▶ Partition function is trace of this tensor

Scalability of this method - difficult though there exists other versions such as **HOTRG**



# Non-Abelian Higgs Model in 2d

$$S_g = -\frac{\beta}{2} \sum_x \text{Tr} \left[ U_{x,1} U_{x+\hat{1},2} U_{x+\hat{2},1}^\dagger U_{x,2}^\dagger \right]$$

Wilson Action

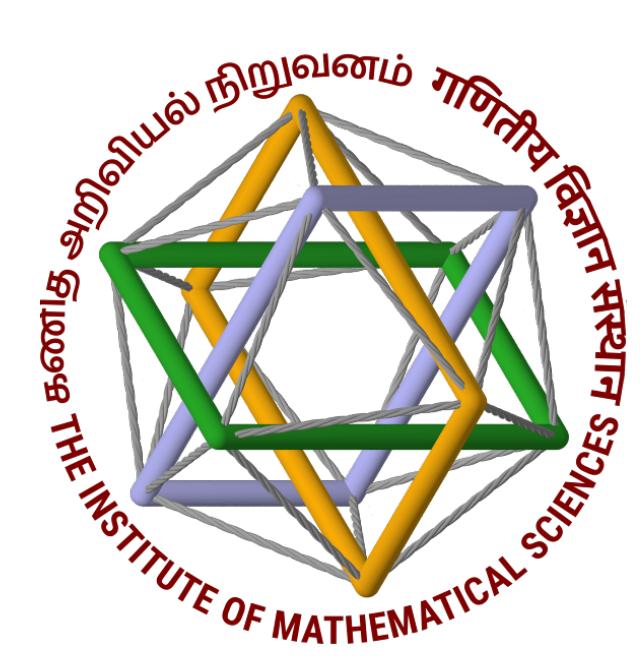
$$S_\Phi = -\frac{\kappa}{2} \sum_x \sum_{\mu=1}^2 \rho_{x+\hat{\mu}} \rho_x \text{Tr} \left[ \alpha_{x+\hat{\mu}}^\dagger U_{x,\mu} \alpha_x \right]$$

Gauge-Matter term

$$V = \sum_x \rho_x^2 + \lambda(\rho_x^2 - 1)^2$$

Potential term

$$Z = \int D[U] D[\rho] D[\alpha] e^{-S_g - S_\Phi - V}$$



# Non-Abelian Higgs Model in 2d

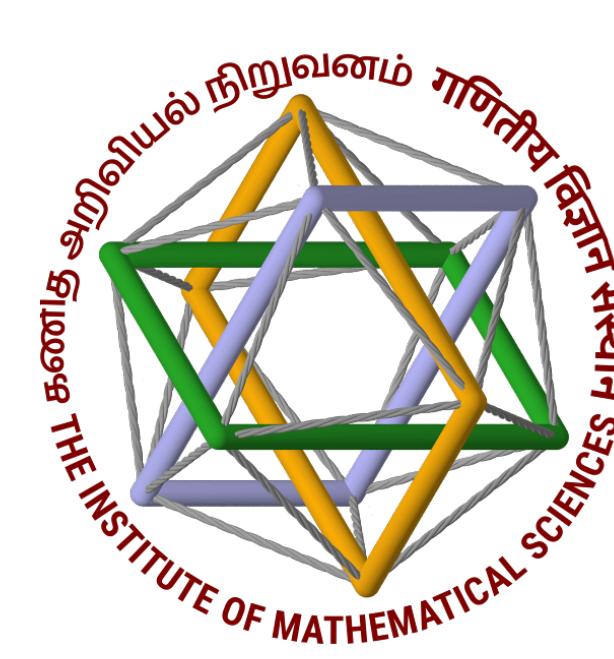
Partition function in terms of character functions

$$f(X \operatorname{Tr}[V]) = \sum_{r=0}^{\infty} F_r(X) \chi^r(V)$$

$$\begin{aligned}\chi^r(U_1 U_2 U_3 \dots U_n) &= D_{nn}^r(U_1 U_2 U_3 \dots U_n) \\ &= D_{ab}^r(U_1) D_{bc}^r(U_2) \dots D_{za}^r(U_n)\end{aligned}$$

$$\begin{aligned}e^{-S_g} &= \exp \left[ \frac{\beta}{2} \sum_x \operatorname{Tr}[U_{x,1} U_{x+\hat{1},2} U_{x+\hat{2},1}^\dagger U_{x,2}^\dagger] \right] \\ &= \prod \sum F_r(\beta) \chi^r(U_{x,1} U_{x+\hat{1},2} U_{x+\hat{2},1}^\dagger U_{x,2}^\dagger)\end{aligned}$$

$$\begin{aligned}\chi^r(U_{x,1} U_{x+\hat{1},2} U_{x+\hat{2},1}^\dagger U_{x,2}^\dagger) &= D_{ab}^r(U_{x,1}) D_{bc}^r(U_{x+\hat{1},2}) D_{cd}^{r\dagger}(U_{x+\hat{2},1}) D_{da}^{r\dagger}(U_{x,2}) \\ &= D_{ab}^r(U_{x,1}) D_{bc}^r(U_{x+\hat{1},2}) D_{dc}^{r*}(U_{x+\hat{2},1}) D_{ad}^{r*}(U_{x,2})\end{aligned}$$



