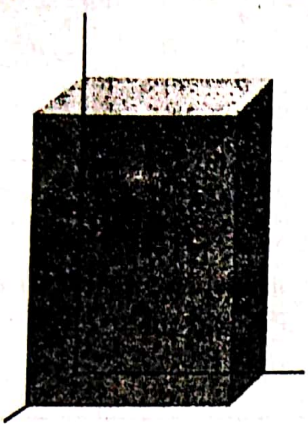

REAL VECTOR SPACES

6.1 VECTOR SPACES*

We have already defined R^n and examined some of its basic properties in Theorem 4.2. We must now study the fundamental structure of R^n. In many applications in mathematics, the sciences, and engineering, the notion of a vector space arises. This idea is merely a carefully constructed generalization of R^n. In studying the properties and structure of a vector space, we can study not only R^n, in particular, but many other important vector spaces. In this section we define the notion of a vector space in general and in later sections we study their structure.

DEFINITION 1* A **real vector space** is a set of elements V together with two operations $\oplus$ and $\odot$ satisfying the following properties:

(α) If $\mathbf{u}$ and $\mathbf{v}$ are any elements of V, then $\mathbf{u} \oplus \mathbf{v}$ is in V (i.e., V is closed under the operation $\oplus$).

(a) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$, for $\mathbf{u}$ and $\mathbf{v}$ in V.

(b) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$, for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V.

(c) There is an element $\mathbf{0}$ in V such that

$$\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}, \quad \text{for all } \mathbf{u} \text{ in } V.$$

(d) For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that

$$\mathbf{u} \oplus -\mathbf{u} = \mathbf{0}.$$

(β) If $\mathbf{u}$ is any element of V and c is any real number, then $c \odot \mathbf{u}$ is in V (i.e., V is closed under the operation $\odot$).

(e) $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$, for all real numbers c and all $\mathbf{u}$ and $\mathbf{v}$ in V.

(f) $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$, for all real numbers c and d, and all $\mathbf{u}$ in V.

(g) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$, for all real numbers c and d and all $\mathbf{u}$ in V.

(h) $1 \odot \mathbf{u} = \mathbf{u}$, for all $\mathbf{u}$ in V.

The elements of V are called **vectors**; the real numbers are called **scalars**. The operation $\oplus$ is called **vector addition**; the operation $\odot$ is called **scalar**

*Although the definitions in this book are not numbered, *this* definition is numbered because it will be referred to a number of times in this chapter.

6.1 Exercises

In Exercises 1 through 4, determine whether the given set V is closed under the operations $\oplus$ and $\odot$.

1. V is the set of all ordered pairs of real numbers (x, y), where $x > 0$ and $y > 0$;

$$(x, y) \oplus (x', y') = (x + x', y + y')$$

and

$$c \odot (x, y) = (cx, cy).$$

2. V is the set of all ordered triples of real numbers of the form $(0, y, z)$;

$$(0, y, z) \oplus (0, y', z') = (0, y + y', z + z')$$

and

$$c \odot (0, y, z) = (0, 0, cz).$$

3. V is the set of all polynomials of the form $at^2 + bt + c$, where a, b, and c are real numbers with $b = a + 1$;

$$(a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$$

and

$$r \odot (at^2 + bt + c) = (ra)t^2 + (rb)t + rc.$$

4. V is the set of all 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a = d$; $\oplus$ is matrix addition and $\odot$ is scalar multiplication.

5. Verify in detail that R^2 is a vector space.

6. Verify in detail that R^3 is a vector space.

7. Verify that the set in Example 2 is a vector space.

8. Verify that all the properties of Definition 1, except property (f), hold for the set in Example 3.

9. Show that the set in Example 5 is a vector space.

10. Show that the space P of all polynomials is a vector space.

In Exercises 11 through 17, determine whether the given set together with the given operations is a vector space. If it is not a vector space, list the properties of Definition 1 that fail to hold.

