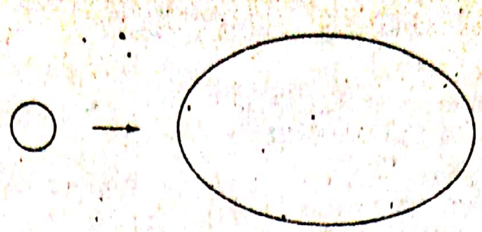
Figure 2.19 ▶



Unit circle   Ellipse

MORTENSON, M. E. *Mathematics for Computer Graphics Applications*, 2nd ed. New York: Industrial Press, Inc., 1999.

ROGERS, D. F., and J. A. ADAMS. *Mathematical Elements for Computer Graphics*, 2nd ed. New York: McGraw-Hill, 1989.

Key Terms

Computer graphics
Computer-aided design
Image
Reflection
Rotation
Shear
Dilation
Contraction

2.3 Exercises

1. Let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

that is, f is a reflection with respect to the y-axis. Find and sketch the image of the rectangle R with vertices $(1, 1)$, $(2, 1)$, $(1, 3)$, and $(2, 3)$.

2. Let R be the rectangle with vertices $(1, 1)$, $(1, 4)$, $(3, 1)$, and $(3, 4)$. Let f be the shear in the x-direction with $k = 3$. Find and sketch the image of R.

H.W

3. A **shear in the y-direction** is the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, and

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix},$$

and k is a scalar. Let R be the rectangle defined in Exercise 2 and let f be the shear in the y-direction with $k = -2$. Find and sketch the image of R.

H.W

4. The matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix};$$

and k is a real number, is called **dilation** if $k > 1$ and a **contraction** if $0 < k < 1$. Thus, dilation stretches a vector, whereas contraction shrinks it. If R is the rectangle defined in Exercise 2, find and sketch the image of R for

(a) $k = 4$ (b) $k = \frac{1}{4}$

5. The matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where Q3(b)

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix},$$

and k is a real number, is called **dilation in the x-direction** if $k > 1$ and a **contraction in the x-direction** if $0 < k < 1$. If R is the unit square and f is dilation in the x-direction with $k = 2$, find and sketch the image of R.

H.W

Exp=4 **6.** The matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix},$$

where k is a real number, is called **dilation in the y-direction** if $k > 1$ and **contraction in the y-direction** if $0 < k < 1$. If R is the unit square and f is the contraction in the y-direction with $k = \frac{1}{2}$, find and sketch the image of R.

7. Let T be the triangle with vertices $(5, 0)$, $(0, 3)$, and $(2, -1)$. Find the coordinates of the vertices of the image of T under the matrix transformation f defined by

$$f(\mathbf{v}) = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\mathbf{v}.$$

H.W

8. Let T be the triangle with vertices $(1, 1)$, $(-3, -3)$, and $(2, -1)$. Find the coordinates of the vertices of the image of T under the matrix transformation defined by

$$f(\mathbf{v}) = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}\mathbf{v}.$$

P2

9. Let f be the counterclockwise rotation through 60°. If T is the triangle defined in Exercise 8, find and sketch the image of T under f:

H.W

10. Let f_1 be reflection with respect to the y-axis and let f_2 be counterclockwise rotation through $\pi/2$ radians. Show that the result of first performing f_2 and then f_1 is not the same as first performing f_1 and then performing f_2.

P2 B5

11. Let A be the singular matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and let T be the triangle defined in Exercise 8. Describe the image of T under the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$.

P2 B3

12. Let f be the matrix transformation defined in Example 5. Find and sketch the image of the rectangle with vertices (0, 0), (1, 0), (1, 1), and (0, 1) for $h = 2$ and $k = 3$.

H.W

13. Let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Find and sketch the image of the rectangle defined in Exercise 12.

In Exercises 14 and 15, let f_1, f_2, f_3, and f_4 be the following matrix transformations:

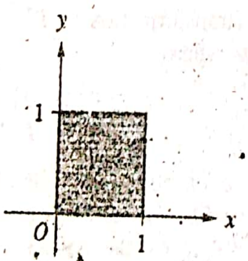
f_1: counterclockwise rotation through the angle ϕ

f_2: reflection with respect to the x-axis

f_3: reflection with respect to the y-axis

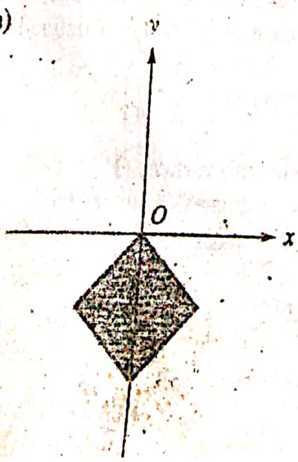
f_4: reflection with respect to the line $y = x$

14. Let S denote the unit square.

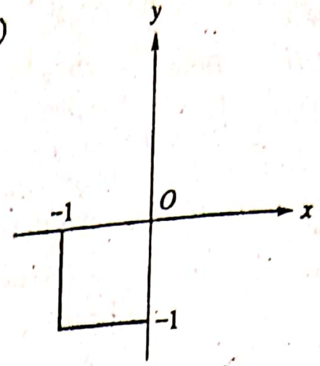


Determine two distinct ways to use the matrix transformations defined on S to obtain the given image. You may apply more than one matrix transformation in succession.

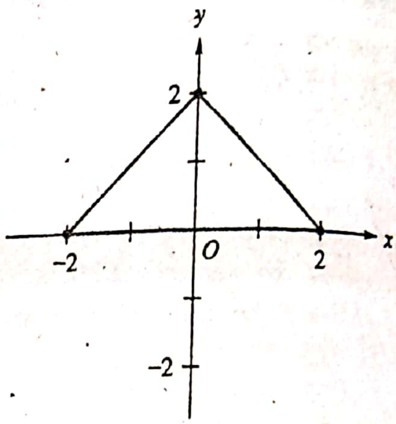
(a)



(b)

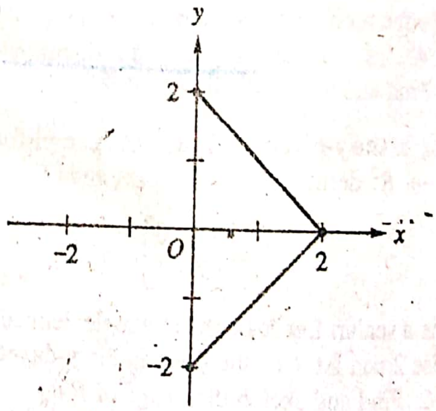


15. Let S denote the triangle shown in the figure.

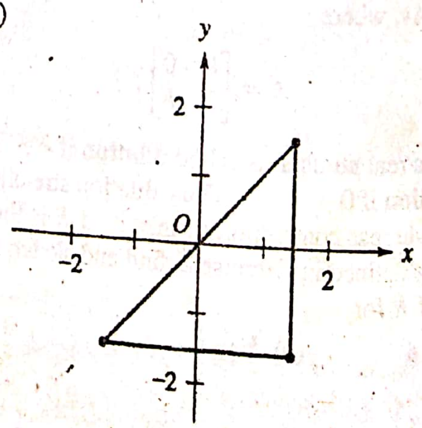


Determine two distinct ways to use the matrix transformations defined on S to obtain the given image. You may apply more than one matrix transformation in succession.

(a)



(b)



Ex 2.3

Q1 $A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ & $V=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 1 & 3 & 3\end{bmatrix}$. (1,1) (2,1) (1,3) (2,3)

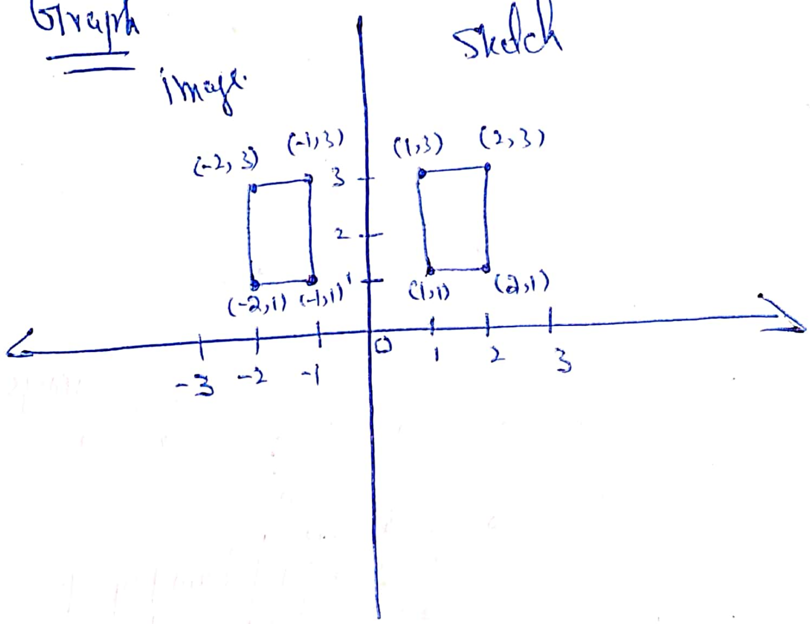
Given $f(v)=AV$

$$=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 1 & 3 & 3\end{bmatrix}.$$

$$=\begin{bmatrix}-1\times1+0\times1 & -1\times2+0\times1 & -1\times1+0\times3 & -1\times2+0\times3\\ 0\times1+1\times1 & 0\times2+1\times1 & 0\times1+1\times3 & 0\times2+1\times3\end{bmatrix}$$

$$f(v)=\begin{bmatrix}-1 & -2 & -1 & -2\\ 1 & 1 & 3 & 3\end{bmatrix}$$ image.

