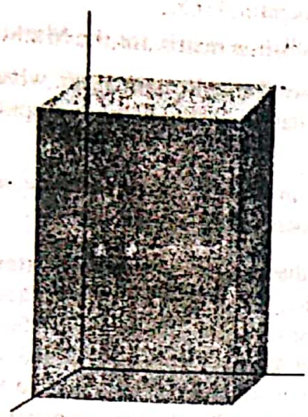
CHAPTER 3



DETERMINANTS

3.1 DEFINITION AND PROPERTIES

In this section we define the notion of a determinant and study some of its properties. Determinants arose first in the solution of linear systems. Although the method given in Chapter 1 for solving such systems is much more efficient than those involving determinants, determinants are useful in other aspects of linear algebra; some of these areas will be considered in Chapter 8. First, we deal rather briefly with permutations, which are used in our definition of determinant. In this chapter, by matrix we mean a square matrix.

DEFINITION Let $S = \{1, 2, \ldots, n\}$ be the set of integers from 1 to n, arranged in ascending order. A rearrangement $j_1 j_2 \cdots j_n$ of the elements of S is called a **permutation** of S.

To illustrate the preceding definition, let $S = \{1, 2, 3, 4\}$. Then 4132 is a permutation of S. It corresponds to the function $f : S \to S$ defined by

$$\begin{aligned} f(1) &= 4 \\ f(2) &= 1 \\ f(3) &= 3 \\ f(4) &= 2. \end{aligned}$$

We can put any one of the n elements of S in first position, any one of the remaining $n-1$ elements in second position, any one of the remaining $n-2$ elements in third position, and so on, until the nth position can only be filled by the last remaining element. Thus there are

$$n(n-1)(n-2)\cdots 2 \cdot 1 \tag{1}$$

permutations of S; we denote the set of all permutations of S by S_n.

The expression in Equation (1) is denoted

$$n!, \quad n \text{ factorial}.$$

We have

$$
\begin{aligned}
1! &= 1\\
2! &= 2 \cdot 1 = 2\\
3! &= 3 \cdot 2 \cdot 1 = 6\\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24\\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\\
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720\\
7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040\\
8! &= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320\\
9! &= 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880.
\end{aligned}
$$

EXAMPLE 1 S_1 consists of only $1! = 1$ permutation of the set $\{1\}$, namely, 1; S_2 consists of $2! = 2 \cdot 1 = 2$ permutations of the set $\{1, 2\}$, namely, 12 and 21; S_3 consists of $3! = 3 \cdot 2 \cdot 1 = 6$ permutations of the set $\{1, 2, 3\}$, namely, 123, 231, 312, 132, 213, and 321. ■

A permutation $j_1 j_2 \cdots j_n$ of $S = \{1, 2, \ldots, n\}$ is said to have an **inversion** if a larger integer j_r precedes a smaller one j_s. A permutation is called **even** or **odd** according to whether the total number of inversions in it is even or odd. Thus, the permutation 4132 of $S = \{1, 2, 3, 4\}$ has four inversions: 4 before 1, 4 before 3, 4 before 2, and 3 before 2. It is then an even permutation.

If $n \geq 2$, it can be shown that S_n has $n!/2$ even permutations and an equal number of odd permutations.

EXAMPLE 2 In S_2, the permutation 12 is even, since it has no inversions; the permutation 21 is odd, since it has one inversion. ■

EXAMPLE 3 The even permutations in S_3 are 123 (no inversions), 231 (two inversions: 21 and 31); and 312 (two inversions: 31 and 32). The odd permutations in S_3 are 132 (one inversion: 32); 213 (one inversion: 21); and 321 (three inversions: 32, 31, and 21). ■

DEFINITION Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. We define the **determinant** of A (written $\det(A)$ or $|A|$) by

$$\det(A) = |A| = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}, \tag{2}$$

where the summation ranges over all permutations $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \ldots, n\}$. The sign is taken as $+$ or $-$ according to whether the permutation $j_1 j_2 \cdots j_n$ is even or odd.

In each term $(\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ of $\det(A)$, the row subscripts are in their natural order, whereas the column subscripts are in the order $j_1 j_2 \cdots j_n$. Since the permutation $j_1 j_2 \cdots j_n$ is merely a rearrangement of the numbers from 1 to n, it has no repeats. Thus each term in $\det(A)$ is a product of n elements of A each with its appropriate sign, with exactly one element from each row and exactly one element from each column. Since we sum over all the permutations of the set, $S = \{1, 2, \ldots, n\}$, $\det(A)$ has $n!$ terms in the sum.

EXAMPLE 4 If $A = \left[a_{11}\right]$ is a 1×1 matrix, then S_1 has only one permutation in it, the identity permutation 1, which is even. Thus $\det(A) = a_{11}$. ■

EXAMPLE 5 If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is a 2×2 matrix, then to obtain $\det(A)$ we write down the terms

$$a_{1-}a_{2-} \quad \text{and} \quad a_{1-}a_{2-},$$

and fill in the blanks with all possible elements of S_2; the subscripts become 12 and 21. Since 12 is an even permutation, the term $a_{11}a_{22}$ has a $+$ sign associated with it; since 21 is an odd permutation, the term $a_{12}a_{21}$ has a $-$ sign associated with it. Hence

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

We can also obtain $\det(A)$ by forming the product of the entries on the line from left to right in the following diagram and subtracting from this number the product of the entries on the line from right to left:

$$\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}.$$

Thus, if

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix},$$

then $\det(A) = (2)(5) - (-3)(4) = 22$. ■

EXAMPLE 6 If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then, to compute $\det(A)$, we write down the six terms

$$a_{1-}a_{2-}a_{3-}, \quad a_{1-}a_{2-}a_{3-}, \quad a_{1-}a_{2-}a_{3-}, \quad a_{1-}a_{2-}a_{3-},$$
$$a_{1-}a_{2-}a_{3-}, \quad \text{and} \quad a_{1-}a_{2-}a_{3-}.$$

All the elements of S_3 are used to fill in the blanks, and if we prefix each term by $+$ or by $-$ according to whether the permutation used is even or odd, we find that

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \tag{3}$$

We can also obtain $\det(A)$ as follows. Repeat the first and second columns of A as shown below. Form the sum of the products of the entries on the lines from left to right, and subtract from this number the products of the entries on the lines from right to left (verify):

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}.$$

■

Warning It should be emphasized that the methods given for evaluating $\det(A)$ in Examples 5 and 6 do not apply for $n \geq 4$.

EXAMPLE 7 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

Evaluate $\det(A)$.

Solution Substituting in (3), we find that

$$\det(A) = (1)(1)(2) + (2)(3)(3) + (3)(2)(1) \\ - (1)(3)(1) - (2)(2)(2) - (3)(1)(3) = 6.$$

We could obtain the same result by using the easy method given previously (verify). ■

It may already have struck the reader that this is an extremely tedious way of computing determinants for a sizable value of n. In fact, $10! = 3.6288 \times 10^6$ and $20! = 2.4329 \times 10^{18}$ are enormous numbers. We shall soon develop a number of properties satisfied by determinants, which will greatly reduce the computational effort.

Permutations are studied to some depth in abstract algebra courses and in courses dealing with group theory. We shall not make use of permutations in our methods for computing determinants. We require the following property of permutations. If we interchange two numbers in the permutation $j_1 j_2 \cdots j_n$, then the number of inversions is either increased or decreased by an odd number (Exercise T.1).

