3.1 INTRODUCTION

The determination of solution of non-linear equations is of interest not only to mathematicians but also to scientists and engineers. Though most of the non-linear equations can be solved algebraically, *i.e.*, analytically, using the techniques of algebra. There are many non-linear equations that cannot be solved algebraically.

As an example, consider the following equation

$$2^x - x - 3 = 0$$

which seems very simple but cannot be solved algebraically.

The solution obtained by algebraic manipulations is known as *algebraic solutions* or *analytical solutions*. There are many instances when the algebraic solution of the non-linear equation does exist but is extremely complicated and impractical for day-to-day purposes. In this chapter, we will consider some methods of obtaining solutions of non-linear equations. These solutions will be numerical not algebraic, and are called *numerical solutions*.

Before describing these methods, let us first review the various types of equations and other issues involved in finding their solutions.

In general, an equation in x is written as

$$f(x) = 0 \qquad \ldots (3.1)$$

The root(s) of this equation, also known as *zero(s) of equation*, is/are those values of x for which the equation is satisfied.

3.2 TYPES OF NON-LINEAR EQUATIONS

The non-linear equations fall in following categories:

- Polynomial
- Transcendental

3.2.1 Polynomial Equations

The polynomials are frequently occurring functions in science and engineering. A polynomial has the general form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0 \qquad \text{where } a_n \neq 0. \qquad \ldots (3.2)$$

It is nth degree polynomial in x and has n roots. These roots may be

- Real and different
- Real and repeated
- Complex

The complex roots always appear in pairs and are of the form $a + ib$ and $a - ib$, where $i = \sqrt{-1}$, is an imaginary number, and a and b are real numbers and represent real and imaginary parts of the roots.

3.2.2 Transcendental Equations

A non-polynomial equation is called transcendental equations. Some examples of transcendental equations are

$$xe^x - x\sin x = 0 \qquad e^x\cos x - 3x = 0$$

$$e^{3x} - \frac{1}{2}x = 0 \qquad 2^x - x - 3 = 0$$

A transcendental equation may have finite/infinite number of real roots or may not have any real root at all.

3.3 METHODS OF FINDING SOLUTIONS OF NON-LINEAR EQUATIONS

In general, there are two kinds of methods to obtain solutions of non-linear equations. These are

- Direct methods
- Iterative methods

3.3.1 Direct Methods

Direct methods give the roots of non-linear equations in a finite number of steps. In addition, these methods are capable of giving all the roots at the same time.

For example, the roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{where } a \neq 0$$

are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.3.2 Iterative Methods

Iterative methods, also known as *trial and error* methods, are based on the idea of successive approximations. They start with one or more initial approximations to the root and obtain a sequence of approximations by repeating a fixed sequence of steps till the solution with reasonable accuracy is obtained. Iterative methods, generally, give one root at a time.

Iterative methods are very cumbersome and time-consuming for solving non-linear equations manually. However, they are best suited for use on computers, due to following reasons:

1. Iterative methods can be concisely expressed as computational algorithms.
2. It is possible to formulate algorithms that can handle class of similar problems. For example, an algorithm can be developed to solve a polynomial equation of degree n.
3. Round-off errors are negligible in iterative methods as compared to direct methods.

3.4 CHOOSING INITIAL APPROXIMATION

The best way to choose an initial approximation to the root of equation

$$f(x) = 0$$

is either to plot the function $f(x)$ or to tabulate it. The roots of the equation are the points where the curve representing the function $f(x)$ intersects the x-axis. Therefore, any point in the interval where the function changes its sign can be taken as the initial approximation.

However, if the equation

$$f(x) = 0$$

can be written as

$$f_1(x) = f_2(x)$$

then the points of intersection of the graphs of equations

$$y = f_1(x) \text{ and } y = f_2(x)$$

give the root of the equation $f(x) = 0$. Therefore, any value in the neighbourhood of this intersection point can be taken as the initial approximation.

Algorithm 3.1: To tabulate an arbitrary function

Let $f(x)$ is the given function, x_{min} and x_{max} are the end points of the interval over which the function $f(x)$ is to be tabulated, and dx is the increment for x.

```
Begin
    read: xmin, xmax, dx
    set x = xmin
    while ( x ≤ xmax ) do
        set y = f(x)
        write: x, y
        set x = x + dx
    endwhile
End.
```

For illustration, consider the equation

$$2^x - x - 3 = 0$$

which has two distinct roots. We are interested to find the intervals that contain the root. In order to do that, the values of

$$f(x) = 2^x - x - 3$$

are tabulated for various values of x. These values are listed in the Table 3.1.

Table 3.1: Tabulated data for function $f(x) = 2^x - x - 3$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	1.0625	0.125	−0.75	−1.5	−2	−2	−1	2	9

The root lies between those points where there is a sign change in $f(x)$. Hence, the root lies in the interval (−3, −2) and (2, 3).

We can state this result more concisely and in a form more suited for use on computers as

> If $f(x)$ is a continuous function and $f(a)\times f(b) < 0$, then the equation $f(x) = 0$ has a root (say μ) lying in the interval (a, b) or in the interval (b, a) if $b < a$.

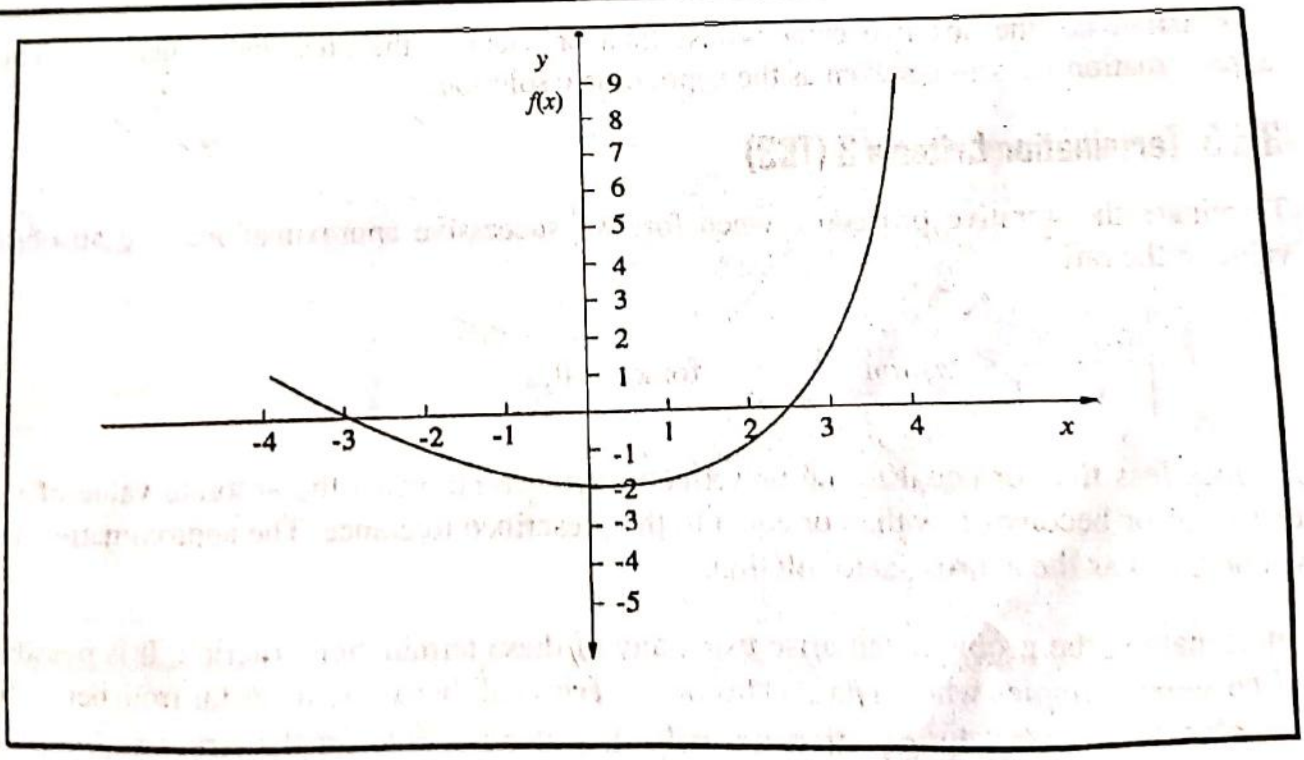

Figure 3.1: Plot of function $f(x) = 2^x - x - 3$

3.5 WHEN TO TERMINATE AN ITERATIVE PROCEDURE?

A very important question that must be addressed as regards to iterative methods is — *When to terminate the iterative procedure?* In general, we can say that iterative procedure is continued till the required degree of accuracy in the solution is achieved. *But how this degree of accuracy can be measured?* In order to obtain more practical answer, assume that the prescribed tolerance in the root is *epsilon*.

3.5.1 Termination Criteria 1 (TC1)

Suppose that starting with x_i as the current approximation, x_{i+1} is the next approximation, the iterative procedure would terminate when the inequality

$$|f(x_{i+1})| \leq epsilon$$

is satisfied. The approximation x_{i+1} will be taken as the approximate solution.

3.5.2 Termination Criteria 2 (TC2)

Terminate the iterative procedure when two successive approximations differ by an amount less than or equal to the prescribe tolerance. If x_i and x_{i+1} are two successive approximation, the iterative procedure would terminate when the inequality

$$|x_{i+1} - x_i| \leq epsilon$$

is satisfied i.e. the absolute error is less than or equal to the prescribed tolerance. The approximation x_{i+1} will be taken as the approximate solution.

