T.12. How many bit 5-vectors are there? How many bit n-vectors are there?

T.13. Make a list of all possible 2×2 bit matrices. How many are there?

T.14. How many 3×3 bit matrices are there?

T.15. How many $n \times n$ bit matrices are there?

T.16. Let 0 represent OFF and 1 represent ON and

$$A = \begin{bmatrix} \text{ON} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{ON} \end{bmatrix}.$$

H.W

Find the ON/OFF matrix B so that $A + B$ is a matrix with each entry OFF.

T.17. Let 0 represent OFF and 1 represent ON and 0 1 1 (c)

$$A = \begin{bmatrix} \text{ON} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{ON} \end{bmatrix}.$$

Find the ON/OFF matrix B so that $A + B$ is a matrix with each entry ON.

T.18. A standard light switch has two positions (or states); either on or off. Let bit matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

represent a bank of light switches where 0 represents OFF and 1 represents ON.

(a) Find a matrix B so that $A + B$ will represent the bank of switches with the state of each switch "reversed."

(b) Let

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Will the matrix B from part (a) also "reverse" that state of the bank of switches represented by C? Verify your answer.

(c) If A is any $m \times n$ bit matrix representing a bank of switches, determine an $m \times n$ bit matrix B so that $A + B$ "reverses" all the states of the switches in A. Give reasons why B will "reverse" the states in A.

MATLAB Exercises

In order to use MATLAB *in this section, you should first read Sections 12.1 and 12.2, which give basic information about* MATLAB *and about matrix operations in* MATLAB. *You are urged to do any examples or illustrations of* MATLAB *commands that appear in Sections 12.1 and 12.2 before trying these exercises.*

ML.1. In MATLAB, enter the following matrices.

$$A = \begin{bmatrix} 5 & 1 & 2 \\ -3 & 0 & 1 \\ 2 & 4 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 4 * 2 & 2/3 \\ 1/201 & 5 - 8.2 \\ 0.00001 & (9 + 4)/3 \end{bmatrix}.$$

Using MATLAB commands, display the following.

(a) a_{23}, b_{32}, b_{12}

(b) $\text{row}_1(A)$, $\text{col}_3(A)$, $\text{row}_2(B)$

(c) Type MATLAB command **format long** and display matrix B. Compare the elements of B from part (a) with the current display. Note that **format short** displays four decimal places rounded. Reset the format to **format short**.

ML.2. In MATLAB, type the command **H = hilb(5)**; (Note that the last character is a semicolon, which suppresses the display of the contents of matrix H. See Section 12.1.) For more information on the **hilb** command, type **help hilb**. Using MATLAB commands, do the following:

(a) Determine the size of H.

(b) Display the contents of H.

(c) Display the contents of H as rational numbers.

(d) Extract as a matrix the first three columns.

(e) Extract as a matrix the last two rows.

Exercises ML.3 through ML.5 use bit matrices and the supplemental instructional commands described in Section 12.9.

ML.3. Use bingen to solve Exercises T.10 and T.11.

ML.4. Use bingen to solve Exercise T.13. (*Hint:* An $n \times n$ matrix contains the same number of entries as an n^2-vector.)

ML.5. Solve Exercise 11 using **binadd**.

1.3 DOT PRODUCT AND MATRIX MULTIPLICATION

In this section we introduce the operation of matrix multiplication. Unlike matrix addition, matrix multiplication has some properties that distinguish it from multiplication of real numbers.

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DEFINITION The **dot product** or **inner product** of the n-vectors **a** and **b** is the sum of the products of corresponding entries. Thus, if

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i.^{\dagger} \tag{1}$$

Similarly, if **a** or **b** (or both) are n-vectors written as a $1 \times n$ matrix, then the dot product $\mathbf{a} \cdot \mathbf{b}$ is given by (1). The dot product of vectors in C^n is defined in Appendix A.2.

The dot product is an important operation that will be used here and in later sections.

EXAMPLE 1 The dot product of

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

is

$$\mathbf{u} \cdot \mathbf{v} = (1)(2) + (-2)(3) + (3)(-2) + (4)(1) = -6.$$

■

EXAMPLE 2 Let $\mathbf{a} = \begin{bmatrix} x & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = -4$, find x.

Solution We have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = 4x + 2 + 6 &= -4 \\ 4x + 8 &= -4 \\ x &= -3. \end{aligned}$$

■

EXAMPLE 3 **(Application: Computing a Course Average)** Suppose that an instructor uses four grades to determine a student's course average: quizzes, two hourly exams, and a final exam. These are weighted as 10%, 30%, 30%, and 30%, respectively. If a student's scores are 78, 84, 62, and 85, respectively, we can compute the course average by letting

$$\mathbf{w} = \begin{bmatrix} 0.10 \\ 0.30 \\ 0.30 \\ 0.30 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 78 \\ 84 \\ 62 \\ 85 \end{bmatrix}$$

and computing

$$\mathbf{w} \cdot \mathbf{g} = (0.10)(78) + (0.30)(84) + (0.30)(62) + (0.30)(85) = 77.1.$$

Thus, the student's course average is 77.1.

■

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†You may already be familiar with this useful notation, the summation notation. It is discussed in detail at the end of this section.

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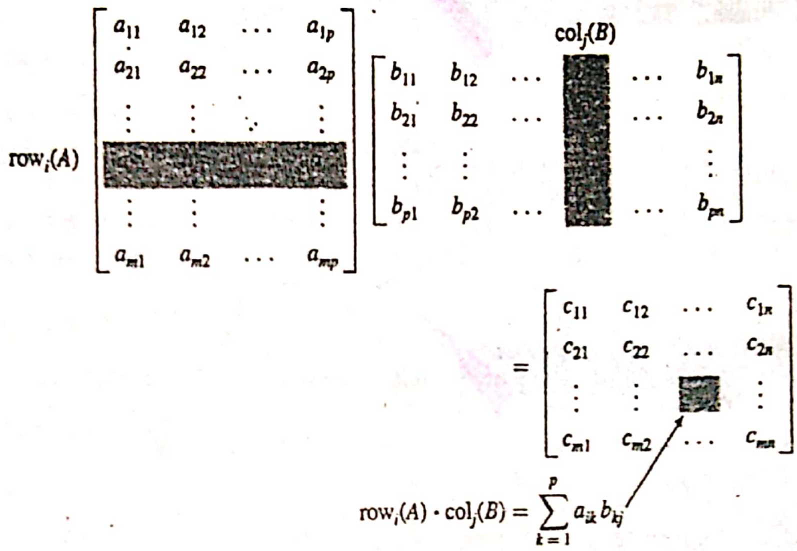
MATRIX MULTIPLICATION

DEFINITION If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then the product of A and B, denoted AB, is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}b_{kj} \quad (1 \le i \le m,\ 1 \le j \le n). \end{aligned} \tag{2}$$

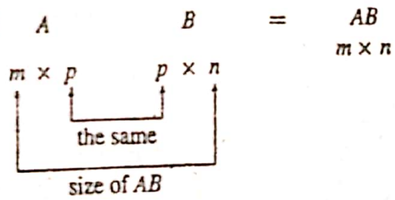
Equation (2) says that the i, jth element in the product matrix is the dot product of the ith row, $\text{row}_i(A)$, and the jth column, $\text{col}_j(B)$, of B; this is shown in Figure 1.4.

Figure 1.4 ►



Observe that the product of A and B is defined only when the number of rows of B is exactly the same as the number of columns of A, as is indicated in Figure 1.5.

Figure 1.5 ►



EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} (1)(-2) + (2)(4) + (-1)(2) & (1)(5) + (2)(-3) + (-1)(1) \\ (3)(-2) + (1)(4) + (4)(2) & (3)(5) + (1)(-3) + (4)(1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}. \end{aligned}$$

■

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EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}.$$

Compute the (3, 2) entry of AB.

Solution If $AB = C$, then the (3, 2) entry of AB is c_{32}, which is $\text{row}_3(A) \cdot \text{col}_2(B)$. We now have

$$\text{row}_3(A) \cdot \text{col}_2(B) = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = -5.$$

EXAMPLE 6 The linear system

$$\begin{aligned} x + 2y - z &= 2 \\ 3x \quad\quad + 4z &= 5 \end{aligned}$$

can be written (verify) using a matrix product as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

EXAMPLE 7 Let

$$A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}.$$

If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, find x and y.

Solution We have

$$AB = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix} = \begin{bmatrix} 2 + 4x + 3y \\ 4 - 4 + y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

Then

$$\begin{aligned} 2 + 4x + 3y &= 12 \\ y &= 6, \end{aligned}$$

so $x = -2$ and $y = 6$.