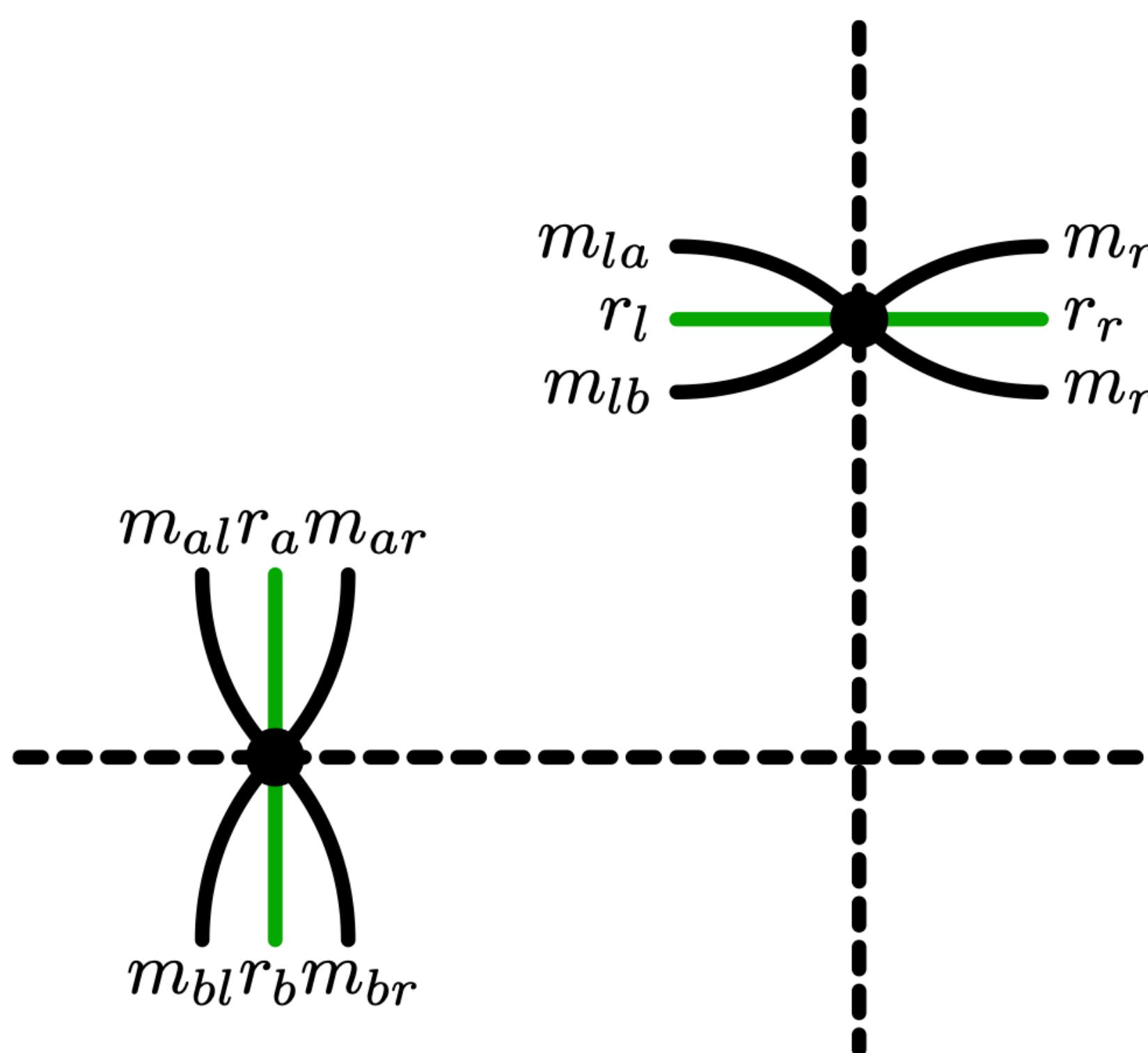
# Non-Abelian Higgs Model in 2d

- ▶ In 2d, two plaquettes associated with single link
- ▶ Another additional variable from gauge-matter term
- ▶ Three matrices associated with each link

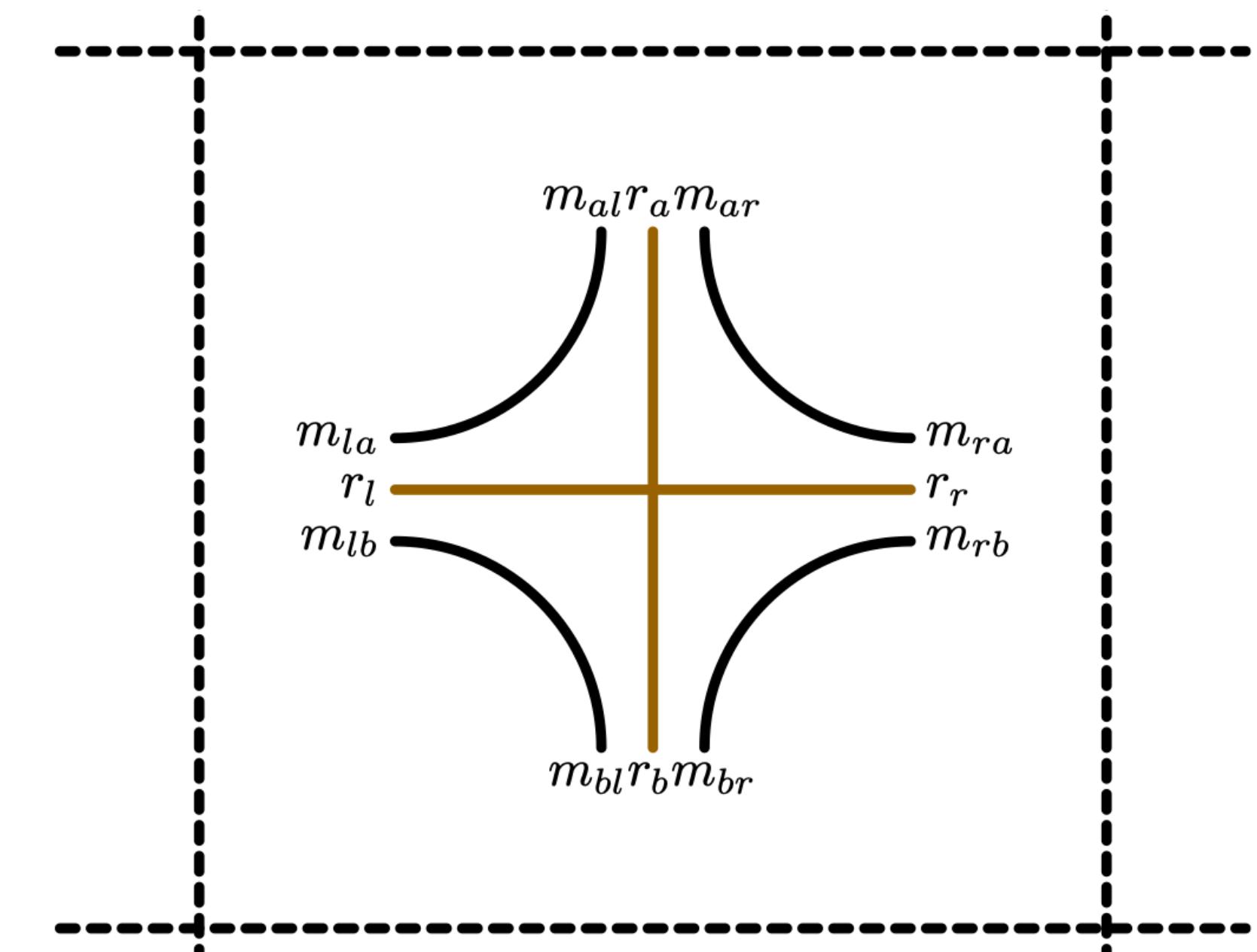
$$\sum_{n=-\sigma}^{\sigma} \int dU D_{m_1 n_1}^{r_1}(U) D_{m_2 n_2}^{r_2 \dagger}(U) D_{nn}^{\sigma}(U)$$
$$= \sum_{n=-\sigma}^{\sigma} \frac{1}{d_{r_2}} C_{r_1 m_1 \sigma n}^{r_2 n_2} C_{r_1 n_1 \sigma n}^{r_2 m_2}.$$