3.5.3 Termination Criteria 3 (TC3)

Terminate the iterative procedure when for two successive approximations, the absolute value of the ratio

$$\left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \leq epsilon \qquad \text{for } x_{i+1} \neq 0$$

becomes less than or equal to the prescribed tolerance i.e. when the absolute value of the relative error becomes less than or equal to the prescribed tolerance. The approximation x_{i+1} will be taken as the approximate solution.

Unfortunately, the problem can arise using any of these termination criteria's. It is possible to find some examples where $|f(x_{i+1})|$ *becomes very small but* x_{i+1} *remains far from being the reasonable solution of the equation* $f(x) = 0$. *It is also possible for the term* $|x_{i+1} - x_i|$ *to become very small without giving reasonable solution of the equation. In general, it is good to use TC3, and it has been proved to be the most reliable of the three termination criteria.*

With this background, we will now discuss some popular iterative methods to find the solution of non-linear equations.

3.6 BISECTION METHOD REVISITED

Bisection method, also known as Bolzano method, is one of the simplest iterative methods. To start with, two initial approximations, say x_1 and x_2 such that $f(x_1)\times f(x_2) < 0$ which ensures that root lies between x_1 and x_2, are taken. The next x-value, say x_3, as the mid point of the interval $[x_1, x_2]$ is computed.

There are three possibilities that can arise:

i) If $f(x_3) = 0$, then we have a root at x_3.

ii) If $f(x_1)$ and $f(x_3)$ are of opposite sign, then the root lies in the interval (x_1, x_3). Thus x_2 is replaced by x_3, and the new interval, which is half of the current interval, is again bisected.

iii) If $f(x_1)$ and $f(x_3)$ are of same sign, then the root lies in the interval (x_3, x_2). Thus x_1 is replaced by x_3, and the new interval is again bisected.

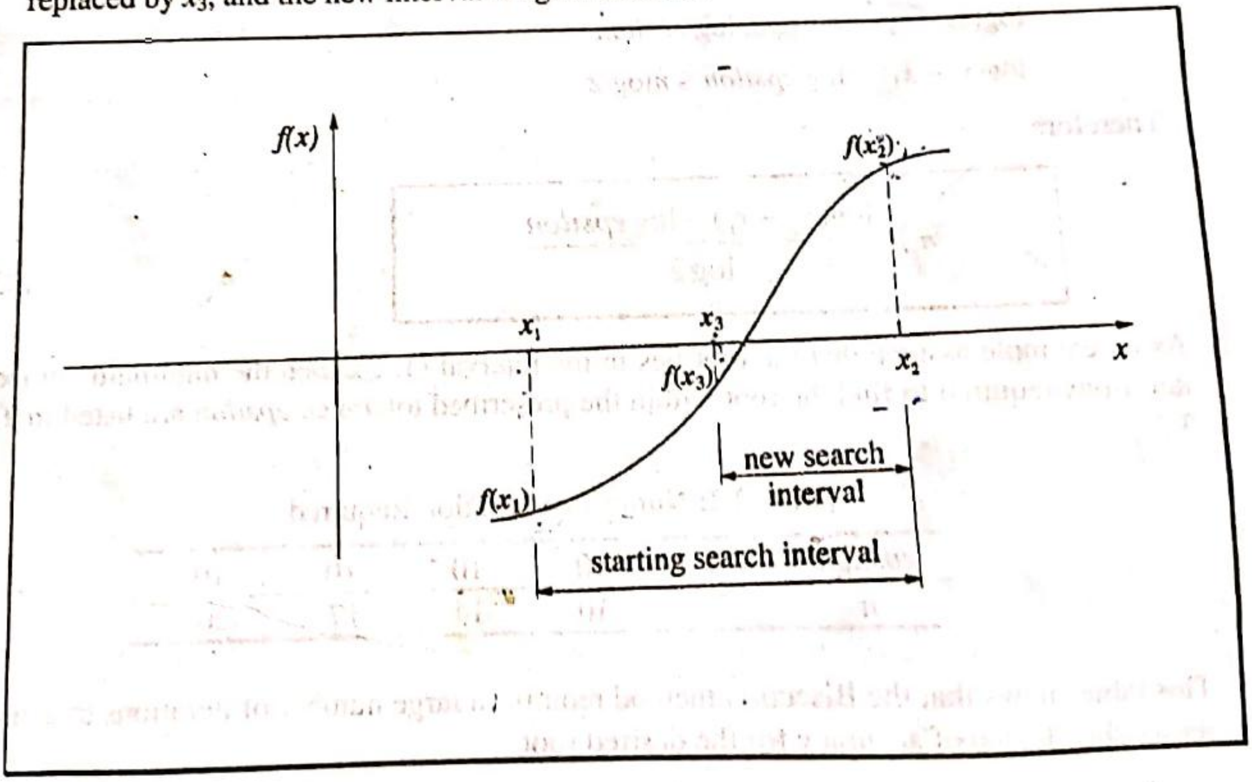

Figure 3.2: Root approximation by Bisection Method

Therefore by repeating this interval bisection procedure, we keep enclosing the root in a new search interval, which is halved in each iteration. This iterative cycle is terminated when the search interval becomes smaller than the prescribed tolerance. Hence, if *epsilon* is the prescribed tolerance in the required root, then the iterative cycle terminates when the absolute error becomes less than or equal to *epsilon* i.e.

$$|x_1 - x_2| \le epsilon$$

After repeating the bisection process n times, the search interval is reduced to length $\frac{x_2 - x_1}{2^n}$ that contains the required root. We take the mid point of the last search interval as the desired approximation to the root.

For prescribed tolerance *epsilon*, the approximate number of iterations required can be determined from the relation

$$\frac{x_2 - x_1}{2^n} \le epsilon$$

as described next.

Taking logarithm on both sides of the inequality, we get

$$\log(x_2 - x_1) - n\log 2 \le \log epsilon$$

$$\log(x_2 - x_1) - \log epsilon \le n\log 2$$

Therefore

$$\boxed{n \ge \frac{\log(x_2 - x_1) - \log epsilon}{\log 2}} \quad \ldots (3.3)$$

As an example assume that the root lies in the interval (1, 2), then the minimum number of iterations required to find the root within the prescribed tolerance *epsilon* are listed in Table 3.2.

Table 3.2: Number of Iteration Required

epsilon	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
n	7	10	14	17	20

This table shows that the Bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the desired root.

However, it requires only one function evaluation for each iteration. If the evaluation of function $f(x)$ is rapid, then the use of Bisection method is strongly recommended.

Example 3.1: Given that one root of the non-linear equation

$$x^3 - 4x - 9 = 0$$

lies between 2.625 and 2.75. Find the root correct to four significant digits.

Solution: Since we want the solution correct to four significant digits, the iterative process will be terminated as soon as the successive iterations produces no change at first four significant positions or the function vanishes at new approximation.

Iteration 1: Starting with $x_1 = 2.625$ and $x_2 = 2.75$

$$f(x_1) = f(2.625) = 2.625^3 - 4 \times 2.625 - 9 = -1.4121$$
$$f(x_2) = f(2.75) = 2.75^3 - 4 \times 2.75 - 9 = 0.7969$$
$$x_3 = (2.625+2.75)/2 = 2.6875$$
$$f(x_3) = f(2.6875) = 2.6875^3 - 4 \times 2.6875 - 9 = -0.3391$$

by mid point $x_3 = \frac{x_1 + x_2}{2}$

Since $f(x_1) \times f(x_3) = -1.4121 \times (-0.3391) > 0$ opposite sign.

Therefore replace x_1 by x_3. The new search interval becomes (2.6875, 2.75).

Iteration 2: Now we take $x_1 = 2.6875$ and $x_2 = 2.75$

$$f(x_1) = f(2.6875) = 2.6875^3 - 4 \times 2.6875 - 9 = -0.3391$$
$$f(x_2) = f(2.75) = 2.75^3 - 4 \times 2.75 - 9 = 0.7969$$
$$x_3 = (2.6875+2.75)/2 = 2.7186$$
$$f(x_3) = f(2.7186) = 2.7186^3 - 4 \times 2.7186 - 9 = 0.2182$$

Since $f(x_1) \times f(x_3) = -0.3391 \times 0.2181 < 0$

Therefore replace x_2 by x_3. The new search interval becomes (2.6875, 2.7186).

Iteration 3: Now we take $x_1 = 2.6875$ and $x_2 = 2.7186$

$$f(x_1) = f(2.6875) = 2.6875^3 - 4 \times 2.6875 - 9 = -0.3391$$
$$f(x_2) = f(2.7186) = 2.7186^3 - 4 \times 2.7186 - 9 = 0.2182$$
$$x_3 = (2.6875+2.7186)/2 = 2.7031$$
$$f(x_3) = f(2.7031) = 2.7031^3 - 4 \times 2.7031 - 9 = -0.0615$$

Since $f(x_1) \times f(x_3) = -0.3391 \times (-0.06215) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.7031, 2.7186).

Iteration 4: Now we take $x_1 = 2.7031$ and $x_2 = 2.7186$

$$f(x_1) = f(2.7031) = 2.7031^3 - 4 \times 2.7031 - 9 = -0.0615$$
$$f(x_2) = f(2.7186) = 2.7186^3 - 4 \times 2.7186 - 9 = 0.2182$$
$$x_3 = (2.7031+2.7186)/2 = 2.7109$$
$$f(x_3) = f(2.7109) = 2.7109^3 - 4 \times 2.7109 - 9 = 0.0787$$

Since $f(x_1) \times f(x_3) = -0.06215 \times 0.0787 < 0$

Therefore replace x_2 by x_3. The new search interval becomes (2.7031, 2.7109).