The basic properties of matrix multiplication will be considered in the following section. However, multiplication of matrices requires much more care than their addition, since the algebraic properties of matrix multiplication differ from those satisfied by the real numbers. Part of the problem is due to the fact that AB is defined only when the number of columns of A is the same as the number of rows of B. Thus, if A is an $m \times p$ matrix and B is a $p \times n$ matrix, then AB is an $m \times n$ matrix. What about BA? Four different situations may occur:

1. BA may not be defined; this will take place if $n \neq m$.
2. If BA is defined, which means that $m = n$, then BA is $p \times p$ while AB is $m \times m$; thus, if $m \neq p$, AB and BA are of different sizes.

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3. If AB and BA are both of the same size, they may be equal.
4. If AB and BA are both of the same size, they may be unequal.

EXAMPLE 8 If A is a 2×3 matrix and B is a 3×4 matrix, then AB is a 2×4 matrix while BA is undefined. ■

EXAMPLE 9 Let A be 2×3 and let B be 3×2. Then AB is 2×2 while BA is 3×3. ■

EXAMPLE 10 Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \quad \text{while} \quad BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}.$$

Thus $AB \neq BA$. ■

One might ask why matrix equality and matrix addition are defined in such a natural way while matrix multiplication appears to be much more complicated. Example 11 provides a motivation for the definition of matrix multiplication.

EXAMPLE 11 (Ecology) Pesticides are sprayed on plants to eliminate harmful insects. However, some of the pesticide is absorbed by the plant. The pesticides are absorbed by herbivores when they eat the plants that have been sprayed. To determine the amount of pesticide absorbed by a herbivore, we proceed as follows. Suppose that we have three pesticides and four plants. Let a_{ij} denote the amount of pesticide i (in milligrams) that has been absorbed by plant j. This information can be represented by the matrix

$$\begin{array}{cc} & \begin{array}{cccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \end{array} \\ A = & \begin{bmatrix} 2 & 3 & 4 & 3 \\ 3 & 2 & 2 & 5 \\ 4 & 1 & 6 & 4 \end{bmatrix} \begin{array}{l} \text{Pesticide 1} \\ \text{Pesticide 2} \\ \text{Pesticide 3} \end{array} \end{array}$$

Now suppose that we have three herbivores, and let b_{ij} denote the number of plants of type i that a herbivore of type j eats per month. This information can be represented by the matrix

$$\begin{array}{cc} & \begin{array}{ccc} \text{Herbivore 1} & \text{Herbivore 2} & \text{Herbivore 3} \end{array} \\ B = & \begin{bmatrix} 20 & 12 & 8 \\ 28 & 15 & 15 \\ 30 & 12 & 10 \\ 40 & 16 & 20 \end{bmatrix} \begin{array}{l} \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array} \end{array}$$

The (i, j) entry in AB gives the amount of pesticide of type i that animal j has absorbed. Thus, if $i = 2$ and $j = 3$, the $(2, 3)$ entry in AB is

$$\begin{aligned} &3(8) + 2(15) + 2(10) + 5(20) \\ &\qquad = 174 \text{ mg of pesticide 2 absorbed by herbivore 3.} \end{aligned}$$

If we now have p carnivores (such as man) who eat the herbivores, we can repeat the analysis to find out how much of each pesticide has been absorbed by each carnivore. ■

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It is sometimes useful to be able to find a column in the matrix product AB without having to multiply the two matrices. It can be shown (Exercise T.9) that the jth column of the matrix product AB is equal to the matrix product $A\text{col}_j(B)$.

EXAMPLE 12 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Then the second column of AB is

$$A\text{col}_2(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix}.$$

■

Remark If $\mathbf{u}$ and $\mathbf{v}$ are n-vectors, it can be shown (Exercise T.14) that if we view them as $n \times 1$ matrices, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}.$$

This observation will be used in Chapter 3. Similarly, if $\mathbf{u}$ and $\mathbf{v}$ are viewed as $1 \times n$ matrices, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T.$$

Finally, if $\mathbf{u}$ is a $1 \times n$ matrix and $\mathbf{v}$ is an $n \times 1$ matrix, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}$.

EXAMPLE 13 Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{v} = 1(2) + 2(-1) + (-3)(1) = -3.$$

Moreover,

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 1(2) + 2(-1) + (-3)(1) = -3.$$

■

THE MATRIX-VECTOR PRODUCT WRITTEN IN TERMS OF COLUMNS

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix and let

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

be an n-vector, that is, an $n \times 1$ matrix. Since A is $m \times n$ and $\mathbf{c}$ is $n \times 1$, the matrix product $A\mathbf{c}$ is the $m \times 1$ matrix

$$A\mathbf{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \text{row}_1(A)\cdot\mathbf{c} \\ \text{row}_2(A)\cdot\mathbf{c} \\ \vdots \\ \text{row}_m(A)\cdot\mathbf{c} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n \\ \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n \end{bmatrix}. \tag{3}$$

The right side of this expression can be written as

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{4}$$

$$= c_1\text{col}_1(A) + c_2\text{col}_2(A) + \cdots + c_n\text{col}_n(A).$$

Thus the product $A\mathbf{c}$ of an $m \times n$ matrix A and an $n \times 1$ matrix $\mathbf{c}$ can be written as a linear combination of the columns of A, where the coefficients are the entries in $\mathbf{c}$.

EXAMPLE 14 Let

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Then the product $A\mathbf{c}$ written as a linear combination of the columns of A is

$$A\mathbf{c} = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}.$$ ■

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, we can then conclude that the jth column of the product AB can be written as a linear combination of the columns of matrix A, where the coefficients are the entries in the jth column of matrix B:

$$\text{col}_j(AB) = A\text{col}_j(B) = b_{1j}\text{col}_1(A) + b_{2j}\text{col}_2(A) + \cdots + b_{pj}\text{col}_p(A).$$

EXAMPLE 15 If A and B are the matrices defined in Example 12, then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 6 \\ 6 & 17 & 16 \\ 17 & 7 & 1 \end{bmatrix}.$$

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The columns of AB as linear combinations of the columns of A are given by

$$\operatorname{col}_1(AB) = \begin{bmatrix} 4 \\ 6 \\ 17 \end{bmatrix} = A\operatorname{col}_1(B) = -2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\operatorname{col}_2(AB) = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix} = A\operatorname{col}_2(B) = 3 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\operatorname{col}_3(AB) = \begin{bmatrix} 6 \\ 16 \\ 1 \end{bmatrix} = A\operatorname{col}_3(B) = 4 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

■

LINEAR SYSTEMS

We now generalize Example 6. Consider the linear system of m equations in n unknowns,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m. \end{array} \tag{5}$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

The entries in the product $A\mathbf{x}$ are merely the left sides of the equations in (5). Hence the linear system (5) can be written in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The matrix A is called the **coefficient matrix** of the linear system (5), and the matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right],$$

obtained by adjoining column $\mathbf{b}$ to A, is called the **augmented matrix** of the linear system (5). The augmented matrix of (5) will be written as $\left[A \,\vdots\, \mathbf{b}\right]$. Conversely, any matrix with more than one column can be thought of as the augmented matrix of a linear system. The coefficient and augmented matrices will play key roles in our method for solving linear systems.

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EXAMPLE 16 Consider the linear system

$$\begin{aligned} -2x \qquad + z &= 5 \\ 2x + 3y - 4z &= 7 \\ 3x + 2y + 2z &= 3. \end{aligned}$$

Letting

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -4 \\ 3 & 2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix},$$

we can write the given linear system in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The coefficient matrix is A and the augmented matrix is

$$\left[\begin{array}{rrr:r} -2 & 0 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right].$$

■

EXAMPLE 17 The matrix

$$\left[\begin{array}{rrr:r} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$$

is the augmented matrix of the linear system

$$\begin{aligned} 2x - y + 3z &= 4 \\ 3x \qquad + 2z &= 5. \end{aligned}$$

■

It follows from our discussion above that the linear system in (5) can be written as a linear combination of the columns of A as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \tag{6}$$

Conversely, an equation as in (6) always describes a linear system as in (5).

PARTITIONED MATRICES (OPTIONAL)

If we start out with an $m \times n$ matrix $A = [a_{ij}]$ and cross out some, but not all of its rows or columns, we obtain a **submatrix** of A.

EXAMPLE 18 Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 4 & -3 & 5 \\ 3 & 0 & 5 & -3 \end{bmatrix}.$$

If we cross out the second row and third column, we obtain the submatrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -3 \end{bmatrix}.$$

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EXAMPLE 29 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ be bit matrices. Then

$$AB = \begin{bmatrix} (1)(0)+(1)(1) & (1)(1)+(1)(1) & (1)(0)+(1)(0) \\ (0)(0)+(1)(1) & (0)(1)+(1)(1) & (0)(0)+(1)(0) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

EXAMPLE 30 Let $A = \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be bit matrices. If $AB = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find x and y.

Solution We have

$$AB = \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y+1+x \\ y+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then $y+1+x = 1$ and $y+1 = 1$. Using base 2 arithmetic, it follows that $y = 0$ and so then $x = 0$.