11. The set of all ordered triples of real numbers (x, y, z) with the operations

$$(x, y, z) \oplus (x', y', z') = (x', y + y', z')$$

and

$$c \odot (x, y, z) = (cx, cy, cz)$$

12. The set of all ordered triples of real numbers (x, y, z) with the operations

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$$

and

$$c \odot (x, y, z) = (x, 1, z)$$

13. The set of all ordered triples of real numbers of the form $(0, 0, z)$ with the operations

$$(0, 0, z) \oplus (0, 0, z') = (0, 0, z + z')$$

and

$$c \odot (0, 0, z) = (0, 0, cz)$$

14. The set of all real numbers with the usual operations of addition and multiplication

15. The set of all ordered pairs of real numbers (x, y), where $x \leq 0$, with the usual operations in R^2

16. The set of all ordered pairs of real numbers (x, y) with the operations $(x, y) \oplus (x', y') = (x + x', y + y')$ and $c \odot (x, y) = (0, 0)$

17. The set of all positive real numbers $\mathbf{u}$ with the operations $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = \mathbf{u}^c$

18. Let V be the set of all real numbers; define $\oplus$ by $\mathbf{u} \oplus \mathbf{v} = 2\mathbf{u} - \mathbf{v}$ and $\odot$ by $c \odot \mathbf{u} = c\mathbf{u}$. Is V a vector space?

19. Let V be the set consisting of a single element $\mathbf{0}$. Let $\mathbf{0} \oplus \mathbf{0} = \mathbf{0}$ and $c \odot \mathbf{0} = \mathbf{0}$. Show that V is a vector space.

20. (a) If V is a vector space that has a nonzero vector, how many vectors are in V?

(b) Describe all vector spaces having a finite number of vectors.

Theoretical Exercises

In Exercises T.1 through T.4, establish the indicated result for a real vector space V.

T.1. Show that $c\mathbf{0} = \mathbf{0}$ for every scalar c.

T.2. Show that $-(-\mathbf{u}) = \mathbf{u}$.

T.3. Show that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

T.4. Show that if $\mathbf{u} \neq \mathbf{0}$ and $a\mathbf{u} = b\mathbf{u}$, then $a = b$.

T.5. Show that a vector space has only one zero vector.

T.6. Show that a vector $\mathbf{u}$ in a vector space has only one negative $-\mathbf{u}$.

T.7. Show that B^n is closed under binary addition of bit n-vectors.

T.8. Show that B^n is closed under scalar multiplication by bits 0 and 1.

T.9. Show that property (h) is valid for all vectors in B^n.

Ex 6.1

Real vector spaces

<u>Definition</u>:- A real vector space is a set of elements V together with two operation $\oplus$ and $\odot$ satisfying the following Properties.

(α) If u and v are any elements of V then $u \oplus v$ is in V (i.e V is closed under the operation $\oplus$).

(a) $u \oplus v = v \oplus u$ for u and v in V

(b) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for u, v and w in V.

(c) There is an element 0 in V such that

$$u \oplus 0 = 0 \oplus u = u \text{ for all } u \text{ in } V$$

(d) For each u in V there is an element $-u$ in V such that $u \oplus -u = 0$.

(β) If u is any element of V and c is any real number then $c \odot u$ is in V (i.e V is closed under the operation $\odot$).

(e) $c \odot (u \oplus v) = c \odot u \oplus c \odot v$, for all real number c and all u and v in V

(f) $(c+d) \odot u = c \odot u \oplus d \odot u$ for all real number c and d and all u in V

(g) $c \odot (d \odot u) = (cd) \odot u$ for all real number c and d and all u in V.

(h) $1 \odot u = u$ for all u in V.

The elements of V are called vector: the real numbers are called scalars The operation $\oplus$ is called vector addition the operation $\odot$ is called scalar.

Ex 6.1

Example 2 consider the set V of all ordered triple of real number of the form $(x, y, 0)$ and define the operations $\oplus$ and $\odot$ by.

$$(x, y, 0) \oplus (x', y', 0) = (x+x', y+y', 0)$$

$$c \odot (x, y, 0) = (cx, cy, 0)$$

Sol:- checking the Properties to show that it is a vector space or not

Let $U = (x, y, 0)$ & $V = (x', y', 0)$.

(a) $U \oplus V = V \oplus U$

$\Rightarrow U \oplus V = (x, y, 0) \oplus (x', y', 0)$.

$= (x+x', y+y', 0) \longrightarrow (i)$

& $V \oplus U = (x', y', 0) \oplus (x, y, 0)$

$= (x'+x, y'+y, 0) — (ii)$

From (i) & (ii) we have

Hence $U \oplus V = V \oplus U$.