EXAMPLE 8 The number of inversions in the permutation 54132 is 8. The number of inversions in the permutation 52134 is 5. The permutation 52134 was obtained from 54132 by interchanging 2 and 4. The number of inversions differs by 3, an odd number. ■

PROPERTIES OF DETERMINANTS

THEOREM 3.1 *The determinants of a matrix and its transpose are equal, that is,* $\det(A^T) = \det(A)$.

Proof Let $A = [a_{ij}]$ and let $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$ $(1 \leq i \leq n, 1 \leq j \leq n)$. Then from (2) we have

$$\det(A^T) = \sum (\pm) b_{1j_1} b_{2j_2} \cdots b_{nj_n} = \sum (\pm) a_{j_1 1} a_{j_2 2} \cdots a_{j_n n}. \qquad (4)$$

We can now rearrange the factors in the term $a_{j_1 1} a_{j_2 2} \cdots a_{j_n n}$ so that the row indices are in their natural order. Thus

$$b_{1j_1} b_{2j_2} \cdots b_{nj_n} = a_{j_1 1} a_{j_2 2} \cdots a_{j_n n} = a_{1k_1} a_{2k_2} \cdots a_{nk_n}.$$

It can be shown, by the properties of permutations discussed in an abstract algebra course,* that the permutation $k_1 k_2 \cdots k_n$, which determines the sign

*See J. Fraleigh, *A First Course in Abstract Algebra*, 7th ed., Reading, Mass.: Addison-Wesley Publishing Company, Inc., 2003; and J. Gallian, *Contemporary Abstract Algebra*, 5th ed., Mass.: Houghton Mifflin, 2002.

associated with $a_{1k_1}a_{2k_2}\cdots a_{nk_n}$, and the permutation $j_1j_2\cdots j_n$, which determines the sign associated with $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$, are both even or both odd. As an example,

$$b_{13}b_{24}b_{35}b_{41}b_{52} = a_{31}a_{42}a_{53}a_{14}a_{25} = a_{14}a_{25}a_{31}a_{42}a_{53};$$

the number of inversions in the permutation 45123 is 6 and the number of inversions in the permutation 34512 is also 6. Since the terms and corresponding signs in (2) and (4) agree, we conclude that $\det(A) = \det(A^T)$. ■

EXAMPLE 9 Let A be the matrix of Example 7. Then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}.$$

Substituting in (3), we find that

$$\begin{aligned}\det(A^T) &= (1)(1)(2) + (2)(1)(3) + (3)(2)(3) \\ &\quad - (1)(1)(3) - (2)(2)(2) - (3)(1)(3) \\ &= 6 = \det(A).\end{aligned}$$ ■

Theorem 3.1 will enable us to replace "row" by "column" in many of the additional properties of determinants; we see how to do this in the following theorem.

THEOREM 3.2 *If matrix B results from matrix A by interchanging two rows (columns) of A, then* $\det(B) = -\det(A)$.

Proof Suppose that B arises from A by interchanging rows r and s of A, say $r < s$. Then we have $b_{rj} = a_{sj}$, $b_{sj} = a_{rj}$, and $b_{ij} = a_{ij}$ for $i \neq r$, $i \neq s$. Now

$$\begin{aligned}\det(B) &= \sum(\pm)b_{1j_1}b_{2j_2}\cdots b_{rj_r}\cdots b_{sj_s}\cdots b_{nj_n} \\ &= \sum(\pm)a_{1j_1}a_{2j_2}\cdots a_{sj_r}\cdots a_{rj_s}\cdots a_{nj_n} \\ &= \sum(\pm)a_{1j_1}a_{2j_2}\cdots a_{rj_s}\cdots a_{sj_r}\cdots a_{nj_n}.\end{aligned}$$

The permutation $j_1j_2\cdots j_s\cdots j_r\cdots j_n$ results from the permutation $j_1j_2\cdots j_r\cdots j_s\cdots j_n$ by an interchange of two numbers; the number of inversions in the former differs by an odd number from the number of inversions in the latter (see Exercise T.1). This means that the sign of each term in $\det(B)$ is the negative of the sign of the corresponding term in $\det(A)$. Hence $\det(B) = -\det(A)$.

Now suppose that B is obtained from A by interchanging two columns of A. Then B^T is obtained from A^T by interchanging two rows of A^T. So $\det(B^T) = -\det(A^T)$, but $\det(B^T) = \det(B)$ and $\det(A^T) = \det(A)$. Hence $\det(B) = -\det(A)$. ■

In the results to follow, proofs will be given only for the rows of A; the proofs of the corresponding column case proceed as at the end of the proof of Theorem 3.2.

EXAMPLE 10 We have

$$\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 7 \quad\text{and}\quad \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7.$$ ■

THEOREM 3.3 *If two rows (columns) of A are equal, then* $\det(A) = 0$.

Proof Suppose that rows r and s of A are equal. Interchange rows r and s of A to obtain a matrix B. Then $\det(B) = -\det(A)$. On the other hand, $B = A$, so $\det(B) = \det(A)$. Thus $\det(A) = -\det(A)$, so $\det(A) = 0$. ■

EXAMPLE 11 Using Theorem 3.3, it follows that

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$

■

THEOREM 3.4 *If a row (column) of A consists entirely of zeros, then* $\det(A) = 0$.

Proof Let the rth row of A consist entirely of zeros. Since each term in the definition for the determinant of A contains a factor from the rth row, each term in $\det(A)$ is zero. Hence $\det(A) = 0$. ■

EXAMPLE 12 Using Theorem 3.4, it follows that

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

■

THEOREM 3.5 *If B is obtained from A by multiplying a row (column) of A by a real number c, then* $\det(B) = c\det(A)$.

Proof Suppose that the rth row of $A = [a_{ij}]$ is multiplied by c to obtain $B = [b_{ij}]$. Then $b_{ij} = a_{ij}$ if $i \neq r$ and $b_{rj} = ca_{rj}$. We obtain $\det(B)$ from Equation (2) as

$$\begin{aligned}
\det(B) &= \sum(\pm)b_{1j_1}b_{2j_2}\cdots b_{rj_r}\cdots b_{nj_n} \\
&= \sum(\pm)a_{1j_1}a_{2j_2}\cdots(ca_{rj_r})\cdots a_{nj_n} \\
&= c\left(\sum(\pm)a_{1j_1}a_{2j_2}\cdots a_{rj_r}\cdots a_{nj_n}\right) = c\det(A).
\end{aligned}$$

■

We can now use Theorem 3.5 to simplify the computation of $\det(A)$ by factoring out common factors from rows and columns of A.

EXAMPLE 13 We have

$$\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 6(4-1) = 18.$$

■

EXAMPLE 14 We have

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 1 & 4 & 3 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 4 & 1 \end{vmatrix} = (2)(3)(0) = 0.$$

Here we first factored out 2 from the third row, then 3 from the third column, and then used Theorem 3.3, since the first and third columns are equal. ■

It is convenient to rewrite these properties in terms of $\det(A)$:

$$\det(A) = -\det(A_{\mathbf{r}_i \leftrightarrow \mathbf{r}_j}), \quad i \neq j$$

$$\det(A) = \frac{1}{k}\det(A_{k\mathbf{r}_i \to \mathbf{r}_i}), \quad k \neq 0$$

$$\det(A) = \det(A_{k\mathbf{r}_i + \mathbf{r}_j \to \mathbf{r}_j}), \quad i \neq j.$$

We proceed similarly for column operations.