Key Terms

Dot product (inner product)
Product of matrices
Coefficient matrix
Augmented matrix
Submatrix
Partitioned matrix
Block multiplication
Summation notation

1.3 Exercises

In Exercises 1 and 2, compute $\mathbf{a} \cdot \mathbf{b}$.

1. (a) $\mathbf{a} = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} -3 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 4 & 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

H.W (d) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. (a) $\mathbf{a} = \begin{bmatrix} 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3. Let $\mathbf{a} = \begin{bmatrix} -3 & 2 & x \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = 17$, find x.

4. Let $\mathbf{w} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. Compute $\mathbf{w} \cdot \mathbf{w}$.

5. Find all values of x so that $\mathbf{v} \cdot \mathbf{v} = 1$, where $\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ x \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$. If $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, find x and y.

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In Exercises 7 and 8, let

$$A=\begin{bmatrix}1 & 2 & -3\\ 4 & 0 & -2\end{bmatrix},\quad B=\begin{bmatrix}3 & 1\\ 2 & 4\\ -1 & 5\end{bmatrix},$$

$$C=\begin{bmatrix}2 & 3 & 1\\ 3 & -4 & 5\\ 1 & -1 & -2\end{bmatrix},\quad D=\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix},$$

$$E=\begin{bmatrix}1 & 0 & -3\\ -2 & 1 & 5\\ 3 & 4 & 2\end{bmatrix},\quad \text{and}\quad F=\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix}.$$

7. If possible, compute:
(a) AB (b) BA (c) $CB+D$
(d) $AB+DF$ (e) $BA+FD$

8. If possible, compute:
(a) $A(BD)$ (b) $(AB)D$ (c) $A(C+E)$
(d) $AC+AE$ (e) $(D+F)A$

9. Let $A=\begin{bmatrix}2 & 3\\ -1 & 4\\ 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1 & 3\\ 1 & 2 & 4\end{bmatrix}$.

Compute the following entries of AB:
(a) The (1, 2) entry (b) The (2, 3) entry
H.W (c) The (3, 1) entry (d) The (3, 3) entry

10. If $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $D=\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}$, compute DI_2 and
H.W I_2D.

11. Let

$$A=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}2 & -1\\ -3 & 4\end{bmatrix}.$$

Show that $AB \neq BA$.

12. If A is the matrix in Example 4 and O is the 3×2 matrix every one of whose entries is zero, compute AO.

In Exercises 13 and 14, let

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 4\\ 4 & -2 & 3\\ 2 & 1 & 5\end{bmatrix}$$

and

$$B=\begin{bmatrix}1 & 0 & -1 & 2\\ 3 & 3 & -3 & 4\\ 4 & 2 & 5 & 1\end{bmatrix}.$$

H.W

13. Using the method in Example 12, compute the following columns of AB:
H.W
(a) The first column (b) The third column

14. Using the method in Example 12, compute the following columns of AB:
(a) The second column (b) The fourth column

15. Let

$$A=\begin{bmatrix}2 & -3 & 4\\ -1 & 2 & 3\\ 5 & -1 & -2\end{bmatrix}\quad\text{and}\quad \mathbf{c}=\begin{bmatrix}2\\ 1\\ 4\end{bmatrix}.$$

H.W Express $A\mathbf{c}$ as a linear combination of the columns of A.

16. Let

$$A=\begin{bmatrix}1 & -2 & -1\\ 2 & 4 & 3\\ 3 & 0 & -2\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}1 & -1\\ 3 & 2\\ 2 & 4\end{bmatrix}.$$

Express the columns of AB as linear combinations of the columns of A.

17. Let $A=\begin{bmatrix}2 & -3 & 1\\ 1 & 2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\\ 2\end{bmatrix}$.

(a) Verify that $AB=3\mathbf{a}_1+5\mathbf{a}_2+2\mathbf{a}_3$, where $\mathbf{a}_j$ is the jth column of A for $j=1,2,3$.

(b) Verify that $AB=\begin{bmatrix}(\text{row}_1(A))B\\ (\text{row}_2(A))B\end{bmatrix}$.

18. Write the linear combination

$$3\begin{bmatrix}-2\\ 3\end{bmatrix}+4\begin{bmatrix}2\\ 5\end{bmatrix}+2\begin{bmatrix}3\\ -1\end{bmatrix}$$

as a product of a 2×3 matrix and a 3-vector.

19. Consider the following linear system:

$$\begin{aligned}2x + w &= 7\\ 3x+2y+3z &= -2\\ 2x+3y-4z &= 3\\ x + 3z &= 5.\end{aligned}$$

(a) Find the coefficient matrix.

(b) Write the linear system in matrix form.

H.W (c) Find the augmented matrix.

20. Write the linear system with augmented matrix

$$\left[\begin{array}{cccc|c}-2 & -1 & 0 & 4 & 5\\ -3 & 2 & 7 & 8 & 3\\ 1 & 0 & 0 & 2 & 4\\ 3 & 0 & 1 & 3 & 6\end{array}\right].$$

H.W

21. Write the linear system with augmented matrix

$$\left[\begin{array}{ccc|c}2 & 0 & -4 & 3\\ 0 & 1 & 2 & 5\\ 1 & 3 & 4 & -1\end{array}\right].$$

H.W

22. Consider the following linear system:

$$\begin{aligned}3x-y+2z&=4\\ 2x+y&=2\\ y+3z&=7\\ 4x-z&=4.\end{aligned}$$

(a) Find the coefficient matrix.

(b) Write the linear system in matrix form.

(c) Find the augmented matrix.

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23. How are the linear systems whose augmented matrices are

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 2 & 3 & 6 & 2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 2 & 3 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

related?

24. Write each of the following as a linear system in matrix form.

(a) $x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} + z\begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

(b) $x\begin{bmatrix}1\\1\\2\end{bmatrix} + y\begin{bmatrix}2\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

25. Write each of the following linear systems as a linear combination of the columns of the coefficient matrix.

(a) $\begin{aligned} x + 2y &= 3 \\ 2x - y &= 5 \end{aligned}$

H.W (b) $\begin{aligned} 2x - 3y + 5z &= -2 \\ x + 4y - z &= 3 \end{aligned}$

26. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. What if anything can you say about the matrix product AB when:

H.W

(a) A has a column consisting entirely of zeros?

(b) B has a row consisting entirely of zeros?

27. (a) Find a value of r so that $AB^T = 0$, where

$$A = \begin{bmatrix} r & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}.$$

H.W

(b) Give an alternate way to write this product.

28. Find a value of r and a value of s so that $AB^T = 0$, where

H.W

$$A = \begin{bmatrix} 1 & r & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 2 & s \end{bmatrix}.$$

29. Formulate the method for adding partitioned matrices and verify your method by partitioning the matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

H.W

in two different ways and finding their sum.

30. Let A and B be the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 1 & 2 & 3 & -1 & 4 \\ 2 & 3 & 2 & 1 & 4 \\ 5 & -1 & 3 & 2 & 6 \\ 3 & 1 & 2 & 4 & 6 \\ 2 & -1 & 3 & 5 & 7 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 1 & 3 & 2 & -1 \\ 1 & 5 & 4 & 2 & 3 \\ 2 & 1 & 3 & 5 & 7 \\ 3 & 2 & 4 & 6 & 1 \end{bmatrix}.$$

Find AB by partitioning A and B in two different ways.

31. *(Manufacturing Costs)* A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \text{Assembly process} & \text{Finishing process} \\ \left[\begin{matrix} 2 \\ 3 \end{matrix}\right. & \left.\begin{matrix} 2 \\ 4 \end{matrix}\right] \end{array}\begin{array}{l} \\ \text{Chair} \\ \text{Table} \end{array}$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$B = \begin{array}{cc} \text{Salt Lake City} & \text{Chicago} \\ \left[\begin{matrix} 9 \\ 10 \end{matrix}\right. & \left.\begin{matrix} 10 \\ 12 \end{matrix}\right] \end{array}\begin{array}{l} \\ \text{Assembly process} \\ \text{Finishing process} \end{array}$$

What do the entries in the matrix product AB tell the

H.W manufacturer?