(b) $U \oplus (V \oplus W) = (U \oplus V) \oplus W$. let $W = (a, b, c)$.

$U \oplus (V \oplus W) = U \oplus [(x', y', 0) \oplus (a, b, c)]$

$= U \oplus [x'+a, y'+b, 0+c]$

$= (x, y, 0) \oplus (x'+a, y'+b+c)$

$= (x+x'+a, y+y'+b, c)$ — (i)

& $(U \oplus V) \oplus W = [(x, y, 0) \oplus (x', y', 0)] \oplus (a, b, c)$

$= (x+x', y+y', 0) \oplus (a, b, c)$

$= (x+x'+a, y+y'+b, c)$ — (ii)

From eq (i) & eq (ii) we have

$U \oplus (V \oplus W) = (U \oplus V) \oplus W.$

(c) $U \oplus 0 = 0 \oplus U = U.$

$U \oplus 0 = (x, y, 0) \oplus 0 = (x+0, y+0, 0+0) = (x, y, 0) = U$

& $0 \oplus U = (0, 0, 0) \oplus (x, y, 0) = (0+x, 0+y, 0+0) = (x, y, 0) = U$

Hence $U \oplus 0 = 0 \oplus U = U.$

(d) $U \oplus -U = 0.$

$(x, y, 0) \oplus -(x, y, 0) = (x, y, 0) \oplus (-x, -y, 0)$

$(x-x, y-y, 0) = (0, 0, 0) = 0$

Hence $U \oplus -U = 0.$

Ex 6.1

Now checking the Properties for scalar multiplication

(e) $c \odot (u \oplus v) = c \odot ((x, y, 0) \oplus (x', y', 0))$

$= c \odot (x+x', y+y', 0)$

$= (c(x+x'), c(y+y'), 0)$ — (i)

& $c \odot u \oplus c \odot v = c \odot (x, y, 0) \oplus c \odot (x', y', 0)$

$= (cx, cy, 0) \oplus (cx', cy', 0)$

$= (cx + cx', cy + cy', 0)$

$= (c(x+x'), c(y+y'), 0)$ — (ii)

From eq (i) & eq (ii) we have.

$c \odot (u \oplus v) = c \odot u \oplus c \odot v$

(f) $(c+d) \odot u = c \odot u \oplus d \odot u$

$(c+d) \odot u = (c+d) \odot (x, y, 0)$

$= (cx, cy, 0 + dx, dy, 0)$

$= (cx + dx, cy + dy, 0)$ — (i)

& $c \odot u \oplus d \odot u = c \odot (x, y, 0) \oplus d \odot (x, y, 0)$

$= (cx, cy, 0) \oplus (dx, dy, 0)$

$= (cx + dx, cy + dy, 0)$ — (ii)

From eq (i) & eq (ii) we have

$(c+d) \odot u = c \odot u \oplus d \odot u$.

(g) $c \odot (d \odot u) = (cd) \odot u$.

$$c \odot (d \odot u) = c \odot (d \odot (x, y, 0))$$
$$= c \odot (dx, dy, 0)$$
$$= (cdx, cdy, 0) \text{ — (i)}$$

$$(cd) \odot u = (cd) \odot (x, y, 0)$$
$$= (cdx, cdy, 0) \text{ — (ii)}$$

From eq (i) & eq (ii) we have

$c \odot (d \odot u) = (cd) \odot u$.

(h) $1 \odot u = u$

$$1 \odot (x, y, 0) = (1 \cdot x, 1 \cdot y, 1 \cdot 0)$$
$$= (x, y, 0)$$
$$1 \odot (x, y, 0) = u$$

Hence it satisfies all the Properties Therefore V is a vector space.

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Ex 6.1

Q1 V is the set of all ordered pairs of real number (x,y) where $x>0$ & $y>0$

$(x,y) \oplus (x',y') = (x+x', y+y')$.

& $c \odot (x,y) = (cx, cy)$.

Sol. Checking the Properties of vector addition $\oplus$

let $U = (x,y)$ & $V = (x',y')$.

(a) $U \oplus V = V \oplus U$

$U \oplus V = (x,y) \oplus (x',y')$

$= (x+x', y+y')$. — (i)

& $V \oplus U = (x',y') \oplus (x,y)$ —

$= (x'+x, y'+y)$ — (ii)

From eq (i) & eq (ii) we have.