Theorems 3.2, 3.5, 3.6, and 3.7 are very useful in the evaluation of $\det(A)$. What we do is transform A by means of our elementary row operations to a triangular matrix. Of course, we must keep track of how the determinant of the resulting matrices changes as we perform the elementary row operations.

EXAMPLE 17 Let $A = \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. Compute $\det(A)$.

Solution We have

$$\det(A) = 2\det(A_{\frac{1}{2}\mathbf{r}_3 \to \mathbf{r}_3}) \qquad \text{Multiply row 3 by } \tfrac{1}{2}.$$

$$= 2\det\left(\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}\right)$$

$$= 2\det\left(\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{\mathbf{r}_1 \leftrightarrow \mathbf{r}_3}\right) \qquad \text{Interchange rows 1 and 3.}$$

$$= (-1)2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix}\right)$$

$$= -2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix}_{\substack{-3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -4\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3}}\right) \qquad \text{Zero out below the } (1, 1) \text{ entry.}$$

$$= -2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix}\right)$$

$$= -2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix}_{-\frac{5}{8}\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3}\right) \qquad \text{Zero out below the } (2, 2) \text{ entry.}$$

$$= -2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & 0 & -\frac{30}{4} \end{bmatrix}\right).$$

Next we compute the determinant of the upper triangular matrix.

$$\det(A) = -2(1)(-8)\left(-\frac{30}{4}\right) = -120 \qquad \text{By Theorem 3.7.}$$

The operations chosen are not the most efficient, but we do avoid fractions during the first few steps. ■

Remark The method used to compute a determinant in Example 17 will be referred to as the **computation via reduction to triangular form.**

We shall omit the proof of the following important theorem.

THEOREM 3.8 *The determinant of a product of two matrices is the product of their determinants; that is,*

$$\det(AB) = \det(A)\det(B).$$

■

EXAMPLE 18 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Then

$$|A| = -2 \quad \text{and} \quad |B| = 5.$$

Also,

$$AB = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}$$

and

$$|AB| = -10 = |A||B|.$$

■

Remark In Example 18 we also have (verify)

$$BA = \begin{bmatrix} -1 & 0 \\ 7 & 10 \end{bmatrix},$$

so $AB \neq BA$. However, $|BA| = |B|\,|A| = -10 = |AB|$.

As an immediate consequence of Theorem 3.8, we can readily compute $\det(A^{-1})$ from $\det(A)$, as the following corollary shows.

COROLLARY 3.2 *If A is nonsingular, then* $\det(A) \neq 0$ *and*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

■

Proof Exercise T.4.

EXAMPLE 19 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then $\det(A) = -2$ and

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Now,

$$\det(A^{-1}) = -\frac{1}{2} = \frac{1}{\det(A)}.$$

■

THE DETERMINANT OF BIT MATRICES (OPTIONAL)

The properties and techniques for the determinant developed in this section apply to bit matrices, where computations are carried out using binary arithmetic.

EXAMPLE 20 The determinant of the 2×2 bit matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

computed by using the technique developed in Example 5 is

$$\det(A) = (1)(1) - (1)(0) = 1.$$

■

EXAMPLE 21 The determinant of the 3×3 bit matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

computed by using the technique developed in Example 6 is

$$\begin{aligned} \det(A) &= (1)(1)(1) + (0)(0)(0) + (1)(1)(1) \\ &\quad - (1)(0)(1) - (1)(0)(1) - (0)(1)(1) \\ &= 1 + 0 + 1 - 0 - 0 - 0 = 1 + 1 = 0. \end{aligned}$$

■

EXAMPLE 22 Use the computation via reduction to triangular form to evaluate the determinant of the bit matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}_{r_1 \leftrightarrow r_2} = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}_{r_1 + r_3 \to r_3} = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

By Theorem 3.3, $\det(A) = 0$. ■

Key Terms

Permutation
n factorial
Inversion
Even permutation
Odd permutation
Determinant
Computation via reduction to triangular form

3.1 Exercises

1. Find the number of inversions in each of the following permutations of $S = \{1, 2, 3, 4, 5\}$.

(a) 52134 (b) 45213 (c) 42135
(d) 13542 (e) 35241 (f) 12345

2. Determine whether each of the following permutations of $S = \{1, 2, 3, 4\}$ is even or odd.

(a) 4213 (b) 1243 (c) 1234
(d) 3214 (e) 1423 (f) 2431

3. Determine the sign associated with each of the following permutations of $S = \{1, 2, 3, 4, 5\}$.
   (a) 25431 (b) 31245 (c) 21345
   (d) 52341 (e) 34125 (f) 41253

4. In each of the following pairs of permutations of $S = \{1, 2, 3, 4, 5, 6\}$, verify that the number of inversions differs by an odd number.
   (a) 436215 and 416235
   (b) 623415 and 523416
   (c) 321564 and 341562
   (d) 123564 and 423561

In Exercises 5 and 6, evaluate the determinants using Equation (2).

5-27

5. (a) $\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -5 \end{vmatrix}$

   (c) $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$ (d) $\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

6. (a) $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{vmatrix}$

   (c) $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ (d) $\begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{vmatrix}$

7. Let $A = [a_{ij}]$ be a 4×4 matrix. Write the general expression for $\det(A)$ using Equation (2).

8. If

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -4,$$

find the determinants of the following matrices:

$$B = \begin{bmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{bmatrix},$$

$$C = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 2c_1 & 2c_2 & 2c_3 \end{bmatrix},$$

and

$$D = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 + 4c_1 & b_2 + 4c_2 & b_3 + 4c_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

9. If

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3,$$

find the determinants of the following matrices: $B =$

$$\begin{bmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

$$C = \begin{bmatrix} a_1 & 3a_2 & a_3 \\ b_1 & 3b_2 & b_3 \\ c_1 & 3c_2 & c_3 \end{bmatrix},$$

and

$$D = \begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

10. If

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 1 \\ 2 & 5 & 1 \end{bmatrix},$$

verify that $\det(A) = \det(A^T)$.

11. Evaluate:
   (a) $\det\left(\begin{bmatrix} \lambda - 1 & 2 \\ 3 & \lambda - 2 \end{bmatrix}\right)$
   (b) $\det(\lambda I_2 - A)$, where $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

12. Evaluate:
   (a) $\det\left(\begin{bmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 2 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\right)$
   (b) $\det(\lambda I_3 - A)$, where $A = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

13. For each of the matrices in Exercise 11, find all values of λ for which the determinant is 0.

14. For each of the matrices in Exercise 12, find all values of λ for which the determinant is 0.

In Exercises 15 and 16, compute the indicated determinant.

15. (a) $\begin{vmatrix} 0 & 2 & -5 \\ 0 & 4 & 6 \\ 0 & 0 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 6 & 6 & 3 & -2 \\ 0 & 4 & 7 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

   (c) $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{vmatrix}$

16. (a) $\begin{vmatrix} 6 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 9 & -3 & 0 \\ 4 & 1 & -3 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

   (c) $\begin{vmatrix} 2 & 6 & -5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{vmatrix}$

In Exercises 17 through 20, evaluate the given determinant via reduction to triangular form.

