10.5 PICARD'S METHOD

Rewriting $\frac{dy}{dx} = f(x,y)$ as $dy = f(x,y)dx$, and integrating this equation, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx \qquad \ldots (10.4)$$

This equation in which the unknown function y appears under the integral sign is called an *integral equation*. Such an equation can be solved by the method of successive approximations in which the first approximation to y is obtained by substituting y_0 for y on the right hand side of Eq. (10.4).

Thus, we write $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$

The integral on the right hand side can now be solved and the resulting y_1 is substituted for y in the integrand of Eq. (10.4) to obtain the second approximation y_2 to y as

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

Proceeding on the similar lines, we can obtain $(i+1)^{th}$ approximation from i^{th} approximations as

$$\boxed{y_{i+1} = y_0 + \int_{x_0}^{x} f(x, y_i)dx} \qquad \ldots (10.5)$$

Note that the Picard's method gives the solution of the ordinary differential equation at a point, say x, as a truncated power series in x as illustrated in Example 10.1.

Picard's method is of considerable theoretical value, but can be applied to a limited class of equations in which the successive integrations can be evaluated easily.

Example 10.1: Given $\dfrac{dy}{dx} = x + y$ with $y(0) = 1$. Find the solution of the given differential equation correct to fifth approximation.

when $x_0=0, y_0=1$

Solution: Since $f(x, y) = x + y$, and $y_0 = 1$, the integral equation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

become

$$y = 1 + \int_{0}^{x} (x + y)dx \qquad \ldots (10.6)$$

First Approximation: Substituting $y = 1$ on the right hand side of Eq. (10.6), we get

$$y_1 = 1 + \int_{0}^{x} (x+1)dx = 1 + x + \frac{x^2}{2}$$

$1+x+\frac{x^2}{2}$

Second Approximation: Substituting $y = 1 + x + \frac{x^2}{2}$ on the right hand side of Eq. (10.6), we get

$$y_2 = 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2} \right) dx = 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2} \right) dx = 1 + x + x^2 + \frac{x^3}{6}$$

Third Approximation: Substituting $y = 1 + x + x^2 + \frac{x^3}{6}$ on the right hand side of Eq. (10.6), we get

$$y_3 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6} \right) dx = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth Approximation: Substituting y $= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$ on the right hand side of Eq. (10.6), we get

$$y_4 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \right) dx = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} + c$$

Fifth Approximation: Substituting $y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$ on the right hand side of Eq. (10.6), we get

$$y_5 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} \right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} + C$$

Example 10.2: Find the solution, correct to three decimal positions, of the first order ordinary differential equation $\frac{dy}{dx} = x + y^2$ for $x = 0.1$ when $y(0) = 1$.

$x_0 = 0$

solve $0 \to 0.1$

Solution: Since $f(x, y) = x + y^2$, $y_0 = 1$, $x_0 = 0$ and $x = 0.1$, the integral equation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \qquad \text{becomes} \quad y = 1 + \int_0^{0.1} (x + y^2)dx \qquad \cdots (10.7)$$

First Approximation: Substituting $y = 1$ on the right hand side of Eq. (10.7), we get

$$y_1 = 1 + \int_0^{0.1} (x + 1^2)dx = 1 + \left|\frac{x^2}{2} + x\right|_0^{0.1} = 1.105$$

Second Approximation: Substituting $y = 1.105$ on the right hand side of Eq. (10.7), we get

$$y_2 = 1 + \int_0^{0.1} (x + (1.105)^2)dx = 1 + \left|\frac{x^2}{2} + (1.105)^2 x\right|_0^{0.1} = 1.127$$

Third Approximation: Substituting $y = 1.127$ on the right hand side of Eq. (10.7), we get

$$y_3 = 1 + \int_0^{0.1} (x + (1.127)^2)dx = 1 + \left|\frac{x^2}{2} + (1.127)^2 x\right|_0^{0.1} = 1.132$$

Fourth Approximation: Substituting $y = 1.132$ on the right hand side of Eq. (10.7), we get

$$y_4 = 1 + \int_0^{0.1} (x + (1.132)^2)dx = 1 + \left|\frac{x^2}{2} + (1.132)^2 x\right|_0^{0.1} = 1.133$$

Fifth Approximation: Substituting $y = 1.133$ on the right hand side of Eq. (10.7), we get

$$y_5 = 1 + \int_0^{0.1} (x + (1.133)^2)dx = 1 + \left|\frac{x^2}{2} + (1.133)^2 x\right|_0^{0.1} = 1.1333$$

By looking at third, fourth, and fifth approximations, we conclude there is no change in the first three decimal places, hence the value of $y(0.1)$ correct to three decimal digits is 1.133 (rounded).

10.6 EULER'S METHOD

The Euler's method is one of the oldest and the simplest method. The Euler's method can be described as a technique of developing a piecewise linear approximation to the solution. In the initial value problem, the starting point of the solution curve and the slope of the curve at the starting point are given. With this information, the method extrapolates the solution curve using the specified step size.

Consider again the following first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition: $y = y_1$ for $x = x_1$

Lets us know find the mathematical formula for the Euler's method.

Recall the mean value theorem states that

If a function is continuous and differentiable between two pints, say (x_1, y_1) and (x_2, y_2), then the slope of the line joining these points is equal to the derivative of the function at least at one other point, say (c, d), between these two points, i.e.

$$y^1(c) = \frac{y(x_2) - y(x_1)}{x_2 - x_1} \qquad \ldots (10.8)$$

If we substitute $c = x_1$ and $h = x_2 - x_1$, then Eq. (10.8) can be written as

$$y(x_2) - y(x_1) = hy^1(x_1)$$

Now from Eq. (10.1), we have

$$y'(x_1) = f(x_1, y_1)$$

therefore

$$y(x_2) - y(x_1) = hf(x_1, y_1)$$
$$y(x_2) = y(x_1) + hf(x_1, y_1)$$
$$y_2 = y_1 + hf(x_1, y_1)$$

Using this equation, we can find the second point on the solution curve as (x_2, y_2).

Similarly, taking (x_2, y_2) as the starting point, we get

$$y_3 = y_2 + hf(x_2, y_2)$$

In general, the $(i+1)^{th}$ point of the solution curve is obtained from the i^{th} point using the following formula

$$\boxed{y_{i+1} = y_i + hf(x_i, y_i)} \quad \ldots (10.9)$$

$y_{i+1} = y_i + hf(x_i, y_i)$

Geometrical Interpretation

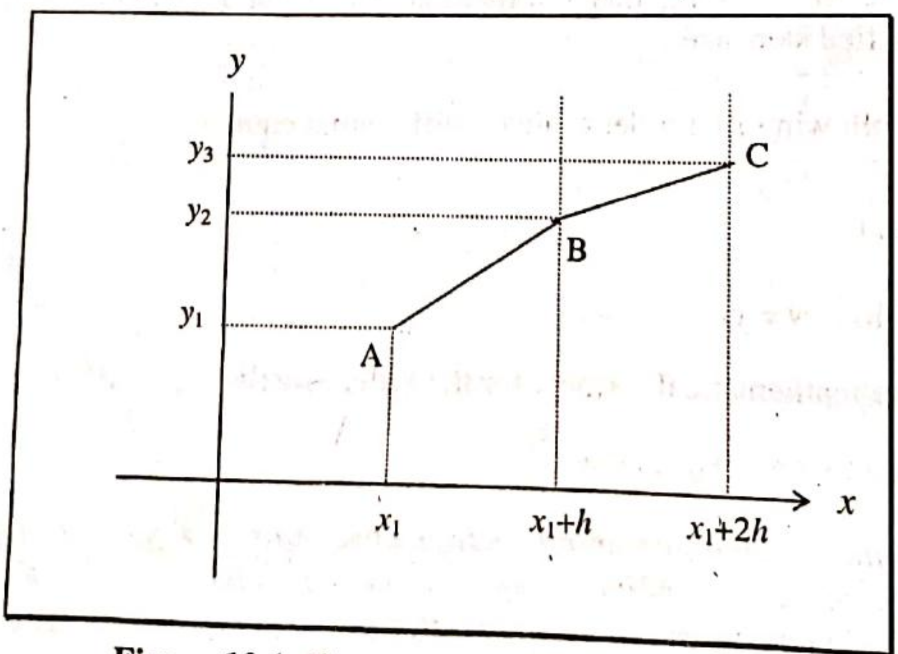

Figure 10.1: Illustration of Euler's method

We are given the initial point (x_1, y_1), say A, of the solution curve, and the slope $f(x_1, y_1)$, say s, of the curve at the initial point. Now from point A draw a line with slope s. The point where this line intersects the vertical line erected at $x_2 = x_1+h$ gives the next value of y. Let it be y_2. Now calculate the slope of the curve at this point, which is given by $f(x_1+h, y_2)$, and draw a line from (x_1+h, y_2) with slope $f(x_1+h, y_2)$; and where this line intersects the vertical line erected at $x_3 = x_1+2h = x_2+h$ gives the next value of y i.e. y_3, and so on. The error involved in each step is of the order of $\frac{h^2}{2}y^2(z)$, where z is a point in the region of computation, and $y^2(z)$ is a second order derivative. Therefore, in order to improve accuracy, the value of the step size h should be as small as possible.