$U \oplus V = V \oplus U$.

b) $U \oplus (V \oplus W) = (U \oplus V) \oplus W$ let $W = (a,b)$

$U \oplus (V \oplus W) = U \oplus ((x', y') \oplus (a, b))$

$= (x, y) \oplus (x'+a, y'+b)$

$= (x+x'+a, y+y'+b)$ — (i)

& $(U \oplus V) \oplus W = ((x, y) \oplus (x', y')) \oplus (a, b)$

$= (x+x', y+y') \oplus (a, b)$

$= (x+x'+a, y+y'+b)$ — (ii)

From eq (i) & eq (ii) we have

$U \oplus (V \oplus W) = (U \oplus V) + W$

(c) $U \oplus 0 = 0 \oplus U = U$

$U \oplus 0 = (x, y) \oplus (0, 0) = (x+0, y+0) = (x, y) = U$

$0 \oplus U = (0, 0) \oplus (x, y) = (0+x, 0+y) = (x, y) = U$

Hence $U \oplus 0 = 0 \oplus U = U$.

(d) $U \oplus -U = 0$

$(x, y) \oplus -(x, y) = (x, y) \oplus (-x, y)$

$(x-x, y-y) = (0, 0) = 0$

Hence $U \oplus -U = 0$. closed under $\oplus$

Ex 6.1

Now checking the Properties for scalar multiplication.

(e) $c \odot (u \oplus v) = c \odot u \oplus c \odot v$

$$c \odot (u \oplus v) = c \odot ((x, y) \oplus (x', y'))$$

$$= c \odot (x + x', y + y')$$

$$= c(x + x'), c(y + y') \quad \text{(i)}$$

& $c \odot u \oplus c \odot v = c \odot (x, y) \oplus c \odot (x', y')$

$$= (cx, cy) \oplus (cx', cy')$$

$$= (cx + cx', cy + cy')$$

$$= c(x + x'), c(y + y') \quad \text{(ii)}$$

From eq (i) & eq (ii) we have

$$c \odot (u \oplus v) = c \odot u \oplus c \odot v$$

But if $c < 0$ then $(x + x') + (y + y') < 0$ So it not closed under $\odot$.

Q2 V is the set of all ordered triples of real numbers of the form $(0, y, z)$.

$$(0, y, z) \oplus (0, y', z') = (0, y+y', z+z')$$

& $$c \odot (0, y, z) = (0, 0, cz).$$

Sol: let $U = (0, y, z)$ & $V = (0, y', z')$

checking the Properties for vector addition (+)

(a) $U \oplus V = V \oplus U$

$$\Rightarrow U \oplus V = (0, y, z) \oplus (0, y', z')$$
$$= (0+0, y+y', z+z')$$
$$= (0, y', z+z'). \quad \text{--- (i)}$$

& $$V \oplus U = (0, y', z') \oplus (0, y, z)$$
$$= (0+0, y'+y, z'+z).$$
$$= (0, y'+y, z'+z). \quad \text{--- (ii)}$$

From eq (i) & eq (ii) we have.

$U \oplus V = V \oplus U.$

Ex 6.1

(b) $U \oplus (V \oplus W) = (U \oplus V) \oplus W$ & let $W = (a,b,c)$

$\Rightarrow U \oplus (V \oplus W) = U \oplus [(0,y',z') \oplus (a,b,c)]$

$= U \oplus (0+a, y'+b, z'+c)$

$= (0,y,z) \oplus (a, y'+b, z'+c)$

$= (a, y+y'+b, z+z'+c)$ —— (i)

& $(U \oplus V) \oplus W = [(0,y,z) \oplus (0,y',z')] \oplus W$

$= (0, y+y', z+z') \oplus W$

$= (0, y+y', z+z') \oplus (a,b,c)$

$= (a, y+y'+b, z+z'+c)$ —— (ii)

From eq (i) & eq (ii) we have.

$U \oplus (V \oplus W) = (U \oplus V) \oplus W$

(c) $U \oplus 0 = 0 \oplus U = U$.

$U \oplus 0 = (0,y,z) \oplus (0,0,0) = (0+0, y+0, z+0) = (0,y,z) = U$

& $0 \oplus U = (0,0,0) \oplus (0,y,z) = (0+0, 0+y, 0+z) = (0,y,z) = U$

Hence $U \oplus 0 = 0 \oplus U = U$.