17. (a) $\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} 4 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

18. (a) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -3 & 1 & 2 \end{vmatrix}$

19. (a) $\begin{vmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ -2 & 0 & 1 & -3 \\ 8 & -2 & 6 & 4 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ -9 & 15 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{vmatrix}$

20. (a) $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 2 & 1 & -5 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ 1 & 4 & 3 \end{vmatrix}$

21. Verify that $\det(AB) = \det(A)\det(B)$ for the following:

(a) $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

22. If $|A| = -4$, find

(a) $|A^2|$ (b) $|A^4|$ (c) $|A^{-1}|$

23. If A and B are $n \times n$ matrices with $|A| = 2$ and $|B| = -3$, calculate $|A^{-1}B^T|$. $-3/2$

In Exercises 24 and 25, evaluate the given determinant of the bit matrices using techniques developed in Examples 5 and 6.

24. (a) $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

25. (a) $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

In Exercises 26 and 27, evaluate the given determinant of the bit matrices via reduction to triangular form.

26. (a) $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$

27. (a) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

Theoretical Exercises

T.1. Show that if we interchange two numbers in the permutation $j_1 j_2 \cdots j_n$, then the number of inversions is either increased or decreased by an odd number. (*Hint*: First show that if two adjacent numbers are interchanged, the number of inversions is either increased or decreased by 1. Then show that an interchange of any two numbers can be achieved by an odd number of successive interchanges of adjacent numbers.)

T.2. Prove Theorem 3.7 for the lower triangular case.

T.3. Show that if c is a real number and A is $n \times n$, then $\det(cA) = c^n \det(A)$.

T.4. Prove Corollary 3.2.

T.5. Show that if $\det(AB) = 0$, then $\det(A) = 0$ or $\det(B) = 0$.

T.6. Is $\det(AB) = \det(BA)$? Justify your answer.

T.7. Show that if A is a matrix such that in each row and in each column one and only one element is $\neq 0$, then $\det(A) \neq 0$.

T.8. Show that if $AB = I_n$, then $\det(A) \neq 0$ and $\det(B) \neq 0$.

T.9. (a) Show that if $A = A^{-1}$, then $\det(A) = \pm 1$.

(b) Show that if $A^T = A^{-1}$, then $\det(A) = \pm 1$.

T.10. Show that if A is a nonsingular matrix such that $A^2 = A$, then $\det(A) = 1$.

T.11. Show that

$$\det(A^T B^T) = \det(A)\det(B^T) = \det(A^T)\det(B).$$

T.12. Show that

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (b-a)(c-a)(b-c).$$

This determinant is called a **Vandermonde*** **determinant.**

T.13. Let $A = [a_{ij}]$ be an upper triangular matrix. Show that A is nonsingular if and only if $a_{ii} \neq 0$, $1 \leq i \leq n$.

T.14. Show that if $\det(A) = 0$, then $\det(AB) = 0$.

T.15. Show that if $A^n = O$, for some positive integer n, then $\det(A) = 0$.

T.16. Show that if A is $n \times n$, with A skew symmetric ($A^T = -A$, see Section 1.4, Exercise T.24), and n odd, then $\det(A) = 0$.

T.17. Prove Corollary 3.1.

T.18. When is a diagonal matrix nonsingular? (*Hint*: See Exercise T.7.)

T.19. Using Exercise T.13 in Section 1.2, determine how many 2×2 bit matrices have determinant 0 and how many have determinant 1.

MATLAB Exercises

In order to use MATLAB *in this section, you should first have read Chapter 12 through Section 12.5.*

ML.1. Use the routine **reduce** to perform row operations and keep track by hand of the changes in the determinant as in Example 17.

(a) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 & 3 & -2 \\ -2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

ML.2. Use routine **reduce** to perform row operations and keep track by hand of the changes in the determinant as in Example 17.

(a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

ML.3. MATLAB has command **det**, which returns the value of the determinant of a matrix. Just type **det(A)**. Find the determinant of each of the following matrices using **det**.

(a) $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

ML.4. Use **det** (see Exercise ML.3) to compute the determinant of each of the following.

(a) **5 ∗ eye(size(A)) − A**, where

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(b) **(3 ∗ eye(size(A)) − A)^2**, where

$$A = \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}.$$

(c) **invert(A) ∗ A**, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

ML.5. Determine a positive integer t so that **det(t ∗ eye(size(A)) − A) = 0**, where

$$A = \begin{bmatrix} 5 & 2 \\ -1 & 2 \end{bmatrix}.$$

---

*Alexandre-Théophile Vandermonde (1735–1796) was born in Paris. His father, a physician, tried to steer him toward a musical career. His published mathematical work consisted of four papers that were presented over a two-year period. He is generally considered the founder of the theory of determinants and also developed formulas for solving general quadratic, cubic, and quartic equations. Vandermonde was a cofounder of the Conservatoire des Arts et Métiers and was its director from 1782. In 1795 he helped to develop a course in political economy. He was an active revolutionary in the French Revolution and was a member of the Commune of Paris and the club of the Jacobins.

3.2 COFACTOR EXPANSION AND APPLICATIONS

So far, we have been evaluating determinants by using Equation (2) of Section 3.1 and the properties established there. We now develop a different method for evaluating the determinant of an $n \times n$ matrix, which reduces the problem to the evaluation of determinants of matrices of order $n - 1$. We can then repeat the process for these $(n-1) \times (n-1)$ matrices until we get to 2×2 matrices.

DEFINITION Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. The determinant $\det(M_{ij})$ is called the **minor** of a_{ij}. The **cofactor** A_{ij} of a_{ij} is defined as

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

EXAMPLE 1 Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}.$$

Then

$$\det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34,$$

$$\det(M_{23}) = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10,$$

and

$$\det(M_{31}) = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -6 - 10 = -16.$$

Also,

$$A_{12} = (-1)^{1+2} \det(M_{12}) = (-1)(-34) = 34,$$
$$A_{23} = (-1)^{2+3} \det(M_{23}) = (-1)(10) = -10,$$

and

$$A_{31} = (-1)^{3+1} \det(M_{31}) = (1)(-16) = -16.$$ ■

If we think of the sign $(-1)^{i+j}$ as being located in position (i, j) of an $n \times n$ matrix, then the signs form a checkerboard pattern that has a $+$ in the $(1, 1)$ position. The patterns for $n = 3$ and $n = 4$ are as follows:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad\qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$n = 3 \qquad\qquad\qquad n = 4$$

The following theorem gives another method of evaluating determinants that is not as computationally efficient as reduction to triangular form.

THEOREM 3.9 *Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then for each $1 \le i \le n$,*

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \qquad (1)$$

(expansion of $\det(A)$ *along the* ith *row);*

and for each $1 \le j \le n$,

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \qquad (2)$$

(expansion of $\det(A)$ *along the* jth *column).*

Proof The first formula follows from the second by Theorem 3.1, that is, from the fact that $\det(A^T) = \det(A)$. We omit the general proof and consider the 3×3 matrix $A = [a_{ij}]$. From (3) in Section 3.1,

$$\begin{aligned}\det(A) = &\; a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.\end{aligned} \qquad (3)$$

We can write this expression as

$$\begin{aligned}\det(A) = &\; a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\ &+ a_{13}(a_{21}a_{32} - a_{22}a_{31}).\end{aligned}$$

Now,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{23}a_{32}),$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{23}a_{31} - a_{21}a_{33}),$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}a_{32} - a_{22}a_{31}).$$

Hence

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},$$

which is the expansion of $\det(A)$ along the first row.