Example 10.4: Given $\frac{dy}{dx} = xy$ with $y(1) = 5$. Find the solution correct to three decimal position in the interval $[1,1.5]$ using step size $h = 0.1$.

Solution: The formula for Euler's method is $y_{i+1} = y_i + hf(x_i, y_i)$

In our example

$y_{i+1} = y_i + hf(x_i, y_i)$

$f(x, y) = xy \qquad x_1 = 1 \qquad y_1 = 5 \qquad h = 0.1$

The Euler's formula can be written as

$$y_{i+1} = y_i + 0.1 \times xy$$

$x_2 = x_1 + h$

$x_2 = 1 + 0.1 = 1.1$

For $i = 1$

$$y_2 = y_1 + 0.1 \times x_1 \times y_1 = 5 + 0.1 \times 1 \times 5 = 5.5$$

Thus, the second solution point is obtained as $(x_2, y_2) = (1.1, 5.5)$.

For $i = 2$

$$y_3 = y_2 + 0.1 \times x_2 \times y_2 = 5.5 + 0.1 \times 1.1 \times 5.5 = 6.105$$

Thus, the third solution point is obtained as $(x_3, y_3) = (1.2, 6.105)$.

For $i = 3$

$$y_4 = y_3 + 0.1 \times x_3 \times y_3 = 6.105 + 0.1 \times 1.2 \times 6.105 = 6.838$$

Thus, the fourth solution point is obtained as $(x_4, y_4) = (1.3, 6.838)$.

For $i = 4$

$$y_5 = y_4 + 0.1 \times x_4 \times y_4 = 6.838 + 0.1 \times 1.3 \times 6.838 = 7.727$$

Thus, the fifth solution point is obtained as $(x_5, y_5) = (1.4, 7.727)$.

For $i = 5$

$$y_6 = y_5 + 0.1 \times x_5 \times y_5 = 7.727 + 0.1 \times 1.4 \times 7.727 = 8.809$$

Thus, the sixth and final solution point is obtained as $(x_6, y_6) = (1.5, 8.809)$.

The complete solution of the given differential equations is given as

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.5	6.105	6.838	7.727	8.809

Algorithm 10.1: Euler's Method

To find the solution of ODE $dy/dx = f(x,y)$ when the solution at the initial point, $x = x_1$, is given, and the solution is desired in the interval $[x_1, x_f]$.

```
Begin
    read: x1, y1, xf
    read: h
    set x = x1
    set y = y1
    set i = 1
    write: i, x, y
    while ( x < xf ) do
        set y = y + hf (x, y)
        set x = x + h
        set i = i + 1
        write: i, x, y
    endwhile
End.
```

10.7 TAYLOR SERIES METHOD

Given $\dfrac{dy}{dx} = f(x,y)$ with an initial condition: $y = y_1$ at $x = x_1$

If the solution curve is expanded in Taylor series around $x = x_1$, we get

$$y(x) = y(x_1) + (x - x_1)y^1(x_1) + \frac{(x - x_1)^2}{2!}y^2(x_1) + \ldots \qquad \ldots (10.10)$$

where $y^1(x_1)$, $y^2(x_1)$, … are first and second order derivatives, and so on.

Then at $x = x_1 + h$, the above equation becomes

$$y(x_1 + h) = y(x_1) + hy^1(x_1) + \frac{h^2}{2!}y^2(x_1) + \ldots$$

Now $y^1(x) = f(x, y)$ and y is a function of x, therefore

$$y^2(x) = \frac{d\left[y^1(x)\right]}{dx} = \frac{d}{dx}\left[f(x,y)\right] = \frac{\partial}{\partial x}f(x,y) + \frac{\partial}{\partial y}f(x,y)\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}f(x,y) + \frac{\partial}{\partial y}f(x,y) \times f(x,y)$$

If we take $f_x(x,y) = \frac{\partial}{\partial x}f(x,y)$ and $f_y(x,y) = \frac{\partial}{\partial y}f(x,y)$

Then $y^2(x) = f_x(x,y) + f_y(x,y) \times f(x,y)$

Thus $y(x_1 + h) = y(x_1) + hf(x_1, y_1) + \frac{h^2}{2!}\left[f_x(x_1,y_1) + f_y(x_1,y_1) \times f(x_1,y_1)\right] + \ldots$

$$\Rightarrow \quad y_2 = y(x_1 + h) = y_1 + hf(x_1,y_1) + \frac{h^2}{2!}\left[f_x(x_1,y_1) + f_y(x_1,y_1) \times f(x_1,y_1)\right] + \ldots \quad \ldots (10.11)$$

Similarly, taking (x_2, y_2) as the starting point, we get

$$y_3 = y_2 + hf(x_2,y_2) + \frac{h^2}{2!}\left[f_x(x_2,y_2) + f_y(x_2,y_2) \times f(x_2,y_2)\right] + \ldots$$

In general

$$\boxed{y_{i+1} = y_i + hf(x_i,y_i) + \frac{h^2}{2!}\left[f_x(x_i,y_i) + f_y(x_i,y_i) \times f(x_i,y_i)\right] + \ldots} \quad \ldots (10.12)$$

This formula is known as the *Taylor series formula* and can be sued repetitively to obtain $y(x)$ for successive values of x. If we reduce the above formula after the third term, the error per step will be $O(h^3)$.

Though the amount of error incurred per step is less that the Euler's method, this method is rarely used on computer as it is not always easy to obtain the partial derivatives of $f(x, y)$, and even if possible, considerable computational effort is required.

10.8 RUNGE-KUTTA METHODS

These methods were devised by *C. Runge* and extended by *W. Kutta* a few years later. The Runge-Kutta methods are actually a family of methods, of which the second order and fourth order methods are widely used. As we have seen in the preceding section, we can improve the accuracy of those methods by taking smaller step sizes. Much greater accuracy can be obtained by using Runge-Kutta methods. These methods are equivalent of approximating the exact solution by matching the first n terms of the Taylor series method. In this text, we will discuss only the second and fourth order methods. In these methods, first the slope at some of the intermediate points is computed, and then weighted average of slopes is used to extrapolate the next solution point.

10.8.1 Runge-Kutta Second Order Methods or Heun's method

The Runge-Kutta second order methods are actually a family of methods, each of that matches the Taylor series method up to the second-degree terms in h, where h is the step size. In these methods the interval $[x_1, x_f]$ is divided into subintervals and a weighted average of derivatives (slopes) at these intervals is used to determine the value of the dependent variable. One advantage of these methods is that they, like Euler's method, are single step methods i.e. in order to evaluate y_{i+1}, we need information only at the preceding point (x_i, y_i).

Consider the following differential equation

$$\frac{dy}{dx} = f(x,y)$$

with an initial condition:

$$y = y_1 \text{ at } x = x_1$$

At the starting point, compute the slope of the curve as $f(x_1, y_1)$. Let it be s_1. Now compute the slope of the curve at point (x_2, y_1+s_1h) as $f(x_2, y_1+s_1h)$, where $x_2 = x_1+h$. Let this new slope be s_2. Find the average of these two slopes, and then compute the value of the dependent variable y from the following equation

$$y_2 = y_1+hs$$

where $s = \dfrac{s_1+s_2}{2}$, $s_1 = f(x_1, y_1)$ and $s_2 = (x_1+h, y_1+s_1h)$

Hence, starting from point $f(x_1, y_1)$, we obtained the second point (x_2, y_2). Similarly, starting from second point, we can obtain the third point. And this process is repeated till we find the solution in the desired interval.