(d) $U \oplus -U = 0$

$(0, y, z) \oplus -(0, y, z)$

$(0, y, z) \oplus (0, -y, -z) = (0, y-y, z-z) = (0, 0, 0) = 0$

Hence $U \oplus -U = 0$

closed under $\oplus$.

Now checking the Properties for Scalar multiplication $\odot$

(e) $c \odot (U+V) = (c \odot U) \oplus (c \odot V)$

$c \odot (U+V) = c \odot [(0, y, z) \oplus (0, y', z')]$

$= c \odot [0+0, y+y', z+z']$

$= c \odot (0, y+y' + z+z')$

$= (0, c(y+y'), c(z+z'))$ —— (i)

$(c \odot U) \oplus (c \odot V) = [c \odot (0, y, z)] \oplus [c \odot (0, y', z')]$

$= (0, cy, cz) \oplus (0, cy', cz')$

$= (0+0, cy+cy', cz+cz')$

$= (0, c(y+y') + c(z+z')$ —— (ii)

From eq (i) & eq (ii) we have.

$c \odot (U+V) = (c \odot U) \oplus (c \odot V)$.

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Ex 6.1

(f) $(c+d)\odot u = c\odot u \oplus d\odot u$.

$(c+d)\odot u = (c+d)\odot(0, y, z)$

$= (0, cy, cz + 0, dy, dz)$

$= (0, cy+dy, cz+dz)$ — (i).

$c\odot u \oplus d\odot u = c\odot(0, y, z)\oplus d\odot(0, y, z)$

$= (0, cy, cz)\oplus(0, dy, dz)$

$= (0, cy+dy, cz+dz)$ — (ii)

From eq (i) & eq (ii) we have.

$(c+d)\odot u = c\odot u\oplus d\odot u$.

(g) $c\odot(d\odot u) = (cd)\odot u$

$\Rightarrow c\odot(d\odot u) = c\odot(d\odot(0, y, z))$

$= c\odot(0, dy, dz)$

$= (0, cdy, cdz)$ — (i)

& $(cd)\odot u = (cd)\odot(0, y, z)$

$= (0, cdy, cdz)$ — (ii)

From eq (i) & eq (ii) we have.

$c\odot(d\odot u) = (cd)\odot u$.

(h) $1 \odot U = U$

$1 \odot (0, y, z) = (0, y, z) = U$

Hence it satisfies all the Properties Therefore V is a vector space.

Q3 V is the set of all Polynomials of the form $at^2 + bt + c$ where a, b and c are real number with $b = a + 1$

$(a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2)$.

$= (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$

& $r \odot (at^2 + bt + c) = (ra)t^2 + (rb)t + rc$.

Sol: Let $U = a_1t^2 + b_1t + c_1$ & $V = a_2t^2 + b_2t + c_2$.

(a) $U \oplus V = V \oplus U$

$\Rightarrow U \oplus V = (a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2)$

$U \oplus V = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$

Put $b_1 = a_1 + 1$, $b_2 = a_2 + 1$

$U \oplus V = (a_1 + a_2)t^2 + (a_1 + 1 + a_2 + 1)t + (c_1 + c_2)$

$= (a_1 + a_2)t^2 + (a_1 + a_2 + 2)t + (c_1 + c_2)$

Ex 6.1

Not closed under $\oplus$ here we get 2 instead of 1.
as given (at^2+bt+c) & $b=a+1$.

Now checking the properties of scalar multiplication

$$r \odot (a_1t^2+b_1t+c_1) = ra_1t^2+rb_1t+rc_1$$
$$= ra_1t^2+r(a_1+1)t+rc_1$$
$$= ra_1t^2+(ra_1+r)t+rc_1$$

Not closed under $\odot$ b/c we get r instead of 1.

Hint

Q4 $U=\begin{bmatrix} a_1, & b_1 \\ c_1 & d_1 \end{bmatrix}$ & $V=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Q5 let $U=(x,y)$, $V=(x',y')$, $w=(a,b)$

Q6 $U=(x,y,z)$, $V=(x',y',z')$, $w=(a,b,c)$.

Q11, Q12 + Q13 same as Q2.

Q16 same as Q1