Integral over each link has form

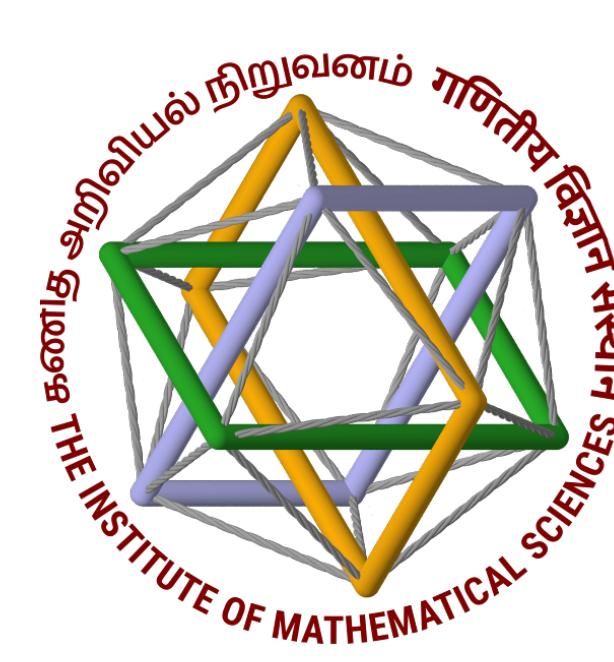
# Non-Abelian Higgs Model in 2d



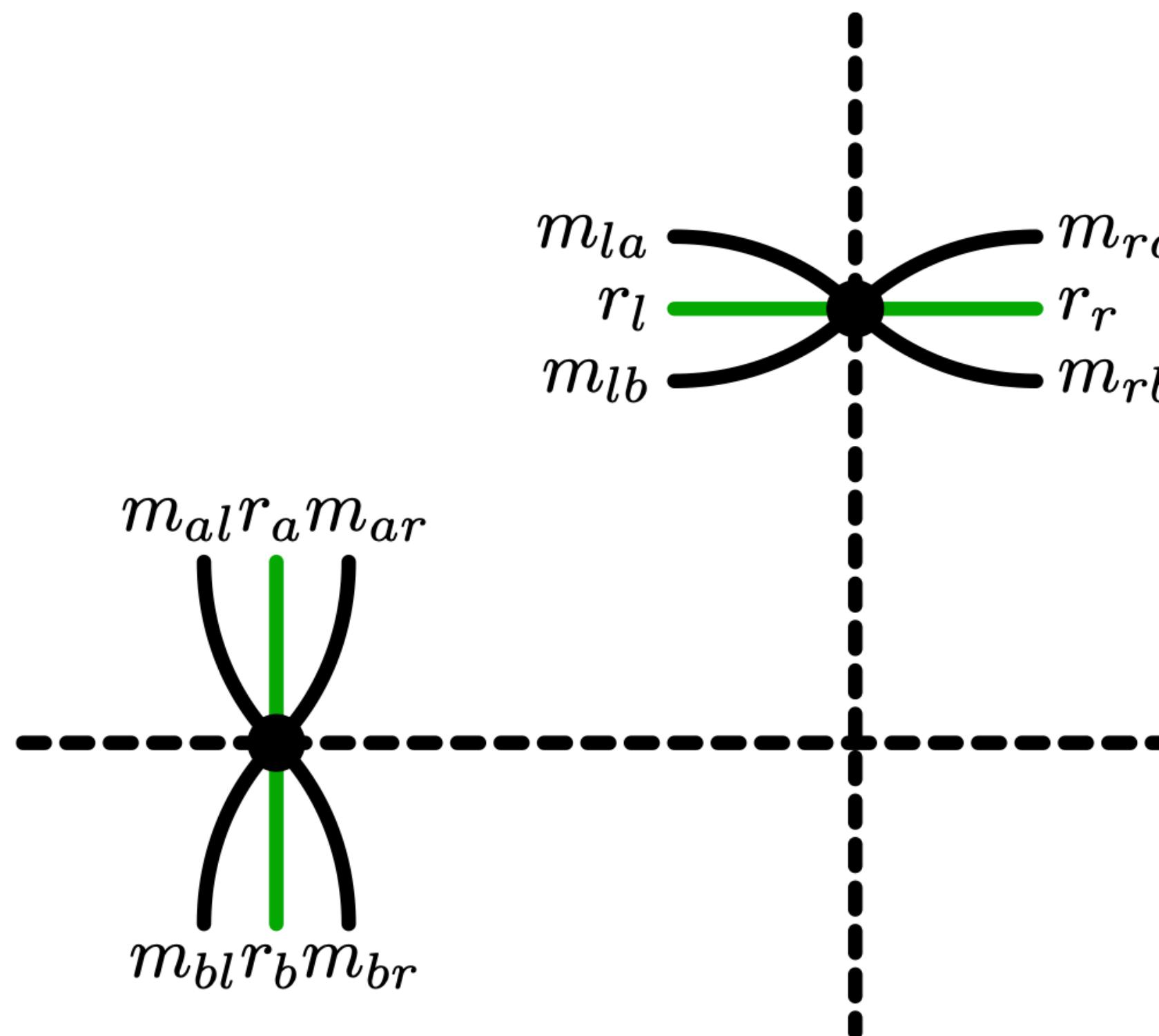
Tensors on link



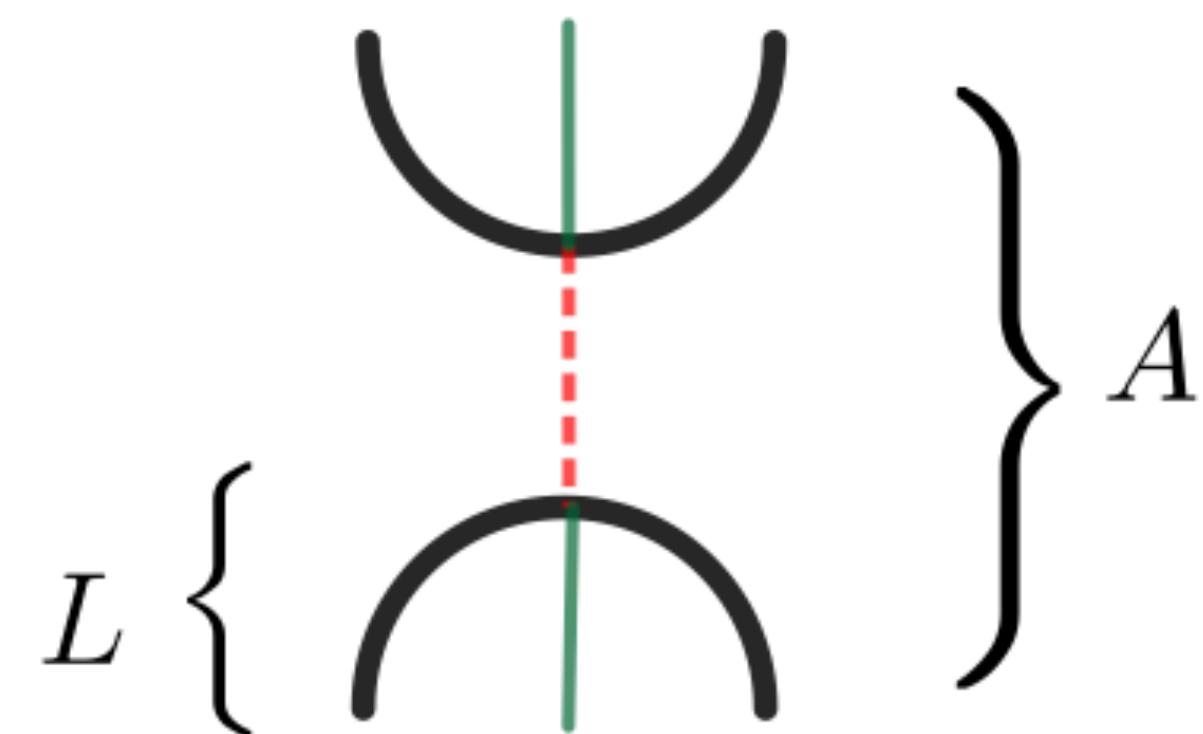
Tensor on Plaquette



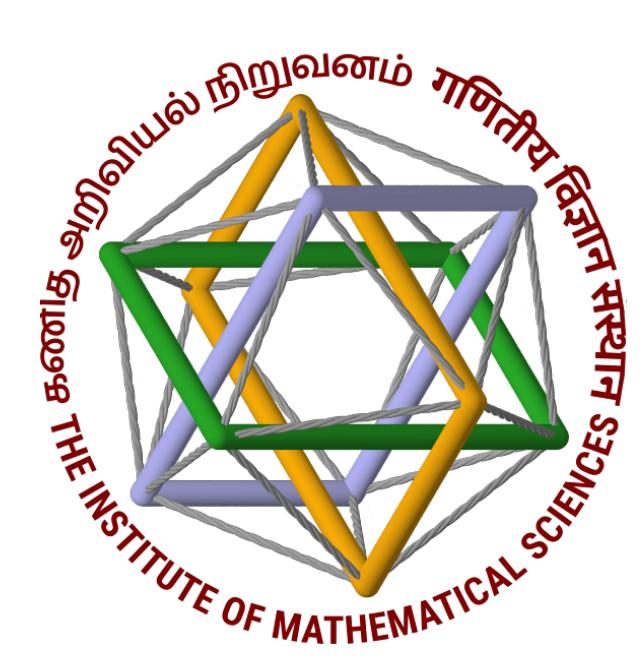
# Non-Abelian Higgs Model in 2d



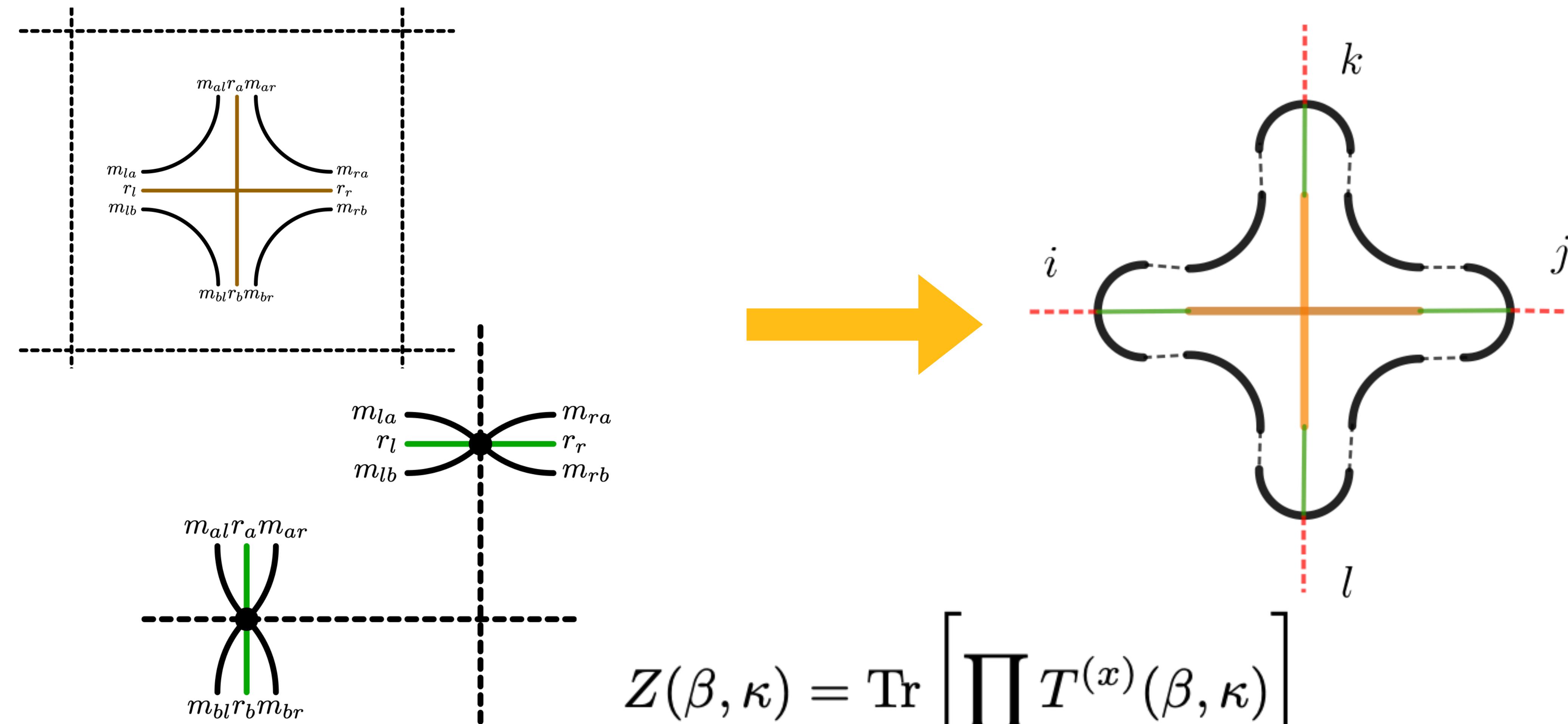
$$A = LL^T$$



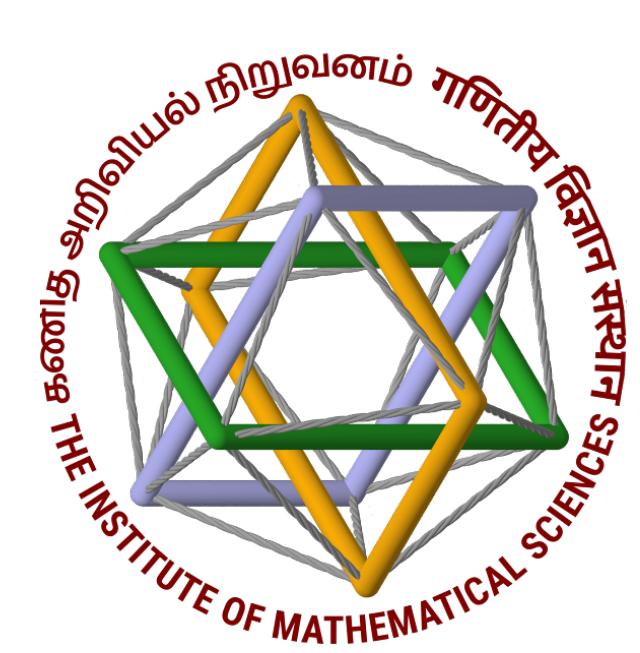
Matrix Factorization



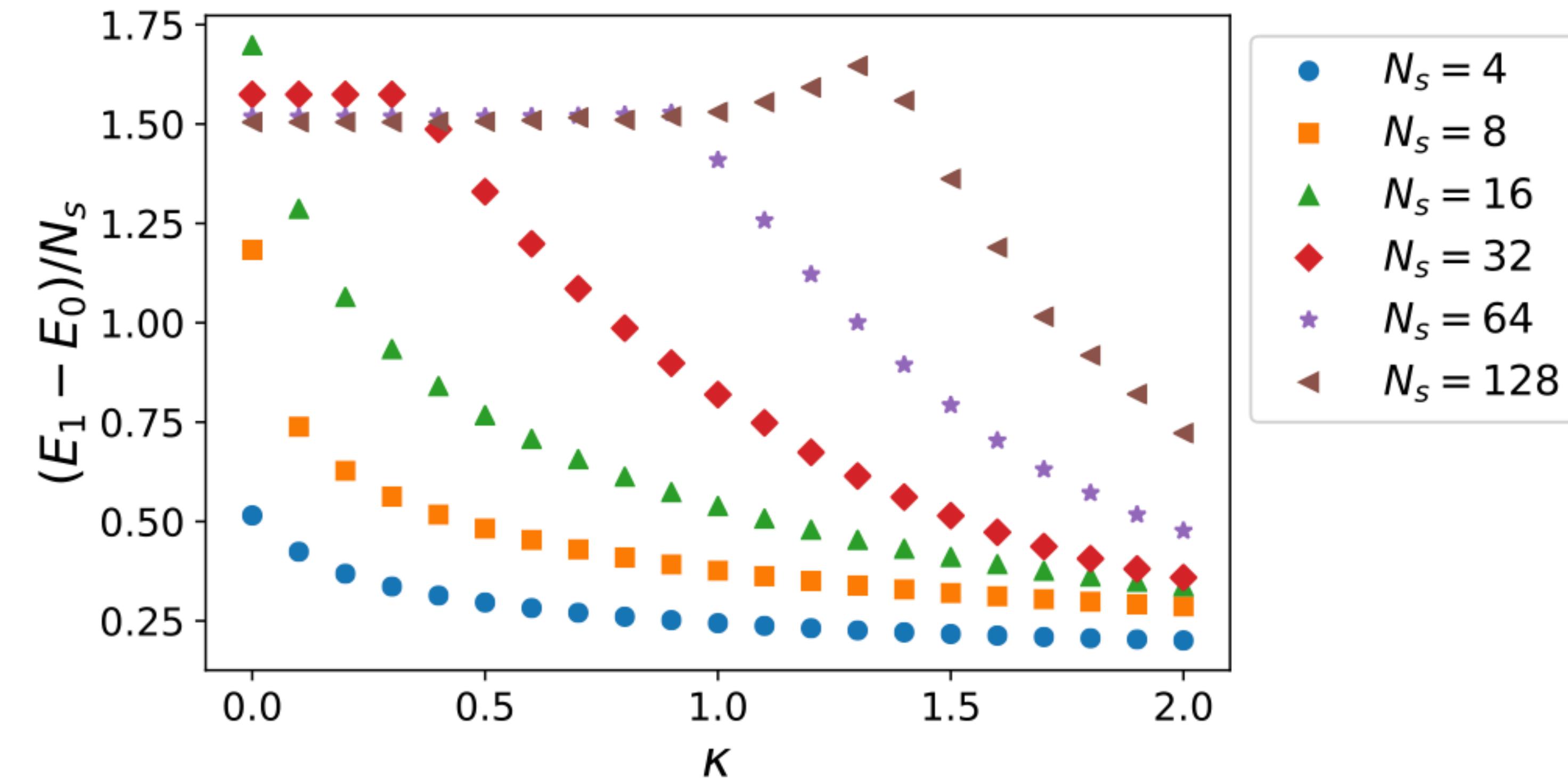
# Non-Abelian Higgs Model in 2d



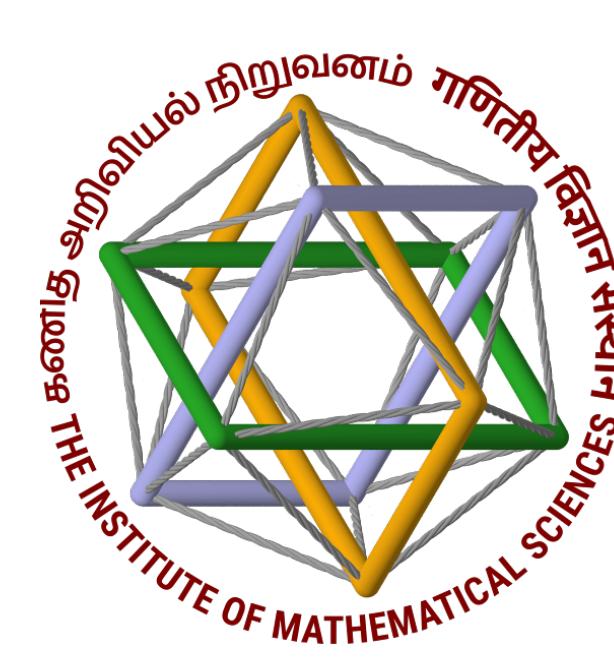
$$Z(\beta, \kappa) = \text{Tr} \left[ \prod_x T^{(x)}(\beta, \kappa) \right]$$



# Non-Abelian Higgs Model in 2d



Mass gap density as function of  $\kappa$  as continuum limit is taken