In general, the value of y for the $(i+1)^{th}$ point on the solution curve is obtained from the i^{th} solution point using the formula

$$\boxed{y_{i+1} = y_i + hs} \qquad \ldots (10.13)$$

where $s = \dfrac{s_i+s_{i+1}}{2}$, $s_i = f(x_i, y_i)$, and $s_{i+1} = f(x_i+h, y_i+s_ih)$

This formula for the Runge-Kutta second order method is also known as *Heun's method.*

Geometric Interpretation

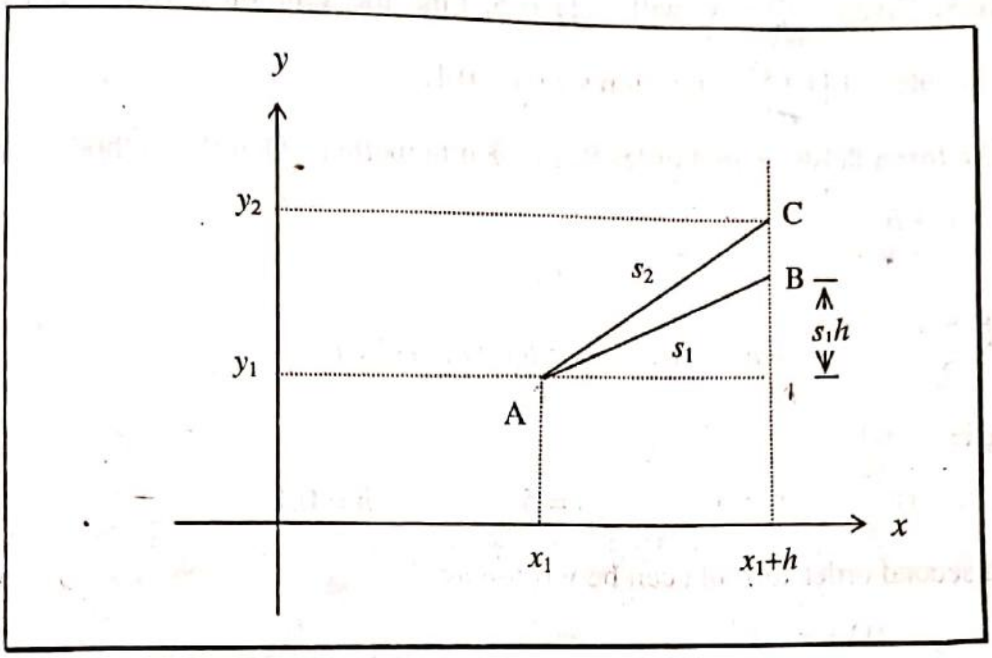

Figure 10.2: Illustration of second order Runge-Kutta method

From the starting point (x_1, y_1) labelled as A, draw a line with slope $s_1(=f(x_1,y_1))$. Let this line intersects the vertical line at x_1+h at a point labelled B, whose co-ordinates are (x_1+h, y_1+s_1h). Now compute the slope of the curve at point B. Let this slope be $s_2(=f(x_1+h, y_1+s_1h))$. Compute the average of these two slopes i.e. s_1 and s_2. Let this average slope be s. Now draw a line from point A with slope s. Let this new line intersects the vertical line at the x_1+h at a point labelled C. The point C, whose co-ordinates are (x_1+h, y_1+sh), is taken as the second solution point. Then in the similar manner we proceed from the second solution point to obtain the third solution point, and so on.

We will now show that Runge-Kutta second order methods are identical to Taylor series method.

Since $y_{i+1} = y_i + \frac{h}{2}(s_i + s_{i+1}) \quad \Rightarrow \quad y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + s_i h)\right]$

Expanding $f(x_i+h, y_i+s_ih)$ in Taylor series, we get

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i, y_i) + hf_x(x_i, y_i) + s_i h f_y(x_i, y_i) + \ldots\right]$$

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i, y_i) + hf_x(x_i, y_i) + hf_y(x_i, y_i) \times f(x_i, y_i) + \ldots\right]$$

$$\Rightarrow \quad y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}\left[f_x(x_i, y_i) + f_y(x_i, y_i) \times f(x_i, y_i) + \ldots\right]$$

which is identical to the Taylor series method, and hence the proof.

Example 10.5: Given $\frac{dy}{dx} = xy$ with $y(1) = 5$. Find the solution correct to three decimal position in the interval $[1,1.5]$ using step size $h = 0.1$.

Solution: The formula for second order Runge-Kutta method (Heun's method) is

$$\boxed{y_{i+1} = y_i + hs}$$

where

$$s = \frac{s_1 + s_2}{2}, \qquad s_1 = f(x_1, y_1), \qquad s_2 = f(x_1+h, y_1+hs_1)$$

In our example

$$f(x, y) = xy \qquad x_1 = 1 \qquad y_1 = 5 \qquad h = 0.1$$

Runge-Kutta second order formula can be written as

$$y_{i+1} = y_i + 0.1 \times s$$

For $i = 1$

$$y_2 = y_1 + 0.1 \times s$$
$$s_1 = f(x_1, y_1) = f(1, 5) = 1 \times 5 = 5$$
$$s_2 = f(x_1+0.1, y_1+0.1 \times s_1) = f(1.1, 5+0.1 \times 5) = f(1.1, 5.5) = 1.1 \times 5.5 = 6.05$$
$$s = (5+6.05)/2 = 5.525$$

Thus $y_2 = 5 + 0.1 \times 5.525 = 5.553$

Thus, the second solution point is obtained as $(x_2, y_2) = (1.1, 5.553)$.

For $i = 2$

$$y_3 = y_2 + 0.1 \times s$$
$$s_1 = f(x_2, y_2) = f(1.1, 5.553) = 1.1 \times 5.553 = 6.108$$
$$s_2 = f(x_2+0.1, y_2+0.1 \times s_1) = f(1.2, 5.553+0.1 \times 6.108) = f(1.2, 6.164)$$
$$= 1.2 \times 6.164 = 7.397$$
$$s = (6.108+7.397)/2 = 6.752$$

Therefore $y_3 = 5.553 + 0.1 \times 6.752 = 6.228$

Thus, the third solution point is obtained as $(x_3, y_3) = (1.2, 6.228)$.

For $i = 3$

$$y_4 = y_3 + 0.1 \times s$$
$$s_1 = f(x_3, y_3) = f(1.2, 6.228) = 1.2 \times 6.228 = 7.474$$

$s_2 = f(x_3+0.1,\ y_3+0.1 \times s_1) = f(1.3,\ 6.228+0.1 \times 7.474) = f(1.3,\ 6.975)$
$\quad = 1.3 \times 6.975 = 9.068$
$s = (7.474+9.068)/2 = 8.271$

Therefore $y_4 = 6.228 + 0.1 \times 8.271 = 7.055$

Thus, the fourth solution point is obtained as $(x_4, y_4) = (1.3, 7.055)$.

For $i = 4$

$y_5 = y_4 + 0.1 \times s$
$s_1 = f(x_4, y_4) = f(1.3, 7.055) = 1.3 \times 7.055 = 9.172$
$s_2 = f(x_4+0.1,\ y_4+0.1 \times s_1) = f(1.4,\ 7.055+0.1 \times 9.172) = f(1.4,\ 7.972)$
$\quad = 1.4 \times 7.972 = 11.161$
$s = (9.172+11.161)/2 = 10.166$

Therefore $y_5 = 7.055 + 0.1 \times 10.166 = 8.072$

Thus, the fifth solution point is obtained as $(x_5, y_5) = (1.4, 8.072)$.

For $i = 5$

$y_6 = y_5 + 0.1 \times s$
$s_1 = f(x_5, y_5) = f(1.4, 8.072) = 1.4 \times 8.072 = 11.301$
$s_2 = f(x_5+0.1,\ y_5+0.1 \times s_1) = f(1.5,\ 8.072+0.1 \times 11.301) = f(1.5,\ 9.202)$
$\quad = 1.5 \times 9.202 = 13.803$
$s = (11.301+13.803)/2 = 12.552$

Therefore $y_6 = 8.072 + 0.1 \times 12.552 = 9.327$

Thus, the sixth and final solution point is obtained as $(x_6, y_6) = (1.5, 9.327)$.

The complete solution of the given differential equations is given as

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.525	6.228	7.055	8.072	9.327

Algorithm 10.2: Runge-Kutta Second Order Method

To find the solution of ODE of type $dy/dx = f(x,y)$ when the solution at the initial point, $x = x_1$, is given, and the solution is desired in the interval $[x_1, x_f]$.

```
Begin
    read: x1, y1, xf
    read: h
    set x = x1
```

```
    set y = y1
    set i = 1
    write: i, x, y
    while ( x < xf ) do
        set s1 = f (x, y)
        set x = x + h
        set s2 = f(x, y+hs1)
        set y = y + h(s1+s2)/2
        set i = i + 1
        write: i, x, y
    endwhile
End.
```

10.8.2 Runge-Kutta Fourth Order Methods

The error in the second order Runge-Kutta methods is $O(h^3)$ per step. However, if more precision is required, then we can use the fourth order Runge-Kutta methods in which the error is $O(h^5)$ per step.

In Runge-Kutta fourth order methods, the slope at four points including the starting point is computed, and then the weighted average of these slopes is computed as

$$s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

Rem

where $s_1 = f(x_1, y_1)$, $s_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}s_1)$

$s_3 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}s_2)$, $s_4 = f(x_1 + h, y_1 + hs_3)$

The value of the dependent variable y is computed as

$$y_2 = y_1 + hs$$

In the similar manner, starting from the second solution point we compute the third point. This process is repeated till we find the solution in the desired interval.