Iteration 5: Now we take $x_1 = 2.7031$ and $x_2 = 2.7109$

$$f(x_1) = f(2.7031) = 2.7031^3 - 4 \times 2.7031 - 9 = -0.0615$$
$$f(x_2) = f(2.7109) = 2.7109^3 - 4 \times 2.7109 - 9 = 0.0787$$
$$x_3 = (2.7031+2.7109)/2 = 2.707$$
$$f(x_3) = f(2.707) = 2.707^3 - 4 \times 2.707 - 9 = 0.0085$$

Since $f(x_1) \times f(x_3) = -0.06215 \times 0.0085 < 0$

Therefore replace x_2 by x_3. The new search interval becomes (2.7031, 2.707).

Iteration 6: Now we take $x_1 = 2.7031$ and $x_2 = 2.707$

$$f(x_1) = f(2.7031) = 2.7031^3 - 4 \times 2.7031 - 9 = -0.0615$$
$$f(x_2) = f(2.707) = 2.707^3 - 4 \times 2.707 - 9 = 0.0085$$
$$x_3 = (2.7031+2.707)/2 = 2.7051$$
$$f(x_3) = f(2.7051) = 2.7051^3 - 4 \times 2.7051 - 9 = -0.0257$$

Since $f(x_1) \times f(x_3) = -0.06215 \times (-0.0257) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.7051, 2.707).

Iteration 7: Now we take $x_1 = 2.7051$ and $x_2 = 2.707$

$$f(x_1) = f(2.7051) = 2.7051^3 - 4 \times 2.7051 - 9 = -0.0257$$
$$f(x_2) = f(2.707) = 2.707^3 - 4 \times 2.707 - 9 = 0.0085$$
$$x_3 = (2.7051+2.707)/2 = 2.7061$$

$f(x_3) = f(2.7061) = 2.7061^3 - 4 \times 2.7061 - 9 = -0.0077$

Since $\quad f(x_1) \times f(x_3) = -0.0257 \times (-0.0077) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.7061, 2.707).

Iteration 8: Now we take $x_1 = 2.7061$ and $x_2 = 2.707$

$f(x_1) = f(2.7061) = 2.7061^3 - 4 \times 2.7061 - 9 = -0.0077$
$f(x_2) = f(2.707) = 2.707^3 - 4 \times 2.707 - 9 = 0.0085$
$x_3 = (2.7061+2.707)/2 = 2.7066$
$f(x_3) = f(2.7066) = 2.7066^3 - 4 \times 2.7066 - 9 = 0.0013$

Since $\quad f(x_1) \times f(x_3) = -0.0077 \times 0.0013 < 0$

Therefore replace x_2 by x_3. The new search interval becomes (2.7061, 2.7066).

Iteration 9: Now we take $x_1 = 2.7061$ and $x_2 = 2.7066$

$f(x_1) = f(2.7061) = 2.7061^3 - 4 \times 2.7061 - 9 = -0.0077$
$f(x_2) = f(2.7066) = 2.7066^3 - 4 \times 2.7066 - 9 = 0.0013$
$x_3 = (2.7061+2.7066)/2 = 2.7064$
$f(x_3) = f(2.7064) = 2.7064^3 - 4 \times 2.7064 - 9 = -0.0023$

Since $\quad f(x_1) \times f(x_3) = -0.0077 \times (-0.0023) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.7064, 2.7066).

Observe that iterations 7, 8 and 9 produce no change at the first four significant positions. Therefore, we take $x = 2.706$ as the desired solution which is correct to four significant digits.

Example 3.2: Find the root of equation

$$x^3 - x - 4 = 0$$

correct to four significant digits using Bisection method.

Solution: Since we want the solution correct to four significant digits, the iterative process will be terminated as soon as the successive iterations produces no change at first four significant positions or the function vanishes at new approximation.

Now $f(1.75) = 1.75^3 - 1.75 - 4 = -0.3906$

$f(1.8) = 1.8^3 - 1.8 - 4 = 0.0320$

Since $f(1.75)$ is –ve and $f(1.8)$ is +ve, therefore one root lies between 1.75 & 1.8.

Iteration 1: Starting with $x_1 = 1.75$ and $x_2 = 1.8$

$x_3 = (1.75+1.8)/2 = 1.775$

$f(x_3) = f(1.775) = 1.775^3 - 1.775 - 4 = -0.1826$

Thus, the first approximation to root is 1.775.

Since $f(x_1) \times f(x_3) = -0.3906 \times (-0.1826) > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (1.775, 1.8).

Iteration 2: Now we take $x_1 = 1.775$ and $x_2 = 1.8$

$f(x_1) = f(1.775) = 1.775^3 - 1.775 - 4 = -0.1826$

$f(x_2) = f(1.8) = 1.8^3 - 1.8 - 4 = 0.0320$

$x_3 = (1.775+1.8)/2 = 1.7875$

$f(x_3) = f(1.7875) = 1.7875^3 - 1.7875 - 4 = -0.0762$

Thus, the second approximation to root is 1.7875.

Since $f(x_1) \times f(x_3) = -0.1826 \times -(0.0762) > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (1.7875, 1.8).

Iteration 3: Now we take $x_1 = 1.7875$ and $x_2 = 1.8$

$f(x_1) = f(1.7875) = 1.7875^3 - 1.7875 - 4 = -0.0762$

$f(x_2) = f(1.8) = 1.8^3 - 1.8 - 4 = 0.0320$

$x_3 = (1.7875+1.8)/2 = 1.7937$

$f(x_3) = f(1.7937) = 1.7937^3 - 1.7937 - 4 = -0.0223$

Thus, the third approximation to root is 1.7937.

Since $f(x_1) \times f(x_3) = -0.0762 \times -(0.0223) > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (1.7937, 1.8).

Iteration 4: Now we take $x_1 = 1.7937$ and $x_2 = 1.8$

$$f(x_1) = f(1.7937) = 1.7937^3 - 1.7937 - 4 = -0.0223$$
$$f(x_2) = f(1.8) = 1.8^3 - 1.8 - 4 = 0.0320$$
$$x_3 = (1.7937+1.8)/2 = 1.7969$$
$$f(x_3) = f(1.7969) = 1.7969^3 - 1.7969 - 4 = 0.0048$$

Thus, the fourth approximation to root is 1.7969.

Since $\quad f(x_1) \times f(x_3) = -0.0223 \times 0.0048 < 0$

Therefore, replace x_2 by x_3. The new search interval becomes (1.7937, 1.7969).

Subsequent iterations will produce the following approximations to the root

1.7953, 1.7961, 1.7965, 1.7963, 1.7964

Observe the last four iterations, they produce no change at the first four significant positions. Therefore, we take $x = 1.796$ as the desired solution which is correct to four significant digits.

Example 3.3: Find the root of equation

$$x^3 - 5x + 3 = 0$$

correct to three decimal positions using Bisection method.

Solution: Since we want the solution correct to three decimal positions, the iterative process will be terminated as soon as the successive iterations produces no change at first three decimal positions or the function vanishes at new approximation.

Now $\quad f(0.5) = 0.5^3 - 5\times0.5 + 3 = 0.625$

$\quad\quad f(1.0) = 1.0^3 - 5\times1.0 + 3 = -1.0$

Since $f(0.5)$ is +ve and $f(1.0)$ is –ve; therefore one root lies between 0.5 and 1.0.

Iteration 1: Starting with $x_1 = 0.5$ and $x_2 = 1.0$

$$x_3 = (0.5+1.0)/2 = 0.75$$
$$f(0.75) = 0.75^3 - 5\times0.75 + 3 = -0.3281$$

Thus, the first approximation to root is 0.75.

Since $\quad f(x_1) \times f(x_3) = 0.625 \times (-0.3281) < 0$

Therefore, replace x_2 by x_3. The new search interval becomes (0.5, 0.75).

Iteration 2: Now we take $x_1 = 0.5$ and $x_2 = 0.75$

$$f(x_1) = f(0.5) = 0.5^3 - 5\times0.5 + 3 = 0.625$$
$$f(x_2) = f(0.75) = 0.75^3 - 5\times0.75 + 3 = -0.3281$$
$$x_3 = (0.5+0.75)/2 = 0.625$$
$$f(0.625) = 0.625^3 - 5\times0.625 + 3 = 0.1191$$

Thus, the second approximation to root is 0.625.

Since $\quad f(x_1) \times f(x_3) = 0.625 \times 0.1191 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.625, 0.75).

Iteration 3: Now we take $x_1 = 0.625$ and $x_2 = 0.75$

$$f(x_1) = f(0.625) = 0.625^3 - 5\times0.625 + 3 = 0.1191$$
$$f(x_2) = f(0.75) = 0.75^3 - 5\times0.75 + 3 = -0.3281$$
$$x_3 = (0.625+0.75)/2 = 0.6875$$
$$f(x_3) = f(0.6875) = 0.6875^3 - 5\times0.6875 + 3 = -0.1125$$

Thus, the third approximation to root is 0.6875.

Since $\quad f(x_1) \times f(x_3) = 0.1191 \times (-0.1125) < 0$

Therefore, replace x_2 by x_3. The new search interval becomes (0.625, 0.6875).

Iteration 4: Now we take $x_1 = 0.625$ and $x_2 = 0.6875$

$$f(x_1) = f(0.625) = 0.625^3 - 5\times0.625 + 3 = 0.1191$$
$$f(x_2) = f(0.6875) = 0.6875^3 - 5\times0.6875 + 3 = -0.1125$$
$$x_3 = (0.625+0.6875)/2 = 0.6562$$
$$f(x_3) = f(0.6562) = 0.6562^3 - 5\times0.6562 + 3 = 0.0014$$

Thus, the fourth approximation to root is 0.6562.

False ⇒ Regula falsi method
Bisection ⇒ Bolzano method.

Since $f(x_1) \times f(x_3) = 0.1191 \times 0.0014 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.6562, 0.6875).