Graph



Mr. Jamal Nasir

Lecturer U.E.T Peshawar

Spring Semester-2012 Electrical Engg:

Q2 Let f be the shear in the x-direction

$$f(v) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} v.$$

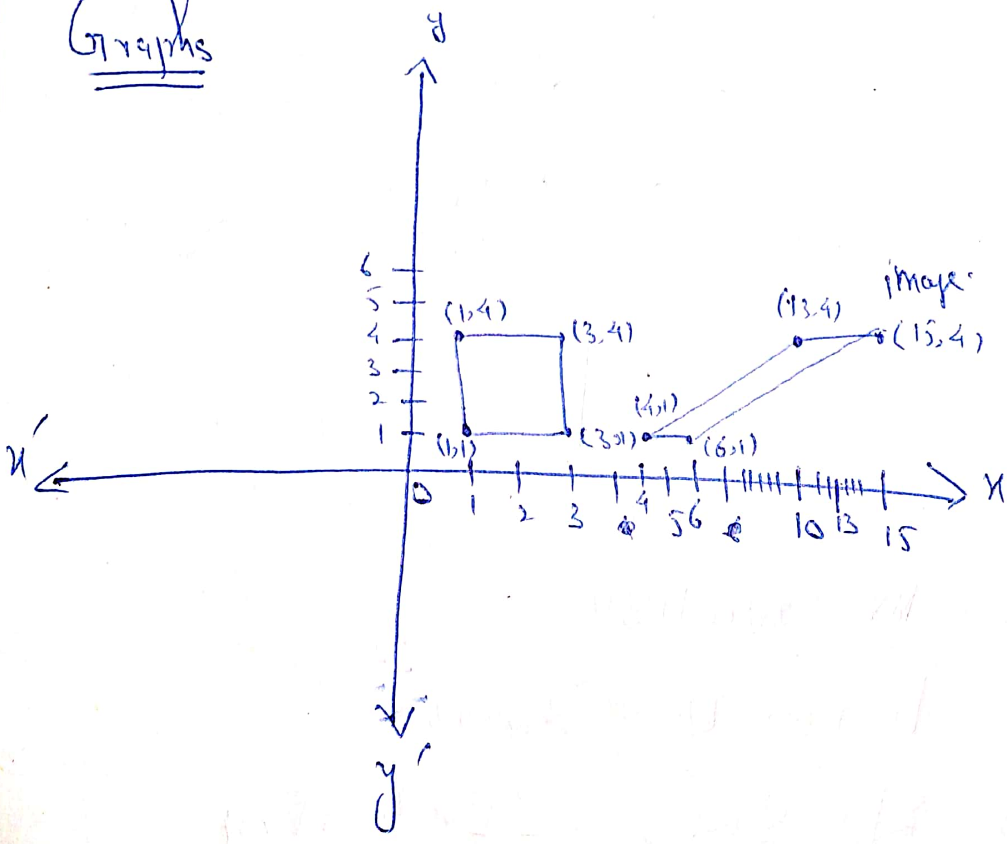
where $k = 3$ & $V = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix}$.

$$f(v) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} V$$

$$\Rightarrow f(v) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 & 6 & 15 \\ 1 & 4 & 1 & 4 \end{bmatrix} \text{ Image}$$

Graphs



Ex 2.3

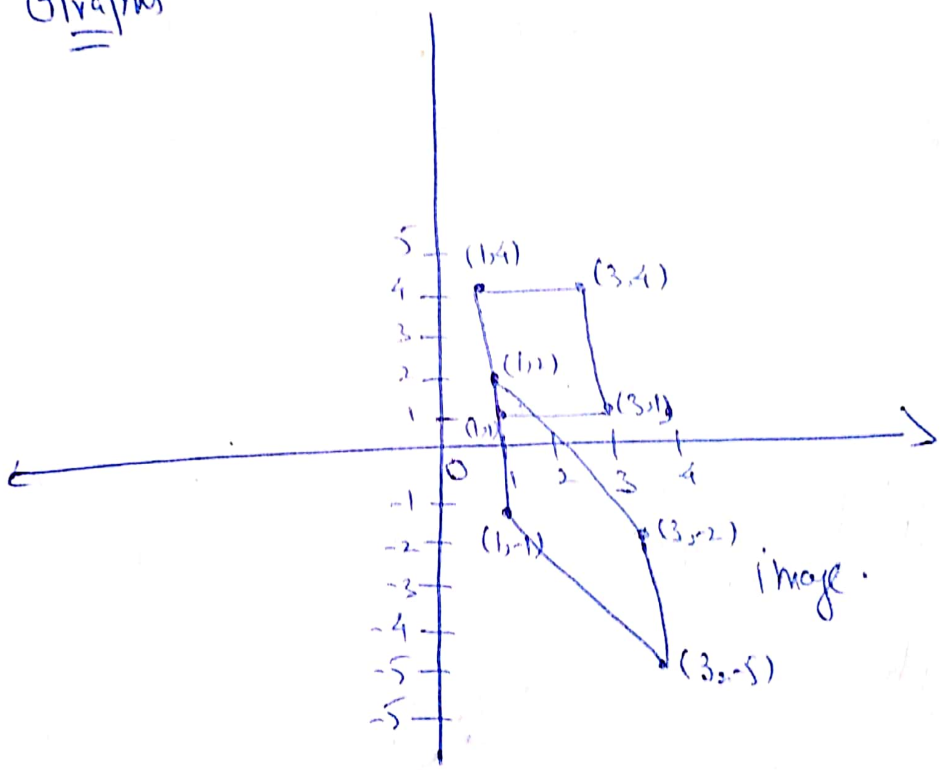
Q3 $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ & $k = -2$ then $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

& $V = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix}$.

Now $f(v) = AV = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix}$.

$= \begin{bmatrix} 1 & 1 & 3 & 3 \\ -1 & 2 & -5 & -2 \end{bmatrix}$ image.

Graphs



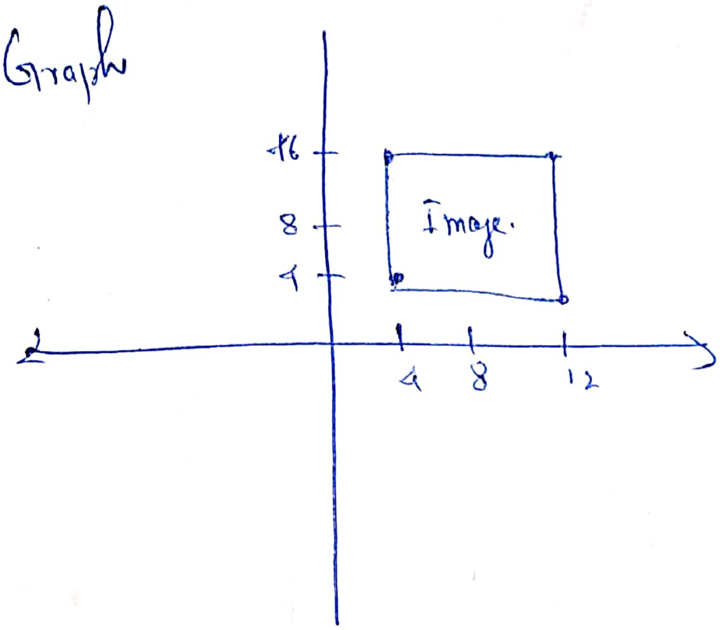
Q4(a) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$. $k = 4$

$V = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix}$.

$f(v) = AV = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix}$.

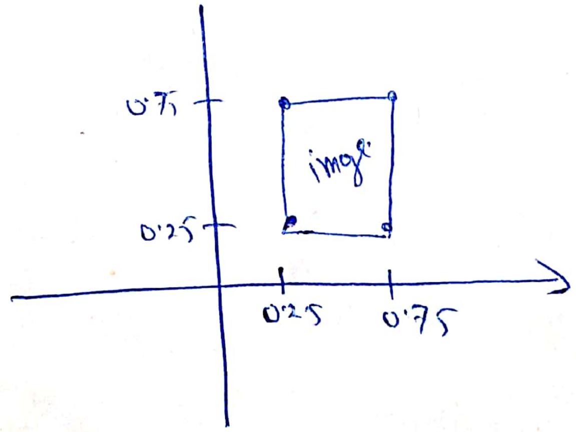
$= \begin{bmatrix} 4 & 4 & 12 & 12 \\ 4 & 16 & 4 & 16 \end{bmatrix}$ image.

Graph



(b) $k = 1/4$ $\begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 4 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.25 & 0.75 & 0.75 \\ 0.25 & 1 & 0.25 & 1 \end{bmatrix}$.