In general, the $(i+1)^{th}$ point of the solution curve is obtained from the i^{th} point using the following equation

$$\boxed{y_{i+1} = y_i + hs} \quad \ldots (10.14)$$

where $s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$, $s_1 = f(x_i, y_i)$, $s_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}s_1)$,

$s_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}s_2)$, $s_4 = f(x_i + h, y_i + hs_3)$

Geometric Interpretation

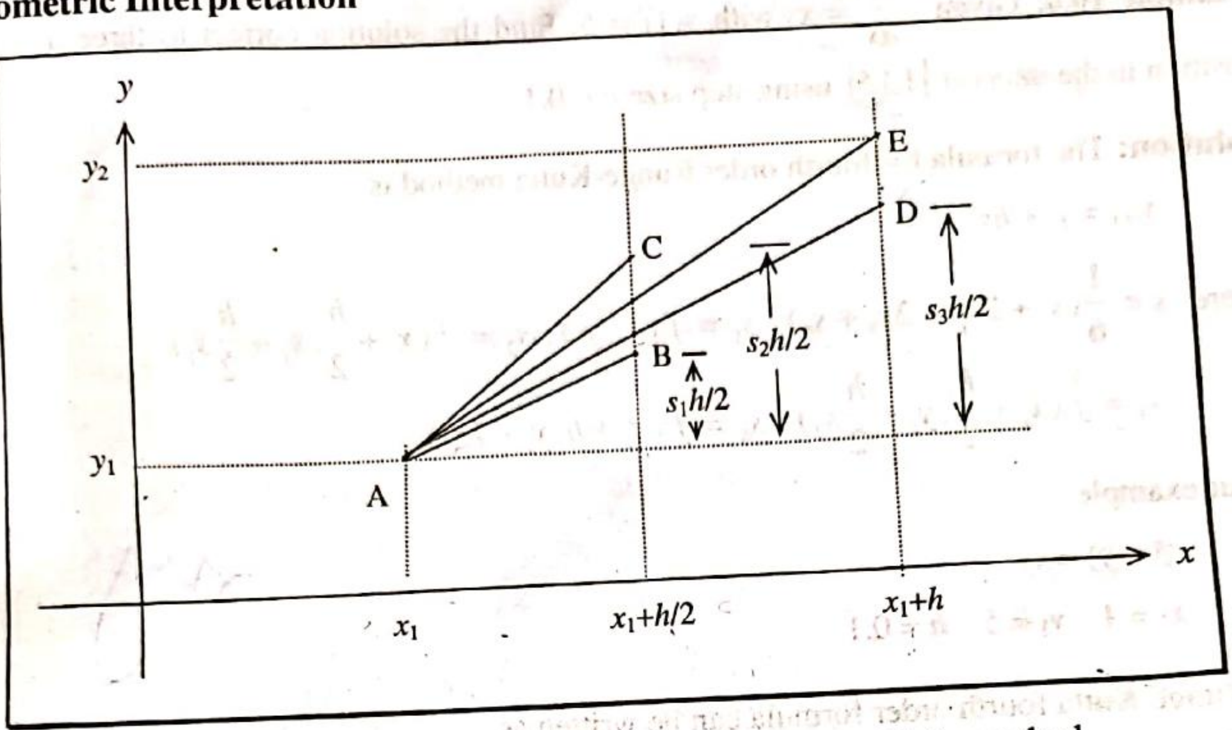

Figure 10.3: Illustration of fourth order Runge-Kutta method

From the starting point (x_1, y_1) labelled as A, draw a line with slope $s_1(=f(x_1,y_1))$. Let this line intersects the vertical line at $x_1+h/2$ at a point labelled B, whose co-ordinates are $(x_1+h/2, y_1+s_1h/2)$. Now compute the slope of the curve at point B. Let this slope be $s_2(=f(x_1+h/2, y_1+s_1h/2))$. Now draw a line from point A with slope s_2. Let this new line intersects the vertical line at $x_1+h/2$ at a point labelled C, whose co-ordinates are $(x_1+h/2, y_1+s_2h/2)$. Now compute the slope of the curve at point C. Let this slope be $s_3(=f(x_1+h/2, y_1+s_2h/2))$. Now draw a line from point A with slope s_3. Let this new line intersects the vertical line at x_1+h at a point labelled D, whose co-ordinates are (x_1+h, y_1+s_3h). Now compute the slope of the curve at point D. Let this slope be $s_4(=f(x_1+h, y_1+s_3h))$. Now compute the weighted slope s as

$$s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

Now draw a line from point A with slope s. Let this new line intersects the vertical line at x_1+h at a point labelled E, whose co-ordinates are (x_1+h, y_1+sh). The point E is taken as the second solution point.

In the similar manner, starting from the second solution point, we obtain the third solution point and so on. This process goes on till we obtain the solution in the desired interval.

The formula for the fourth order Runge-Kutta method is identical to the Taylor series truncated after the term containing h^4. The proof is left as an exercise for the students.

$dx = 1.03$
$h = .1$

Example 10.6: Given $\dfrac{dy}{dx} = xy$ with $y(1) = 5$. Find the solution correct to three decimal position in the interval $[1, 1.5]$ using step size $h = 0.1$.

Solution: The formula for fourth order Runge-Kutta method is

$$y_{i+1} = y_i + hs$$

where $s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$, $s_1 = f(x_i, y_i)$, $s_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}s_1)$

$$s_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}s_2), \quad s_4 = f(x_i + h, y_i + hs_3)$$

In our example

$$f(x, y) = xy$$

$$x_1 = 1 \quad y_1 = 5 \quad h = 0.1$$

The Runge-Kutta fourth order formula can be written as

$$y_{i+1} = y_i + 0.1 \times s$$

For $i = 1$

$$y_2 = y_1 + 0.1 \times s$$

$$s_1 = f(x_1, y_1) = f(1, 5) = 1 \times 5 = 5$$

$$s_2 = f(x_1 + 0.05, y_1 + 0.05 \times s_1) = f(1.05, 5 + 0.05 \times 5) = f(1.05, 5.25)$$
$$= 1.05 \times 5.25 = 5.513$$

$$s_3 = f(x_1 + 0.05, y_1 + 0.05 \times s_2) = f(1.05, 5 + 0.05 \times 5.513) = f(1.05, 5.276)$$
$$= 1.05 \times 5.276 = 5.540$$

$$s_4 = f(x_1 + 0.1, y_1 + 0.1 \times s_3) = f(1.1, 5 + 0.1 \times 5.540) = f(1.1, 5.554)$$
$$= 1.1 \times 5.554 = 6.109$$

$$s = \frac{1}{6}[5 + 2 \times 5.513 + 2 \times 5.540 + 6.109] = 5.536$$

Therefore $y_2 = 5 + 0.1 \times 5.536 = 5.554$

Thus, the second solution point is obtained as $(x_2, y_2) = (1.1, 5.554)$.

For $i = 2$

$$y_3 = y_2 + 0.1 \times s$$

$$s_1 = f(x_2, y_2) = f(1.1, 5.554) = 1.1 \times 5.554 = 6.109$$

$s_2 = f(x_2+0.05,\ y_2+0.05 \times s_1) = f(1.15,\ 5.554+0.05\times6.109) = f(1.15,\ 5.859)$
$= 1.15 \times 5.859 = 6.738$

$s_3 = f(x_2+0.05,\ y_2+0.05 \times s_2) = f(1.15,\ 5.554+0.05\times6.738) = f(1.15,\ 5.891)$
$= 1.15 \times 5.891 = 6.775$

$s_4 = f(x_2+0.1,\ y_2+0.1 \times s_3) = f(1.2,\ 5.554+0.1\times6.775) = f(1.2,\ 6.232)$
$= 1.2 \times 6.232 = 7.478$

$$s = \frac{1}{6}[6.109 + 2 \times 6.738 + 2 \times 6.775 + 7.478] = 6.769$$

Therefore $y_3 = 5.554 + 0.1 \times 6.769 = 6.231$

Thus, the third solution point is obtained as $(x_3, y_3) = (1.2, 6.231)$.

For $i = 3$

$y_4 = y_3 + 0.1 \times s$

$s_1 = f(x_3, y_3) = f(1.2, 6.231) = 1.2 \times 6.231 = 7.477$

$s_2 = f(x_3+0.05,\ y_3+0.05 \times s_1) = f(1.25,\ 6.231+0.05\times7.477) = f(1.25,\ 6.604)$
$= 1.25 \times 6.604 = 8.256$

$s_3 = f(x_3+0.05,\ y_3+0.05 \times s_2) = f(1.25,\ 6.231+0.05\times8.256) = f(1.25,\ 6.644)$
$= 1.25 \times 6.644 = 8.305$

$s_4 = f(x_3+0.1,\ y_3+0.1 \times s_3) = f(1.3,\ 6.231+0.1\times8.305) = f(1.3,\ 7.062)$
$= 1.3 \times 7.062 = 9.181$

$$s = \frac{1}{6}[7.477 + 2\times8.256 + 2\times8.305 + 9.181] = 8.297$$

Therefore $y_4 = 6.231 + 0.1 \times 8.297 = 7.061$

Thus, the fourth solution point is obtained as $(x_4, y_4) = (1.3, 7.061)$.