Subsequent iterations will produce the following approximations to the root

0.6719, 0.6641, 0.6602, 0.6682, 0.6572, 0.6567, 0.6565. 0.6566

Observe the last three iterations, they produce no change at the first three decimal positions. Therefore, we take $x = 0.656$ as the desired solution which is correct to three decimal positions.

Algorithm 3.2: Bisection method imp

To find a root of $f(x) = 0$ within a prescribed tolerance say *epsilon*. Given values x_1 and x_2 such that $f(x_1) \times f(x_2) < 0$. The variable x_3 is used to store mid point of the interval.

```
Begin
  read: x1, x2                          // input value for x1and x2
  read: epsilon                         // input the prescribed tolerance
  do
    set x3 = ( x1+x2)/2                 // compute the mid point
    if (f(x1)×f(x3)) < 0) then          // select the appropriate subinterval
       set x2 = x3
    else
       set x1 = x3
    endif
  while ( | x1 - x2 | > epsilon )
  write: x3, "as the approximate root"     // output the computed root
End.
```

3.7 FALSE POSITION METHOD

Though the bisection method guarantees that iterative process will converge, but its convergence is slow. The false position method, also known as *regula-falsi* or *method of linear interpolation*, is similar to the bisection method but faster than it. It also starts with two initial approximations to the root, say x_1 and x_2, for which $f(x)$ has opposite signs, and then by linear interpolation the next approximation is determined.

To describe its working assume that x_1 and x_2 are two initial approximations to root for which $f(x)$ has opposite signs. Join the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ by a straight line. The point where this line intersects the x-axis is the next approximation to the root. Let us suppose that the line intersects the x-axis at x_3. This is illustrated in Figure 3.3.

There are three possibilities:

i) If $f(x_3) = 0$, then we have a root at x_3.

ii) If $f(x_1)$ and $f(x_3)$ are of opposite sign, then the root lies in the interval (x_1, x_3). Thus x_2 is replaced by x_3 and the iterative procedure is repeated.

iii) If $f(x_1)$ and $f(x_3)$) are of same sign, then the root lies in the interval (x_3, x_2). Thus x_1 is replaced by x_3, and the iterative procedure is repeated.

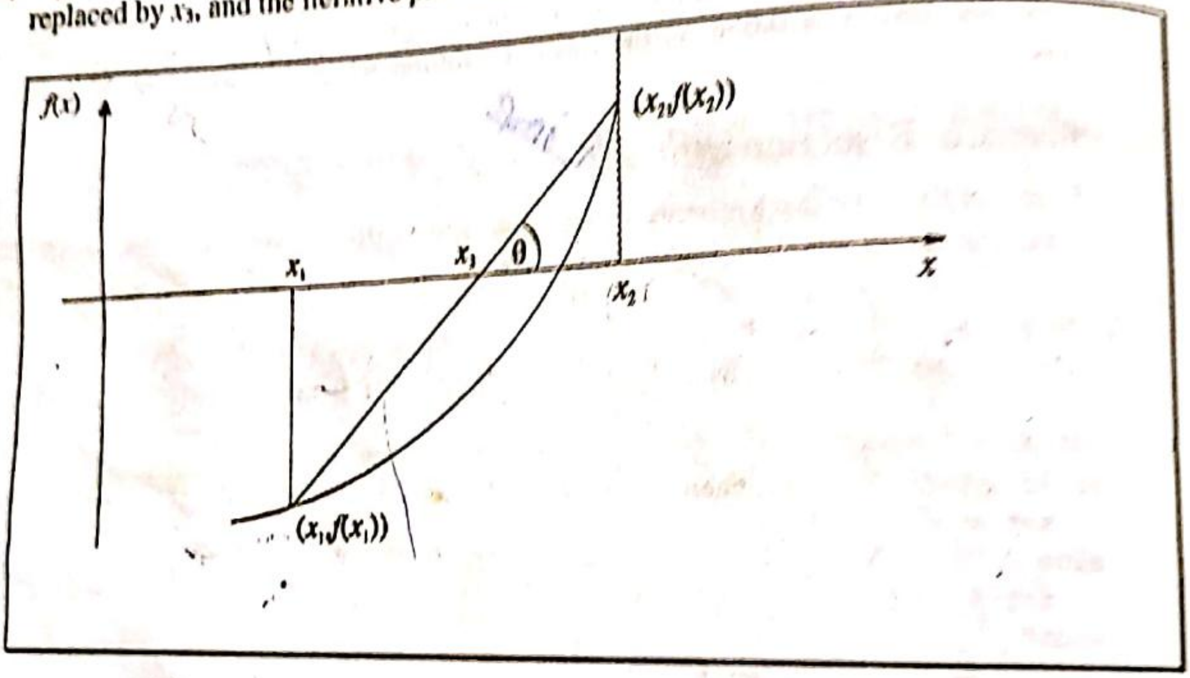

Figure 3.3: Root approximation by false position method

The iterative procedure terminates when the size of the search interval becomes less than the prescribed tolerance.

For derivation of the computational formula to find the intersection point, refer to Figure 3.3.

Slope of the line joining the points $(x_3, 0)$ and $(x_2, f(x_2))$ is given by

$$\tan\theta = \frac{f(x_2)}{x_2 - x_3} \qquad \ldots (3.4)$$

Slope of the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$\tan\theta = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \ldots (3.5)$$

Since the line joining the points $(x_3, 0)$ and $(x_2, f(x_2))$ is a part of the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, therefore

$x^2 - 16 = 0$

3 5

$$\frac{f(x_2)}{x_2 - x_3} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Solving for x_3, we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \qquad \ldots (3.6)$$

$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

which gives the next approximation to the root.

In general, the $(i+1)^{\text{th}}$ approximation to the root is given by the formula

$n = 2$

where

$$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}} \qquad \ldots (3.7)$$

The expression $f(x_i) - f(x_{i-1})$ represents the slope of the curve in current interval. And if the value of this expression becomes very small, it will introduce a large amount of error in the new approximation.

Therefore, for computer implementation, we must put a limit on the permissible slope of the curve so that the iterative procedure does not fall in an endless loop.

Example 3.4: Given that one of the roots of the non-linear equation

$$x^3 - 2x - 5 = 0$$

lies in the interval (1.75, 2.5). Find the root correct to four significant digits.

Solution: Since we want the solution correct to four significant digits, the iterative process will be terminated as soon as the successive iterations produces no change at first four significant positions or the function vanishes at new approximation.

Iteration 1: Starting with $x_1 = 1.75$ and $x_2 = 2.5$

$$f(x_1) = f(1.75) = 1.75^3 - 2 \times 1.75 - 5 = -3.1406$$

$$f(x_2) = f(2.5) = 2.5^3 - 2 \times 2.5 - 5 = 5.625$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{1.75 \times 5.625 - 2.5 \times (-3.1406)}{5.625 - (-3.1406)} = 2.0187$$

$$f(x_3) = f(2.0187) = 2.0187^3 - 2 \times 2.0187 - 5 = -0.8109$$

Since $\quad f(x_1) \times f(x_3) = -3.1406 \times (-0.8109) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.0187, 2.5).

Iteration 2: Now we take $x_1 = 2.0187$ and $x_2 = 2.5$

$$f(x_1) = f(2.0187) = 2.0187^3 - 2 \times 2.0187 - 5 = -0.8109$$
$$f(x_2) = f(2.5) = 2.5^3 - 2 \times 2.5 - 5 = 5.625$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.0187 \times 5.625 - 2.5 \times (-0.8109)}{5.625 - (-0.8109)} = 2.0793$$
$$f(x_3) = f(2.0793) = 2.0793^3 - 2 \times 2.0793 - 5 = -0.1688$$

Since $\quad f(x_1) \times f(x_3) = -0.8109 \times (-0.1688) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.0793, 2.5).

Iteration 3: Now we take $x_1 = 2.0793$ and $x_2 = 2.5$

$$f(x_1) = f(2.0793) = 2.0793^3 - 2 \times 2.0793 - 5 = -0.1688$$
$$f(x_2) = f(2.5) = 2.5^3 - 2 \times 2.5 - 5 = 5.625$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.0793 \times 5.625 - 2.5 \times (-0.1688)}{5.625 - (-0.1688)} = 2.0916$$
$$f(x_3) = f(2.0916) = 2.0916^3 - 2 \times 2.0916 - 5 = -0.0329$$

Since $\quad f(x_1) \times f(x_3) = -0.1688 \times (-0.0329) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.0916, 2.5).

Iteration 4: Now we take $x_1 = 2.0916$ and $x_2 = 2.5$

$$f(x_1) = f(2.0916) = 2.0916^3 - 2 \times 2.0916 - 5 = -0.0329$$
$$f(x_2) = f(2.5) = 2.5^3 - 2 \times 2.5 - 5 = 5.625$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.0916 \times 5.625 - 2.5 \times (-0.0329)}{5.625 - (-0.0329)} = 2.0940$$
$$f(x_3) = f(2.0940) = 2.0940^3 - 2 \times 2.0940 - 5 = -0.0062$$

Since $\quad f(x_1) \times f(x_3) = -0.03299 \times (-0.0062) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.0940, 2.5).

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Iteration 5: Now we take $x_1 = 2.0940$ and $x_2 = 2.5$

$$f(x_1) = f(2.0940) = 2.0940^3 - 2 \times 2.0940 - 5 = -0.0062$$
$$f(x_2) = f(2.5) = 2.5^3 - 2 \times 2.5 - 5 = 5.625$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.0940 \times 5.625 - 2.5 \times (-0.0062)}{5.625 - (-0.0062)} = 2.0944$$
$$f(x_3) = f(2.0944) = 2.0944^3 - 2 \times 2.0944 - 5 = -0.0017$$

Since $\quad f(x_1) \times f(x_3) = -0.0062 \times (-0.0017) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.0944, 2.5).

