CHAPTER 1

Linear Equations and Matrices

1.1 LINEAR SYSTEMS

A good many problems in the natural and social sciences as well as in engineering and the physical sciences deal with equations relating two sets of variables. An equation of the type Sets

$$ax = b,$$

expressing the variable b in terms of the variable x and the constant a, is called a **linear equation**. The word *linear* is used here because the graph of the equation above is a straight line. Similarly, the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{1}$$

expressing b in terms of the variables $x_1, x_2, \ldots, x_n$ and the known constants $a_1, a_2, \ldots, a_n$, is called a **linear equation**. In many applications we are given b and the constants $a_1, a_2, \ldots, a_n$ and must find numbers $x_1, x_2, \ldots, x_n$, called **unknowns**, satisfying (1).

A **solution** to a linear equation (1) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted in (1).

Thus $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

This is not the only solution to the given linear equation, since $x_1 = 3$, $x_2 = 1$, and $x_3 = -7$ is another solution.

More generally, a **system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$**, or simply a **linear system**, is a set of m linear equations each in n unknowns. A linear system can be conveniently denoted by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned} \tag{2}$$

As you have probably already observed, the method of elimination has been described, so far, in general terms. Thus we have not indicated any rules for selecting the unknowns to be eliminated. Before providing a systematic description of the method of elimination, we introduce, in the next section, the notion of a matrix, which will greatly simplify our notation and will enable us to develop tools to solve many important problems.

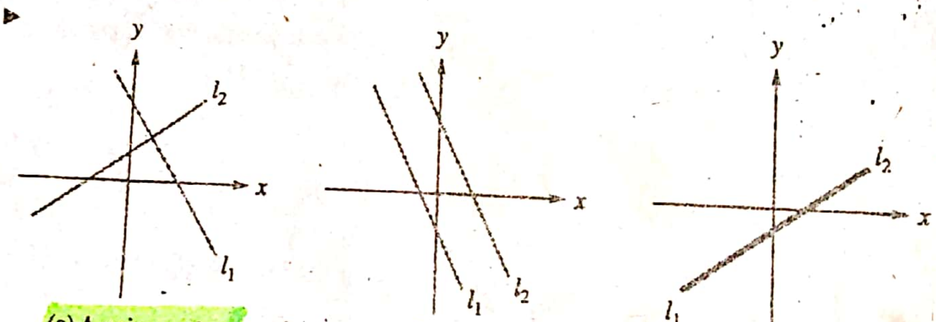
Consider now a linear system of two equations in the unknowns x and y:

$$\begin{aligned} a_1x + a_2y &= c_1 \\ b_1x + b_2y &= c_2. \end{aligned} \tag{15}$$

The graph of each of these equations is a straight line, which we denote by l_1 and l_2, respectively. If $x = s_1$, $y = s_2$ is a solution to the linear system (15), then the point (s_1, s_2) lies on both lines l_1 and l_2. Conversely, if the point (s_1, s_2) lies on both lines l_1 and l_2, then $x = s_1$, $y = s_2$ is a solution to the linear system (15). (See Figure 1.1.) Thus we are led geometrically to the same three possibilities mentioned previously.

1. The system has a unique solution; that is, the lines l_1 and l_2 intersect at exactly one point.
2. The system has no solution; that is, the lines l_1 and l_2 do not intersect.
3. The system has infinitely many solutions; that is, the lines l_1 and l_2 coincide.

Figure 1.1 ▶



(a) A unique solution (b) No solution (c) Infinitely many solutions

Next, consider a linear system of three equations in the unknowns x, y, and z:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{16}$$

The graph of each of these equations is a plane, denoted by P_1, P_2, and P_3, respectively. As in the case of a linear system of two equations in two unknowns, the linear system in (16) can have a unique solution, no solution, or infinitely many solutions. These situations are illustrated in Figure 1.2. For a more concrete illustration of some of the possible cases, the walls (planes) of a room intersect in a unique point, a corner of the room, so the linear system has a unique solution. Next, think of the planes as pages of a book. Three pages of a book (when held open) intersect in a straight line, the spine. Thus, the linear system has infinitely many solutions. On the other hand, when the book is closed, three pages of a book appear to be parallel and do not intersect, so the linear system has no solution.

since we must have $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. When $x_3 = 10$, we have

$$x_1 = 5, \qquad x_2 = 10, \qquad x_3 = 10$$

while

$$x_1 = \tfrac{13}{2}, \qquad x_2 = 13, \qquad x_3 = 7$$

when $x_3 = 7$. The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

Key Terms

Linear equation
Unknowns
Solution to a linear equation
Linear system
Solution to a linear system
Method of elimination
Unique solution
No solution
Infinitely many solutions
Manipulations on a linear system

1.1 Exercises

In Exercises 1 through 14, solve the given linear system by the method of elimination.