For $i = 4$

$y_5 = y_4 + 0.1 \times s$

$s_1 = f(x_4, y_4) = f(1.3, 7.061) = 1.3 \times 7.061 = 9.179$

$s_2 = f(x_4+0.05,\ y_4+0.05 \times s_1) = f(1.35,\ 7.061+0.05\times9.179) = f(1.35,\ 7.520)$
$= 1.35 \times 7.520 = 10.152$

$s_3 = f(x_4+0.05,\ y_4+0.05 \times s_2) = f(1.35,\ 7.061+0.05\times10.152) = f(1.35,\ 7.569)$
$= 1.35 \times 7.569 = 10.218$

$s_4 = f(x_4+0.1,\ y_4+0.1 \times s_3) = f(1.4,\ 7.061+0.1\times10.218) = f(1.4,\ 9.402)$
$= 1.4 \times 9.402 = 13.163$

$$s = \frac{1}{6}[9.179 + 2 \times 10.152 + 2 \times 10.218 + 14.305] = 10.514$$

Therefore $y_5 = 7.061 + 0.1 \times 10.514 = 8.112$

Thus, the fifth solution point is obtained as $(x_5, y_5) = (1.4, 8.112)$.

For $i = 5$

$y_6 = y_5 + 0.1 \times s$

$s_1 = f(x_5, y_5) = f(1.4, 8.112) = 1.4 \times 8.112 = 11.357$

$s_2 = f(x_5+0.05, y_5+0.05 \times s_1) = f(1.45, 8.112+0.05 \times 11.357) = f(1.45, 8.680)$
$= 1.45 \times 8.680 = 12.586$

$s_3 = f(x_5+0.05, y_5+0.05 \times s_2) = f(1.45, 8.112+0.05 \times 12.586) = f(1.45, 8.741)$
$= 1.45 \times 8.741 = 12.675$

$s_4 = f(x_5+0.1, y_5+0.1 \times s_3) = f(1.5, 8.112+0.1 \times 12.675) = f(1.5, 9.380)$
$= 1.5 \times 9.380 = 14.070$

$$s = \frac{1}{6}[11.357 + 2 \times 12.586 + 2 \times 12.675 + 14.070] = 12.658$$

Therefore $y_6 = 8.112 + 0.1 \times 12.658 = 9.378$

Thus, the sixth and final solution point is obtained as $(x_6, y_6) = (1.5, 9.378)$.

The complete solution of the given differential equations is given as

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.554	6.231	7.061	8.112	9.378

Algorithm 10.3: Runge-Kutta Fourth Order Method

To find the solution of ODE of type $dy/dx = f(x,y)$ when the solution at the initial point, $x = x_1$, is given, and the solution is desired in the interval $[x_1, x_f]$.

```
Begin
   read: x1, y1, xf
   read: h
   set x = x1
   set y = y1
   set i = 1
   write: i, x, y
   while ( x < xf ) do
      set s1 = f (x, y)
      set s2 = f (x +h/2, y +hs1/2)
```

```
        set s3 = f (x +h/2, y +hs2/2)
        set s4 = f (x +h, y +hs3)
        set s = (s1+2s2+2s3+s4)/6
        set y = y  + hs
        set x = x  + h
        set i = i  + 1
        write: i, x, y
    endwhile
End.
```

10.9 PREDICTOR CORRECTOR METHODS

The methods discussed so far are *single step methods, i.e.*, in order to extrapolate the solution curve they use only information available at the previous point. In contrast, the *multiple step methods* use the past information of the curve to extrapolate the solution curve. This class of methods is grouped under the name *predictor-corrector methods*. Some of the predictor corrector methods use the information about the solution curve at two previous points, some use at three points, still others use at more. But the only problem with predictor corrector methods, excluding *Modified Euler's method*, is that they are not self starting, *i.e.*, the values at first few points are computed using some other methods and then from there on these methods can take on.

10.9.1 Modified Euler's Method

This method is also known as *modified Euler's method*. As we have seen in the basic Euler's method, we use the slope at the starting point of the solution curve to determine the next point of the solution curve. This technique will work correctly only if the function is linear. The alternate approach is to use the average slope within the interval. This can be approximated by the mean of the slopes at both end points of the interval.

Suppose we use interval bounded by the point $x = x_i$ and $x = x_{i+1}$, then the average slope is

$$\frac{y^1(x_i) + y^1(x_{i+1})}{2}$$

Then we have

$$y_{i+1} = y_i + \frac{h}{2}\left[y^1(x_i) + y^1(x_{i+1})\right]$$

$$\Rightarrow \quad y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right] \qquad \ldots(10.15)$$

This is an improved estimate for y_{i+1} at x_{i+1}. But we are unable to apply Eq. (10.15) directly since we cannot evaluate $f(x_{i+1}, y_{i+1})$ as the value of y_{i+1} is unknown. This method works by estimating or *predicting* the value of y_{i+1} by basic Euler's method, Eq. (10.9). Then it uses this value to compute $f(x_{i+1}, y_{i+1})$ giving an improved estimate (a corrected value) for y_{i+1}. Thus the value of y_{i+1} is predicted using the equation

$$\boxed{y_{i+1}^{p} = y_i + hf(x_i, y_i)} \quad \ldots (10.16)$$

This equation is known as the *predictor formula*. The superscript p indicates the predicted value of y_{i+1}.

Using this predicted value of y i.e. y_{i+1}^{p}, a more accurate value of y_{i+1} is computed using the equation

$$\boxed{y_{i+1}^{c} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{p})\right]} \quad \ldots (10.17)$$

This equation is known as the *corrector formula*. The superscript c indicates the corrected value of y_{i+1}. The corrector formula can be used repeatedly to improve the accuracy of the corrected value of y_{i+1}.

These two equations i.e. Eq. (10.16) and Eq. (10.17), constitute the *Euler's predictor corrector method*, and is a two step method comprising of following steps:

Step 1. Predict y_{i+1} using the predictor formula.

Step 2. Correct y_{i+1} using the corrector formula.

If we need a higher precision, the step 2 can be executed repeatedly till the difference between two successive values of y_{i+1} agrees with some prescribed precision.

However, if more than two re-corrections are required, it is more efficient to reduce the step (interval) size or use a different method.

Example 10.7: Given $\dfrac{dy}{dx} = xy$ with $y(1) = 5$. Find the solution in the interval $[1, 1.5]$ using step size $h = 0.1$.

Solution: The formula for predictor corrector method is

$$y_{i+1}^{p} = y_i + hf(x_i, y_i)$$

$$y_{i+1}^{c} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{p})\right]$$

For $i = 1$

Starting with initial conditions, we first predict the value of y_2 using the predictor formula

$$y_2^{p} = y_1 + hf(x_1, y_1) = 5 + 0.1 \times f(1, 5) = 5 + 0.1 \times 1 \times 5 = 5.5$$

Using the predicted value of y_2, we make the correction on it using the corrector formula

$$y_2^c = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^p)\right]$$
$$= 5 + \frac{0.1}{2}\left[f(1,5) + f(1.1, 5.5)\right] = 5 + \frac{0.1}{2}[1 \times 5 + 1.1 \times 5.5] = 5.553$$

Thus, the second solution point is obtained as $(x_2, y_2) = (1.1, 5.553)$.

For $i = 2$

Starting with second solution point $(x_2, y_2) = (1.1, 5.553)$, we first predict the value of y_3 using the predictor formula

$$y_3^p = y_2 + hf(x_2, y_2) = 5.553 + 0.1 \times f(1.1, 5.553)$$
$$= 5.553 + 0.1 \times 1.1 \times 5.553 = 6.164$$

Using the predicted value of y_3, we make the correction on it using the corrector formula

$$y_3^c = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^p)\right] = 5.553 + \frac{0.1}{2}\left[f(1.1, 5.553) + f(1.2, 6.164)\right]$$
$$= 5.553 + \frac{0.1}{2}[1.1 \times 5.553 + 1.2 \times 6.164] = 6.231$$

Thus, the third solution point is obtained as $(x_3, y_3) = (1.2, 6.231)$.

For $i = 3$

Starting with third solution point $(x_3, y_3) = (1.2, 6.231)$, we first predict the value of y_4 using the predictor formula

$$y_4^p = y_3 + hf(x_3, y_3) = 6.231 + 0.1 \times f(1.2, 6.231)$$
$$= 6.231 + 0.1 \times 1.2 \times 6.231 = 6.979$$

Using the predicted value of y_4, we make the correction on it using the corrector formula

$$y_4^c = y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_4, y_4^p)\right] = 6.231 + \frac{0.1}{2}\left[f(1.2, 6.231) + f(1.3, 6.979)\right]$$
$$= 6.231 + \frac{0.1}{2}[1.2 \times 6.231 + 1.3 \times 6.979] = 7.059$$

Thus, the fourth solution point is obtained as $(x_4, y_4) = (1.3, 7.059)$.