Iteration 6: Now we take $x_1 = 2.0944$ and $x_2 = 2.5$

$$f(x_2) = f(2.0944) = 2.0944^3 - 2 \times 2.0944 - 5 = -0.0017$$
$$f(x_2) = f(2.5) = 2.5^3 - 2 \times 2.5 - 5 = 5.625$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.0944 \times 5.625 - 2.5 \times (-0.0017)}{5.625 - (-0.0017)} = 2.0945$$
$$f(x_2) = f(2.0945) = 2.0945^3 - 2 \times 2.0945 - 5 = -0.0006$$

Since $\quad f(x_1) \times f(x_3) = -0.0017 \times (-0.0006) > 0$

Therefore replace x_1 by x_3. The new search interval becomes (2.0945, 2.5).

Observe that iterations 4, 5 and 6 produce no change at the first four significant positions in the successive approximations to the root. Therefore, we take $x = 2.094$ as the desired solution correct to four significant digits.

Example 3.5: Find the root of equation

$$\cos x - 3x + 1 = 0$$

correct to three decimal positions using False position method.

Solution: Since we want the solution correct to three decimal positions, the iterative process will be terminated as soon as the successive iterations produces no change at first three decimal positions or the function vanishes at new approximation.

Now $\quad f(0) = \cos(0) - 3\times0 + 1 = 2.0$

$\quad f(1) = \cos(1) - 3\times1 + 1 = -1.4597$

Since $f(0)$ is +ve and $f(1)$ is −ve, therefore one root lies between 0 and 1.

Iteration 1: Starting with $x_1 = 0$ and $x_2 = 1$

$$f(0) = \cos(0) - 3\times 0 + 1 = 2.0$$
$$f(1) = \cos(1) - 3\times 1 + 1 = -1.4597$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0\times(-1.4597) - 1\times 2.0}{-1.4597 - 2.0} = 0.5781$$
$$f(x_3) = f(0.5781) = \cos(0.5781) - 3\times 0.5781 + 1 = 0.1033$$

Thus, the first approximation to the root is 0.5781.

Since $\quad f(x_1) \times f(x_3) = 2.0 \times 0.1033 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.5781, 1.0).

Iteration 2: Now we take $x_1 = 0.5781$ and $x_2 = 1.0$

$$f(x_1) = f(0.5781) = \cos(0.5781) - 3\times 0.5781 + 1 = 0.1033$$
$$f(x_2) = f(1) = \cos(1) - 3\times 1 + 1 = -1.4597$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.5781\times(-1.4597) - 1\times 0.1033}{-1.4597 - 0.1033} = 0.6060$$
$$f(x_3) = f(0.6060) = \cos(0.6060) - 3\times 0.6060 + 1 = 0.0041$$

Thus, the second approximation to the root is 0.6060.

Since $\quad f(x_1) \times f(x_3) = 0.1033 \times 0.0041 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.6060, 1.0).

Iteration 3: Now we take $x_1 = 0.6060$ and $x_2 = 1.0$

$$f(x_1) = f(0.6060) = \cos(0.6060) - 3\times 0.6060 + 1 = 0.0041$$
$$f(x_2) = f(1) = \cos(1) - 3\times 1 + 1 = -1.4597$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.6060\times(-1.4597) - 1\times 0.0041}{-1.4597 - 0.0041} = 0.6071$$
$$f(x_3) = f(0.6071) = \cos(0.6071) - 3\times 0.6071 + 1 = 0.0002$$

Thus, the third approximation to the root is 0.6071.

Since $f(x_1) \times f(x_3) = 0.0041 \times 0.0002 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.6071, 1.0).

Iteration 4: Now we take $x_1 = 0.6071$ and $x_2 = 1.0$

$$f(x_1) = f(0.6071) = \cos(0.6071) - 3\times0.6071 + 1 = 0.0002$$
$$f(x_2) = f(1) = \cos(1) - 3\times1 + 1 = -1.4597$$
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.6071 \times (-1.4597) - 1 \times 0.0002}{-1.4597 - 0.0002} = 0.6072$$
$$f(x_3) = f(0.6071) = \cos(0.6071) - 3\times0.6071 + 1 = 0.0001$$

Thus, the fourth approximation to the root is 0.6072.

Since last two iterations produce no change at the first three decimal positions, therefore, we take $x = 0.607$ as the desired solution correct to three decimal positions.

Example 3.6: Find the root of equation

$$\cos x - xe^x = 0$$

correct to three decimal places using False position method.

Solution: Since we want the solution correct to three decimal places, the iterative process will be terminated as soon as the successive iterations produces no change at three decimal places or the function vanishes at new approximation.

Now $f(0.5) = \cos(0.5) - 0.5 \times e^{0.5} = 0.0532$

$$f(0.75) = \cos(0.75) - 0.75 \times e^{0.75} = -0.8561$$

Since $f(0.5)$ is +ve and $f(0.75)$ is –ve, therefore one root lies between 0.5 & 0.75.

Iteration 1: Starting with $x_1 = 0.5$ and $x_2 = 0.75$

0.487

$$f(x_1) = f(0.5) = \cos(0.5) - 0.5 \times e^{0.5} = 0.0532$$
$$f(x_2) = f(0.75) = \cos(0.75) - 0.75 \times e^{0.75} = -0.8561$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.5 \times (-0.8561) - 0.75 \times 0.0532}{-0.8561 - 0.0532} = 0.5146$$

$$f(x_3) = f(0.5146) = \cos(0.5146) - 0.5146 \times e^{0.5146} = 0.0095$$

Thus, the first approximation to the root is 0.5146.

Since $f(x_1) \times f(x_3) = 0.0532 \times 0.0095 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.5146, 0.75).

Iteration 2: Now we take $x_1 = 0.5146$ and $x_2 = 0.75$

$$f(x_1) = f(0.5146) = \cos(0.5146) - 0.5146 \times e^{0.5146} = 0.0095$$

$$f(x_2) = f(0.75) = \cos(0.75) - 0.75 \times e^{0.75} = -0.8561$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.5146 \times (-0.8561) - 0.75 \times 0.0095}{-0.8561 - 0.0095} = 0.5172$$

$$f(x_3) = f(0.5172) = \cos(0.5172) - 0.5172 \times e^{0.5172} = 0.0017$$

Thus, the second approximation to the root is 0.5172.

Since $f(x_1) \times f(x_3) = 0.0532 \times 0.0017 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.5172, 0.75).

Iteration 3: Now we take $x_1 = 0.5172$ and $x_2 = 0.75$

$$f(x_1) = f(0.5172) = \cos(0.5172) - 0.5172 \times e^{0.5172} = 0.0017$$

$$f(x_2) = f(0.75) = \cos(0.75) - 0.75 \times e^{0.75} = -0.8561$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.5172 \times (-0.8561) - 0.75 \times 0.0017}{-0.8561 - 0.0017} = 0.5177$$

$$f(x_3) = f(0.5177) = \cos(0.5177) - 0.5177 \times e^{0.5177} = 0.0003$$

Thus, the third approximation to the root is 0.5177.

Since $f(x_1) \times f(x_3) = 0.0532 \times 0.0003 > 0$

Therefore, replace x_1 by x_3. The new search interval becomes (0.5177, 0.75).

Iteration 4: Now we take $x_1 = 0.5177$ and $x_2 = 0.75$

$$f(x_1) = f(0.5177) = \cos(0.5177) - 0.5177 \times e^{0.5177} = 0.0003$$

$$f(x_2) = f(0.75) = \cos(0.75) - 0.75 \times e^{0.75} = -0.8561$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0.5177 \times (-0.8561) - 0.75 \times 0.0003}{-0.8561 - 0.0003} = 0.5178$$

$$f(x_3) = f(0.5178) = \cos(0.5178) - 0.5178 \times e^{0.5178} = 0.0000$$

Thus, the fourth approximation to the root is 0.5178.

Since, the last three iterations produce no change in first three decimal positions, and also the function $f(x)$ vanishes after fourth iteration. Therefore, we take $x = 0.517$ as solution correct to three decimal positions.

Algorithm 3.3: False Position Method

Here $f(x)$ is given function, x_1 and x_2 are two initial approximations, *epsilon* is the prescribed tolerance in the root, and *delta* is the prescribed lower bound for the slope of $f(x)$.

```
Begin
    read: x1, x2, epsilon, delta
    set f1 = f(x1)
    set f2 = f(x2)
    if ( f1 × f2 > 0 ) then
        write: "Initial approximation are unsuitable"
        exit
    endif
    do
        if ( | f2 - f1 | ≤ delta ) then
            write: "Slope of the function becomes too small"
            exit
        endif
        set x3 = (x1×f2 - x2×f1) / (f2 - f1)
        set f3 = f(x3)
        if ( f1 × f3 < 0 ) then
            set x2 = x3
            set f2 = f3
        else
            set x1 = x3
            set f1 = f3
        endif
    while ( | x1 - x2 | > epsilon )
    write: x3, "as the approximate root"
End.
```

3.9 NEWTON-RAPHSON METHOD

The Newton-Raphson method, also known as Newton's method of tangents, is one of the fastest iterative methods. This method begins with one initial approximation. Here one have to take due care while selecting the initial approximation, as it is very sensitive to the initial approximation. Once proper choice is made for the initial approximation, it converges faster than false position method and the secant method.

newton

$x^3-2x^2-70x-2020$

3.9.1 Analytical Derivation

-3 -2 $3-4x+9$

$x_0 = 25$

4.885

-2.706

6. 4

Let $f(x) = 0$

be the given non-linear equation, and x_0 be the initial approximation, and h the correction to x_0 so that

$$f(x_0+h) = 0$$

Expanding $f(x_0+h)$ by Taylor's series, we get

$$f(x_0+h) = f(x_0) + hf^1(x_0) + \frac{h^2}{2!}f^2(x_0) + \ldots + \frac{h^n}{n!}f^n(x_0) \qquad \ldots (3.9)$$

where $f^1(x_0), f^2(x_0), \ldots$ are 1^{st} order, 2^{nd} order, ... derivatives of $f(x)$ at x_0.