Graph



Ex 2.3

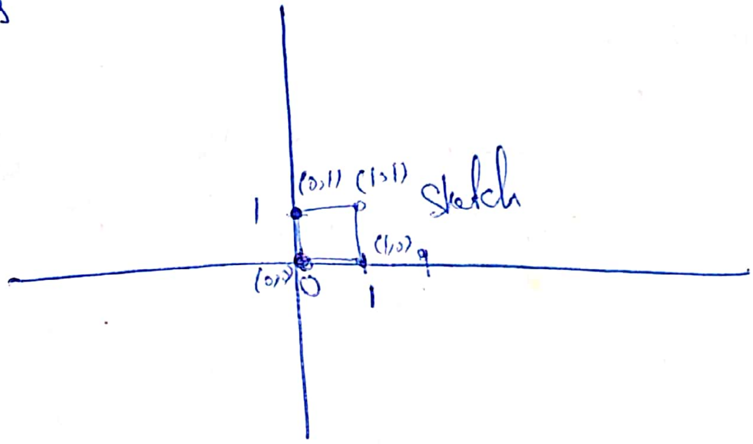
Q5 $k=2$. $A=\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

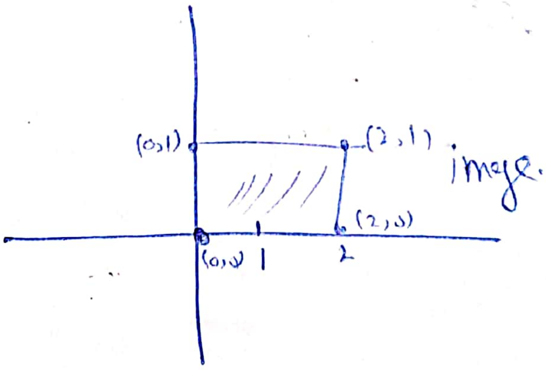
$$V=\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$f(V)=AV=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}.\ \text{image}$$

Graphs





Q6 is similarly to Q5

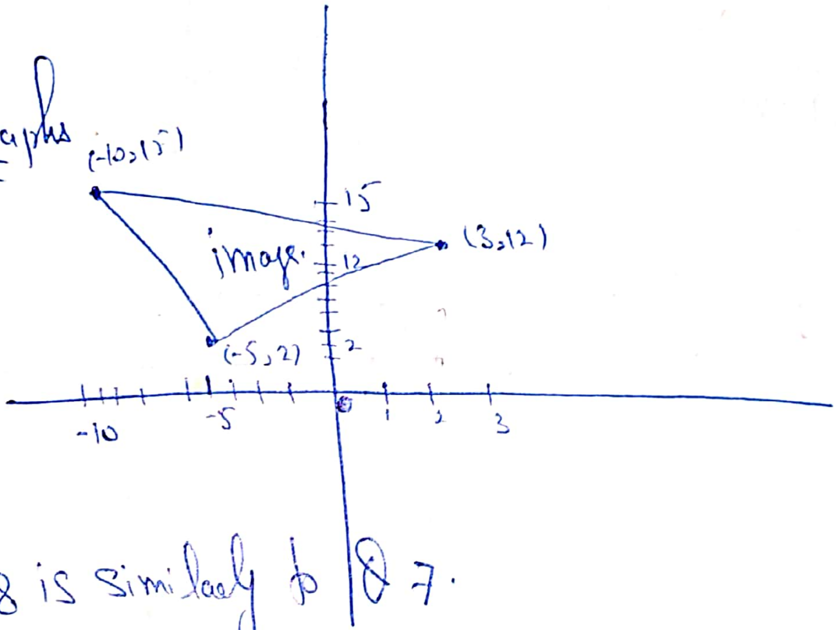
Q7 $f(v) = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} v$

$T = V' = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & -1 \end{bmatrix}.$

$f(v) = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} -10 & 3 & -5 \\ 15 & 12 & 2 \end{bmatrix}.$

Graphs



Q8 is similarly to Q 7.

Ex 2.3

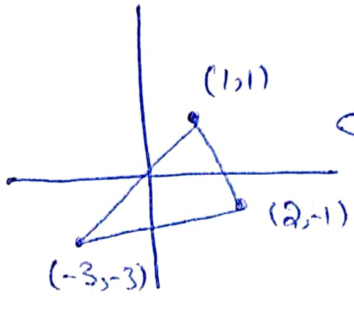
Q9 $T=\begin{bmatrix}1 & -3 & 2\\ 1 & -3 & -1\end{bmatrix}$ & $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ & $\theta=60^{\circ}$

$$A=\begin{bmatrix}\cos 60 & -\sin 60\\ \sin 60 & \cos 60\end{bmatrix}=\begin{bmatrix}0.5 & -0.866\\ 0.866 & 0.5\end{bmatrix}$$ (Exp 9 Ex 1.5)

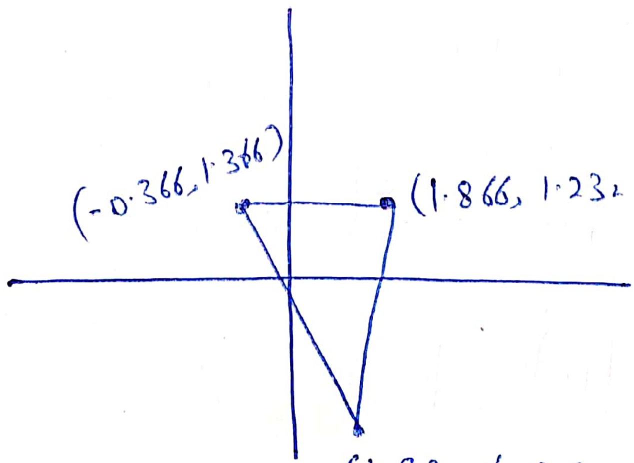
$$f(T)=AT=\begin{bmatrix}0.5 & -0.866\\ 0.8666 & 0.5\end{bmatrix}\begin{bmatrix}1 & -3 & 2\\ 1 & -3 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-0.366 & 1.098 & 1.866\\ 1.366 & -4.098 & 1.232\end{bmatrix}$$

Graph



Sketch



Q10 (Assg:)

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (exps exis) & $u = \begin{bmatrix} x \\ y \end{bmatrix}$.

$f_2(u) = Au = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$.

$f_1(f_2(u)) = f_1(Au) = B(Au)$

$B = \begin{bmatrix} \cos\pi/2 & \sin\pi/2 \\ -\sin\pi/2 & \cos\pi/2 \end{bmatrix}$ (for y-axis).

$B(Au) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$.

Now

$f_1(u) = \begin{bmatrix} \cos\pi/2 & \sin\pi/2 \\ -\sin\pi/2 & \cos\pi/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$.

$f_2(f_1(u)) = f_2(Bu) = A(Bu)$

$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$.

Hu $f_1(f_2 u) \neq f_2(f_1(u))$.

Q11 $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, $T=\begin{bmatrix}1 & -3 & 2\\ 1 & -3 & -1\end{bmatrix}$.

$$f(v)=Av=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & -3 & 2\\ 1 & -3 & -1\end{bmatrix}.$$

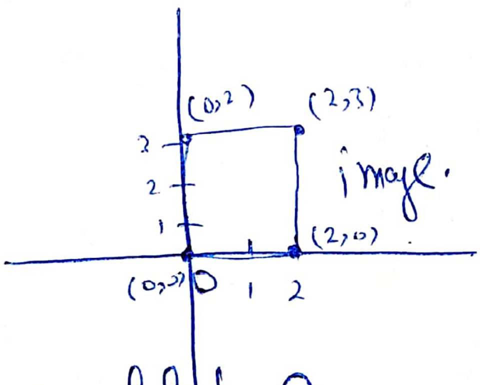
$$=\begin{bmatrix}3 & -9 & 0\\ 6 & -18 & 0\end{bmatrix}.$$

Q12 $A=\begin{bmatrix}h & 0\\ 0 & k\end{bmatrix}$ & $h=2,\ k=3$ (Given)

$A=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$. $V=\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$.

$$AV=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}.$$

$$=\begin{bmatrix}0 & 2 & 2 & 0\\ 0 & 0 & 3 & 3\end{bmatrix}.$$

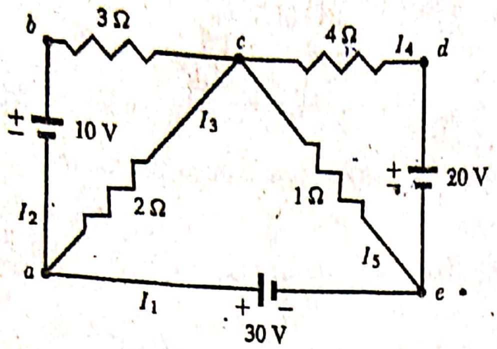


Q13 is similarly to Q12.

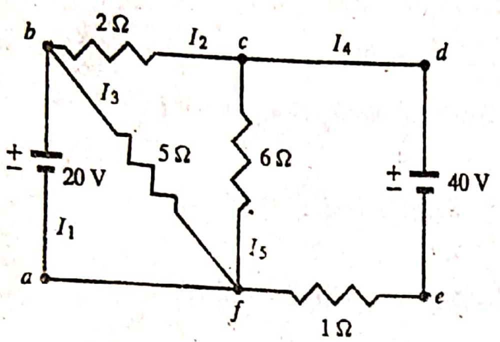
2.4 Exercises

In Exercises 1 through 4, determine the unknown currents in the given circuit.

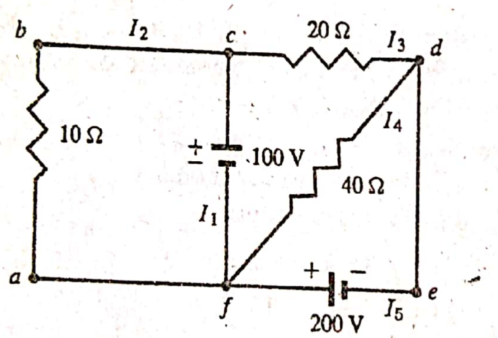
1.