For $i = 4$

Starting with fourth solution point $(x_4, y_4) = (1.3, 7.059)$, we first predict the value of y_5 using the predictor formula

$$y_5^p = y_4 + hf(x_4, y_4) = 7.059 + 0.1 \times f(1.3, 7.059)$$
$$= 7.059 + 0.1 \times 1.3 \times 7.059 = 7.977$$

Using the predicted value of y_5, we make the correction on it using the corrector formula

$$y_5^c = y_4 + \frac{h}{2}\left[f(x_4, y_4) + f(x_5, y_5^p)\right]$$

$$= 7.059 + \frac{0.1}{2}\left[f(1.3, 7.059) + f(1.4, 7.977)\right]$$

$$= 7.059 + \frac{0.1}{2}[1.3 \times 7.059 + 1.4 \times 7.977] = 8.076$$

Thus, the fifth solution point is obtained as $(x_5, y_5) = (1.4, 8.076)$.

For $i = 5$

Starting with fifth solution point $(x_5, y_5) = (1.4, 8.076)$, we first predict the value of y_6 using the predictor formula

$$y_6^p = y_5 + hf(x_5, y_5) = 8.076 + 0.1 \times f(1.4, 8.076)$$

$$= 8.076 + 0.1 \times 1.4 \times 8.076 = 9.207$$

Using the predicted value of y_6, we make the correction on it using the corrector formula

$$y_6^c = y_5 + \frac{h}{2}\left[f(x_5, y_5) + f(x_6, y_6^p)\right]$$

$$= 8.076 + \frac{0.1}{2}\left[f(1.4, 8.076) + f(1.5, 9.207)\right]$$

$$= 8.076 + \frac{0.1}{2}[1.4 \times 8.076 + 1.5 \times 9.207] = 9.332$$

Thus, the sixth and final solution point is obtained as $(x_6, y_6) = (1.5, 9.332)$.

The complete solution of the given differential equations is given as

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.553	6.231	7.059	8.076	9.332

Algorithm 10.3: Modified Euler's Method

To find the solution of ODE of type $dy/dx = f(x,y)$ when the solution at the initial point, $x = x_1$, is given, and the solution is desired in the interval $[x_1, x_f]$.

```
Begin
   read: x1, y1, xf
   read: h, epsilon
   set x = x1
   set y = y1
   set i = 1
   write: i, x, y
```

```
while ( x < x_f ) do
    set y2 = y + hf (x, y)
    do
        set y1 = y2
        set y2 = y + (h/2)[f (x, y) + f (x +h, y2)]
    while ( |(y2 - y1)/y2) | > epsilon )
    set y = y2
    set x = x +h
    set i = i  +1
    write: i, x, y
endwhile
End.
```

9.2.1 Trapezoidal Rule

The Trapezoidal rule approximates the area under a curve by connecting successive points on the curve to form trapezoids of uniform width, and then summing the area under these trapezoids to obtain the approximate area under the curve.

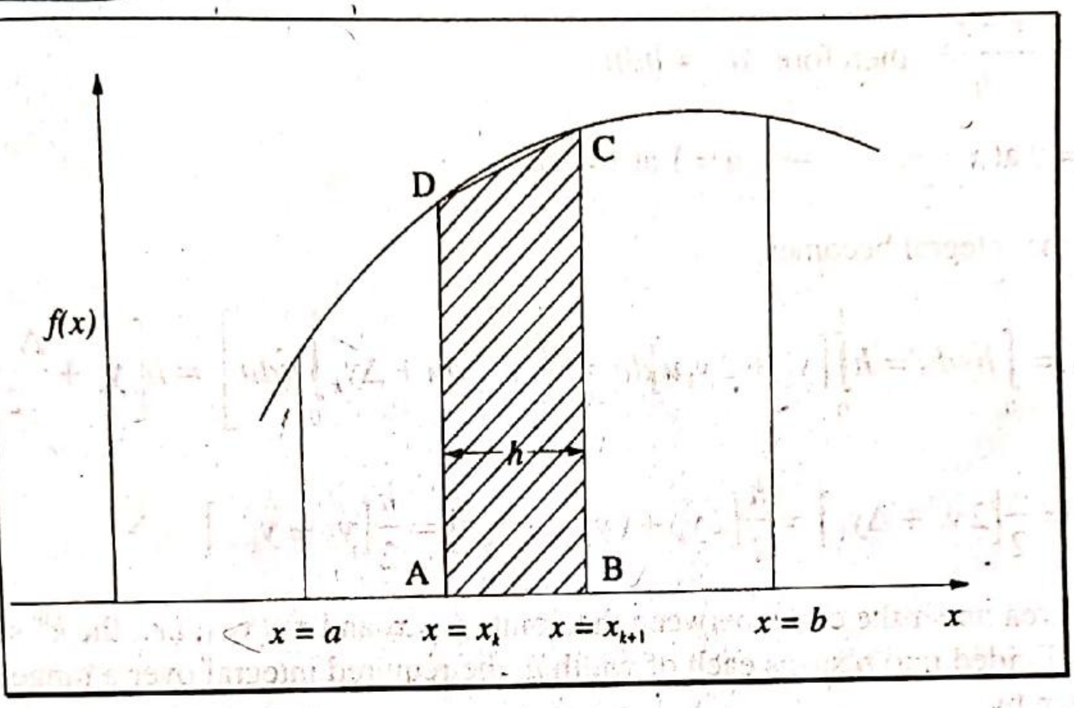

Figure 9.3: Approximation of area by Trapezoidal rule

Consider the function $f(x)$, whose graph between $x = a$ and $x = b$ is shown in Figure 9.2. An approximation to the area under the curve is obtained by dividing the interval $[a, b]$ into n strips of width h each, and approximating the area of each strip by that of a trapezoid as shown by the shaded area.

For the derivation of the formula for the trapezoid rule, we assume that the function $f(x)$ is given in the following form

x	x_1	x_1+h	x_1+2h	...	x_1+nh
$y = f(x)$	y_1	y_2	y_3	...	y_{n+1}

We consider the trapezoid formed by connecting the points (x_k, y_k) and (x_{k+1}, y_{k+1}). Further, consider the first two terms of the Newton's forward difference interpolating polynomial to represent the straight line function $f(x)$ as shown below:

$$y = f(x) = y_k + \Delta y_k u \qquad \ldots (9.3)$$

where $\Delta y_k = y_{k+1} - y_k \qquad u = \dfrac{x - x_k}{h} \qquad h = x_{k+1} - x_k$

Then the area of the k^{th} strip is given by evaluating the following integral

$$I_k = \int_{x_k}^{x_{k+1}} y\,dx$$

Since $u = \dfrac{x - x_k}{h}$, therefore $dx = h\,du$

And $u = 0$ at $x = x_k$ $u = 1$ at $x = x_{k+1}$

With this, the integral becomes

$$I_k = \int_0^1 hy\,du = h\int_0^1 [y_k + \Delta y_k u]du = h\left[y_k \int_0^1 du + \Delta y_k \int_0^1 u\,du\right] = h\left[y_k + \frac{\Delta y_k}{2}\right]$$

$$= \frac{h}{2}[2y_k + \Delta y_k] = \frac{h}{2}[2y_k + (y_{k+1} - y_k)] = \frac{h}{2}[y_k + y_{k+1}]$$

This is the area under the curve between the points $x = x_k$ and $x = x_{k+1}$, *i.e.*, the k^{th} strip. If the function is divided into n strips each of width h, the required integral over a range $x = a$ and $x = b$ is given by

$$I = \sum_{k=1}^{n} I_k$$

$$= I_1 + I_2 + I_3 + \ldots + I_n$$

$$= \frac{h}{2}[y_1 + y_2] + \frac{h}{2}[y_2 + y_3] + \frac{h}{2}[y_3 + y_4] + \ldots + \frac{h}{2}[y_n + y_{n+1}]$$

$$\boxed{I = \frac{h}{2}[y_1 + 2y_2 + 2y_3 + 2y_4 + .. + 2y_n + y_{n+1}]} \qquad \ldots (9.4)$$

This is the formula for the trapezoidal rule. It gives the correct value of the integral only if $f(x)$ is a linear function.

Example 9.1: The function $f(x)$ is given as follows

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0

Compute the integral of $f(x)$ between $x = 0$ and $x = 1.0$.