32. *(Ecology-Pollution)* A manufacturer makes two kinds of products, P and Q, at each of two plants, X and Y. In making these products, the pollutants sulfur dioxide, nitric oxide, and particulate matter are produced. The amounts of pollutants produced are given (in kilograms) by the matrix

$$A = \begin{array}{ccc} \text{Sulfur dioxide} & \text{Nitric oxide} & \text{Particulate matter} \\ \left[\begin{matrix} 300 \\ 200 \end{matrix}\right. & \begin{matrix} 100 \\ 250 \end{matrix} & \left.\begin{matrix} 150 \\ 400 \end{matrix}\right] \end{array}\begin{array}{l} \\ \text{Product } P \\ \text{Product } Q \end{array}$$

State and federal ordinances require that these pollutants be removed. The daily cost of removing each kilogram of pollutant is given (in dollars) by the matrix

$$B = \begin{array}{cc} \text{Plant } X & \text{Plant } Y \\ \left[\begin{matrix} 8 \\ 7 \\ 15 \end{matrix}\right. & \left.\begin{matrix} 12 \\ 9 \\ 10 \end{matrix}\right] \end{array}\begin{array}{l} \\ \text{Sulfur dioxide} \\ \text{Nitric oxide} \\ \text{Particulate matter} \end{array}$$

H.W What do the entries in the matrix product AB tell the manufacturer?

33. *(Medicine)* A diet research project consists of adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{cc} \text{Adults} & \text{Children} \\ \left[\begin{matrix} 80 \\ 100 \end{matrix}\right. & \left.\begin{matrix} 120 \\ 200 \end{matrix}\right] \end{array}\begin{array}{l} \\ \text{Male} \\ \text{Female} \end{array}$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{ccc} \text{Protein} & \text{Fat} & \text{Carbo-hydrate} \\ \left[\begin{matrix} 20 \\ 10 \end{matrix}\right. & \begin{matrix} 20 \\ 20 \end{matrix} & \left.\begin{matrix} 20 \\ 30 \end{matrix}\right] \end{array}\begin{array}{l} \\ \text{Adult} \\ \text{Child} \end{array}$$

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(a) How many grams of protein are consumed daily by the males in the project?

(b) How many grams of fat are consumed daily by the females in the project?

34. (*Business*) A photography business has a store in each of the following cities: New York, Denver, and Los Angeles. A particular make of camera is available in automatic, semiautomatic, and nonautomatic models. Moreover, each camera has a matched flash unit and a camera is usually sold together with the corresponding flash unit. The selling prices of the cameras and flash units are given (in dollars) by the matrix

$$A = \begin{array}{c} \begin{array}{ccc} \text{Auto-matic} & \text{Semi-automatic} & \text{Non-automatic} \end{array} \\ \begin{bmatrix} 200 & 150 & 120 \\ 50 & 40 & 25 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Camera} \\ \text{Flash unit} \end{array}$$

The number of sets (camera and flash unit) available at each store is given by the matrix

$$B = \begin{array}{c} \begin{array}{ccc} \text{New York} & \text{Denver} & \text{Los Angeles} \end{array} \\ \begin{bmatrix} 220 & 180 & 100 \\ 300 & 250 & 120 \\ 120 & 320 & 250 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Automatic} \\ \text{Semiautomatic} \\ \text{Nonautomatic} \end{array}$$

(a) What is the total value of the cameras in New York?

(b) What is the total value of the flash units in Los Angeles?

35. Let $\mathbf{s}_1 = \begin{bmatrix} 18.95 & 14.75 & 8.98 \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} 17.80 & 13.50 & 10.79 \end{bmatrix}$ be 3-vectors denoting the current prices of three items at stores A and B, respectively.

(a) Obtain a 2×3 matrix representing the combined information about the prices of the three items at the two stores.

(b) Suppose that each store announces a sale so that the price of each item is reduced by 20%. Obtain a 2×3 matrix representing the sale prices at the two stores.

Exercises 36 through 41 involve bit matrices.

36. For bit vectors $\mathbf{a}$ and $\mathbf{b}$ compute $\mathbf{a} \cdot \mathbf{b}$.

(a) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

H.W

37. For bit vectors $\mathbf{a}$ and $\mathbf{b}$ compute $\mathbf{a} \cdot \mathbf{b}$.

(a) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

38. Let $\mathbf{a} = \begin{bmatrix} 1 & x & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} x \\ 1 \\ 1 \end{bmatrix}$ be bit vectors. If $\mathbf{a} \cdot \mathbf{b} = 0$, find all possible values of x.

39. Let $A = \begin{bmatrix} 1 & 1 & x \\ 0 & y & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be bit matrices. If

H.W $AB = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find x and y.

40. For bit matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

compute AB and BA.

41. For bit matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, determine a 2×2 bit matrix B so that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Theoretical Exercises

T.1. Let $\mathbf{x}$ be an n-vector.

(a) Is it possible for $\mathbf{x} \cdot \mathbf{x}$ to be negative? Explain.

(b) If $\mathbf{x} \cdot \mathbf{x} = 0$, what is $\mathbf{x}$?

T.2. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be n-vectors and let k be a real number.

(a) Show that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(b) Show that $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.

(c) Show that $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$.

T.3. (a) Show that if A has a row of zeros, then AB has a row of zeros.

(b) Show that if B has a column of zeros, then AB has a column of zeros.

T.4. Show that the product of two diagonal matrices is a diagonal matrix.

T.5. Show that the product of two scalar matrices is a scalar matrix.

T.6. (a) Show that the product of two upper triangular matrices is upper triangular.

(b) Show that the product of two lower triangular matrices is lower triangular.

T.7. Let A and B be $n \times n$ diagonal matrices. Is $AB = BA$? Justify your answer.

T.8. (a) Let $\mathbf{a}$ be a $1 \times n$ matrix and B an $n \times p$ matrix. Show that the matrix product $\mathbf{a}B$ can be written as

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Ex: 1.3

Q1 (a) $a = [1, 2]$, $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Find $a \cdot b$

Sol: $a \cdot b = [1\ 2] \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$= 1 \times 4 + 2(-1)$

$= 4 - 2 = 2$ Ans.

Q2 is similarly to Q1

Q3 let $a = [-3, 2\ x]$ and $b = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$ if $a \cdot b = 17$ find x.

Sol: Given that $a \cdot b = 17$ — (i)

$a \cdot b = [-3\ 2\ x] \cdot \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$

$= -3 \times -3 + 2(2) + x \cdot x$

$= 9 + 4 + x^2$

$a \cdot b = 13 + x^2$ — (ii)

comparing eq (i) & eq (ii)

$17 = 13 + x^2$

$\Rightarrow x^2 = 17 - 13 = 4$

$x^2 = 4$

$\boxed{x = \pm 2}$ A.

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?

Q4 Let $W = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. compute $W \cdot W$

Sol. $W \cdot W = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$

$$= \sin^2\theta + \cos^2\theta$$

$\boxed{W \cdot W = 1}$ An.

Q5 Find all values of x so that $V \cdot V = 1$ where $V = \begin{bmatrix} 1/2 \\ -1/2 \\ x \end{bmatrix}$.

Sol.-

$$V \cdot V = 1$$

$$\begin{bmatrix} 1/2 \\ -1/2 \\ x \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ x \end{bmatrix} = 1$$

$$\left(\frac{1}{2}\right)(1/2) + (-1/2)(-1/2) + x \cdot x = 1$$

$$1/4 + 1/4 + x^2 = 1$$

$$\frac{1+1}{4} - 1 = -x^2$$

$$\frac{\cancel{2}}{\cancel{4}_2} - 1 = -x^2$$

$$\frac{1-2}{2} = -x^2$$

$$\cancel{-}1/2 = \cancel{-}x^2$$

$$x^2 = 1/2 = \frac{2}{4}$$

$\boxed{x = \pm\frac{\sqrt{2}}{2}}$ An.

Ex = 1.3

Q6 Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$ if $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ find x & y.

Sol: Given that $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ —— (i)

Now $A \cdot B = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$

$= \begin{bmatrix} y + 2x + x \\ 3y - x + 2 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} y + 3x \\ 3y - x + 2 \end{bmatrix}$ —— (ii)

comparing eq (i) & eq (ii)

$\begin{bmatrix} y + 3x \\ 3y - x + 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$y + 3x = 6$ — (i) $\quad 3y - x + 2 = 8$ — (ii)

Xing eq (i) by 3 then subtr from (ii)

$3y + 9x = 18$

$\underline{-3y \mp x = -6}$

$10x = 12 \Rightarrow \boxed{x = 6/5}$

eq (i) $\Rightarrow y + 3(6/5) = 6$

$y = 6 - 18/5 = \frac{30 - 18}{5} = 12/5$

$\boxed{y = 12/5}$

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In Exercises 7 and 8 let

$$A=\begin{bmatrix}1 & 2 & -3\\ 4 & 0 & -2\end{bmatrix} \quad B=\begin{bmatrix}3 & 1\\ 2 & 4\\ -1 & 5\end{bmatrix} \quad C=\begin{bmatrix}2 & 3 & 1\\ 3 & -4 & 5\\ 1 & -1 & -2\end{bmatrix}.$$

$$D=\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix} \quad E=\begin{bmatrix}1 & 0 & -3\\ -2 & 1 & 5\\ 3 & 4 & 2\end{bmatrix} \text{ and } F=\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix}.$$