If the values of h and $f^n(x_0)$ are such that

$$\boxed{\lim_{n\to\infty} \frac{h^n}{n!} f^n(x_0) = 0} \qquad \ldots (3.10)$$

then neglecting the higher order terms in h, i.e., $O(h^2)$, we get

$$f(x_0) + hf^1(x_0) = 0 \Rightarrow h = -\frac{f(x_0)}{f^1(x_0)} \qquad \ldots (3.11)$$

Therefore if x_0 is the initial approximation, by adding to this the value of h, we get the next approximation to the root i.e.

$$x_1 = x_0 + h$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} \qquad \cdots (3.12)$$

Similarly, taking x_1 as the starting value, we get

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

In general, from the i^{th} approximation, we obtain the $(i+1)^{th}$ approximation as

nth apr

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}} \qquad \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad \cdots (3.13)$$

The iterative procedure terminates when the relative error for two successive approximations becomes less than or equal to the prescribed tolerance.

3.9.2 Geometric Derivation and Interpretation

Consider that the initial approximation is x_0. Now draw a tangent to the curve at the point $(x_0, f(x_0))$. The point, say $(x_1, 0)$, where this tangent intersects the x-axis gives the next approximation to the root. Starting with x_1 as the current approximation, we again draw a tangent to the curve at the point $(x_1, f(x_1))$. The intersection point of these tangent gives, says x_2, as the next approximation to the root. This iterative procedure is repeated till the relative error between two successive approximations is less than or equal to the prescribed tolerance.

If the proper choice of the initial approximation is not made, the method may diverge. This causes the iterative procedure to fall in endless loop. Therefore, to provide an exit in such circumstances, we put a limit on the maximum number of iterations permitted.

The slope of the curve at point $(x_0, f(x_0))$ is given by

$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f^1(x_0)$$

Solving for x_1, we get

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} \qquad \ldots (3.14)$$

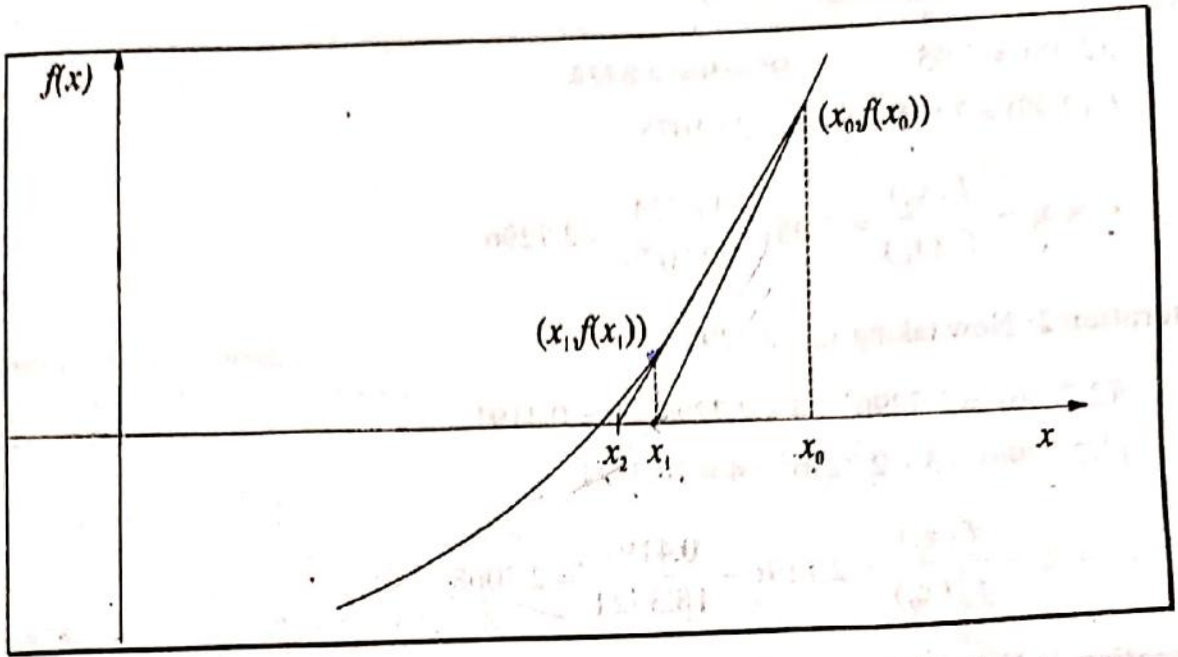

Figure 3.5: Root approximation by Newton-Raphson method

The next approximation would be

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$

In general, the $(i+1)^{th}$ approximation is obtained from the i^{th} approximation as

$(n+1)^{th}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

In the above formula, we assume that $f^1(x_i)$, the slope of the curve, is not equal to or nearly equal to zero. However, if the slope of the curve at x_i is too small, it may complicate the computations. Therefore, to avoid this we put a limit on the lower bound of the slope.

Example 3.10: Given that one of the roots of the non-linear equation

$$x^3 - 4x - 9 = 0$$

lies between 2.625 and 3.0. Find the root correct to four significant digits.

Solution: Since we want the solution correct to three significant digits, the iterative process will be terminated as soon as the successive iterations produces no change at first four significant positions or the function vanishes at new approximation.

Given $\quad f(x) = x^3 - 4x - 9$

$\frac{2.625+3.0}{2}$

therefore

$$f^1(x) = 3x^2 - 4$$

2.8125 $= \frac{5.625}{2} =$

Iteration 1: We start with $x_0 = 2.95$.

$$f(2.95) = 2.95^3 - 4 \times 2.95 - 9 = 4.8724$$
$$f^1(2.95) = 3 \times 2.95^2 - 4 = 22.1075$$
$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2.95 - \frac{4.8724}{22.1075} = 2.7296$$

Iteration 2: Now taking $x_0 = 2.7296$

$$f(2.7296) = 2.7296^3 - 4 \times 2.7296 - 9 = 0.4191$$
$$f^1(2.7296) = 3 \times 2.7296^2 - 4 = 18.3521$$
$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2.7296 - \frac{0.4191}{18.3521} = 2.7068$$

Iteration 3: Now taking $x_0 = 2.7068$

$$f(2.7068) = 2.7068^3 - 4 \times 2.7068 - 9 = 0.0049$$
$$f^1(2.7068) = 3 \times 2.7068^2 - 4 = 17.9803$$
$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2.7068 - \frac{0.0049}{17.9803} = 2.7065$$

Iteration 4: Now taking $x_0 = 2.7065$

$$f(2.7065) = 2.7065^3 - 4 \times 2.7065 - 9 = -0.0005$$
$$f^1(2.7065) = 3 \times 2.7065^2 - 4 = 17.9754$$
$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2.7065 - \frac{-0.0005}{17.9754} = 2.7065$$

Observe that iterations 2, 3, and 4 produce no change at first four significant digits in the successive approximations to the root. Therefore, we take $x = 2.706$ as the desired solution correct to four significant digits.

Example 3.11: Find the root of equation

$$x^3 - x - 4 = 0$$

correct to three decimal places using Newton-Raphson method.

Solution: Since we want the solution correct to three decimal places, the iterative process will be terminated as soon as the successive iterations produces no change at first three decimal places or the function vanishes at new approximation.

Given $f(x) = x^3 - x - 4$

2.625
3.0
5.1625
2

therefore

$$f^1(x) = 3x^2 - 1$$

Iteration 1: We start with $x_0 = 2$

$$f(2) = 2^3 - 2 - 4 = 2$$

2.8

$$f^1(2) = 3 \times 2^2 - 1 = 11$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2 - \frac{2}{11} = 1.8182$$

Thus, the first approximation to the root is 1.8182.

Iteration 2: Now taking $x_0 = 1.8182$

$$f(1.8182) = 1.8182^3 - 1.8182 - 4 = 0.1923$$

$$f^1(1.8182) = 3 \times 1.8182^2 - 1 = 8.9173$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 1.8183 - \frac{0.1923}{8.9173} = 1.7966$$

Thus, the second approximation to the root is 1.7966.

Iteration 3: Now taking $x_0 = 1.7966$

$$f(1.7966) = 1.7976^3 - 1.7966 - 4 = 0.0025$$

$$f^1(1.7966) = 3 \times 1.7966^2 - 1 = 8.6834$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 1.7966 - \frac{0.0025}{8.6834} = 1.7963$$

Thus, the third approximation to the root is 1.7963.

Since for $x_0 = 1.7963$

$$f(1.7963) = 1.7973^3 - 1.7963 - 4 = 0.0000$$

We terminate the iterative procedure here and take $x = 1.796$ as the desired solution correct to three decimal places.

Example 3.12: Evaluate $\sqrt{12}$ correct to three decimal places using Newton-Raphson method.

Solution: Let $\quad x = \sqrt{12} \quad \Rightarrow \quad x^2 = 12 \quad \Rightarrow \quad x^2 - 12 = 0$

Thus, the problem of evaluating $\sqrt{12}$ reduces to the problem of finding the root of equation $x^2 - 12 = 0$.

Since we want the solution correct to three decimal places, the iterative process will be terminated as soon as the successive iterations produces no change at first three decimal places or the function vanishes at new approximation.