Solution: Given $h = 0.1$ and $n = 10$

Therefore formula of Eq. (9.4) becomes

$$I = \frac{h}{2}[y_1 + 2(y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10}) + y_{11}]$$

Substituting the values of y_i's and h, we get

$$I = \frac{0.1}{2}[1 + 2 \times (1.2 + 1.4 + 1.6 + 1.8 + 2.0 + 2.2 + 2.4 + 2.6 + 2.8) + 3.0]$$

$$= 2.00$$

Example 9.2: Evaluate $\int_1^2 e^{-\frac{1}{2}x}\,dx$ using four intervals. 0·6063

Solution: With four intervals, the interval size $h = 0.25$, the function is tabulated as (rounded to three decimal digits)

x	1	1.25	1.50	1.75	2.0
$f(x)$	0.607	0.535	0.472	0.417	0.368

Substituting these values, we get

$$I = \frac{h}{2}[y_1 + 2(y_2 + y_3 + y_4) + y_5]$$

$$= \frac{0.25}{2}[0.607 + 2(0535 + 0.472 + 0.417) + 0.368]$$

.607
.368
.975

$$= 0.478 \qquad \text{(rounded to three decimal places)}$$

Algorithm 9.1: Trapezoidal rule for tabulated functions

One dimensional arrays named x and y are used to hold the table of $(n+1)$ values for tabulated function. The interval size is h.

```
Begin
    read: n, h
    for i = 1 to (n+1) by 1 do
        read: xi, yi
    endfor
    set sum = (y1 + yn+1)/2
    for i = 2 to n by 1 do
        set sum = sum + yi
    endfor
    set integral = h × sum
```

Sum + f(1+1.25)
Sum f(1.25,

```
    write: "Value of the integral = ", integral
End.
```

Algorithm 9.2: Trapezoidal rule for known functions

The function $f(x)$ is to be integrated from a to b by dividing the integration range in to n intervals..

```
Begin
    read: a, b, n
    set h = (b-a)/n
    set sum = (f(a)+f(b))/2
    for i = 1 to (n-1) by 1 do
        set sum = sum + f(a+ixh)
    endfor
    set integral = h X sum
    write: "Value of the integral = ", integral
End.
```

(1 + 1 × .25
2.25

So 438.5
2.25
440.75
3.25
433.95
4.25
438.20
0.25

.2.2 Simpson's 1/3rd Rule

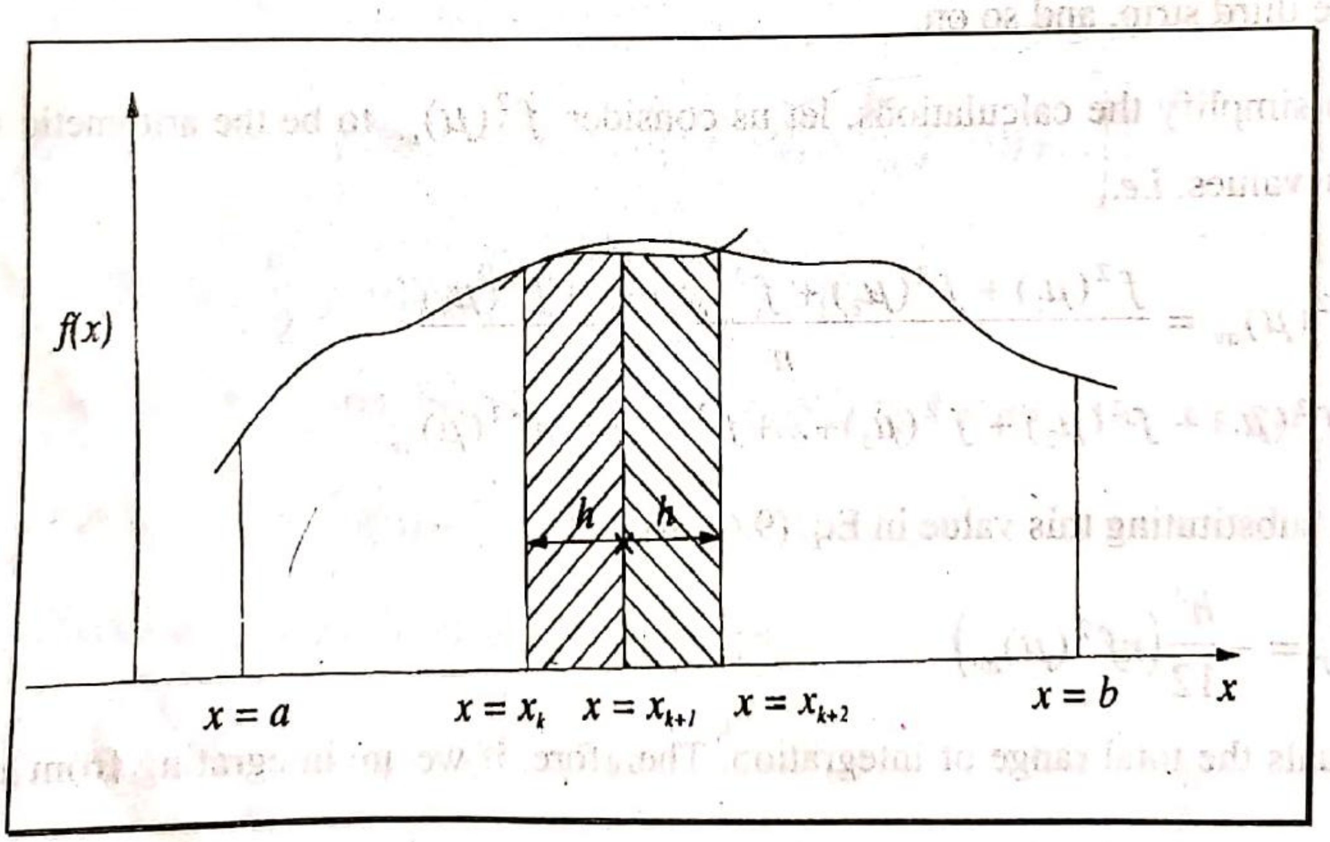

Figure 9.4: Approximation of area by Simpson's 1/3rd rule

Simpson's 1/3rd rule gives more accurate approximation of the integral value since it connects three points on the curve by second order parabolas, and then sums the areas under the parabolas to obtain the approximate area under the curve.

Consider the function $f(x)$, whose graph between $x = a$ and $x = b$ is shown in Figure 9.4. An approximation to the area under the curve is obtained by dividing it into n strips of width h each, and approximating the area of two strips by that of the area under the parabolas as shown by the shaded area in Figure 9.4.

For the derivation of the formula for the Simpson's 1/3rd rule, we assume that the function is given in the following form:

x	x_1	x_1+h	x_1+2h	...	x_1+nh
$y = f(x)$	y_1	y_2	y_3	...	y_{n+1}

We consider that the parabola passes through the points (x_k, y_k), (x_{k+1}, y_{k+1}) and (x_{k+2}, y_{k+2}). Further consider the first two terms of the Newton's forward difference interpolating polynomial to represent parabolic function $f(x)$ as shown:

$$y = f(x) = y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1) \qquad \text{... (9.8)}$$

where $\Delta y_k = y_{k+1} - y_k$, $\Delta^2 y_k = y_{k+2} - 2y_{k+1} + y_k$, $u = \dfrac{x - x_k}{h}$, $h = x_{k+1} - x_k$

The area under the parabola is given by evaluating the following integral

$$I_k = \int_{x_k}^{x_{k+2}} y\,dx$$

Since $u = \dfrac{x - x_k}{h}$, therefore $dx = h\,du$ and $u = 0$ at $x = x_k$, $u = 2$ at $x = x_{k+2}$

With this, the integral becomes

$$I_k = \int_0^2 hy\,du = h\int_0^2 \left[y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1) \right] du$$

$$= h\left[y_k \int_0^2 du + \Delta y_k \int_0^2 u\,du + \frac{\Delta^2 y_k}{2!} \int_0^2 u(u-1)\,du \right]$$

$$= h\left[2y_k + 2\Delta y_k + \frac{\Delta^2 y_k}{3} \right] = \frac{h}{3}\left[6y_k + 6\Delta y_k + \Delta^2 y_k \right]$$

$$= \frac{h}{3}\left[6y_k + 6(y_{k+1} - y_k) + (y_{k+2} - 2y_{k+1} + y_k) \right] = \frac{h}{3}\left[y_k + 4y_{k+1} + y_{k+2} \right]$$

$u^2 - u$

$\frac{u^3}{3} - \frac{u^2}{2}$

$\frac{8}{3}$

16 =

6

This is the area under the curve between the points $x = x_k$ and $x = x_{k+2}$. This covers two intervals of width h each. If we use this process repetitively $n/2$ times, we can get the area under the curve for n intervals. Thus, if the range of integration is divided into n intervals, each of width h, the required integral over a range $x = a$ and $x = b$ is given by

$$I = \sum_{k=1}^{n/2} I_{2k-1} = I_1 + I_3 + I_5 + \ldots + I_{n-1}$$

$$= \frac{h}{3}[y_1 + 4y_2 + y_3] + \frac{h}{3}[y_3 + 4y_4 + y_5] + \frac{h}{3}[y_5 + 4y_6 + y_7] +$$

$$\ldots + \frac{h}{3}[y_{n-1} + 4y_n + y_{n+1}]$$

$$\boxed{I = \frac{h}{3}[y_1 + 4y_2 + 2y_3 + 4y_4 + \ldots + 2y_{n-1} + 4y_n + y_{n+1}]} \qquad \ldots (9.9)$$

This is the formula for the Simpson's 1/3rd rule. It gives the correct value of the integral only if $f(x)$ is a second order (quadratic) function.