Q7 If Possible compute

a) AB

∴ if $C_1 = R_1$ then $\times$ing Possible

$$A.B=\begin{bmatrix}1 & 2 & -3\\ 4 & 0 & -2\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 4\\ -1 & 5\end{bmatrix}$$

$$=\begin{bmatrix}\cancel{3} 1\times3+2\times2+(-3)(-1) & 1\times1+2\times4+(-3)(5)\\ 4\times3+0\times2+(-2)(-1) & 4\times1+0\times4+(-2)(5)\end{bmatrix}$$

$$=\begin{bmatrix}3+4+3 & 1+8-15\\ 12+0+2 & 4+0-10\end{bmatrix}$$

$$=\begin{bmatrix}10 & -6\\ 14 & -6\end{bmatrix} \text{ A.}$$

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Ex: 1.3

Q9 Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

(a) The (1,2) entry

Sol: (1,2) means = 1st row and 2nd column of AB matrix.

$$\begin{bmatrix} 2, 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 + 3 \times 2 \end{bmatrix} = \begin{bmatrix} -2 + 6 \end{bmatrix} = 4. \text{ A.}$$

(b) (2,3) entry.

$$\begin{bmatrix} -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 + 16 \end{bmatrix} = 13 \text{ A.}$$

(c) (3,1) entry.

$$\begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0 + 3 = 3 \text{ A.}$$

(d) (3,3) entry

$$\begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 12 \text{ A.}$$

Q10 If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ compute DI_2 & I_2D.

Sol.
$$I_2D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2+0 & 3-0 \\ 0-1 & 0-2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}.$$

$$\& \; DI_2 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}.$$

So $I_2D = DI_2 = D$ A

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Q11 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

Show that $AB \neq BA$.

Sol

$$A.B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2-6 & -1+8 \\ 6-6 & -3+8 \end{bmatrix} = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix}.$$

$$B.A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2-3 & 4-2 \\ -3+12 & -6+8 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix}.$$

So $AB \neq BA$.

Q12 $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}_{2\times3}$ $\quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times2}$

$$AO = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}.$$

Q13 a) The 1st column AB $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & -3 & 4 \\ 4 & 2 & 5 & 1 \end{bmatrix}$.

Sol

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1-3+8 \\ 3+6+16 \\ 4-6+12 \\ 2+3+20 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ 10 \\ 25 \end{bmatrix}$$

b) The 3rd colum.

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

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Ex: 1.3

Q14 is similarly Q13.

Q15 let $A = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix}$ and $c = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

$$c_1 \, col_1 A + c_2 \, col_2 A + c_3 \, col_3 A$$

$$= 2\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}.$$

Q16 $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}$.

$$col_1(AB) = A \, col_1(B)$$

$$= 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

$$col_2(AB) = A \, col_2(B)$$

$$= -1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

Q17 $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

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a) Verify $AB = 3a_1 + 5a_2 + 2a_3$.

$$AB = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-15+2 \\ 3+10+8 \end{bmatrix} = \begin{bmatrix} -7 \\ 21 \end{bmatrix} \quad \text{--- (1)}$$

Now $3a_1 + 5a_2 + 2a_3 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

$$= \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -15 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 6-15+2 \\ 3+10+8 \end{bmatrix} = \begin{bmatrix} -7 \\ 21 \end{bmatrix} \quad (2)$$

from eq (1) & eq (2) we have

$AB = 3a_1 + 5a_2 + 2a_3$ Hence verified.

(b) Verify that $AB = \begin{bmatrix} (\text{row}_1(A))B \\ (\text{row}_2(A))B \end{bmatrix}$.

from Part a $AB = \begin{bmatrix} -7 \\ 21 \end{bmatrix}$ --- (1)

$[\text{row}_1(A)]B = [2 \ -3 \ 1]\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = [6-15+2] = -7$.

$[\text{row}_2(A)]B = [1 \ 2 \ 4]\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = [3+10+8] = 21$

Now $\begin{bmatrix} (\text{row}_1(A))B \\ (\text{row}_2(A))B \end{bmatrix} = \begin{bmatrix} -7 \\ 21 \end{bmatrix}$ --- (2)

from (1) & (2) Hence verified.

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Ex: 1.3

Q18 Sol.
$$\begin{bmatrix} -2 & 2 & 3 \\ 3 & 5 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}.$$

Q19 a)
$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 \\ 2 & 3 & -4 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$
Which is coefficient matrix

b)
$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 \\ 2 & 3 & -4 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}$$
Which is the linear system in matrix form. 2

c)
$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & 7 \\ 3 & 2 & 3 & 0 & -2 \\ 2 & 3 & -4 & 0 & 3 \\ 1 & 0 & 3 & 0 & 5 \end{array}\right]$$
Which is augmented matrix.

Q20 a) Sol.
$$\begin{aligned} -2x - y + 0z + 4w &= 5 \\ -3x + 2y + 7z + 8w &= 3 \\ x + 0y + 0z + 2w &= 4 \\ 3x + 0y + z + 3w &= 6 \end{aligned}$$

Q21 similarly Q20

Q22 similarly Q19.

Q23 Sol:- ~~These~~ They are equivalent as third row have zero entry.

Q24. a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Q25 a) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -3 & 5 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -3 \\ 4 \end{bmatrix} + z\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Q26 a) ~~can~~ Say nothing.

b) can say nothing.

Q27 a) $AB^t = 0$

$\begin{bmatrix} r & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 0$

$1(r) + 3(1) - 1(-2) = 0$

$r + 3 + 2 = 0$

$\boxed{r = -5}$.

Q28 Similarly Q27.

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Ex 1.3

Q31

$$A = \begin{array}{c} \begin{array}{cc} A.P & F.P \end{array} \\ \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \end{array} \begin{array}{l} \text{Chair} \\ \text{Table} \end{array} \qquad B = \begin{array}{c} \begin{array}{cc} S.L.C & \text{Chicago} \end{array} \\ \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} \end{array} \begin{array}{l} A.P \\ F.P \end{array}$$

$$AB = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} S.L.C & \text{Chicago} \end{array} \\ \begin{bmatrix} 38 & 44 \\ 67 & 78 \end{bmatrix} \end{array} \begin{array}{l} \text{chair} \\ \text{table} \end{array}$$

where AB gives total cost of Producing each kind of Product in each city.

Q32

$$A = \begin{array}{c} \begin{array}{ccc} S.D & N.O & P.M \end{array} \\ \begin{bmatrix} 300 & 100 & 150 \\ 200 & 250 & 400 \end{bmatrix} \end{array} \begin{array}{l} \text{Product } P \\ \text{Product } Q. \end{array}$$

$$B = \begin{array}{c} \begin{array}{cc} \text{Plant } X & \text{Plant } Y \end{array} \\ \begin{bmatrix} 8 & 12 \\ 7 & 9 \\ 15 & 10 \end{bmatrix} \end{array} \begin{array}{l} S.D \\ N.O \\ P.M \end{array}$$

$$A.B = \begin{bmatrix} 300 & 100 & 150 \\ 200 & 250 & 400 \end{bmatrix}\begin{bmatrix} 8 & 12 \\ 7 & 9 \\ 15 & 10 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} \text{Plant } X & \text{Plant } Y \end{array} \\ \begin{bmatrix} 5350 & 6000 \\ 9350 & 8650 \end{bmatrix} \end{array} \begin{array}{l} \text{Product } P \\ \text{〃} \quad Q. \end{array}$$

where AB gives each Product quantity in each Plant.

Q33

$$A = \begin{array}{cc} \text{Adults} & \text{Children} \end{array}$$
$$A = \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \begin{array}{l} \text{Male} \\ \text{Female} \end{array}$$

$$\begin{array}{ccc} P & F & C \end{array}$$
$$B = \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix} \begin{array}{l} \text{Adult} \\ \text{child} \end{array}$$

a) $\begin{bmatrix} 20 \\ 10 \end{bmatrix} \begin{bmatrix} 80 & 120 \end{bmatrix} = 80 \times 20 + 10 \times 120 = 2800g.$

b) $\begin{bmatrix} 20 \\ 20 \end{bmatrix} \begin{bmatrix} 100 & 200 \end{bmatrix} = 20 \times 100 + 20 \times 200 = 6000g.$

Q34 (a) $\begin{bmatrix} 220 \\ 330 \\ 120 \end{bmatrix} \begin{bmatrix} 200 & 150 & 120 \end{bmatrix} = 103400$

(b) $\begin{bmatrix} 50 & 40 & 25 \end{bmatrix} \begin{bmatrix} 100 \\ 120 \\ 250 \end{bmatrix} = 16050.$

Q35 a) $S_1 = \begin{bmatrix} 18.95 & 14.75 & 8.98 \end{bmatrix}$

$S_2 = \begin{bmatrix} 17.80 & 13.50 & 10.79 \end{bmatrix}$

$$\varphi = \begin{bmatrix} 18.95 & 14.75 & 8.98 \\ 17.80 & 13.50 & 10.79 \end{bmatrix}$$

(b) 20% Reduced.

$$\varphi = \begin{bmatrix} 18.95 \times 80\% & 14.75 \times 80\% & 8.98 \times 80\% \\ 17.80 \times 80\% & 13.50 \times 80\% & 10.79 \times 80\% \end{bmatrix}.$$

$$= \begin{bmatrix} 15.16 & 11.8 & 7.18 \\ 14.24 & 10.8 & 8.63 \end{bmatrix}.$$

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Ex 1.3

Q36 (b) $a \cdot b = [0\ 1\ 1\ 0]\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = 0$

(a) $a \cdot b = [1\ 1\ 0]\begin{bmatrix}0\\1\\1\end{bmatrix} = 1$

Q37 similarly to Q36.