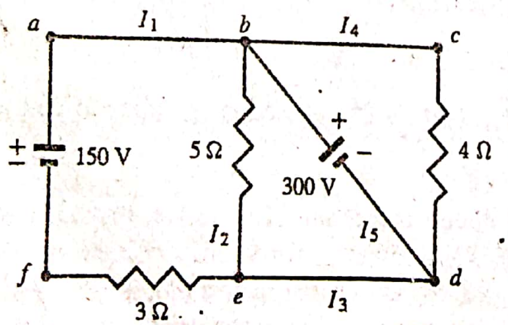
2.



3.

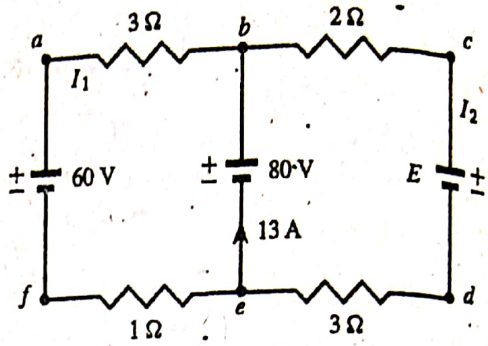


4.

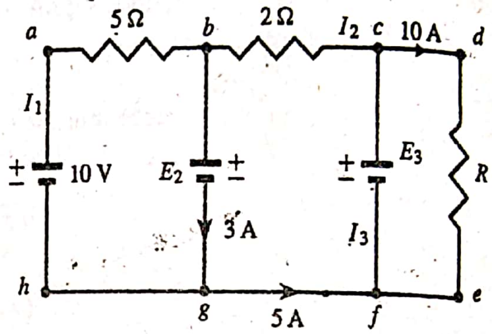


In Exercises 5 through 8, determine the unknowns in the given circuit.

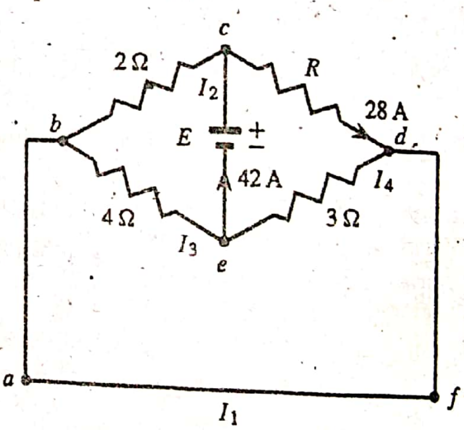
5.



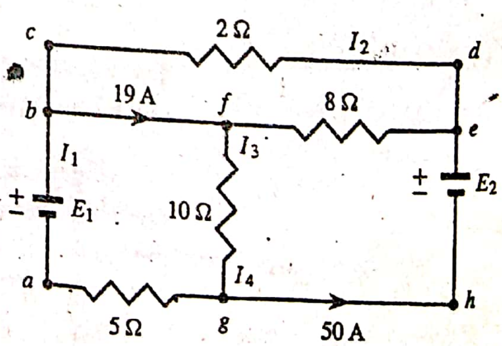
6.



7.

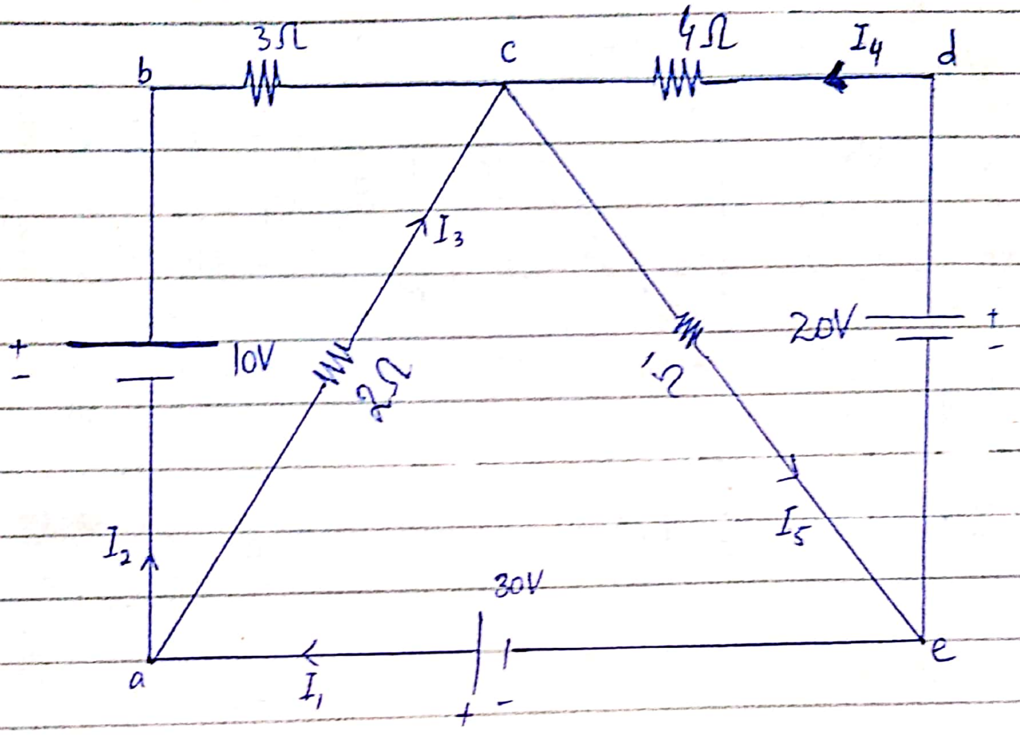


8.



Q.1

Determine the unknown currents in the given circuits.



→ Nodes

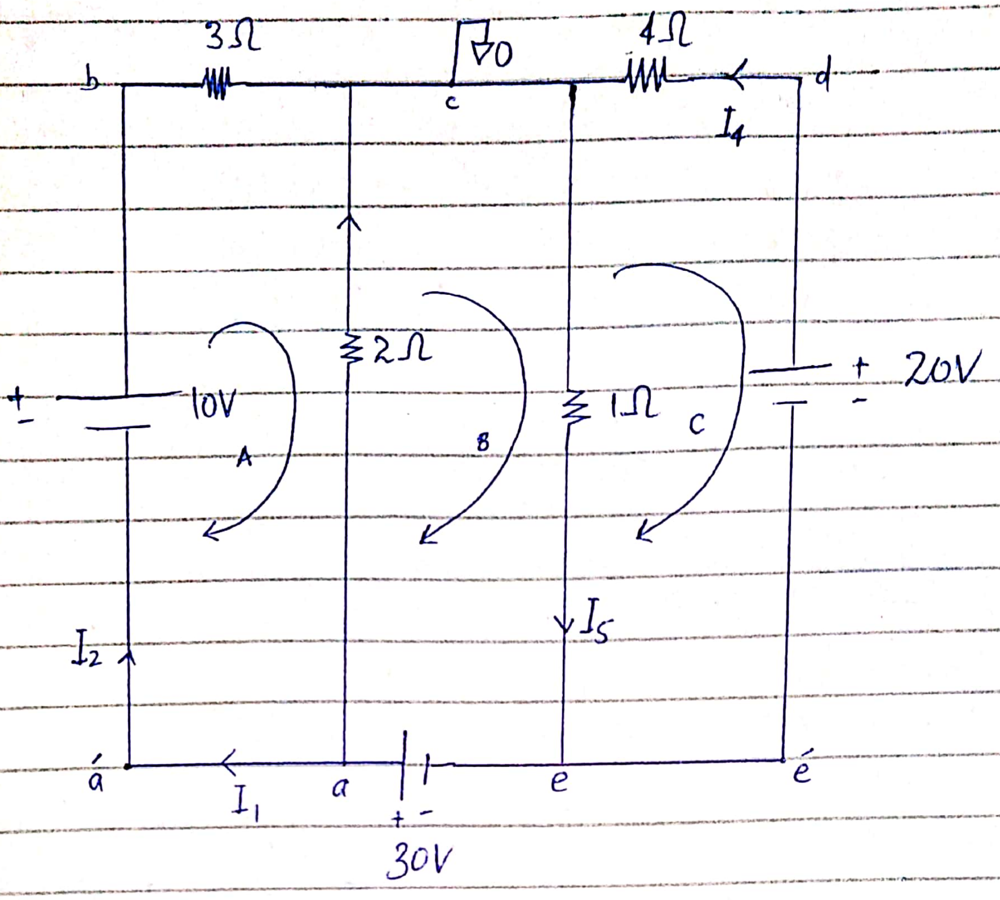
Given circuit has 3 Nodes (a, c, e)

Node is a point where current divides.

→ To solve circuit, we will ground one (Any one) node. This will help in simplification of circuit

→ Let "c" Node is grounded.

→ SIMPLIFIED CIRCUIT.