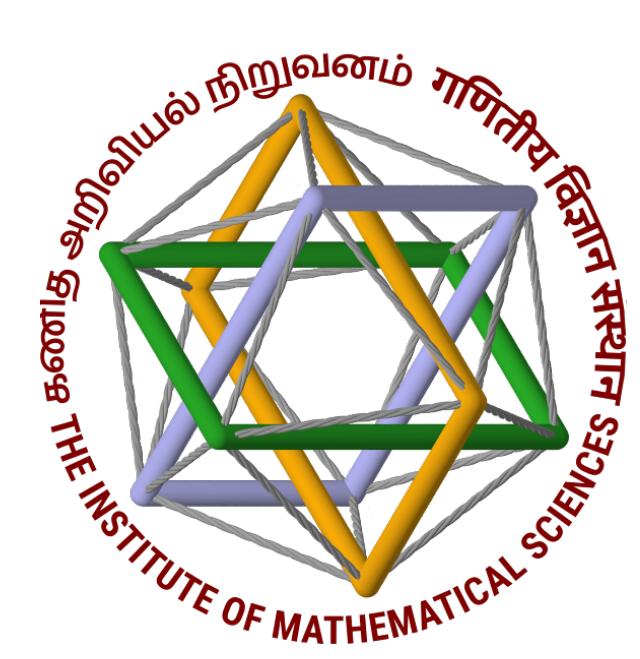
# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

Naive discretisation → Fermion Doubling

Staggered fermions — Does not erase the extra fermion copies completely but reduces the number

Since both spinor (not with staggered) and color index is involved with fermions - more tensor legs

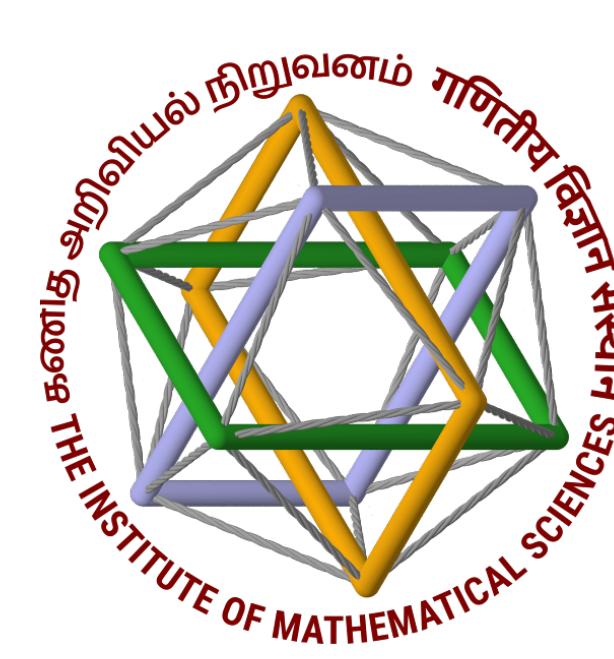
○ Computationally challenging



# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

$$S_F[U] = \sum_n \left[ m\bar{\psi}_n \psi_n + \sum_{\mu=1}^2 \frac{\eta_{n,\mu}}{2} (\bar{\psi}_n U_{n,\mu} \psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}} U_{n,\mu}^\dagger \psi_n) \right].$$