Example 9.3: The function $f(x)$ is given as follows

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1	1.01	1.04	1.09	1.16	1.25	1.36	1.49	1.64	1.81	2.0
		4		4		4		4		4	

Compute the integral of $f(x)$ between $x = 0$ and $x = 1.0$.

Solution: Given $h = 0.1$ and $n = 10$. Therefore formula of Eq. (9.9) becomes

$$I = \frac{h}{3}[y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + 2y_7 + 4y_8 + 2y_9 + 4y_{10} + y_{11}]$$

Substituting the values of y_i's and h, we get

$$I = \frac{0.1}{3}\begin{bmatrix} 1 + 4\times(1.01 + 1.09 + 1.25 + 1.49 + 1.81) \\ +2\times(1.04 + 1.16 + 1.36 + 1.64) + 2.0 \end{bmatrix}$$

$= 1.333$ (rounded to three decimal places)

Example 9.4: Evaluate $\int_1^2 e^{-\frac{1}{2}x} dx$ using four intervals.

Solution: With four intervals, the interval size $h = 0.25$, the function is evaluated as (rounded to three decimal digits)

x	1	1.25	1.50	1.75	2.0
$f(x)$	0.607	0.535	0.472	0.417	0.368

Substituting these values, we get

$$I = \frac{h}{3}[y_1 + 4y_2 + 2y_3 + 4y_4 + y_5]$$

$$= \frac{0.25}{3}[0.607 + 4 \times 0.535 + 2 \times 0.472 + 4 \times 0.417 + 0.368]$$

$$= 0.477 \qquad \text{(rounded to three decimal places)}$$

Algorithm 9.3: Simpson's 1/3rd rule for tabulated functions

One dimensional arrays named x and y are used to hold the table of $(n+1)$ values for tabulated function. The interval size is h.

```
Begin
    read: n, h
    for i = 1 to (n+1) by 1 do
        read: x_i, y_i
    endfor
    set s = y_1 + y_(n+1)
    set s_2 = 0
    set s_4 = 0
    for i = 2 to n by 2 do
        set s_4 = s_4 + y_i
    endfor
    for i = 3 to (n-1) by 2 do
        set s_2 = s_2 + y_i
    endfor
    set integral = (h/3) × (s + 2×s_2 + 4×s_4)
    write: "Value of the integral = ", integral
End.
```

Algorithm 9.4: Simpson's 1/3rd rule for known functions

The function $f(x)$ is to be integrated from a to b by dividing the integration range in to n intervals.

```
Begin
    read: a, b, n
    set h = (b - a)/n
    set s = f(a)+f(b)
    set s_2 = 0
    set s_4 = 0
    for i = 1 to n by 2 do
        set s_4 = s_4 + f(a+i×h)
```

```
    endfor
    for i = 2 to (n-1) by 2 do
        set s2 = s2 + f(a+i×h)
    endfor
    set integral = (h/3) × (s + 2×s2 + 4×s4)
    write: "Value of the integral = ", integral
End.
```

9.2.3 Simpson's 3/8th Rule

Assume that the tabulated function can be approximated by a third order polynomial. In this case, we can use the first four terms of the Newton's forward difference interpolation polynomial to represent $f(x)$ as shown below:

$$y = f(x) = y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1) + \frac{\Delta^3 y_k}{3!} u(u-1)(u-2) \quad \ldots (9.13)$$

where Δy_k, $\Delta^2 y_k$, $\Delta^3 y_k$ are forward differences at $x = x_k$, and

$$u = \frac{x - x_k}{h} \quad \text{, and } h = x_{k+1} - x_k$$

Since $f(x)$ is a third order polynomial, we assume it passes through the points (x_k, y_k), (x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}) and (x_{k+3}, y_{k+3}), then

$$I_k = \int_{x_k}^{x_{k+3}} y\,dx$$

Since $u = \frac{x - x_k}{h}$, therefore $dx = h\,du$ and $u = 0$ at $x = x_k$, $u = 3$ at $x = x_{k+3}$

With this, the integral becomes

$$I_k = \int_0^3 hy\,du$$

After simplification, which is left as an exercise for the readers, we get

$$I_k = \frac{3h}{8}\left[y_k + 3y_{k+1} + 3y_{k+2} + y_{k+3}\right]$$

This is the area under the curve between the points $x = x_k$ and $x = x_{k+3}$. This covers three intervals of width h each. If we use this process repetitively $n/3$ times, we can get the area under the curve for n intervals.

$y_n + \Delta y_k u + \frac{\Delta^2 y_n}{2!} u(u-1) + \frac{\Delta^3 y_n}{3!} u(u-1)(u-2)$

Thus, if the range of integration is divided into n intervals, each of width h, the required integral over a range $x = a$ and $x = b$ is given by

$$I = \sum_{k=1}^{n/3} I_{3k-2} = I_1 + I_4 + I_7 + \ldots + I_{n-2}$$

$$= \frac{3h}{8}\left[y_1 + 3y_2 + 3y_3 + y_4\right] + \frac{3h}{8}\left[y_4 + 3y_5 + 3y_6 + y_7\right]$$

$$+ \frac{3h}{8}\left[y_7 + 3y_8 + 3y_9 + y_{10}\right] + \ldots + \frac{3h}{8}\left[y_{n-2} + 3y_{n-1} + 3y_n + y_{n+1}\right]$$

$$\boxed{I = \frac{3h}{8}\begin{bmatrix} y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + 3y_6 + 2y_7 + \ldots \\ +2y_{n-2} + 3y_{n-1} + 3y_n + y_{n+1} \end{bmatrix}} \qquad \ldots (9.14)$$

This is the formula for Simpson's 3/8th rule. It gives the correct value of the integral only if $f(x)$ is a cubic. The truncation error per step, *i.e.*, local error, in using Simpson's 3/8th rule is $O(h^5)$, whereas the total truncation error, *i.e.*, global error, is $O(h^4)$. It is left as an exercise for the readers to prove.

Example 9.5: The function $f(x)$ is given as follows

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.001	1.008	1.027	1.064	1.125	1.216	1.343	1.512	1.729	2.0
		3	3		3	3		3	3	

Compute the integral of $f(x)$ between $x = 0.1$ and $x = 1.0$.

Solution: Given $h = 0.1$ and $n = 9$

Therefore formula of Eq. (9.14) becomes

$$I = \frac{3h}{8}\left[y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + 3y_6 + 2y_7 + 3y_8 + 3y_9 + y_{10}\right]$$

Substituting the values of y_i's and h, we get

$$I = \frac{3 \times 0.1}{8}[1.001 + 3 \times 1.008 + 3 \times 1.027 + 2 \times 1.064 + 3 \times 1.125$$

$$+3 \times 1.216 + 2 \times 1.343 + 3 \times 1.512 + 3 \times 1.729 + 2.0]$$

$= 1.150$ (rounded to three decimal places)

Algorithm 9.5: Simpson's 3/8th rule for tabulated functions

One dimensional arrays named x and y are used to hold the table of $(n+1)$ values for tabulated function. The interval size is h.

```
Begin
    read: n, h
    for i = 1 to (n+1) by 1 do
        read: x_i, y_i
    endfor
    set s1 = y1 + y(n+1)
    set s2 = 0
    set s3 = 0
    for i = 2 to (n-1) by 3 do
        set s3 = s3 + y_i + y(i+1)
    endfor
    for i = 4 to (n-2) by 3 do
        set s2 = s2 + y_i
    endfor
    set integral = (3×h/8) × (s1 + 2×s2 + 3×s3)
    write: "Value of the integral = ", integral
End.
```

Algorithm 9.6: Simpson's 3/8th rule for known functions

The function $f(x)$ is to be integrated from a to b by dividing the integration range into n intervals.

```
Begin
    read: a, b, n
    set h = (b-a)/n
    set s1 = f(a)+f(b)
    set s2 = 0
    set s3 = 0
    for i = 1 to n by 3 do
        set x = a+i×h
        set s3 = s3 + f(x) + f(x+h)
    endfor
    for i = 3 to n by 3 do
        set s2 = s2 + f(a+i×h)
    endfor
    set integral = (3×h/8) × (s1 + 2×s2 + 3×s3)
    write: "Value of the integral = ", integral
End.
```

8(1.25) + 8(1.50) +