Given $\quad f(x) = x^2 - 12 \quad$ therefore $\quad f^1(x) = 2x$

Iteration 1: We start with $x_0 = 3$

$$f(3) = 3^2 - 12 = -3$$

$$f^1(3) = 2 \times 3 = 6$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 3 - \frac{-3}{6} = 3.5$$

Thus, the first approximation to the root is 3.5.

Iteration 2: Now taking $x_0 = 3.5$

$$f(3.5) = 3.5^2 - 12 = 0.25$$

$$f^1(3.5) = 2 \times 3.5 = 7$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 3.5 - \frac{0.25}{7} = 3.4643$$

Thus, the second approximation to the root is 3.4643.

Iteration 3: Now taking $x_0 = 3.4643$

$$f(3.4643) = 3.4643^2 - 12 = 0.0013$$

$$f^1(3.4643) = 2 \times 3.4643 = 6.9286$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 3.4643 - \frac{0.0013}{6.9286} = 3.4641$$

Thus, the third approximation to the root is 3.4641.

Since for $x_0 = 3.4641$

$$f(3.4641) = 3.4641^2 - 12 = 0.0000$$

We terminate the iterative procedure here as function vanishes at the new approximation and take $x = 3.464$ as the desired solution correct to three decimal places of equation $x^2 - 12 = 0$, hence the value of $\sqrt{12}$ correct to three decimal places is 3.464.

Algorithm 3.5: Newton-Raphson Method

Here $f(x)$ is the given function, $f^1(x)$ is first order derivative of function $f(x)$, x_0 is the initial approximations, *epsilon* is the prescribed tolerance, *delta* is the prescribed lower bound for the slope of $f(x)$, and n is the maximum number of iterations permitted.

```
Begin
    read: x0, epsilon, delta, n
    for i = 1 to n by 1 do
        if ( | f 1(x0) | ≤ delta ) then
            write: "Slope of f(x) becomes too small near x = ", x0
            exit
        endif

        set x1 = x0 - f(x0) / f 1(x0)
        set relative_error = | ((x1 - x0)/x1) |
        set x0 = x1

        if ( relative_error ≤ epsilon ) then
            write: x1, "as the approximate root"
            exit
        endif
    endfor
    write: "Solution does not converge in", n, "iterations"
End.
```

3.10 METHOD OF SUCCESSIVE APPROXIMATIONS

This method also known as the *direct substitution method* or *method of iterations* or *method of fixed iterations*, is applicable if the equation

$$f(x) = 0$$

can be expressed as

$$x = g(x)$$

If x_1 is the initial approximation to the root, then next approximation to the root is given by

$$x_2 = g(x_1)$$

and the next approximation will be

$$x_3 = g(x_2)$$

In general

$$\boxed{x_{i+1} = g(x_i)} \qquad \ldots (3.17)$$

The iterative cycle will terminate when the relative error in the new approximation is within the prescribed tolerance.

As we know that the solution of the equation

$$f(x) = 0$$

is given by the intersection point of the graph $f(x)$ and the x-axis.

We may regard such a solution as the point of intersection of two curves:

- One having the equation $y = 0$, which is the equation of line representing the x-axis.
- Other having the equation $y = f(x)$.

In the method of successive approximation, we may still regard the solution as being the intersection of two curves, but now the curves are:

- One having the equation $y = x$, which is the equation of line with slope +1.
- Other having the equation $y = g(x)$.

If the graphs of $y = x$ and $y = g(x)$ are as shown below, then at the point of intersection P, equations $y = x$ and $y = g(x)$ are satisfied simultaneously. At this point, we have $x = g(x)$, hence the x co-ordinate of this point is the required solution.

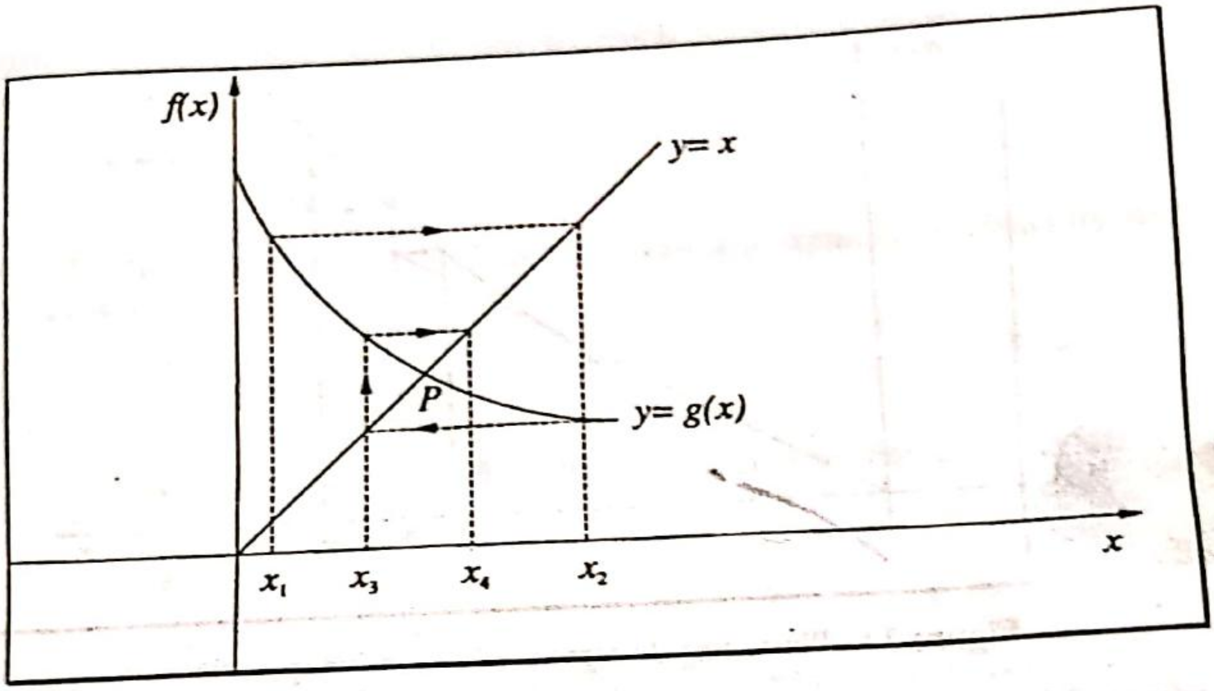

Figure 3.6: Root approximation by method of successive approximations

However, this procedure converge if and only if $|g^1(x_i)| < 1$.