APPLY K.C.L. AT NODE (a)

$$I_1 = I_2 + I_3$$

$$I_1 - I_2 - I_3 = 0 \quad \dashrightarrow (1)$$

APPLY KCL AT NODE (e)

$$I_5 = I_1 + I_4$$

$$I_5 - I_1 - I_4 = 0 \quad \dashrightarrow (2)$$

Simplified circuit has 3 loops (A, B, C)

Loop (A) or $\bar{a}bc\bar{a}$ loop

$$10 = 3I_2 - 2I_3$$

$$3I_2 - 2I_3 = 10 \longrightarrow (3)$$

Loop (B) or $a'cea'$ loop

$$30 = 2I_3 + I_5 \longrightarrow (4)$$

Loop (C) or dced loop:-

$$20 = 4I_4 + I_5 \longrightarrow (5)$$

Write eq 1, 2, 3, 4, 5 in Matrix form

$$\begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & +1 & 0 \\ 0 & 3 & -2 & 0 & 0 & 10 \\ 0 & 0 & 2 & 0 & 1 & 30 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

Scanned with CamScanner

Converting it to reduced echelon form.

$$R_2 + R_1 \quad \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & +1 & 0 \\ 0 & 3 & -2 & 0 & 0 & 10 \\ 0 & 0 & 2 & 0 & 1 & 30 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

$$-1\,(R_2) \quad \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 & 0 & 10 \\ 0 & 0 & 2 & 0 & 1 & 30 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

$$\begin{array}{c} R_1 + R_2 \\ \& \\ R_3 - 3R_2 \\ -\frac{1}{5} \times R_3 \end{array} \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3/5 & -3/5 & -2 \\ 0 & 0 & 2 & 0 & 1 & 30 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

$$R_2 \div R_3 \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2/5 & -2/5 & 2 \\ 0 & 0 & 1 & 3/5 & -3/5 & -2 \\ 0 & 0 & 2 & 0 & 1 & 30 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

$$R_4 - 2R_3 \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2/5 & -2/5 & 2 \\ 0 & 0 & 1 & 3/5 & -3/5 & -2 \\ 0 & 0 & 0 & -6/5 & 11/5 & 34 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

$$-\frac{5}{6} \times R_4 \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2/5 & -2/5 & 2 \\ 0 & 0 & 1 & 3/5 & -3/5 & -2 \\ 0 & 0 & 0 & -6/5 & 11/5 & 34 \\ 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_4 \\ R_2 - \frac{2}{5} R_4 \\ R_3 - \frac{3}{5} R_4 \\ R_5 - 4R_4 \end{array} \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 5/6 & 85/3 \\ 0 & 1 & 0 & 0 & 1/3 & 40/3 \\ 0 & 0 & 1 & 0 & 1/2 & 15 \\ 0 & 0 & 0 & 1 & -11/6 & -85/3 \\ 0 & 0 & 0 & 0 & 25/3 & 400/3 \end{array}\right]$$

$$\frac{3}{25} \times R_5 \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 5/6 & 85/3 \\ 0 & 1 & 0 & 0 & 1/3 & 40/3 \\ 0 & 0 & 1 & 0 & 1/2 & 15 \\ 0 & 0 & 0 & 1 & -11/6 & -85/3 \\ 0 & 0 & 0 & 0 & 1 & 16 \end{array}\right]$$

$$\begin{array}{l} R_1 - \frac{5}{6}R_5 \\ R_2 - \frac{1}{3}R_5 \\ R_3 - \frac{1}{2}R_5 \\ R_4 + \frac{11}{6}R_5 \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 16 \end{array}\right]$$

Now in linear System

$I_1 = 15$ Ampere

$I_2 = 8$ Ampere

$I_3 = 7$ Ampere

$I_4 = 1$ Ampere

$I_5 = 15$ Ampere.

$$\begin{array}{l} R_1 - \frac{5}{6}R_5 \\ \\ R_2 - \frac{1}{3}R_5 \\ R_3 - \frac{1}{2}R_5 \\ R_4 + \frac{11}{6}R_5 \end{array} \quad \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 16 \end{array}$$

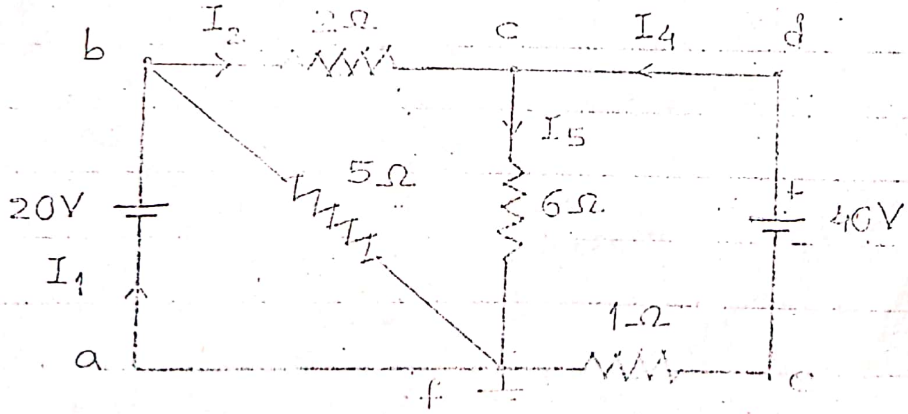
now in linear system of equations:-

$$I_1 = 15$$
$$I_2 = 8$$
$$I_3 = 7$$
$$I_4 = 1$$
$$I_5 = 15$$

Hence,

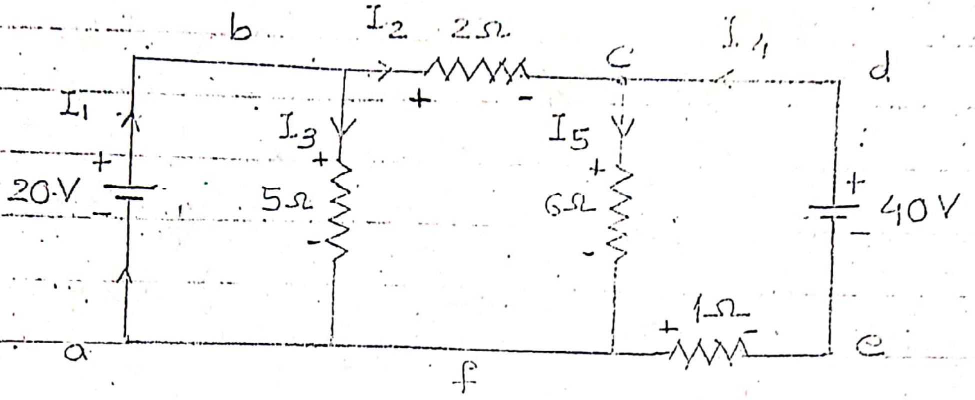
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 7 \\ 1 \\ 15 \end{bmatrix}$$

QUESTION # 2 : Determine unknown currents :



SOLUTION :

Simplified circuit is :-



Now, applying KCL at nodes :

Node (b):

$$I_1 = I_2 + I_3$$

$$I_1 - I_2 - I_3 = 0 \qquad \text{— (1)}$$

Node (c):

$$I_2 + I_4 = I_5$$

$$I_2 + I_4 - I_5 = 0 \qquad \text{— (2)}$$

Now applying KVL on 3 loops :

Loop 1 or mesh abfa :

$$5 I_3 = 20 \qquad \text{— (3)}$$

Loop 2 or mesh bcfb :

$$2 I_2 + 6 I_5 - 5 I_3 = 0 \qquad \text{— (4)}$$

Loop 3 or mesh cdefc :

$$I_4 + 6 I_5 = 40 \qquad \text{— (5)}$$

In matrix form :-

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 2 & -5 & 0 & 6 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 0 \\ 40 \end{bmatrix}$$

now reducing it to reduced echelon form :-

$R_1 + R_2$
$R_4 - 2R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 20 \\ 0 & 0 & -5 & -2 & 8 & 0 \\ 0 & 0 & 0 & 1 & 6 & 40 \end{array}\right]$$

$\frac{1}{5} \times R_3$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & -5 & -2 & 8 & 0 \\ 0 & 0 & 0 & 1 & 6 & 40 \end{array}\right]$$

$R_1 + R_3$
$R_4 + 5R_3$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & -2 & 8 & 20 \\ 0 & 0 & 0 & 1 & 6 & 40 \end{array}\right]$$

9

$\frac{-1}{2} \times R_4$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & -4 & -10 \\ 0 & 0 & 0 & 1 & 6 & 40 \end{array}\right]$$

$R_1 - R_4$
$R_2 - R_4$
$R_5 - R_4$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & +3 & 14 \\ 0 & 1 & 0 & 0 & 3 & 10 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & -4 & -10 \\ 0 & 0 & 0 & 0 & 10 & 50 \end{array}\right]$$

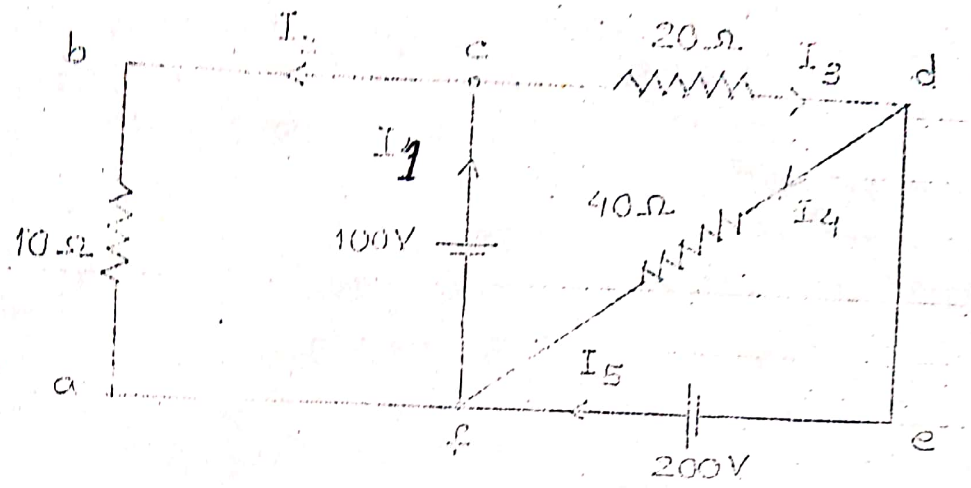
$\frac{1}{10} \times R_5$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 14 \\ 0 & 1 & 0 & 0 & 3 & 10 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & -4 & -10 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 - 3R_5$
$R_2 - 3R_5$
$R_4 + 4R_5$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Hence,
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 4 \\ 10 \\ 5 \end{bmatrix}$$