If we now write (3) as

$$\begin{aligned}\det(A) = &\; a_{13}(a_{21}a_{32} - a_{22}a_{31}) + a_{23}(a_{12}a_{31} - a_{11}a_{32}) \\ &+ a_{33}(a_{11}a_{22} - a_{12}a_{21}),\end{aligned}$$

we can easily verify that

$$\det(A) = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33},$$

which is the expansion of $\det(A)$ along the third column. ■

EXAMPLE 2 To evaluate the determinant

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix},$$

we note that it is best to expand along either the second column or the third row because they each have two zeros. Obviously, the optimal course of action is to expand along a row or column having the largest number of zeros because in that case the cofactors A_{ij} of those a_{ij} that are zero do not have to be evaluated, since $a_{ij}A_{ij} = (0)(A_{ij}) = 0$. Thus, expanding along the third row, we have

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$$

$$= (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{3+2}(0)\begin{vmatrix} 1 & -3 & 4 \\ -4 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix} \tag{4}$$

$$+ (-1)^{3+3}(0)\begin{vmatrix} 1 & 2 & 4 \\ -4 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} + (-1)^{3+4}(-3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}.$$

We now evaluate

$$\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix}$$

by expanding along the first column, obtaining

$$(-1)^{1+1}(2)\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} + (-1)^{2+1}(2)\begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix}$$

$$= (1)(2)(9) + (-1)(2)(-1) = 20.$$

Similarly, we evaluate

$$\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

by expanding along the third row, obtaining

$$(-1)^{3+1}(2)\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} + (-1)^{3+3}(-2)\begin{vmatrix} 1 & 2 \\ -4 & 2 \end{vmatrix}$$

$$= (1)(2)(8) + (1)(-2)(10) = -4.$$

Substituting in Equation (4), we find the value of the given determinant as

$$(+1)(3)(20) + 0 + 0 + (-1)(-3)(-4) = 48.$$

On the other hand, evaluating the given determinant by expanding along the first column, we have

$$(-1)^{1+1}(1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{2+1}(-4)\begin{vmatrix} 2 & -3 & 4 \\ 0 & 0 & -3 \\ 0 & -2 & 3 \end{vmatrix}$$

$$+ (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{4+1}(2)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= (1)(1)(-12) + (-1)(-4)(-12) + (1)(3)(20) + (-1)(2)(-24) = 48. \blacksquare$$

We can use the properties of Section 3.1 to introduce many zeros in a given row or column and then expand along that row or column. This is illustrated in the following example.

EXAMPLE 3 Consider the determinant of Example 2. We have

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}_{c_4+c_1\to c_4} = \begin{vmatrix} 1 & 2 & -3 & 5 \\ -4 & 2 & 1 & -1 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 5 \end{vmatrix}$$

$$= (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 5 \\ 2 & 1 & -1 \\ 0 & -2 & 5 \end{vmatrix}_{r_1-r_2\to r_1}$$

$$= (-1)^4(3)\begin{vmatrix} 0 & -4 & 6 \\ 2 & 1 & -1 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= (-1)^4(3)(-2)(-8) = 48.$$

■

THE INVERSE OF A MATRIX

It is interesting to ask what $a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn}$ is for $i \neq k$ because, as soon as we answer this question, we shall obtain another method for finding the inverse of a nonsingular matrix.

THEOREM 3.10 *If $A = [a_{ij}]$ is an $n \times n$ matrix, then*

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = 0 \quad \text{for } i \neq k; \tag{5}$$

$$a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = 0 \quad \text{for } j \neq k. \tag{6}$$

Proof We prove only the first formula. The second follows from the first one by Theorem 3.1.

Consider the matrix B obtained from A by replacing the kth row of A by its ith row. Thus B is a matrix having two identical rows—the ith and kth rows. Then $\det(B) = 0$. Now expand $\det(B)$ along the kth row. The elements of the kth row of B are $a_{i1}, a_{i2}, \ldots, a_{in}$. The cofactors of the kth row are $A_{k1}, A_{k2}, \ldots, A_{kn}$. Thus from Equation (1) we have

$$0 = \det(B) = a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn},$$

which is what we wanted to show. ■

This theorem says that if we sum the products of the elements of any row (column) times the corresponding cofactors of any other row (column), then we obtain zero.

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & 5 & -2 \end{bmatrix}.$$

Then

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = 19, \qquad A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -14,$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 3.$$

Now

$$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = (4)(19) + (5)(-14) + (-2)(3) = 0$$

and

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = (1)(19) + (2)(-14) + (3)(3) = 0. \quad \blacksquare$$

We may combine (1) and (5) as

$$\begin{aligned} a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} &= \det(A) \quad \text{if } i = k \\ &= 0 \quad \text{if } i \neq k. \end{aligned} \tag{7}$$

Similarly, we may combine (2) and (6) as

$$\begin{aligned} a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} &= \det(A) \quad \text{if } j = k \\ &= 0 \quad \text{if } j \neq k. \end{aligned} \tag{8}$$

DEFINITION Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix adj A, called the **adjoint** of A, is the matrix whose i, jth element is the cofactor A_{ji} of a_{ji}. Thus

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

Remarks

1. The adjoint of A is formed by taking the transpose of the matrix of cofactors of the elements of A.
2. It should be noted that the term *adjoint* has other meanings in linear algebra in addition to its use in the above definition.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}.$$

Compute adj A.

Solution The cofactors of A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18; \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17;$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6;$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6; \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10;$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2;$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10; \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1;$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28.$$

Then

$$\operatorname{adj} A = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}.$$

■

THEOREM 3.11 *If $A = [a_{ij}]$ is an $n \times n$ matrix, then*

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = \det(A)I_n.$$

Proof We have

$$A(\operatorname{adj} A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{jn} & \cdots & A_{nn} \end{bmatrix}.$$

The i, jth element in the product matrix $A(\operatorname{adj} A)$ is, by (7),

$$\begin{aligned} a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} &= \det(A) && \text{if } i = j \\ &= 0 && \text{if } i \neq j. \end{aligned}$$

This means that

$$A(\operatorname{adj} A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \det(A) \end{bmatrix} = \det(A)I_n.$$

The i, jth element in the product matrix $(\operatorname{adj} A)A$ is, by (8),

$$\begin{aligned} A_{1i}a_{1j} + A_{2i}a_{2j} + \cdots + A_{ni}a_{nj} &= \det(A) && \text{if } i = j \\ &= 0 && \text{if } i \neq j. \end{aligned}$$

Thus $(\operatorname{adj} A)A = \det(A)I_n$. ■

EXAMPLE 6 Consider the matrix of Example 5. Then

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} = \begin{bmatrix} -94 & 0 & 0 \\ 0 & -94 & 0 \\ 0 & 0 & -94 \end{bmatrix}$$

$$= -94 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} = -94 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \blacksquare$$

We now have a new method for finding the inverse of a nonsingular matrix, and we state this result as the following corollary.