1. $\begin{aligned} x + 2y &= 8 \\ 3x - 4y &= 4 \end{aligned}$

2. $\begin{aligned} 2x - 3y + 4z &= -12 \\ x - 2y + z &= -5 \\ 3x + y + 2z &= 1 \end{aligned}$

3. $\begin{aligned} 3x + 2y + z &= 2 \\ 4x + 2y + 2z &= 8 \\ x - y + z &= 4 \end{aligned}$

4. $\begin{aligned} x + y &= 5 \\ 3x + 3y &= 10 \end{aligned}$

5. $\begin{aligned} 2x + 4y + 6z &= -12 \\ 2x - 3y - 4z &= 15 \\ 3x + 4y + 5z &= -8 \end{aligned}$

6. $\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{aligned}$

7. $\begin{aligned} x + 4y - z &= 12 \\ 3x + 8y - 2z &= 4 \end{aligned}$

8. $\begin{aligned} 3x + 4y - z &= 8 \\ 6x + 8y - 2z &= 3 \end{aligned}$

9. $\begin{aligned} x + y + 3z &= 12 \\ 2x + 2y + 6z &= 6 \end{aligned}$

10. $\begin{aligned} x + y &= 1 \\ 2x - y &= 5 \\ 3x + 4y &= 2 \end{aligned}$

11. $\begin{aligned} 2x + 3y &= 13 \\ x - 2y &= 3 \\ 5x + 2y &= 27 \end{aligned}$

12. $\begin{aligned} x - 5y &= 6 \\ 3x + 2y &= 1 \\ 5x + 2y &= 1 \end{aligned}$

13. $\begin{aligned} x + 3y &= -4 \\ 2x + 5y &= -8 \\ x + 3y &= -5 \end{aligned}$

14. $\begin{aligned} 2x + 3y - z &= 6 \\ 2x - y + 2z &= -8 \\ 3x - y + z &= -7 \end{aligned}$

15. Given the linear system

$$\begin{aligned} 2x - y &= 5 \\ 4x - 2y &= t, \end{aligned}$$

(a) determine a value of t so that the system has a solution.

(b) determine a value of t so that the system has no solution.

(c) how many different values of t can be selected in part (b)?

H.W

16. Given the linear system

$$\begin{aligned} 2x + 3y - z &= 0 \\ x - 4y + 5z &= 0, \end{aligned}$$

(a) verify that $x_1 = 1$, $y_1 = -1$, $z_1 = -1$ is a solution.

(b) verify that $x_2 = -2$, $y_2 = 2$, $z_2 = 2$ is a solution.

(c) is $x = x_1 + x_2 = -1$, $y = y_1 + y_2 = 1$, and $z = z_1 + z_2 = 1$ a solution to the linear system?

(d) is $3x$, $3y$, $3z$, where x, y, and z are as in part (c), a solution to the linear system?

17. Without using the method of elimination, solve the linear system

$$\begin{aligned} 2x + y - 2z &= -5 \\ 3y + z &= 7 \\ z &= 4. \end{aligned}$$

H.W

18. Without using the method of elimination, solve the linear system

$$\begin{aligned} 4x \qquad\qquad &= 8 \\ -2x + 3y \qquad &= -1 \\ 3x + 5y - 2z &= 11. \end{aligned}$$

19. Is there a value of r so that $x = 1$, $y = 2$, $z = r$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 2x + 3y - z &= 11 \\ x - y + 2z &= -7 \\ 4x + y - 2z &= 12 \end{aligned}$$

Ex: 1.1 8^{th} Edition

Linear Algebra Bernard Kolman

Q1 $x+2y=8$ ——— (1)

$3x-4y=4$ ——— (2)

Xing Equation (1) by 3 and subtring from (2)

$3x+6y=24$

$\underline{3x \mp 4y = \underline{-}4}$

$10y = 20 \Rightarrow \boxed{y=2}$

eq (1) $\Rightarrow x+2(2)=8 \Rightarrow x=8-4=4 \Rightarrow \boxed{x=4}$

Hence $(x,y)=(4,2)$ which is unique solution.

Q3 $3x+2y+z=2$ ——— (i)

$4x+2y+2z=8$ ——— (ii)

$x-y+z=4$ ——— (iii)

Solving eq (i) & (iii)

Xing eq (iii) by 3 then subtrcing from (i)

$3x-3y+3z=12$

$\underline{3x+2y+z=2}$

$-5y+2z=10$ ——— (iv).

Now solving Equation (iii) & (ii)

Xing Equation (iii) by 4 then Subtracting from (ii)

$$4x - 4y + 4z = 16$$
$$\underline{\cancel{4x} + 2y + 2z = 8}$$
$$- \quad - \quad - \quad -$$

$$-6y + 2z = 8 \text{ —— (iv)}$$

Now solving eq (iv) & (v).