Q38 $a = [1\ x\ 0]$ & $b = \begin{bmatrix}x\\1\\1\end{bmatrix}$ find $x = ?$ if $a.b = 0$

$a \cdot b = [1\ x\ 0]\begin{bmatrix}x\\1\\1\end{bmatrix} = 0$

$= 1[x] + 1[x] + 0[1] = 0$

$= x + x = 0$

$2x = 0 \quad \Rightarrow \boxed{x = 0}$

Q39 $AB = \begin{bmatrix}1 & 1 & x\\0 & y & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

$= \begin{bmatrix}x+0\\y+1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \boxed{x = 0,\ y = -1 = 1}$ for Bit $\cdot$ M

Q41 $AB = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\begin{bmatrix}a+c & b+d\\c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\boxed{c = 0},\ \boxed{d = 1}$

$a + c = 1 \Rightarrow a + 0 = 1 \Rightarrow \boxed{a = 1}$

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$b+d=0 \Rightarrow \cancel{1+d=} \cdot b+1=0 \Rightarrow b=-1=1$ (for bit. m

$$B=\begin{bmatrix} 1 & +1 \\ 0 & 1 \end{bmatrix}$$

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Mr. Jamal Nasir
Lecturer U.e.T Peshawar

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ML.5. Use the MATLAB command **diag** to form each of the following diagonal matrices. Using **diag** we can form diagonal matrices without typing in all the entries. (To refresh your memory about command **diag**, use MATLAB's help feature.)

(a) The 4×4 diagonal matrix with main diagonal $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$.

(b) The 5×5 diagonal matrix with main diagonal $\begin{bmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$.

(c) The 5×5 scalar matrix with all 5's on the diagonal.

ML.6. In MATLAB the dot product of a pair of vectors can be computed using the **dot** command. If the vectors **v** and **w** have been entered into MATLAB as either rows or columns, their dot product is computed from the MATLAB command **dot(v, w)**. If the vectors do not have the same number of elements, an error message is displayed.

(a) Use **dot** to compute the dot product of each of the following vectors.

(i) $\mathbf{v} = \begin{bmatrix} 1 & 4 & -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 7 & 2 & 0 \end{bmatrix}$

(ii) $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 6 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -1 \end{bmatrix}$

(b) Let $\mathbf{a} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. Find a value for k so that the dot product of **a** with $\mathbf{b} = \begin{bmatrix} k & 1 & 4 \end{bmatrix}$ is zero. Verify your results in MATLAB.

(c) For each of the following vectors **v**, compute **dot(v,v)** in MATLAB.

(i) $\mathbf{v} = \begin{bmatrix} 4 & 2 & -3 \end{bmatrix}$

(ii) $\mathbf{v} = \begin{bmatrix} -9 & 3 & 1 & 0 & 6 \end{bmatrix}$

(iii) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -5 \\ -3 \end{bmatrix}$

What sign is each of these dot products? Explain why this is true for almost all vectors **v**. When is it not true?

Exercises ML.7 through ML.11 use bit matrices and the supplemental instructional commands described in Section 12.9.

ML.7. Use **binprod** to solve Exercise 40.

ML.8. Given the bit vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, use **binprod** to compute $\mathbf{a} \cdot \mathbf{b}$.

ML.9. (a) Use **bingen** to generate a matrix B whose columns are all possible bit 3-vectors.

(b) Define A = **ones(3)** and compute AB using **binprod**.

(c) Describe why AB contains only columns of all zeros or all ones. (*Hint*: Look for a pattern based on the columns of B.)

ML.10. Repeat Exercise ML.9 with 4-vectors and A = ones(4).

ML.11. Let B be the $n \times n$ matrix of all ones. Compute BB for $n = 2, 3, 4$, and 5. What is BB for $n = k$, where k is any positive integer?

1.4 PROPERTIES OF MATRIX OPERATIONS

In this section we consider the algebraic properties of the matrix operations just defined. Many of these properties are similar to familiar properties of the real numbers. However, there will be striking differences between the set of real numbers and the set of matrices in their algebraic behavior under certain operations, for example, under multiplication (as seen in Section 1.3). Most of the properties will be stated as theorems, whose proofs will be left as exercises.

THEOREM 1.1 (Properties of Matrix Addition) *Let A, B, C, and D be $m \times n$ matrices.*

(a) $A + B = B + A$.

(b) $A + (B + C) = (A + B) + C$.

(c) *There is a unique $m \times n$ matrix O such that*

$$A + O = A \tag{1}$$

for any $m \times n$ matrix A. The matrix O is called the $m \times n$ **additive identity** *or* **zero matrix**.

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EXAMPLE 2 To illustrate (d) of Theorem 1.1, let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -4 & 5 & -2 \end{bmatrix}.$$

Then

$$-A = \begin{bmatrix} -2 & -3 & -4 \\ 4 & -5 & 2 \end{bmatrix}.$$

We now have $A + (-A) = O$. ■

EXAMPLE 3 Let

$$A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 2 \\ -3 & 4 & 6 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 3-2 & -2-3 & 5-2 \\ -1+3 & 2-4 & 3-6 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -2 & -3 \end{bmatrix}.$$ ■

THEOREM 1.2 (Properties of Matrix Multiplication)

(a) *If A, B, and C are of the appropriate sizes, then*

$$A(BC) = (AB)C.$$

(b) *If A, B, and C are of the appropriate sizes, then*

$$A(B + C) = AB + AC.$$

(c) *If A, B, and C are of the appropriate sizes, then*

$$(A + B)C = AC + BC.$$

Proof (a) We omit a general proof here. Exercise T.2 asks the reader to prove the result for a specific case.

(b) Exercise T.3.

(c) Exercise T.3. ■

EXAMPLE 4 Let

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}.$$

Then

$$A(BC) = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$$

and

$$(AB)C = \begin{bmatrix} 19 & -1 & 6 & 13 \\ 16 & -8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}.$$ ■

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Then

$$\tfrac{1}{4}a + \tfrac{1}{3}b = a$$
$$\tfrac{3}{4}a + \tfrac{2}{3}b = b$$

or

$$\begin{aligned} -\tfrac{3}{4}a + \tfrac{1}{3}b &= 0 \\ \tfrac{3}{4}a - \tfrac{1}{3}b &= 0. \end{aligned} \tag{4}$$

Observe that the two equations in (4) are the same. Using Equation (3) and one of the equations in (4), we find (verify) that

$$a = \tfrac{4}{13} \quad \text{and} \quad b = \tfrac{9}{13}.$$ ■

The problem described is an example of a **Markov chain.** We shall return to this topic in Section 2.5.

THEOREM 1.3 **(Properties of Scalar Multiplication)** *If r and s are real numbers and A and B are matrices, then*

(a) $r(sA) = (rs)A$
(b) $(r + s)A = rA + sA$
(c) $r(A + B) = rA + rB$
(d) $A(rB) = r(AB) = (rA)B$

Proof Exercise T.12. ■

EXAMPLE 10 Let $r = -2$,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 0 & -2 \end{bmatrix}.$$

Then

$$A(rB) = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ -2 & -8 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 8 & 0 \end{bmatrix}$$

and

$$r(AB) = (-2)\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 8 & 0 \end{bmatrix},$$

which illustrates (d) of Theorem 1.3. ■

It is easy to show that $(-1)A = -A$ (Exercise T.13).