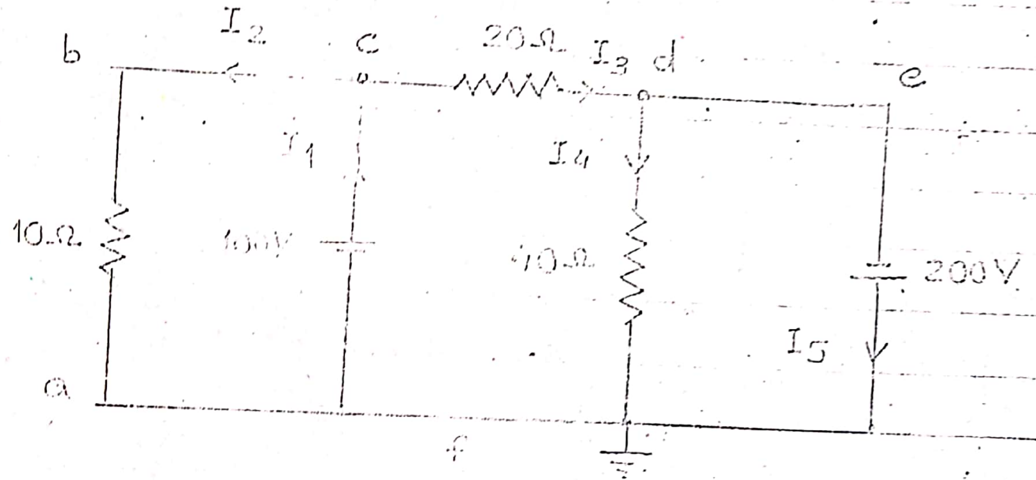
QUESTION # 3 :

Determine unknown currents in the circuit :



SOLUTION :

Simplified circuit is :



Now applying KCL at nodes :

Node (c) :

$$I_1 = I_2 + I_3$$

$$I_1 - I_2 - I_3 = 0 \qquad \text{—— (1)}$$

Node (d):

$$I_3 = I_4 + I_5$$

$$I_3 - I_4 - I_5 = 0 \qquad \text{--- (2)}$$

Applying KVL to 3 loops:-

Mesh abcfa:

$$10 I_2 = 100 \qquad \text{--- (3)}$$

Mesh cdfc:

$$20 I_3 + 40 I_4 = 100$$

$$I_3 + 2 I_4 = 5 \qquad \text{--- (4)}$$

Mesh defd:

$$40 I_4 = -200$$

$$I_4 = -5 \qquad \text{--- (5)}$$

In the matrix form:-

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 5 \\ -5 \end{bmatrix}$$

Reducing to reduced echelon form:-

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & -5 \end{array}\right]$$

2)

$R_1 + R_2$ $\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & -5 \end{array}\right]$

$R_1 + R_3$
$R_4 - R_3$ $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -1 & 10 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & -5 \end{array}\right]$

$R_4 - 2R_5$ $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -1 & 10 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 15 \\ 0 & 0 & 0 & 1 & 0 & -5 \end{array}\right]$

$R_1 + R_4$
$R_3 + R_4$
$R_5 - R_4$ $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 25 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 1 & 1 & 15 \\ 0 & 0 & 0 & 0 & -1 & -20 \end{array}\right]$

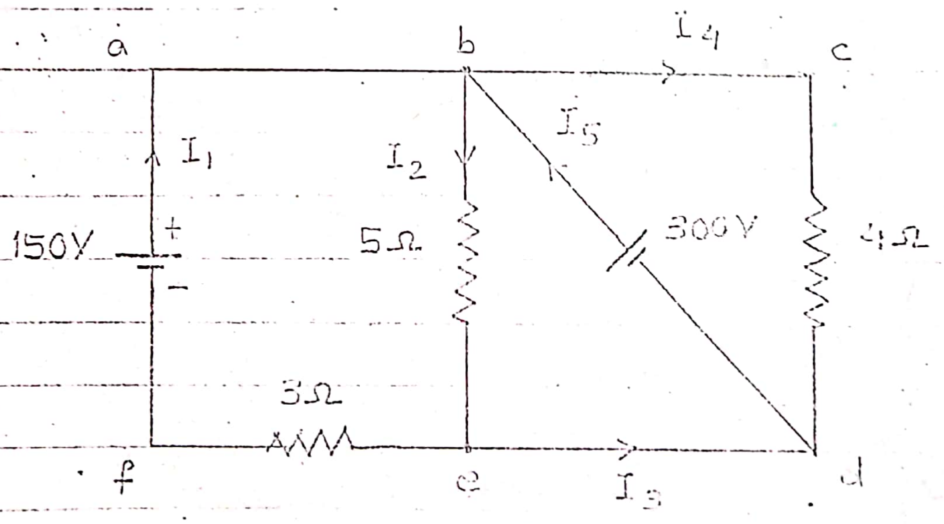
$-1 \times R_5$ $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 25 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 1 & 1 & 15 \\ 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right]$

$$R_4 - R_5 \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 25 \\ 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right]$$

Hence,
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 25 \\ 10 \\ 15 \\ -5 \\ 20 \end{bmatrix}$$

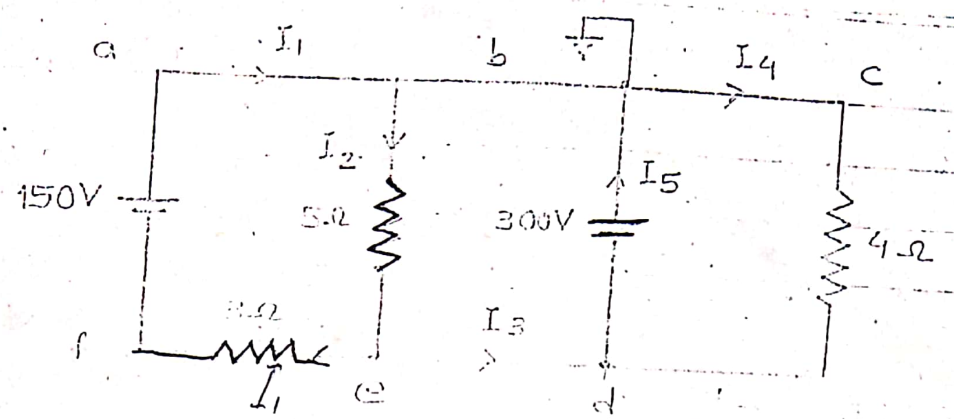
QUESTION # 4:

Determine the unknown currents:



SOLUTION:

Simplied circuit is :



Applying KCL at nodes:

Node (d):

$$I_4 + I_3 = I_5$$

$$I_3 + I_4 - I_5 = 0 \quad \text{—— (1)}$$

Node (e):

$$I_1 + I_3 = I_2$$

$$I_1 - I_2 + I_3 = 0 \quad \text{—— (2)}$$

Now, applying KVL on 3 meshes.

Mesh abefa:

$$150 = 5I_2 + 3I_1$$

$$3I_1 + 5I_2 = 150 \quad \text{—— (3)}$$

Mesh bdeb:

$$300 = 5I_2$$

$$I_2 = 60 \quad \text{—— (4)}$$

Mesh bcdb:

$$300 = 4I_4$$

$$I_4 = 75 \quad \text{—— (5)}$$

In the matrix form:-

$$\begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & +1 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 150 \\ 60 \\ 75 \end{bmatrix}$$

USING REDUCED ECHELON FORM:-
(GAUSS JORDAN REDUCTION METHOD)

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & -1 & +1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 150 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$R_1 \Leftrightarrow R_2$

$$\left[\begin{array}{ccccc|c} 1 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 3 & 5 & 0 & 0 & 0 & 150 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$R_2 \Leftrightarrow R_4$

$$\left[\begin{array}{ccccc|c} 1 & -1 & +1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 3 & 5 & 0 & 0 & 0 & 150 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$R_3 -$

$R_1 + R$

$R_3 - 8$

$-\frac{1}{3} \times R$

$R_1 + R_3$

$R_4 - R_3$

$R_5 \Leftrightarrow R$

$R_3 - 3R_1$
$$\left[\begin{array}{ccccc|c} 1 & -1 & +1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 8 & -3 & 0 & 0 & 150 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$R_1 + R_2$
$R_3 - 8R_2$
$$\left[\begin{array}{ccccc|c} 1 & 0 & +1 & 0 & 0 & 60 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & -3 & 0 & 0 & -330 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$\frac{-1}{3} \times R_3$
$$\left[\begin{array}{ccccc|c} 1 & 0 & +1 & 0 & 0 & 60 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & +1 & 0 & 0 & +110 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$R_1 + R_3$
$R_4 - R_3$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 0 & 110 \\ 0 & 0 & 0 & 1 & -1 & -110 \\ 0 & 0 & 0 & 1 & 0 & 75 \end{array}\right]$$