COROLLARY 3.3 *If A is an $n \times n$ matrix and* $\det(A) \neq 0$, *then*

$$A^{-1} = \frac{1}{\det(A)}(\operatorname{adj} A) = \begin{bmatrix} \dfrac{A_{11}}{\det(A)} & \dfrac{A_{21}}{\det(A)} & \cdots & \dfrac{A_{n1}}{\det(A)} \\ \dfrac{A_{12}}{\det(A)} & \dfrac{A_{22}}{\det(A)} & \cdots & \dfrac{A_{n2}}{\det(A)} \\ \vdots & \vdots & & \vdots \\ \dfrac{A_{1n}}{\det(A)} & \dfrac{A_{2n}}{\det(A)} & \cdots & \dfrac{A_{nn}}{\det(A)} \end{bmatrix}.$$

Proof By Theorem 3.11, $A(\operatorname{adj} A) = \det(A) I_n$, so if $\det(A) \neq 0$, then

$$A \frac{1}{\det(A)}(\operatorname{adj} A) = \frac{1}{\det(A)}\left[A(\operatorname{adj} A)\right] = \frac{1}{\det(A)}(\det(A) I_n) = I_n.$$

Hence

$$A^{-1} = \frac{1}{\det(A)}(\operatorname{adj} A). \quad \blacksquare$$

EXAMPLE 7 Again consider the matrix of Example 5. Then $\det(A) = -94$, and

$$A^{-1} = \frac{1}{\det(A)}(\operatorname{adj} A) = \begin{bmatrix} \frac{18}{94} & \frac{6}{94} & \frac{10}{94} \\ -\frac{17}{94} & \frac{10}{94} & \frac{1}{94} \\ \frac{6}{94} & \frac{2}{94} & -\frac{28}{94} \end{bmatrix}. \quad \blacksquare$$

POLYNOMIAL INTERPOLATION REVISITED

At the end of Section 1.7 we discussed the problem of finding a quadratic polynomial that interpolates the points (x_1, y_1), (x_2, y_2), (x_3, y_3), $x_1 \neq x_2$, $x_1 \neq x_3$, and $x_2 \neq x_3$. Thus, the polynomial has the form

$$y = a_2x^2 + a_1x + a_0 \tag{9}$$

[this was Equation (15) in Section 1.6]. Substituting the given points in (9), we obtain the linear system

$$\begin{aligned} a_2x_1^2 + a_1x_1 + a_0 &= y_1 \\ a_2x_2^2 + a_1x_2 + a_0 &= y_2 \\ a_2x_3^2 + a_1x_3 + a_0 &= y_3. \end{aligned} \tag{10}$$

The coefficient matrix of this linear system is

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}$$

whose determinant is the Vandermonde determinant (see Exercise T.12 in Section 3.1), which has the value

$$(x_2 - x_1)(x_3 - x_1)(x_2 - x_3).$$

Since the three given points are *distinct*, the Vandermonde determinant is not zero. Hence, the coefficient matrix of the linear system in (10) is nonsingular, which implies that the linear system has a unique solution. Thus there is a unique interpolating quadratic polynomial. The general proof for n points is similar.

OTHER APPLICATIONS OF DETERMINANTS

In Section 4.1 we use determinants to compute the area of a triangle and in Section 5.1 to compute the area of a parallelepiped.

Key Terms

Minor
Cofactor
Adjoint

3.2 Exercises

1. Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}.$$

Compute all the cofactors.

2. Let

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & 4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}.$$

Compute all the cofactors of the elements in the second row and all the cofactors of the elements in the third column.

In Exercises 3 through 6, evaluate the determinants using Theorem 3.8.

Q3 - 6 Det

3. (a) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 4 & -2 & 0 \\ 0 & 2 & 4 \\ -1 & -1 & -3 \end{vmatrix}$

4. (a) $\begin{vmatrix} 2 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 4 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 1 & -2 \\ -1 & 3 & 1 \\ 2 & -2 & 3 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 1 & -3 \\ 0 & 1 & 2 \\ -4 & 2 & 1 \end{vmatrix}$

5. (a) $\begin{vmatrix} 3 & 1 & 2 & -1 \\ 2 & 0 & 3 & -7 \\ 1 & 3 & 4 & -5 \\ 0 & -1 & 1 & -5 \end{vmatrix}$

(b) $\begin{vmatrix} 3 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & -3 & 0 \\ 2 & 0 & 2 \\ 2 & 1 & -3 \end{vmatrix}$

6. (a) $\begin{vmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & 2 & 0 \\ 1 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix}$

(c) $\begin{vmatrix} -1 & 2 & -1 \\ 3 & 2 & 1 \\ 1 & 4 & 2 \end{vmatrix}$

7. Verify Theorem 3.10 for the matrix

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 4 & 1 & -3 \\ 2 & 0 & 1 \end{bmatrix}$$

by computing $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32}$.

8. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}.$$

(a) Find adj A.

(b) Compute $\det(A)$.

(c) Verify Theorem 3.11; that is, show that

$$A(\text{adj } A) = (\text{adj } A)A = \det(A)I_3.$$

9. Let

$$A = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

(a) Find adj A.

(b) Compute $\det(A)$.

(c) Verify Theorem 3.11; that is, show that

$$A(\text{adj } A) = (\text{adj } A)A = \det(A)I_3.$$

In Exercises 10 through 13, compute the inverses of the given matrices, if they exist, using Corollary 3.3.

10. (a) $\begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 7 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

11. (a) $\begin{bmatrix} 1 & 2 & -3 \\ -4 & -5 & 2 \\ -1 & 1 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 4 \\ 0 & 1 & -2 \end{bmatrix}$

12. (a) $\begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -1 \\ 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 \\ 1 & -5 & 6 \\ 3 & -1 & 2 \end{bmatrix}$

13. (a) $\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ -2 & 1 & 5 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

14. Use Theorem 3.12 to determine which of the following matrices are nonsingular.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & -7 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$

15. Use Theorem 3.12 to determine which of the following matrices are nonsingular.

(a) $\begin{bmatrix} 4 & 3 & -5 \\ -2 & -1 & 3 \\ 4 & 6 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & -6 & 4 & 1 \\ 3 & 5 & -1 & 3 \\ 4 & -6 & 5 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 & -4 \\ 1 & 5 & 2 \\ 3 & 7 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix}$

16. Find all values of λ for which

(a) $\det\left(\begin{bmatrix} \lambda-2 & 2 \\ 3 & \lambda-3 \end{bmatrix}\right) = 0$

(b) $\det(\lambda I_3 - A) = 0$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

17. Find all values of λ for which

(a) $\det\left(\begin{bmatrix} \lambda-1 & -4 \\ 0 & \lambda-4 \end{bmatrix}\right) = 0$

(b) $\det(\lambda I_3 - A) = 0$, where

$$A = \begin{bmatrix} -3 & -1 & -3 \\ 0 & 3 & 0 \\ -2 & -1 & -2 \end{bmatrix}$$

18. Use Corollary 3.4 to find whether the following homogeneous systems have nontrivial solutions.

(a) $\begin{aligned} x - 2y + z &= 0 \\ 2x + 3y + z &= 0 \\ 3x + y + 2z &= 0 \end{aligned}$

(b) $\begin{aligned} x + 2y + w &= 0 \\ x + 2y + 3z &= 0 \\ z + 2w &= 0 \\ y + 2z - w &= 0 \end{aligned}$

19. Repeat Exercise 18 for the following homogeneous systems.

(a) $\begin{aligned} x + 2y - z &= 0 \\ 2x + y + 2z &= 0 \\ 3x - y + z &= 0 \end{aligned}$

(b) $\begin{aligned} x + y + 2z + w &= 0 \\ 2x - y + z - w &= 0 \\ 3x + y + 2z + 3w &= 0 \\ 2x - y - z + w &= 0 \end{aligned}$

In Exercises 20 through 23, if possible, solve the given linear system by Cramer's rule.

20. $\begin{aligned} 2x + 4y + 6z &= 2 \\ x + 2z &= 0 \\ 2x + 3y - z &= -5 \end{aligned}$

21. $\begin{aligned} x + y + z - 2w &= -4 \\ 2y + z + 3w &= 4 \\ 2x + y - z + 2w &= 5 \\ x - y + w &= 4 \end{aligned}$

22. $\begin{aligned} 2x + y + z &= 6 \\ 3x + 2y - 2z &= -2 \\ x + y + 2z &= 4 \end{aligned}$

23. $\begin{aligned} 2x + 3y + 7z &= 2 \\ -2x - 4z &= 0 \\ x + 2y + 4z &= 0 \end{aligned}$

In Exercises 24 and 25, determine which of the following bit matrices are nonsingular using any of the techniques in the list of nonsingular equivalences.

24. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

25. (a) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

Theoretical Exercises

T.1. Show by a column (row) expansion that if $A = [a_{ij}]$ is upper (lower) triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

T.2. If $A = [a_{ij}]$ is a 3×3 matrix, develop the general expression for $\det(A)$ by expanding (a) along the second column, and (b) along the third row. Compare these answers with those obtained for Example 6 in Section 3.1.

T.3. Show that if A is symmetric, then adj A is also symmetric.

T.4. Show that if A is a nonsingular upper triangular matrix, then A^{-1} is also upper triangular.

T.5. Show that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$. If this condition is satisfied, use Corollary 3.3 to find A^{-1}.

T.6. Using Corollary 3.3, find the inverse of

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

[*Hint*: See Exercise T.12 in Section 3.1, where $\det(A)$ was computed.]

T.7. Show that if A is singular, then adj A is singular. [*Hint*: Show that if A is singular, then $A(\text{adj } A) = O$.]

T.8. Show that if A is an $n \times n$ matrix, then $\det(\text{adj } A) = [\det(A)]^{n-1}$.

T.9. Do Exercise T.10 in Section 1.6 using determinants.

T.10. Let $AB = AC$. Show that if $\det(A) \neq 0$, then $B = C$.

T.11. Let A be an $n \times n$ matrix all of whose entries are integers. Show that if $\det(A) = \pm 1$, then all entries of A^{-1} are integers.

T.12. Show that if A is nonsingular, then adj A is nonsingular and

$$(\text{adj } A)^{-1} = \frac{1}{\det(A)} A = \text{adj }(A^{-1}).$$

T.13. Let A be a 2×2 bit matrix such that A is row equivalent to I_2. Determine all possible such matrices A. (*Hint*: See Exercise T.19 in Section 3.1 or Exercise T.11 in Section 1.6.)

MATLAB Exercises

ML.1. In MATLAB there is a routine **cofactor** that computes the (i, j) cofactor of a matrix. For directions on using this routine, type **help cofactor**. Use **cofactor** to check your hand computations for the matrix A in Exercise 1.

ML.2. Use the **cofactor** routine (see Exercise ML.1) to compute the cofactor of the elements in the second row of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & -1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

ML.3. Use the **cofactor** routine to evaluate the determinant of A using Theorem 3.9.

$$A = \begin{bmatrix} 4 & 0 & -1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{bmatrix}$$

ML.4. Use the **cofactor** routine to evaluate the determinant of A using Theorem 3.9.

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

ML.5. In MATLAB there is a routine **adjoint**, which computes the adjoint of a matrix. For directions on using this routine, type **help adjoint**. Use **adjoint** to aid in computing the inverses of the matrices in Exercise 11.

3.3 DETERMINANTS FROM A COMPUTATIONAL POINT OF VIEW

In this book we have, by now, developed two methods for solving a linear system of n equations in n unknowns: Gauss–Jordan reduction and Cramer's rule. We also have two methods for finding the inverse of a nonsingular matrix: the method involving determinants and the method discussed in Section 1.7. In this section we discuss criteria to be considered when selecting one or another of these methods.

Most sizable problems in linear algebra are solved on computers so that it is natural to compare two methods by estimating their computing time for the same problem. Since addition is so much faster than multiplication, the number of multiplications is often used as a basis of comparison for two numerical procedures.

Consider the linear system $A\mathbf{x} = \mathbf{b}$, where A is 25×25. If we find $\mathbf{x}$ by Cramer's rule, we must first obtain $\det(A)$. We can find $\det(A)$ by cofactor expansion, say $\det(A) = a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{n1}A_{n1}$, where we have expanded $\det(A)$ along the first column. Note that if each cofactor is available, we require 25 multiplications. Now each cofactor A_{ij} is plus or minus the determinant of a 24×24 matrix, which can be expanded along a given row or column, requiring 24 multiplications. Thus the computation of $\det(A)$ requires more than $25 \times 24 \times \cdots \times 2 \times 1 = 25!$ multiplications. Even if we were to use a futuristic (not very far into the future) computer capable of performing 10 trillion (1×10^{12}) multiplications *per second* (3.15×10^{19} *per year*), it would take *about 49,000 years* to evaluate $\det(A)$. However, Gauss–Jordan reduction takes about $25^3/3$ multiplications, and we would find the solution in less than *one second*. Of course, we can compute $\det(A)$ in a much more efficient way by using elementary row operations to reduce A to triangular form and then use Theorem 3.7 (see Example 17 in Section 3.1). When implemented this way for an $n \times n$ matrix, Cramer's rule will require approximately n^4 multiplications, compared to $n^3/3$ multiplications for Gauss–Jordan reduction. Thus Gauss–Jordan reduction is still much faster.

In general, if we are seeking numerical answers, then any method involving determinants can be used for $n \leq 4$. For $n \geq 5$, determinant-dependent methods are much less efficient than Gauss–Jordan reduction or the method of Section 1.7. for inverting a matrix.

The importance of determinants obviously does not lie in their computational use. Note that methods involving determinants enable one to express the inverse of a matrix and the solution to a linear system of n equations in n unknowns by means of expressions or formulas. Gauss–Jordan reduction and the method for finding A^{-1} given in Section 1.6 do not yield a *formula* for the answer; we must proceed numerically to obtain the answer. Sometimes we do not need numerical answers but an expression for the answer because we may wish to further manipulate the answer. Another important reason for studying determinants is that they play a key role in the study of eigenvalues and eigenvectors, which will be undertaken in Chapter 8.

Key Ideas for Review

- **Theorem 3.1.** $\det(A^T) = \det(A)$.
- **Theorem 3.2.** If B results from A by interchanging two rows (columns) of A, then $\det(B) = -\det(A)$.
- **Theorem 3.3.** If two rows (columns) of A are equal, then $\det(A) = 0$.
- **Theorem 3.4.** If a row (column) of A consists entirely of zeros, then $\det(A) = 0$.
- **Theorem 3.5.** If B is obtained from A by multiplying a row (column) of A by a real number c, then $\det(B) = c\det(A)$.
- **Theorem 3.6.** If B is obtained from A by adding a multiple of a row (column) of A to another row (column) of A, then $\det(B) = \det(A)$.
- **Theorem 3.7.** If $A = [a_{ij}]$ is upper (lower) triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.
- **Theorem 3.8.** $\det(AB) = \det(A)\det(B)$.
- **Theorem 3.9 (Cofactor Expansion).** If $A = [a_{ij}]$, then

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

and

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

- **Corollary 3.3.** If $\det(A) \neq 0$, then

$$A^{-1} = \begin{bmatrix} \dfrac{A_{11}}{\det(A)} & \dfrac{A_{21}}{\det(A)} & \cdots & \dfrac{A_{n1}}{\det(A)} \\ \dfrac{A_{12}}{\det(A)} & \dfrac{A_{22}}{\det(A)} & \cdots & \dfrac{A_{n2}}{\det(A)} \\ \vdots & \vdots & & \vdots \\ \dfrac{A_{1n}}{\det(A)} & \dfrac{A_{2n}}{\det(A)} & \cdots & \dfrac{A_{nn}}{\det(A)} \end{bmatrix}.$$

- **Theorem 3.12.** A is nonsingular if and only if $\det(A) \neq 0$.
- **Corollary 3.4.** If A is an $n \times n$ matrix, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det(A) = 0$.
- **Theorem 3.13 (Cramer's Rule).** Let $A\mathbf{x} = \mathbf{b}$ be a linear system of n equations in n unknowns. If $\det(A) \neq 0$, then the system has the unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where A_i is the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$.