$$\begin{aligned} Z_F[U] &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \prod_n e^{-S_F[U]} \\ &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \prod_n \prod_{a=1}^2 \sum_{s_n^a=0}^1 (-m\bar{\psi}_n^a \psi_n^a)^{s_n^a} \\ &\quad \cdot \prod_{a,b=1}^2 \sum_{x_{n,1}=0}^1 \left( -\frac{\eta_{n,1}}{2} \bar{\psi}_n^a U_{n,1}^{ab} \psi_{n+1}^b \right)^{x_{n,1}^{ab}} \sum_{x_{n,2}=0}^1 \left( \frac{\eta_{n,1}}{2} \bar{\psi}_{n+1}^a U_{n,1}^{ba*} \psi_n^b \right)^{x_{n,2}^{ab}} \\ &\quad \cdot \sum_{t_{n,1}=0}^1 \left( -\frac{\eta_{n,2}}{2} \bar{\psi}_n^a U_{n,2}^{ab} \psi_{n+2}^b \right)^{t_{n,1}^{ab}} \sum_{t_{n,2}=0}^1 \left( \frac{\eta_{n,2}}{2} \bar{\psi}_{n+2}^a U_{n,2}^{ba*} \psi_n^b \right)^{t_{n,2}^{ab}} \end{aligned}$$