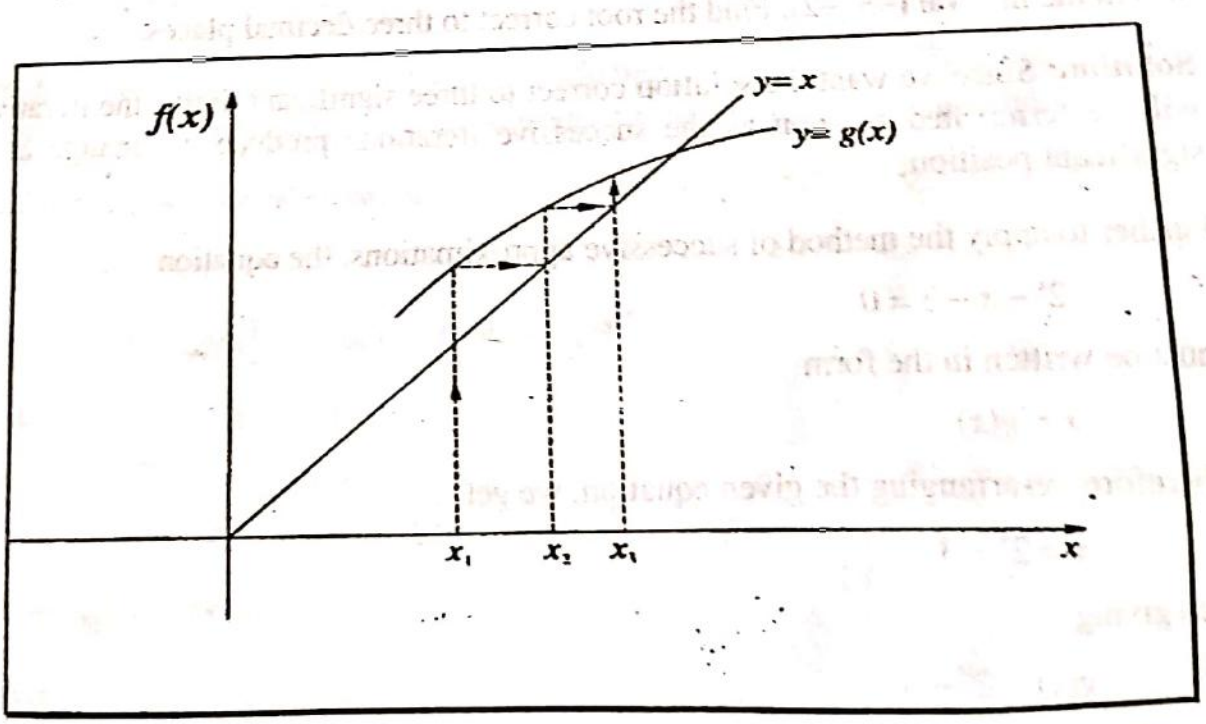

Figure 3.7: Illustrating convergence of successive approximation

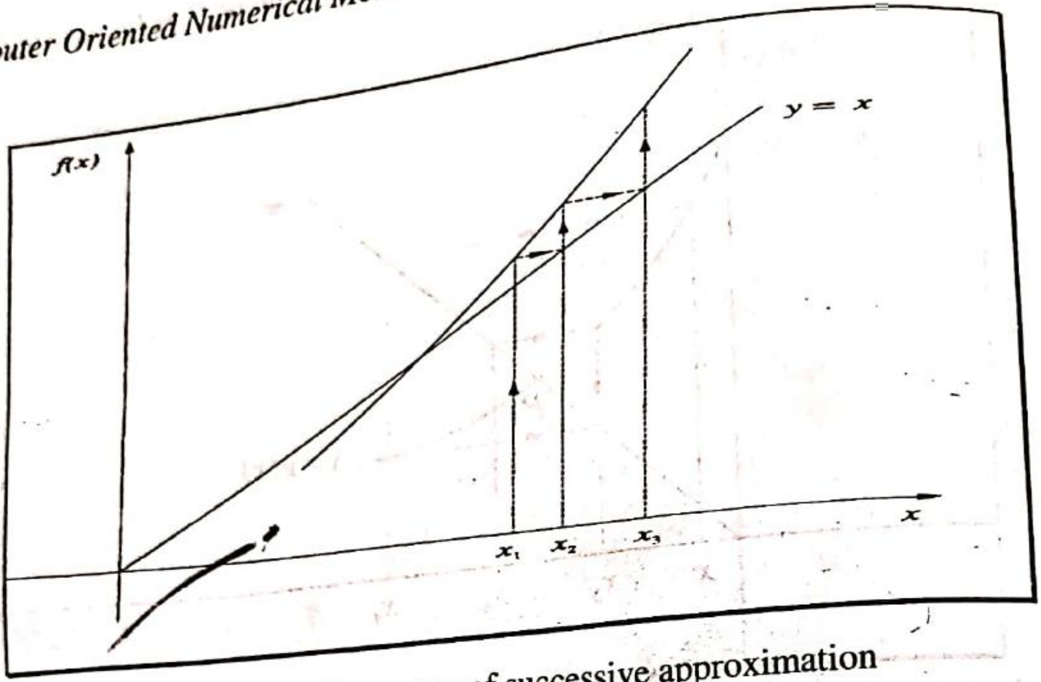

Figure 3.8: Illustrating divergence of successive approximation

Example 3.13: Given that one root of the non-linear equation

$$2^x - x - 3 = 0$$

lies in the interval (–3, –2). Find the root correct to three decimal places.

Solution: Since we want the solution correct to three significant digits, the iterative process will be terminated as soon as the successive iterations produce no change at first four significant positions.

Further to apply the method of successive approximations, the equation

$$2^x - x - 3 = 0$$

must be written in the form

$$x = g(x)$$

Therefore, re-arranging the given equation, we get

$$x = 2^x - 3$$

thus giving

$$g(x) = 2^x - 3$$

Iteration 1: Let us start with initial approximation $x_1 = -3$.

$$x_2 = g(x_1)$$

$$= 2^{-3} - 3 = -2.875$$

Iteration 2: Now we take $x_2 = -2.875$ as the current approximation to obtain the next approximation as

$$x_3 = g(x_2)$$
$$= 2^{-2.875} - 3 = -2.8637$$

Iteration 3: Now we take $x_3 = -2.8637$ as the current approximation to obtain the next approximation as

$$x_4 = g(x_3)$$
$$= 2^{-2.8637} - 3 = -2.8626$$

Iteration 4: Now we take $x_4 = -2.8626$ as the current approximation to obtain the next approximation as

$$x_5 = g(x_4)$$
$$= 2^{-2.8626} - 3 = -2.8625$$

Iteration 5: Now we take $x_5 = -2.8625$ as the current approximation to obtain the next approximation as

$$x_6 = g(x_5)$$
$$= 2^{-2.8625} - 3 = -2.8625$$

We see that after iteration number 3, there is no change at the first four significant digits. Therefore we take $x = -2.862$ as the desired solution correct to four significant digits.

3.11 CONCEPT OF SYNTHETIC DIVISION

Division of a polynomial $f(x)$ by a factor $(x-r)$ can be carried rapidly by synthetic division. The remainder of the above division will give the value of the polynomial for $x = r$. However, if r happens to the root of the polynomial $f(x)$, then the remainder resulting from the division will be zero.

Similarly, the value of the derivative $f^{\,1}(x)$ of the polynomial $f(x)$ for $x = r$ is equal to the remainder obtained by second synthetic division of the result obtained from the first synthetic division, which will be a polynomial of degree one less than the original polynomial.

To understand the synthetic division, let us consider the following example.

Example 3.16: Divide the polynomial

$$x^5 - 3x^4 - 10x^3 + 5x^2 + 22x + 16$$

by $(x-4)$.

Solution: The division by a factor $(x-4)$ is carried out as shown below:

$$\begin{array}{r|rrrrrr}
4 & 1 & -3 & -10 & 5 & 22 & 16 \\
 & & 4 & 4 & -24 & -76 & -216 \\
\hline
 & 1 & 1 & -6 & -19 & -54 & (-200)
\end{array}$$

The first row consists of the coefficients of the given polynomial. The second and third rows are generated from left to right using the following procedure:

The first coefficient is written as such in the third row. The first element in the third row is multiplied by 4. The product gives the first element of the second row and is written under the second element of the first row. Now the second element of first row and the first element of the second row are added. This sum gives the second element of the third row. Next the second element of the third row is multiplied by 4. This product gives the second element of the second row and is written under the third element of the first row. Then the third element of first row and the second element of the second row are added. This sum gives the third element of the third row. This procedure is repeated till the last element in the third row is obtained. This way, the second and third rows are generated from left to right.

The last element of the third row is the value of the polynomial at $x = 4$, i.e.

$$f(4) = -200$$

The reduced polynomial, a polynomial of degree one less than the given is obtained from the third row barring the last element as

$$x^4 + x^3 - 6x^2 - 19x - 54$$

Example 3.17: Obtain the derivative of the polynomial

$$x^5 - 3x^4 - 10x^3 + 5x^2 + 22x + 16$$

at $x = 4$.

Solution: As seen in Example 3.6, the first synthetic division gives polynomial

$$x^4 + x^3 - 6x^2 - 19x - 54$$

as quotient. The division of the reduced polynomial gives the value of the derivative of the original polynomial by a factor $(x-4)$ as shown below.

4	1	1	−6	−19	−54
		4	20	56	148
	1	5	14	37	(94)

The remainder 94 is the value of $f^1(4)$.

Similarly, we can obtain the values of higher derivatives as well.

Algorithm 3.7: Synthetic Division by Linear Factor $(x-r)$

Here array a represent the coefficients of the given polynomial of degree n of type

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

and array b represents the coefficients of the polynomial obtained as quotient after the division, and r is the point where value of the polynomial (obtained in b_0) is desired.

```
Begin
   read: n, r
   for i = n to 0 by -1 do
      read: a_i
   endfor
   set b_n = a_n
```

```
    for i = (n - 1) to 0 by -1 do
        set b_i = a_i + r×b_{i+1}
    endfor
    write: "Coefficients of the reduced polynomial"
    for i = n to 1 by -1 do
        write: b_i
    endfor
    write: b_0, "as the value of the polynomial"
End.
```

Now we consider the synthetic division by a quadratic factor.

Example 3.18: Suppose the polynomial is

$$x^5 - 3x^4 - 10x^3 + 10x^2 + 44x + 48$$

is to be divided by the quadratic factor $\cancel{x^2 + 3x + 1}$. $x^2 + 2x + 1$

Solution: The division by a factor $x^2 + 2x + 1$ is carried out as shown below:

−2	1	−3	−10	10	44	48
−1		−2	10	2	−34	−22
			−1	5	1	−17
	1	−5	−1	17	11	9

The first row consists of the coefficients of the given polynomial. The second, third and fourth rows are generated from left to right using the following procedure:

The first coefficient is written as such in the fourth row. The first element in the fourth row is multiplied by −2. The product gives the first element of the second row and is written under the second element of the first row. Now the second element of first row and the first element of the second row are added. This sum gives the second element of the fourth row. Next the second element of the fourth row is multiplied by −2. This product gives the second element of the second row and is written under the third element of the first row. The first element of fourth row is multiplied by −1. This product gives the first element of third row and is written in the third column. Now the elements in the third column are added. This sum gives the third element of the fourth row. Next the third element of the fourth row is multiplied by −2.

This product gives the third element of the second row and is written under the fourth element, *i.e.*, fourth column, of the first row. The second element of fourth row is multiplied by −1. This product gives the second element of third row and is written in the fourth column. Now the elements in the fourth column are added. This sum gives the fourth element of the fourth row. This procedure is repeated till the last element in the fourth row is obtained. This way, the second, third and fourth rows are generated from left to right.

The reduced polynomial, a polynomial of degree less than by a factor of two is obtained from the fourth row barring the last two elements as

$$x^3 - 5x^2 - x + 17$$

The remainder as a result of above division is

$$(x+2)11 + 9 = 11x + 31$$

Algorithm 3.8: Synthetic Division by Quadratic Factor (x^2+ux+v)

Here array *a* represent the coefficients of the given polynomial of degree *n* of type

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0.$$

and array *b* represents the coefficients of the polynomial obtained as quotient after the division, *u* and *v* are the coefficients of the quadratic factor.

```
Begin
    read: n
    for i = n to 0 by -1 do
        read: a_i
    endfor
    read: u, v
    set b_n = a_n
    set b_n-1 = a_n-1 - uxb_n
    for i = (n - 2) to 0 by -1 do
        set b_i = a_i - uxb_i+1 - vxb_i+2
    endfor
    write: "Coefficients of the reduced polynomial"
    for i = n to 2 by -1 do
        write: b_i
    endfor
    write: (x + u)b_1 + b_0, "as the remainder"
End.
```