■ **List of Nonsingular Equivalences.** The following statements are equivalent:

1. A is nonsingular.
2. $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\det(A) \neq 0$.

Supplementary Exercises

1. Evaluate the following determinants using Equation (2) of Section 3.1.

(a) $\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & -3 \\ 4 & 0 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 3 & 2 & -1 & -4 \end{vmatrix}$

2. If

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5,$$

find the determinants of the following matrices:

(a) $B = \begin{bmatrix} \frac{1}{2}a_1 & \frac{1}{2}a_2 & \frac{1}{2}a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(b) $C = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 2c_1 & 2c_2 & 2c_3 \end{bmatrix}$

3. Let A be 4×4 and suppose that $|A| = 5$. Compute

(a) $|A^{-1}|$ (b) $|2A|$ (c) $|2A^{-1}|$

(d) $|(2A)^{-1}|$

4. Let $|A| = 3$ and $|B| = 4$. Compute

(a) $|AB|$ (b) $|ABA^T|$ (c) $|B^{-1}AB|$

5. Find all values of λ for which

$$\det\left(\begin{bmatrix} \lambda+2 & -1 & 3 \\ 2 & \lambda-1 & 2 \\ 0 & 0 & \lambda+4 \end{bmatrix}\right) = 0.$$

6. Evaluate

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 5 \\ -2 & -3 & -2 \end{vmatrix}.$$

7. Evaluate

$$\begin{vmatrix} 3 & 2 & -1 & 1 \\ 4 & 1 & 1 & 0 \\ -1 & 2 & 3 & 4 \\ -2 & 3 & 5 & 1 \end{vmatrix}.$$

8. Compute all the cofactors of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \\ -3 & -4 & 6 \end{bmatrix}.$$

9. Evaluate

$$\begin{vmatrix} 3 & 2 & -1 & 0 \\ -1 & 0 & 3 & 2 \\ 4 & 1 & 5 & -2 \\ 1 & 3 & 2 & -3 \end{vmatrix}$$

by cofactor expansion.

10. Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \\ 1 & 3 & 2 \end{bmatrix}.$$

(a) Find adj A.

(b) Compute $\det(A)$.

(c) Show that $A(\operatorname{adj} A) = \det(A)I_3$.

11. Compute the inverse of the following matrix, if it exists, using Corollary 3.3:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}.$$

12. Find all values of λ for which

$$\begin{bmatrix} \lambda-3 & 0 & 3 \\ 0 & \lambda+2 & 0 \\ -5 & 0 & \lambda+5 \end{bmatrix}$$

is singular.

13. If

$$A = \begin{bmatrix} \lambda & 0 & 1 \\ 1 & \lambda-1 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix},$$

find all values of λ for which the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

14. If possible, solve the following linear system by Cramer's rule:

$$\begin{aligned} 3x + 2y - z &= -1 \\ x - y - z &= 0 \\ 2x + y - 2z &= 3. \end{aligned}$$

15. Using only elementary row or elementary column operations and Theorems 3.2, 3.5, and 3.6 (do not expand the determinants), verify the following.

(a) $\begin{vmatrix} a-b & 1 & a \\ b-c & 1 & b \\ c-a & 1 & c \end{vmatrix} = \begin{vmatrix} a & 1 & b \\ b & 1 & c \\ c & 1 & a \end{vmatrix}$

(b) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

16. Find all values of a for which the linear system

$$\begin{aligned} 2x + ay &= 0 \\ ax + 2y &= 0 \end{aligned}$$

has (a) a unique solution; (b) infinitely many solutions.

17. Find all values of a for which the matrix

$$\begin{bmatrix} a-2 & 2 \\ a-2 & a+2 \end{bmatrix}$$

is nonsingular.

18. Use Cramer's rule to find all values of a for which the linear system

$$\begin{aligned} x - 2y + 2z &= 9 \\ 2x + y &= a \\ 3x - y - z &= -10 \end{aligned}$$

has the solution in which $y = 1$.

Theoretical Exercises

T.1. Show that if two rows (columns) of the $n \times n$ matrix A are proportional, then $\det(A) = 0$.

T.2. Show that if A is an $n \times n$ matrix, then $\det(AA^T) \geq 0$.

T.3. Let Q be an $n \times n$ matrix in which each entry is 1. Show that $\det(Q - nI_n) = 0$.

T.4. Let P be an invertible matrix. Show that if $B = PAP^{-1}$, then $\det(B) = \det(A)$.

T.5. Show that if A is a singular $n \times n$ matrix, then $A(\operatorname{adj} A) = O$. (*Hint*: See Theorem 3.11.)

T.6. Show that if A is a singular $n \times n$ matrix, then AB is singular for any $n \times n$ matrix B.

T.7. Show that if A and B are square matrices, then

$$\det\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right) = \det(A)\det(B).$$

T.8. Show that if A, B, and C are square matrices, then

$$\det\left(\begin{bmatrix} A & O \\ C & B \end{bmatrix}\right) = \det(A)\det(B).$$

T.9. Let A be an $n \times n$ matrix with integer entries and $\det(A) = \pm 1$. Show that if $\mathbf{b}$ has all integer entries, then every solution to $A\mathbf{x} = \mathbf{b}$ consists of integers.

Chapter Test

1. Evaluate

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ -1 & 2 & -3 & 4 \\ 0 & 5 & 0 & -2 \end{vmatrix}.$$

2. Let A be 3×3 and suppose that $|A| = 2$. Compute
   (a) $|3A|$ (b) $|3A^{-1}|$ (c) $|(3A)^{-1}|$

3. For what value of a is

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & a \end{vmatrix} + \begin{vmatrix} 0 & a & 1 \\ 1 & 3a & 0 \\ -2 & a & 2 \end{vmatrix} = 14?$$

4. Find all values of a for which the matrix

$$\begin{bmatrix} a^2 & 0 & 3 \\ 5 & a & 2 \\ 3 & 0 & 1 \end{bmatrix}$$

is singular.

5. Solve the following linear system by Cramer's rule.

$$\begin{aligned} x - y + z &= -1 \\ 2x + y - 3z &= 8 \\ x - 2y + 3z &= -5. \end{aligned}$$

6. Answer each of the following as true or false. Justify your answer.
   (a) $\det(AA^T) = \det(A^2)$.
   (b) $\det(-A) = -\det(A)$.
   (c) If $A^T = A^{-1}$, then $\det(A) = 1$.
   (d) If $\det(A) = 0$, then $A = O$.
   (e) If $\det(A) = 7$, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
   (f) The sign of the term $a_{15}a_{23}a_{31}a_{42}a_{54}$ in the expansion of the determinant of a 5×5 matrix is $+$.
   (g) If $\det(A) = 0$, then $\det(\operatorname{adj} A) = 0$.
   (h) If $B = PAP^{-1}$, and P is nonsingular, then $\det(B) = \det(A)$.
   (i) If $A^4 = I_n$, then $\det(A) = 1$.
   (j) If $A^2 = A$ and $A \neq I_n$, then $\det(A) = 0$.