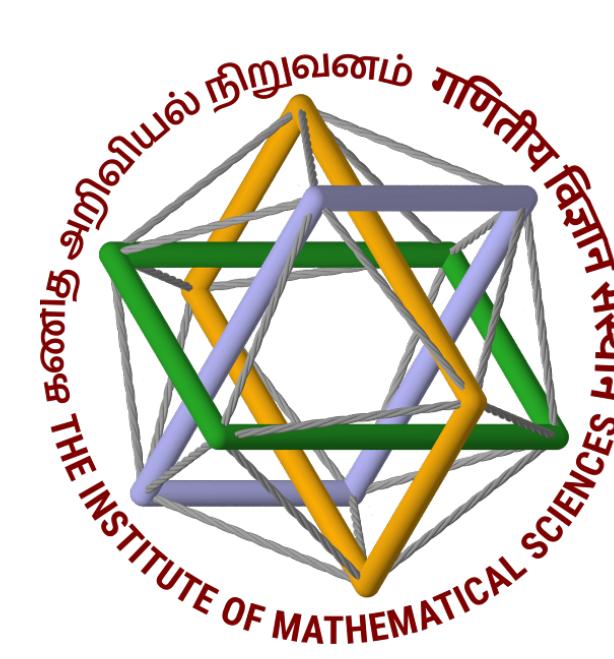


# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

- $\psi, \bar{\psi}$
- Color index
- Hopping index

We are not even considering the bond dimension from gauge sector as yet

Bond dimension  $2^{2 \times 2 \times 2} = 256$  with each fermion link



# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

Solution the paper provides is reduced staggered fermions

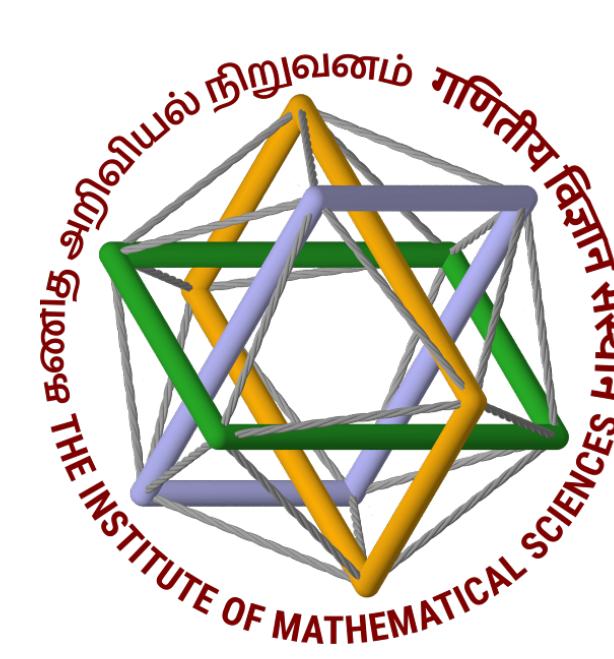
Place  $\psi$  and  $\bar{\psi}$  on odd and even sites respectively

$$\psi_n \rightarrow (1 - \epsilon_n) \psi_n / 2$$

$$\bar{\psi}_n \rightarrow (1 + \epsilon_n) \psi_n / 2$$

$$S_F[U] = \sum_n \sum_{\mu=1}^2 \frac{\eta_{n,\mu}}{2} \psi_n^T \mathcal{U}_{n,\mu} \psi_{n+\hat{\mu}}$$