THEOREM 1.4 **(Properties of Transpose)** *If r is a scalar and A and B are matrices, then*

(a) $(A^T)^T = A$
(b) $(A + B)^T = A^T + B^T$
(c) $(AB)^T = B^T A^T$
(d) $(rA)^T = rA^T$

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Key Terms

Properties of matrix addition
Additive identity or zero matrix
Additive inverse or negative of a matrix

Properties of matrix multiplication
Identity matrix
Powers of a matrix

Properties of transpose
Symmetric matrix
Skew symmetric matrix

GIST

1.4 Exercises

1. Verify Theorem 1.1 for
$$A=\begin{bmatrix}1&2&-2\\3&4&5\end{bmatrix},\quad B=\begin{bmatrix}2&0&1\\3&-2&5\end{bmatrix},$$
and
$$C=\begin{bmatrix}-4&-6&1\\2&3&0\end{bmatrix}.$$

2. Verify (a) of Theorem 1.2 for
$$A=\begin{bmatrix}1&3\\2&-1\end{bmatrix},\quad B=\begin{bmatrix}-1&3&2\\1&-3&4\end{bmatrix},$$
and
$$C=\begin{bmatrix}1&0\\3&-1\\1&2\end{bmatrix}.$$

3. Verify (b) of Theorem 1.2 for
$$A=\begin{bmatrix}1&-3\\-3&4\end{bmatrix},\quad B=\begin{bmatrix}2&-3&2\\3&-1&-2\end{bmatrix},$$
and
$$C=\begin{bmatrix}0&1&2\\1&3&-2\end{bmatrix}.$$

4. Verify (a), (b), and (c) of Theorem 1.3 for $r=6$, $s=-2$, and
$$A=\begin{bmatrix}4&2\\1&-3\end{bmatrix},\quad B=\begin{bmatrix}0&2\\-4&3\end{bmatrix}.$$

5. Verify (d) of Theorem 1.3 for $r=-3$ and
$$A=\begin{bmatrix}1&3\\2&-1\end{bmatrix},\quad B=\begin{bmatrix}-1&3&2\\1&-3&4\end{bmatrix}.$$

6. Verify (b) and (d) of Theorem 1.4 for $r=-4$ and
$$A=\begin{bmatrix}1&3&2\\2&1&-3\end{bmatrix},\quad B=\begin{bmatrix}4&2&-1\\-2&1&5\end{bmatrix}.$$

7. Verify (c) of Theorem 1.4 for
$$A=\begin{bmatrix}1&3&2\\2&1&-3\end{bmatrix},\quad B=\begin{bmatrix}3&-1\\2&4\\1&2\end{bmatrix}.$$

In Exercises 8 and 9, let
$$A=\begin{bmatrix}2&1&-2\\3&2&5\end{bmatrix},\quad B=\begin{bmatrix}2&-1\\3&4\\1&-2\end{bmatrix},$$
$$C=\begin{bmatrix}2&1&3\\-1&2&4\\3&1&0\end{bmatrix},\quad D=\begin{bmatrix}2&-1\\-3&2\end{bmatrix},$$
$$E=\begin{bmatrix}1&1&2\\2&-1&3\\-3&2&-1\end{bmatrix},\quad \textit{and}\quad F=\begin{bmatrix}1&0\\2&-3\end{bmatrix}.$$

8. If possible, compute:
(a) $(AB)^T$ (b) B^TA^T (c) A^TB^T
(d) BB^T (e) B^TB

9. If possible, compute:
(a) $(3C-2E)^TB$ (b) $A^T(D+F)$
(c) B^TC+A (d) $(2E)A^T$
(e) $(B^T+A)C$

10. If
$$A=\begin{bmatrix}-2&3\\2&-3\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}3&6\\2&4\end{bmatrix},$$
show that $AB=O$.

11. If
$$A=\begin{bmatrix}-2&3\\2&-3\end{bmatrix},\quad B=\begin{bmatrix}-1&3\\2&0\end{bmatrix},$$
and
$$C=\begin{bmatrix}-4&-3\\0&-4\end{bmatrix},$$
show that $AB=AC$.

12. If $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, show that $A^2=I_2$.

13. Let $A=\begin{bmatrix}4&2\\1&3\end{bmatrix}$. Find
(a) A^2+3A
(b) $2A^3+3A^2+4A+5I_2$

14. Let $A=\begin{bmatrix}1&-1\\2&3\end{bmatrix}$. Find
(a) A^2-2A
(b) $3A^3-2A^2+5A-4I_2$

A1

15. Determine a scalar r such that $A\mathbf{x}=r\mathbf{x}$, where
$$A=\begin{bmatrix}2&1\\1&2\end{bmatrix}\quad\text{and}\quad \mathbf{x}=\begin{bmatrix}1\\1\end{bmatrix}.$$

16. Determine a constant k such that $(kA)^T(kA)=1$, where
$$A=\begin{bmatrix}-2\\1\\-1\end{bmatrix}.$$
Is there more than one value of k that could be used?

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17. Let

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix}$$

and $\mathbf{a}_j = \text{col}_j(A)$, $j = 1, 2, 3$. Verify that

$$A^TA = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \mathbf{a}_1 \cdot \mathbf{a}_3 \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \mathbf{a}_2 \cdot \mathbf{a}_3 \\ \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{a}_3 \cdot \mathbf{a}_2 & \mathbf{a}_3 \cdot \mathbf{a}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \mathbf{a}_1^T\mathbf{a}_3 \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \mathbf{a}_2^T\mathbf{a}_3 \\ \mathbf{a}_3^T\mathbf{a}_1 & \mathbf{a}_3^T\mathbf{a}_2 & \mathbf{a}_3^T\mathbf{a}_3 \end{bmatrix}.$$

Exercises 18 through 21 deal with Markov chains, an area that will be studied in greater detail in Section 2.5.

18. Suppose that the matrix A in Example 9 is

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{3}{5} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(a) Find the distribution of the market after 1 year.

(b) Find the stable distribution of the market.

19. Consider two quick food companies, M and N. Each year, company M keeps $\frac{1}{3}$ of its customers, while $\frac{2}{3}$ switch to N. Each year, N keeps $\frac{1}{2}$ of its customers, while $\frac{1}{2}$ switch to M. Suppose that the initial distribution of the market is given by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

(a) Find the distribution of the market after 1 year.

(b) Find the stable distribution of the market.

20. Suppose that ~~in Example 9~~ there were three rival companies R, S, and T so that the pattern of customer retention and switching is given by the information in the matrix A where

$$A = \begin{array}{c} \begin{matrix} R & S & T \end{matrix} \\ \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{array} \begin{matrix} R \\ S \\ T \end{matrix}$$

(a) If the initial market distribution is given by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix},$$

then determine the market distribution after 1 year; after 2 years.

(b) Show that the stable market distribution is given by

$$\mathbf{x} = \begin{bmatrix} \frac{21}{53} \\ \frac{24}{53} \\ \frac{8}{53} \end{bmatrix}.$$

(c) Which company R, S, or T will gain the most market share over a long period of time (assuming that the retention and switching patterns remain the same)? Approximately what percent of the market was gained by this company?

21. Suppose that in Exercise 20 the matrix A was given by

$$A = \begin{array}{c} \begin{matrix} R & S & T \end{matrix} \\ \begin{bmatrix} 0.4 & 0 & 0.4 \\ 0 & 0.5 & 0.4 \\ 0.6 & 0.5 & 0.2 \end{bmatrix} \end{array} \begin{matrix} R \\ S \\ T \end{matrix}$$

(a) If the initial market distribution is given by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix},$$

then determine the market distribution after 1 year; after 2 years.

(b) Show that the stable market distribution is given by

$$\mathbf{x} = \begin{bmatrix} \frac{10}{37} \\ \frac{12}{37} \\ \frac{15}{37} \end{bmatrix}.$$

(c) Which company R, S, or T will gain the most market share over a long period of time (assuming that the retention and switching patterns remain the same)? Approximately what percent of the market was gained by this company?

Exercises 22 through 25 involve bit matrices.

22. If bit matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, show that $A^2 = O$.

23. If bit matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, show that $A^2 = I_2$.

24. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ be a bit matrix. Find

(a) $A^2 + A$ (b) $A^3 + A^2 + A$

25. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ be a bit matrix. Find

(a) $A^2 + A$ (b) $A^4 + A^3 + A^2$

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Ex: 1.4

Q1 verify theorem 1.1

(a) $A+B = B+A$

$$A+B = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 2 & 10 \end{bmatrix} \text{ and } B+A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 2 & 10 \end{bmatrix}.$$

(b) $A+(B+C) = (A+B)+C.$

$$B+C = \begin{bmatrix} -2 & -6 & 2 \\ 5 & 1 & 5 \end{bmatrix}.$$

$$A+(B+C) = \begin{bmatrix} -1 & -4 & 0 \\ 8 & 5 & 10 \end{bmatrix}.$$

Similarly

$$A+B = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 2 & 10 \end{bmatrix}$$

$$(A+B)+C = \begin{bmatrix} -1 & -4 & 0 \\ 8 & 5 & 10 \end{bmatrix}.$$

$\therefore$ Hence $\boxed{A+(B+C) = (A+B)+C}$

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Q2 Verify (a) of Theorem 1.2 for

$$A(BC) = (AB)C.$$

Sol

$$BC = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1+9+2 & 0-3+4 \\ 1-9+4 & 0+3+8 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ -4 & 11 \end{bmatrix}.$$

$$A(BC) = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 10 & 1 \\ -4 & 11 \end{bmatrix} = \begin{bmatrix} 10-12 & 1+33 \\ 20+4 & 2-11 \end{bmatrix} = \begin{bmatrix} -2 & 34 \\ 24 & -9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix}.$$

$$= \begin{bmatrix} -1+3 & 3-9 & 2+12 \\ -2-1 & 6+3 & 4-4 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 14 \\ -3 & 9 & 0 \end{bmatrix}.$$

$$(AB)C = \begin{bmatrix} 2 & -6 & 14 \\ -3 & 9 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 34 \\ 24 & -9 \end{bmatrix}.$$

Hence $\boxed{A(BC) = (AB)C.}$

Q3 Verify (b) of Th: 1.2

$$A(B+C) = AB + AC$$

S.Y.S.