(iv) − (v)

$$-5y + 2\cancel{z} = 10$$
$$\underline{-6y + \cancel{2z} = 8}$$
$$+ \quad - \quad -$$

$$\boxed{y = 2}$$

eq (iv) $\Rightarrow -5(2) + 2z = 10$

$$-10 + 2z = 10$$

$$2z = 10 + 10 \Rightarrow \cancel{2}z = \cancel{20}^{10} = \boxed{z = 10}$$

Putting the value of y & z in eq ①

$$3x + 2(2) + 10 = 2$$

$$3x = 2 - 14 = -12$$

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{-\cancel{12}^{4}}{\cancel{3}} = \boxed{x = -4}$$

Hence $(x, y, z) = (-4, 2, 10)$ which is unique Solution.

Ex: 1.1

Q7

$x + 4y - z = 12$ —— (i)

$3x + 8y - 2z = 4$ —— (ii)

xing Equ: (i) by 3 then sub: from (ii)

$$\begin{array}{r} \cancel{3x} + 12y - 3z = 36 \\ \underline{\underset{-}{\cancel{3x}} \underset{-}{+} 8y \underset{+}{-} 2z = \underset{-}{4}} \\ 4y - z = 32 \end{array}$$

$\Rightarrow 4y = 32 + z \Rightarrow y = 8 + z/4$ —— (iii)

where z is any real no then $z = r$.

eq (iii) $\Rightarrow y = 8 + r/4$.

eq (i) $\Rightarrow x = 12 - 4y + z$

$x = 12 - 4(8 + z/4) + z$

$x = 12 - 32 - \frac{\cancel{4}z}{\cancel{4}} + z$

$x = 12 - 32 - \cancel{z} + \cancel{z}$

$x = -20$

Hence $x = -20$, $y = \frac{1}{4}r + 8$, $z = r$, where r is any real number.

Q9 $x+y+3z=12$ —— (i)

$2x+2y+6z=6$ —— (ii)

Xing Eq (i) by (2) then subrt: from eq: (ii)

$$\cancel{2x}+\cancel{2y}+\cancel{6z}=24$$
$$\underline{\cancel{2x}+\cancel{2y}+\cancel{6z}=6}$$
$$\; \;\;\;\; -\;\;\;\;\;\;\;\; -\;\;\;\;\;\;\;\; -\;\;\;\;\;\;\;\; -$$
$$0 = 18$$

No Solution.

Q11 $2x+3y=13$ —— (i)

$x-2y=3$ —— (ii)

$5x+2y=27$ —— (iii)

Xing eq (ii) by (2) and Substr: from (i)

$$2x+3y=13$$
$$\underline{\cancel{2x}-4y=6}$$
$$-\;\;\;\;+\;\;\;\;\;\;\;\;\;\;-$$
$$7y=7 \Rightarrow \boxed{y=1}$$

Xing eq (ii) by 5 then subtr: from (iii)

$$\cancel{5x}-10y=15$$
$$\underline{\cancel{5x}+2y=27}$$
$$-\;\;\;\;-\;\;\;\;\;\;\;\;\;\;-$$
$$-12y=-12 \Rightarrow \boxed{y=1}$$

Ex:1.1

Xing eq (i) by 5 & eq (iii) by 2 then subtr:

$$10x + 15y = 65$$
$$\underline{\cancel{10x} \pm 2y = \pm 54}$$
$$11y = 11 \Rightarrow \boxed{y = 1}$$

Put y = in (i) then

$2x + 3(1) = 13 \Rightarrow 2x = 13 - 3 = 10 \Rightarrow \boxed{x = 5}$

Hence $y = 1$ & $x = 5$ which is unique solution

—— x ——

Q17 Without using the method of elimination solve the linear system

$$2x + y - 2z = -5 \text{ —— (i)}$$
$$3y + z = 7 \text{ —— (ii)}$$
$$z = 4 \text{ —— (iii)}$$

Sol: $z = 4$ then eq (ii) $\Rightarrow 3y + 4 = 7 \Rightarrow 3y = 7 - 4 = 3$

$3y = 3 \Rightarrow \boxed{y = 1}$

If $z = 4$ & $y = 1$ eq (i) $\Rightarrow$

$$2x + 1 - 2(4) = -5$$
$$2x = -5 + 7 = 2 \Rightarrow \cancel{2}x = \cancel{2} \; \boxed{x = 1}$$

Hence $x = 1$, $y = 1$ & $z = 4$ Ans.

Q19 Is there a value of r so that $x=1, y=2, z=r$ is a solution to the following linear system? If there is, find it

$$2x+3y-z=11 \quad \text{---(i)}$$
$$x-y+2z=-7 \quad \text{---(ii)}$$
$$4x+y-2z=12 \quad \text{---(iii)}$$

Given that $x=1, y=2, z=r$

eq (i) $\Rightarrow 2(1)+3(2)-r=11$

$2+6-r=11 \Rightarrow -r=11-8=3 \Rightarrow \boxed{r=-3}$

eq (ii) $\Rightarrow 1-2+2r=-7$

$-1+2r=-7 \Rightarrow 2r=-7+1=-6$

$\Rightarrow \boxed{r=-3}$

eq (iii) $\Rightarrow 4(1)+2-2r=12$

$-2r=12-6$

$-2r=6$

$\boxed{r=-3}$

Hence $\boxed{r=-3}$ Ans.