Bond dimension reduced to  $2^{2 \times 2} = 16$



# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

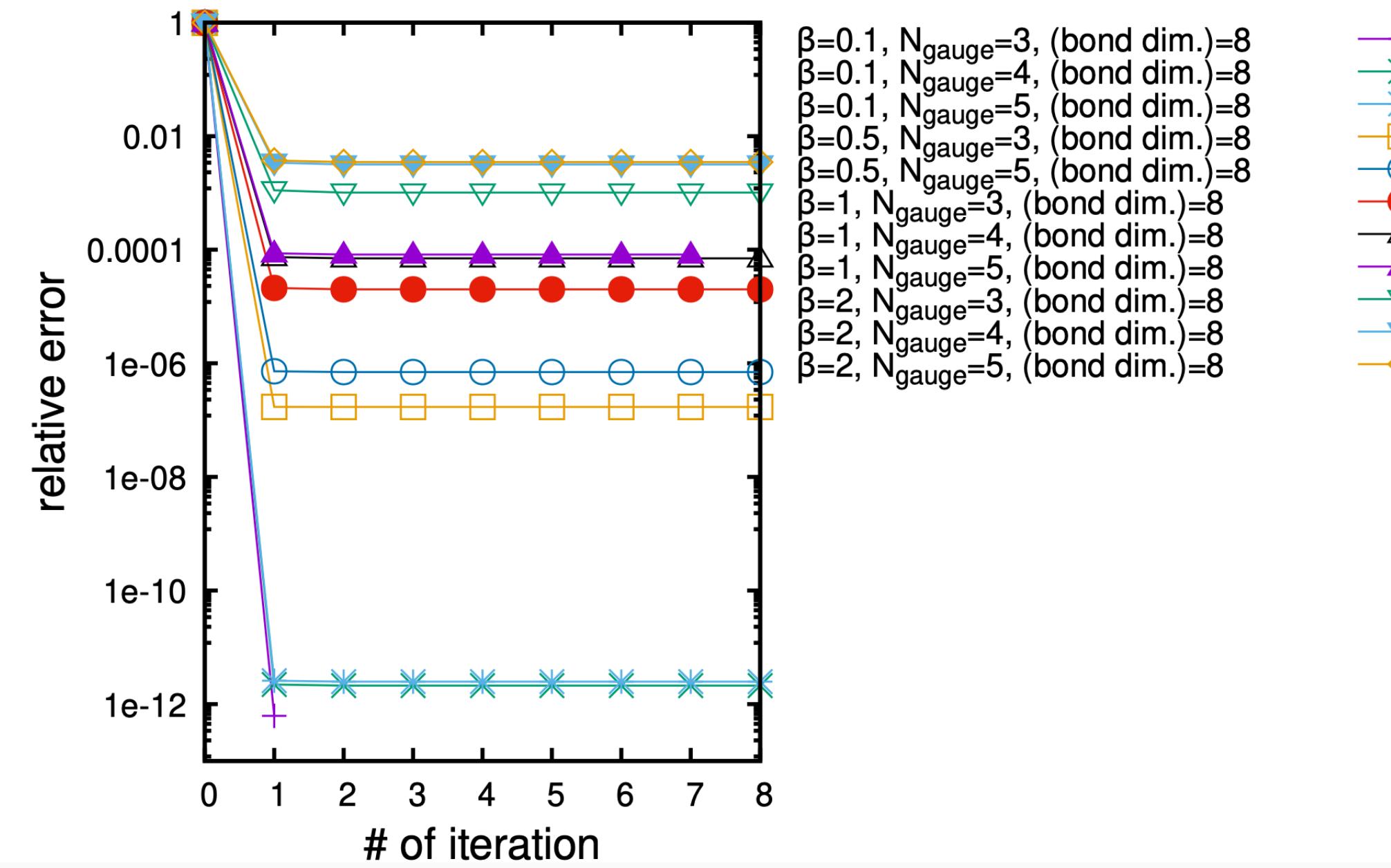
Proceeded with character expansion

Instead of using HOSVD for plaquette tensor they used HOOI

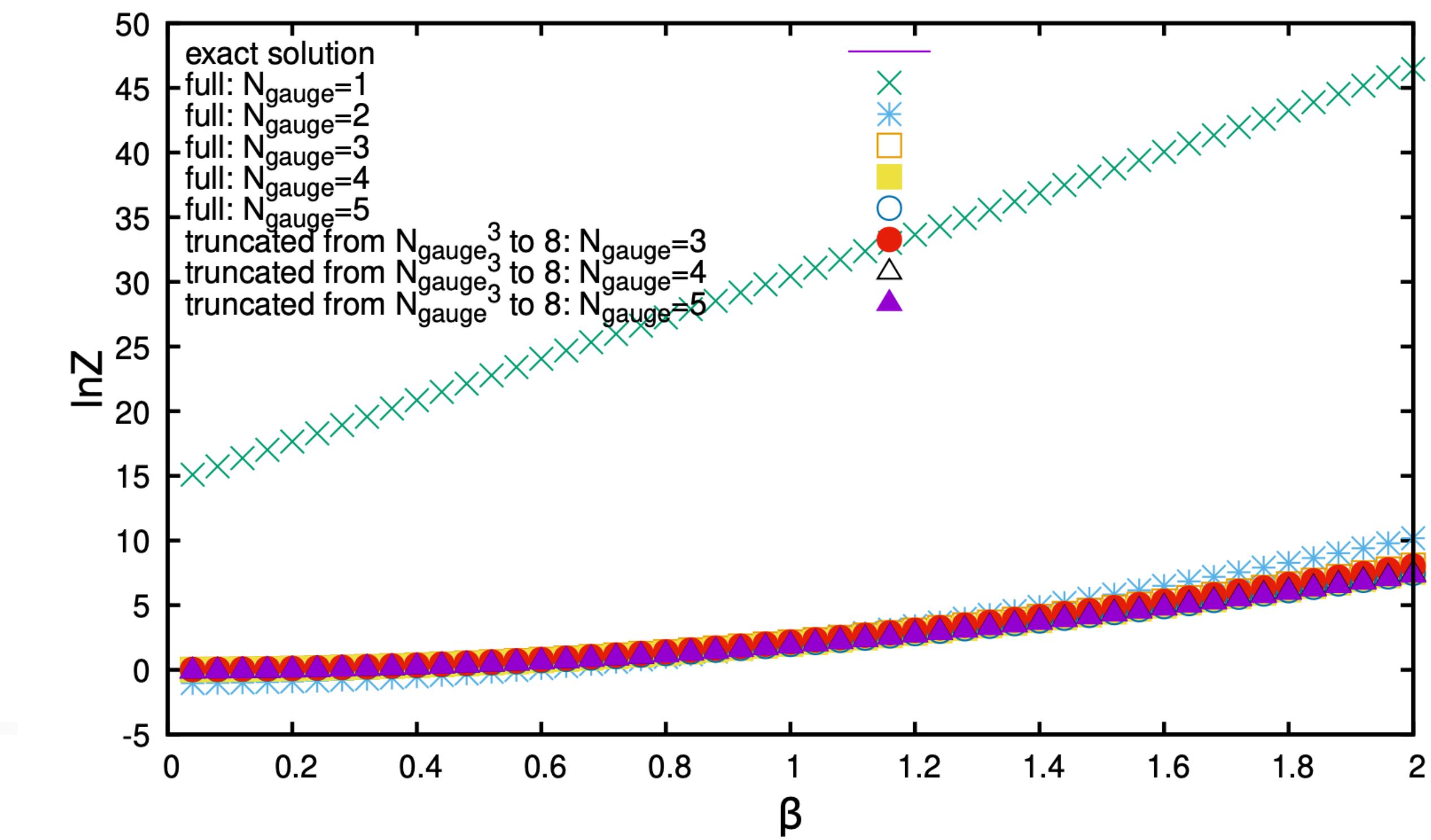
Advantage both in terms of memory and CPU

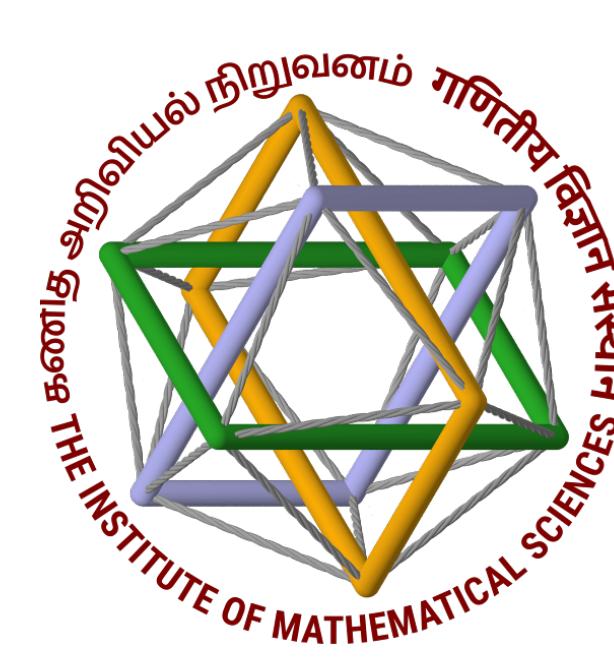
Process is to obtain a tensor with bond dimension less than original tensor

# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

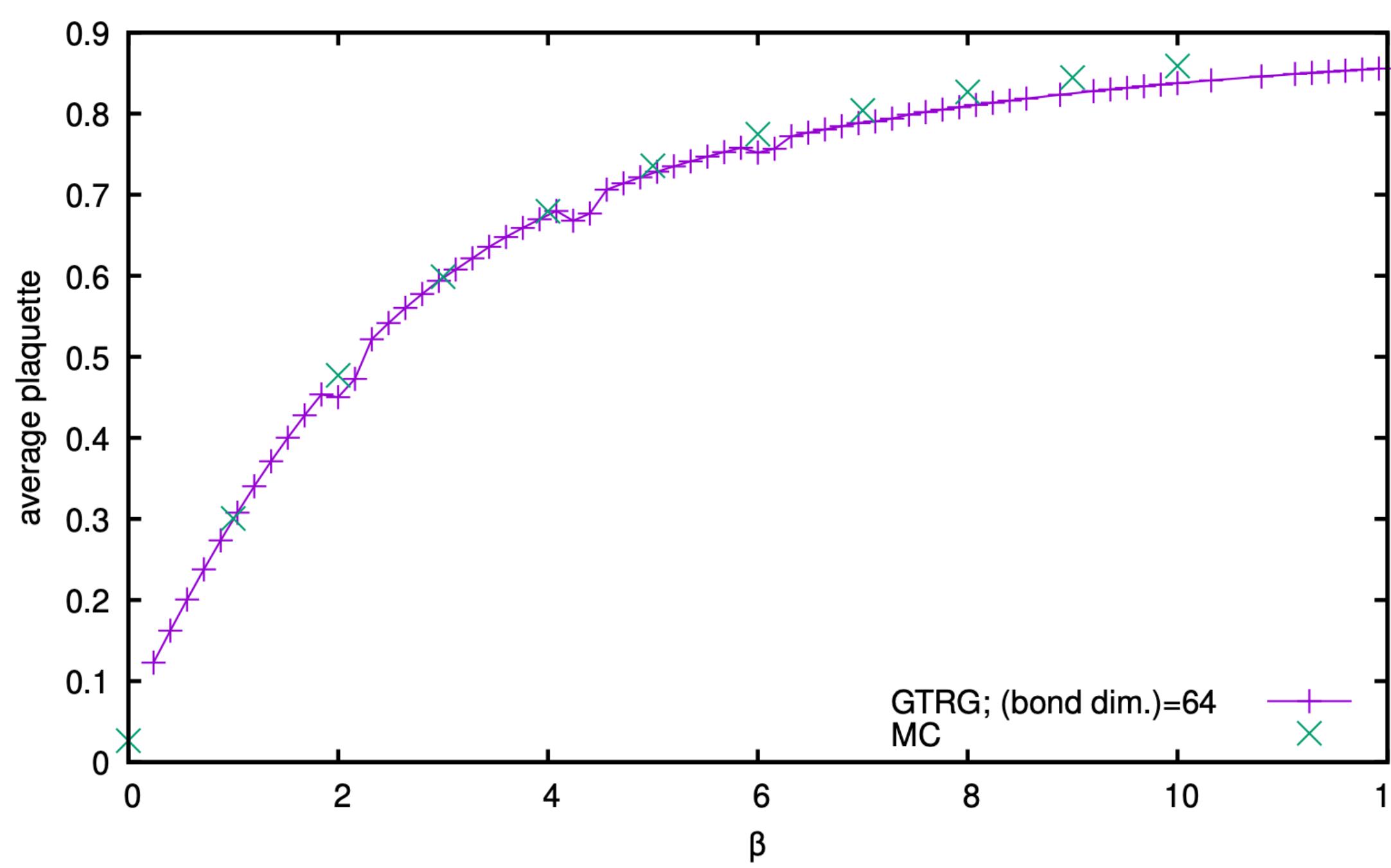


Without fermions

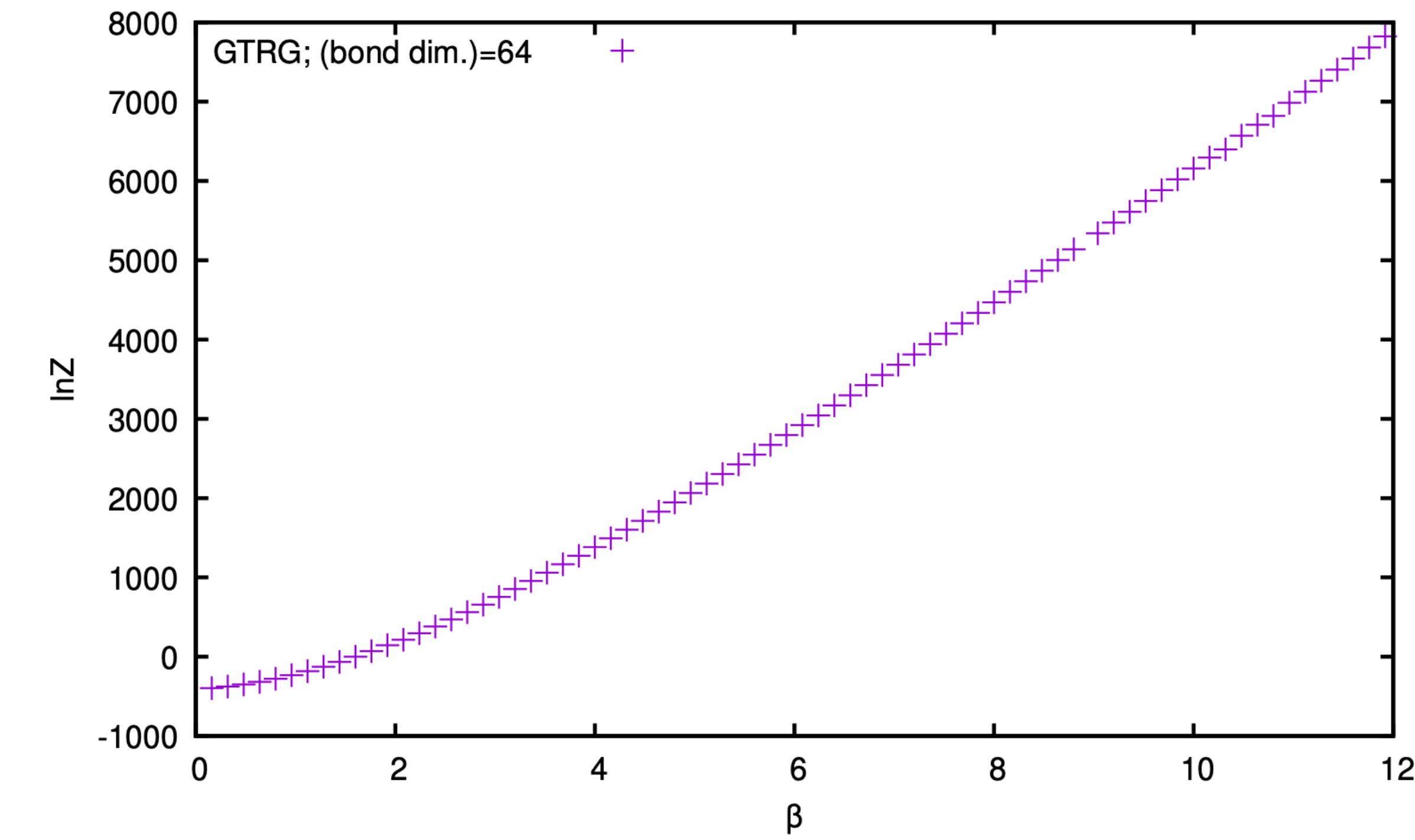




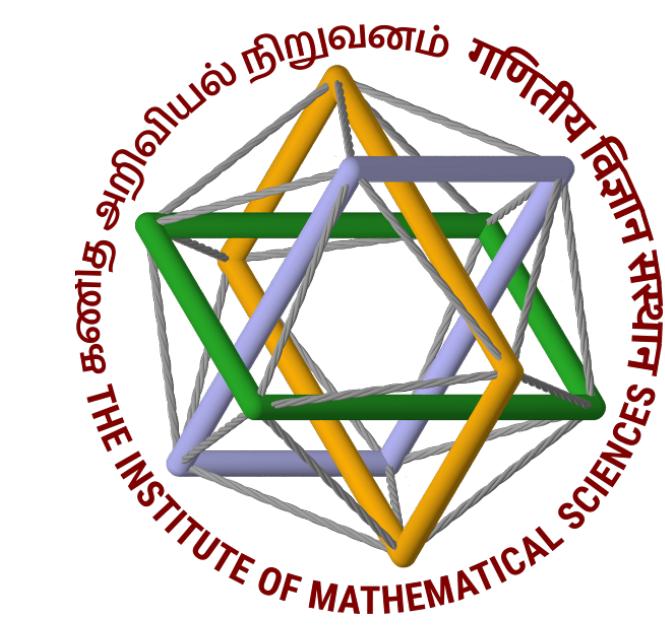
# Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions



With fermions



# THANK YOU



Navdeep Singh Dhindsa 15/02/2024