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Ex: 1.4

Q4 Verify (a) (b) and (c) of theorem 1.3 for $r=6$, $s=-2$.

(a) $r(sA)=(rs)A$

(b) $(r+s)A=rA+sA$

(c) $r(A+B)=rA+rB$

~~(d) A(rB)~~

(a) $r(sA)=(rs)A$

$$sA=-2\begin{bmatrix}4&2\\1&-3\end{bmatrix}=\begin{bmatrix}-8&-4\\-2&6\end{bmatrix}.$$

$$r(sA)=6\begin{bmatrix}-8&-4\\-2&6\end{bmatrix}=\begin{bmatrix}-48&-24\\-12&36\end{bmatrix}.$$

$(rs)=(-2)(6)=-12$

$$(rs)A=-12\begin{bmatrix}4&2\\1&-3\end{bmatrix}=\begin{bmatrix}-48&-24\\-12&36\end{bmatrix}.$$

Hence $\boxed{r(sA)=(rs)A}$ verified.

$\boxed{\text{(b) \& (c) Similarly to a.}}$

Q5 verify (d) of thm: 1.3 for $\gamma = -3$.

(d) $A(\gamma B) = \gamma(AB) = (\gamma A)B$.

S.Y.S.

Q6 verify (b) & (d) of thm: 1.4 for $\gamma = -4$

(b) $(A+B)^T = A^T + B^T$

(d) $(\gamma A)^T = \gamma A^T$

S.Y.S.

Q7 verify (c) of Thm: 1.4 for

(c) $(AB)^T = B^T A^T$

S.Y.S.

Q10, Q11 S.Y.S

Q12 If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ show $A^2 = I_2$

$$A.A = A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Hen $A^2 = I_2$

Q13 & Q14 S.Y.S.

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Ex: 1.4

Q15 $Ax = \gamma x.$

$$Ax = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

$$\gamma x = \gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}$$

given $Ax = \gamma x$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} \Rightarrow \boxed{\gamma = 3}$$

Q16 $(kA)^T(kA) = 1$

$$kA = k\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2k \\ k \\ -k \end{bmatrix}.$$

$$(kA)^T = [-2k \;\; k \;\; -k].$$

$$(kA)^T(kA) = [-2k \;\; k \;\; -k]\begin{bmatrix} -2k \\ k \\ -k \end{bmatrix} = 4k^2 + k^2 + k^2 = 1$$

$$6k^2 = 1$$

$$k^2 = 1/6$$

$$\boxed{k = \pm 1/\sqrt{6}} \; d.$$

Q18 a) $Ax_0 = \begin{bmatrix} 1/3 & 2/5 \\ 2/3 & 3/5 \end{bmatrix}\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

$Ax_0 \begin{bmatrix} 2/9 + 2/15 \\ 4/9 + 3/15 \end{bmatrix}$.

b) For stable distribution

$Ax_0 = x_0$

$\begin{bmatrix} 1/3 & 2/5 \\ 2/3 & 3/5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

$= \begin{bmatrix} 1/3\,a + 2/5\,b \\ 2/3\,a + 3/5\,b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

$1/3\,a + 2/5\,b = a$ —— (i).

$2/3\,a + 3/5\,b = b$ —— (ii).

eq (i) $\Rightarrow \frac{1}{3}a - a + 2/5\,b = 0 \Rightarrow$

$-\frac{2}{3}a + 2/5\,b = 0$

$-10a + 6b = 0$ ———— (iii)

eq (ii) $\Rightarrow \frac{2}{3}a + 3/5\,b - b = 0$

$\Rightarrow 10a - 6b = 0$ ———— (iv)

Solving eq (iii) & (iv)

$a = 3/8$ & $b = 5/8$ $x_0 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$.

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Ex: 14

Q19

$$A = \begin{array}{c c} \begin{array}{cc} M & N \end{array} & \\ \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} & \begin{array}{c} M \\ N \end{array} \end{array} \qquad x_0 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

(a) $Ax_0 = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/9 + 2/6 \\ 2/9 + 2/6 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}.$

(b) $Ax_0 = x_0$

$$\begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

$\frac{1}{3}a + 1/2\, b = a$ —— (i)

$2/3\, a + 1/2\, b = b$ —— (ii)

S.Y.S

$a = 3/7$ & $b = 4/7$

$$x_0 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}.$$

Q20 (a) $x_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ $\quad A = \begin{array}{c c} \begin{array}{ccc} R & S & T \end{array} & \\ \begin{bmatrix} 1/3 & 1/2 & 1/4 \\ 2/3 & 1/4 & 1/2 \\ 0 & 1/4 & 1/4 \end{bmatrix} & \begin{array}{c} R \\ S \\ T \end{array} \end{array}.$

$$Ax_0 = \begin{bmatrix} 1/3 & 1/2 & 1/4 \\ 2/3 & 1/4 & 1/2 \\ 0 & 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 13/36 \\ 17/36 \\ 1/6 \end{bmatrix}.$$

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after 2 years

$x_2 = Ax_1 = A(Ax_0)$

$$= \begin{bmatrix} 1/3 & 1/2 & 1/4 \\ 2/3 & 1/4 & 1/2 \\ 0 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 13/36 \\ 17/36 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 43/108 \\ 191/432 \\ 23/144 \end{bmatrix}.$$

(b) $Ax_0 = x_0$, $x_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$a+b+c = 1$ —— (1)

$$\begin{bmatrix} 1/3 & 1/2 & 1/4 \\ 2/3 & 1/4 & 1/2 \\ 0 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$\frac{1}{3}a + 1/2\, b + 1/4\, c = a$ —— (2)

$2/3\, a + 1/4\, b + 1/2\, c = b$ —— (3)

$1/4\, b + 1/4\, c = c$ —— (4)

eq (2) $\Rightarrow \dfrac{4a+6b+3c}{12} = a$

$4a+6b+3c = 12a$

$\Rightarrow -8a+6b+3c$ —— (5)

eq (3) $\Rightarrow \dfrac{8a+3b+6c}{12} = b$

$8a+3b+6c = 12b$

$\Rightarrow 8a-9b+6c = 0$ —— (6)

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Ex: 1.4

$\therefore (4) \Rightarrow b + c = 4c$

$\Rightarrow b - 3c = 0$ — (7)

$b = 3c$

$\therefore (5) \Rightarrow -8a + 6(3c) + 3c = 0$

$\Rightarrow \boxed{a = 21/8\, c}$ — (8)

$\therefore (1) \Rightarrow 21/8\, c + 3c + c = 1$

$\Rightarrow \boxed{c = 8/53}$

$\therefore (8) \Rightarrow a = \frac{21}{8} \times \frac{8}{53} = \boxed{21/53 = a}$

$\therefore (7) \Rightarrow b = 3 \cdot 8/53 = \boxed{24/53 = b}$

So $x_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 21/53 \\ 24/53 \\ 8/53 \end{bmatrix}$ Ans.

Q2 (a) $x_1 = A x_0$

$x_1 = \begin{bmatrix} 0.4 & 0 & 0.4 \\ 0 & 0.5 & 0.4 \\ 0.6 & 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$

$x_1 = \begin{bmatrix} 4/15 \\ 3/10 \\ 13/30 \end{bmatrix}.$

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After 2 years.

$Ax_1 = A(Ax_0)$

$$= \begin{bmatrix} 0.4 & 0 & 0.4 \\ 0 & 0.5 & 0.4 \\ 0.6 & 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 4/15 \\ 3/10 \\ 13/30 \end{bmatrix}$$

$$= \begin{bmatrix} 84/300 \\ 97/300 \\ 119/300 \end{bmatrix}.$$

(b) $Ax_0 = x_0 \qquad x_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$a+b+c = 1$ —— (1)

New

$$\begin{bmatrix} 0.4 & 0 & 0.4 \\ 0 & 0.5 & 0.4 \\ 0.6 & 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

S.V.S as (1) 2 (2).

$$x_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10/37 \\ 12/37 \\ 15/37 \end{bmatrix}.$$

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