$R_5 \leftrightarrow R_4$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 0 & 110 \\ 0 & 0 & 0 & 1 & 0 & 75 \\ 0 & 0 & 0 & 1 & -1 & -110 \end{array}\right]$$

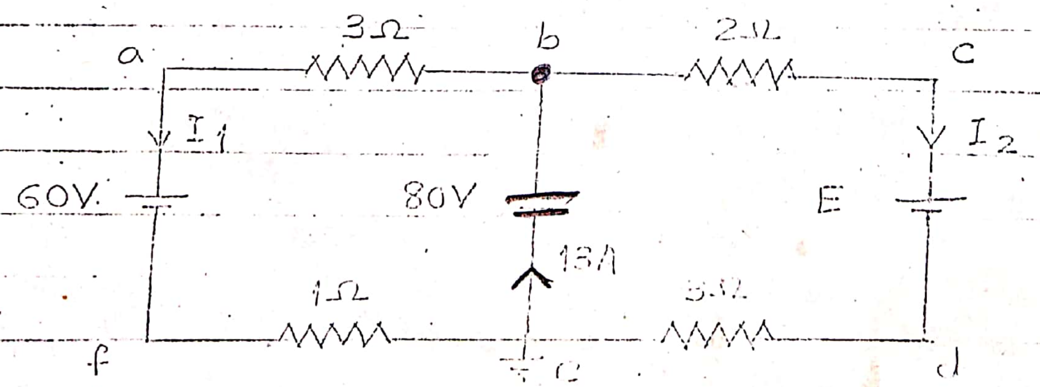
$R_5 - R_4$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 0 & 110 \\ 0 & 0 & 0 & 1 & 0 & 75 \\ 0 & 0 & 0 & 0 & -1 & -185 \end{array}\right]$$

$-1 \times R_5$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 0 & 110 \\ 0 & 0 & 0 & 1 & 0 & 75 \\ 0 & 0 & 0 & 0 & 1 & 185 \end{array}\right]$$

Hence,
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} -50 \\ 60 \\ 110 \\ 75 \\ 185 \end{bmatrix}$$

QUESTION # 5:

Determine unknown currents & voltage E.



SOLUTION:

Applying KCL at a single node:-

Node (b):

$$I_1 + I_2 = 13 \quad \text{--- (1)}$$

Applying KVL to two loops:-

Mesh abefa:

$$80 - 60 = (3+1) I_1$$
$$20 = 4 I_1$$
$$I_1 = 5 \quad \text{--- (2)}$$

Mesh bcdeb:

$$80 - E = (2+3) I_2$$
$$80 - E = 5 I_2 \quad \text{--- (3)}$$

since, $I_1 = 5$ A

now, equation (1) implies:

$$I_2 = 13 - I_1$$
$$I_2 = 13 - 5 = 8\text{ A}$$

now, equation becomes:-

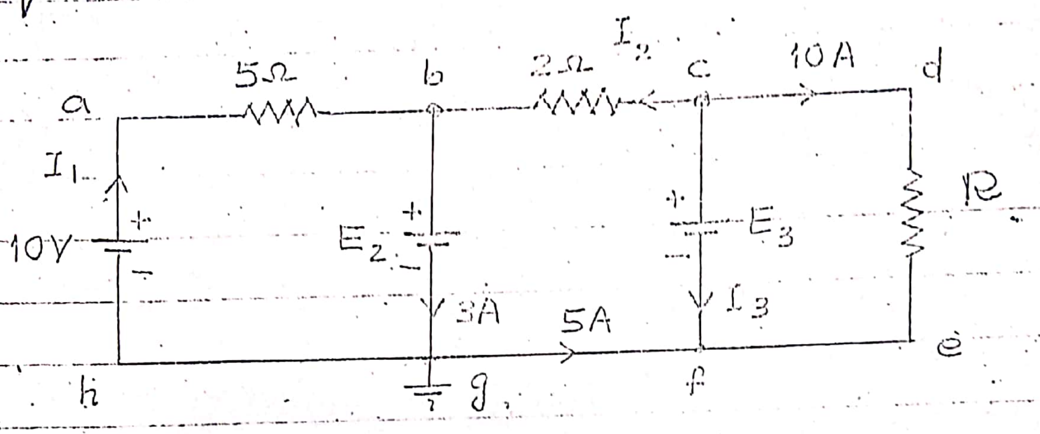
$$80 - E = 5(8)$$
$$80 - 40 = E$$
$$E = 40\text{ V}$$

Hence, $I_1 = 5$ A

$I_2 = 8$ A

$E = 40V$

QUESTION # 6 : Determine unknown quantities :



SOLUTION :

Applying KCL at nodes :-

Node (b) :

$$I_1 + I_2 = 3 \quad \text{——} \quad (1)$$

Node (c) :

$$I_2 + 10 = I_3$$

$$I_2 + I_3 = -10 \quad \text{——} \quad (2)$$

Node (f) :

$$I_3 + 3A + 5A$$

$$I_3 = -15 \quad \text{——} \quad (3)$$

In the matrix form

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ -15 \end{bmatrix}$$

USING GAUSS JORDAN METHOD

$R_1 - R_2$ $$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 13 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -15 \end{array}\right]$$

$R_1 + R_3$
$R_2 - R_3$ $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -15 \end{array}\right]$$

Hence, $$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -15 \end{bmatrix}$$

Now, applying KVL to 3 loops:-

Mesh abgha:

$$10 - E_2 = 5I_1$$

since, $I_2 = 5$:

$$10 - E_2 = 5(-2)$$
$$E_2 = 10 + 10$$
$$E_2 = 20V$$

Mesh bcfgh :

$$E_3 - E_2 = 2I_2$$

$$E_3 - 20 = 2(5)$$
$$E_3 = 30\ V$$

Now,

$$R = \frac{E_3}{10}$$

$$R = \frac{30}{10}$$

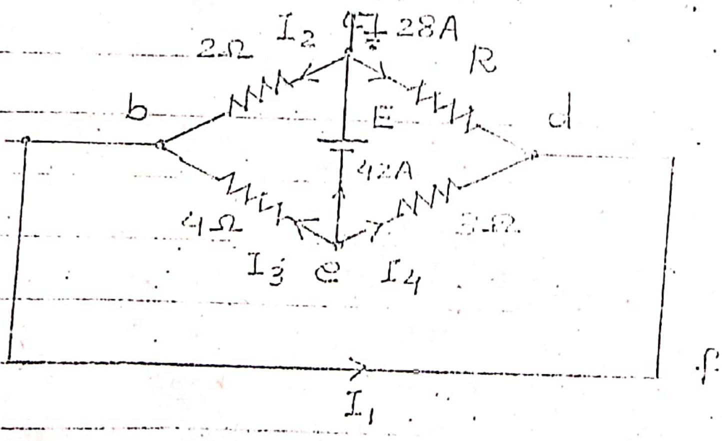
$$R = 3\ \Omega$$

Hence, other unknown quantities are:-

$$E_2 = 20V$$
$$E_3 = 30\ V$$
$$R = 3\ \Omega$$

QUESTION # 7:

Determine unknown quantities:



SOLUTION:

Applying KCL at nodes b, d, e:

Node (b):

$$I_1 = I_2 + I_3$$

$$I_1 - I_2 - I_3 = 0 \quad \text{--- (1)}$$

Node (d):

$$I_1 + 28 + I_4 = 0$$

$$I_1 + I_4 = -28 \quad \text{--- (2)}$$

Node (e):

$$I_3 + I_4 + 42 = 0$$

$$I_3 + I_4 = -42 \quad \text{--- (3)}$$

Applying KVL at loop abedfar

$$-4I_3 + 3I_4 = 0 \quad \text{--- (4)}$$

In the matrix form:-

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -28 \\ -42 \\ 0 \end{bmatrix}$$

USING GJ REDUCTION METHOD

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -28 \\ 0 & 0 & 1 & 1 & -42 \\ 0 & 0 & -4 & 3 & 0 \end{array}\right]$$

$R_2 - R_1$
$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -28 \\ 0 & 0 & 1 & 1 & -42 \\ 0 & 0 & -4 & +3 & 0 \end{array}\right]$$

$R_1 + R_2$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -28 \\ 0 & 1 & 1 & 1 & -28 \\ 0 & 0 & 1 & 1 & -42 \\ 0 & 0 & -4 & 3 & 0 \end{array}\right]$$

$R_2 - R_3$
$R_4 + 4R_3$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -28 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & 1 & -42 \\ 0 & 0 & 0 & 7 & -168 \end{array}\right]$$

$\frac{1}{7} \times R_4$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -28 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & 1 & -42 \\ 0 & 0 & 0 & 1 & -24 \end{array}\right]$$

$R_1 - R_4$
$R_3 - R_4$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & -24 \end{array}\right]$$

Hence,
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \\ -18 \\ -24 \end{bmatrix}$$

Now, considering loop bceb :-

$$E = 2I_2 - 4I_3$$
$$E = 2(14) - 4(-18)$$
$$E = 28 + 72$$
$$E = 100V$$

now, unknown resistance R, consider loop cdec :

$$R = \frac{E - 3(I_4)}{28}$$

$$R = \frac{100 - 2(-24)}{28}$$

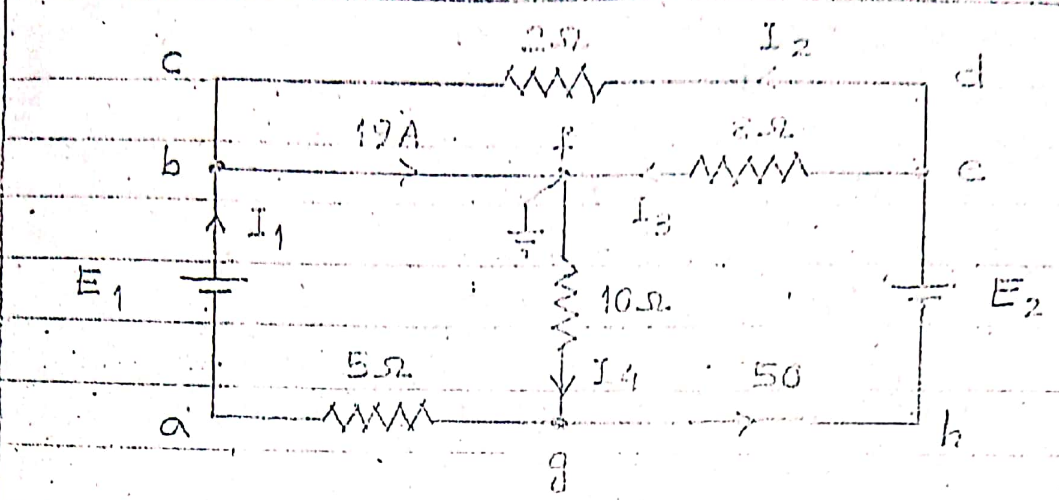
$$R = 28/28 = 1\Omega$$

Hence, $E = 100\,V$
$R = 1\Omega$

QUESTION # 8 :

Determine the unknown quantities :

→



SOLUTION:

Applying KCL at nodes b, e, g :-

Node (b):

$$I_1 + I_2 = 19A \quad \text{——— (1)}$$

Node (e):

$$I_2 + I_3 = 50 \quad \text{——— (2)}$$

Node (g):

$$I_1 + 50 = I_4$$

$$I_1 - I_4 = -50 \quad \text{——— (3)}$$

Applying KVL on loop bcdefb :-

$$2I_2 - 8I_3 = 0 \quad \text{——— (4)}$$

In the matrix form:-

$$\left[\begin{array}{cccc|c|c} 1 & 1 & 0 & 0 & I_1 & 19 \\ 0 & 1 & 1 & 0 & I_2 & 50 \\ 1 & 0 & 0 & -1 & I_3 & -50 \\ 0 & 2 & -8 & 0 & I_4 & 0 \end{array}\right]$$

USING GAUSS JORDAN REDUCTION METHOD:-

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 19 \\ 0 & 1 & 1 & 0 & 50 \\ 1 & 0 & 0 & -1 & -50 \\ 0 & 2 & -8 & 0 & 0 \end{array}\right]$$

$$R_3 - R_1 \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 19 \\ 0 & 1 & 1 & 0 & 50 \\ 0 & -1 & 0 & -1 & -69 \\ 0 & 2 & -8 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ R_4 - 2R_2 \\ R_3 + R_2 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -31 \\ 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & 1 & -1 & -19 \\ 0 & 0 & -10 & 0 & -100 \end{array}\right]$$

$$R_3 \leftrightarrow R_4 \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -31 \\ 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & -10 & 0 & -100 \\ 0 & 0 & 1 & -1 & -19 \end{array}\right]$$

$$\frac{-1}{10} \times R_3 \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -31 \\ 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -1 & -19 \end{array}\right]$$

$$\begin{array}{l} R_1+R_3 \\ R_2-R_3 \\ R_4-R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -21 \\ 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & -1 & -29 \end{array}\right]$$

Hi

$$-1 \times R_4 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -21 \\ 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & +29 \end{array}\right]$$

Hence,

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} -21 \\ 40 \\ 10 \\ +29 \end{bmatrix}$$

now, loop equation of loop abfga :-

$$E_1 = 5I_1 + 10 I_4$$
$$E_1 = 5(-21) + 10(+29)$$
$$E_1 = -105 + 290$$
$$E_1 = +185 \text{ V}$$

now, for E_2, consider loop fehgf :-

$$E_2 = 8I_3 + 10 I_4$$
$$E_2 = 8(10) + 10(29)$$
$$E_2 = 80 + 290$$
$$E_2 = 370 \text{ V}$$

$E_1 = 185 V$

$E_2 = 370 V$

2.5 Exercises

P-1 **1.** Which of the following can be transition matrices of a Markov process?

(a) $\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.8 & 0.5 & 0.7 \\ 0.0 & 0.2 & 0.2 \end{bmatrix}$

(c) $\begin{bmatrix} 0.55 & 0.33 \\ 0.45 & 0.67 \end{bmatrix}$ (d) $\begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.0 & 0.8 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$

P-2 **2.** Which of the following are probability vectors?

(a) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{1}{10} \\ \frac{2}{10} \end{bmatrix}$

In Exercises 3 and 4, determine a value of each missing entry, denoted by □, so that the matrix will be a transition matrix of a Markov chain. In some cases there may be more than one correct answer.

P-2 BS **3.** $\begin{bmatrix} \square & 0.4 & 0.3 \\ 0.3 & \square & 0.5 \\ \square & 0.2 & \square \end{bmatrix}$ **4.** $\begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.3 & \square & 0.5 \\ \square & \square & \square \end{bmatrix}$

P-4 BS **5.** Consider the transition matrix

$$T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

(a) If $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ to three decimal places.

H.W (b) Show that T is regular and find its steady-state vector.

6. Consider the transition matrix

$$T = \begin{bmatrix} 0 & 0.2 & 0.0 \\ 0 & 0.3 & 0.3 \\ 1 & 0.5 & 0.7 \end{bmatrix}.$$

(a) If

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

compute $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$, and $\mathbf{x}^{(4)}$ to three decimal places.

P-2 (b) Show that T is regular and find its steady-state vector.

7. Which of the following transition matrices are regular?

(a) $\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

H.W (c) $\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ H.W (d) $\begin{bmatrix} \frac{1}{4} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{2}{5} & \frac{1}{2} \end{bmatrix}$

H.W **8.** Show that each of the following transition matrices reaches a state of equilibrium.

(a) $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$

P-2 BS (c) $\begin{bmatrix} \frac{1}{3} & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.0 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$

9. Let

$$T = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

(a) Show that T is not regular.

(b) Show that $T^n\mathbf{x} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for any probability vector $\mathbf{x}$.

Thus a Markov chain may have a unique steady-state vector even though its transition matrix is not regular.

P-4 BS **10.** Find the steady-state vector of each of the following regular matrices.

(a) $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix}$ H.W (b) $\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.9 \end{bmatrix}$

H.W (c) $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{2}{3} \\ \frac{3}{4} & 0 & 0 \end{bmatrix}$ H.W (d) $\begin{bmatrix} 0.4 & 0.0 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.6 \end{bmatrix}$

P-3 **11.** (*Psychology*) A behavioral psychologist places a rat each day in a cage with two doors, A and B. The rat can go through door A, where it receives an electric shock, or through door B, where it receives some food. A record is made of the door through which the rat passes. At the start of the experiment, on a Monday, the rat is equally likely to go through door A as through door B. After going through door A, and receiving a shock, the probability of going through the same door on the next day is 0.3. After going through door B, and receiving food, the probability of going through the same door on the next day is 0.6.

(a) Write the transition matrix for the Markov process.

(b) What is the probability of the rat going through door A on Thursday (the third day after starting the experiment)?

(c) What is the steady-state vector?

12. (*Business*). The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine, while others do not. From this mailing list, 60% of those who already subscribe will subscribe again, while 25% of those who do not now subscribe will subscribe.

(a) Write the transition matrix for this Markov process.

(b) On the last letter it was found that 40% of those receiving it, ordered a subscription. What percentage of those receiving the current letter can be expected to order a subscription?

13. (*Sociology*) A study has determined that the occupation of a boy, as an adult, depends upon the occupation of his father and is given by the following transition matrix, where P = professional, F = farmer, and L = laborer.

$$\begin{array}{cc} & \text{Father's occupation} \\ & \begin{array}{ccc} \quad P & \quad F & \quad L \end{array} \\ \text{Son's occupation} \begin{array}{c} P \\ F \\ L \end{array} & \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

Thus the probability that the son of a professional will also be a professional is 0.8, and so on.

(a) What is the probability that the grandchild of a professional will also be a professional?

HW (b) In the long run, what proportion of the population will be farmers?

14. (*Genetics*) Consider a plant that can have red flowers (R), pink flowers (P), or white flowers (W), depending upon the genotypes RR, RW, and WW. When we cross each of these genotypes with genotype RW, we obtain the transition matrix

$$\begin{array}{cc} & \text{Flowers of parent plant} \\ & \begin{array}{ccc} \quad R & \quad P & \quad W \end{array} \\ \text{Flowers of offspring plant} \begin{array}{c} R \\ P \\ W \end{array} & \begin{bmatrix} 0.5 & 0.25 & 0.0 \\ 0.5 & 0.50 & 0.5 \\ 0.0 & 0.25 & 0.5 \end{bmatrix} \end{array}.$$

Suppose that each successive generation is produced by crossing only with plants of RW genotype. When the process reaches equilibrium, what percentage of the plants will have red, pink, or white flowers?

HW **15.** (*Mass Transit*) A new mass transit system has just gone into operation. The transit authority has made studies that predict the percentage of commuters who will change to mass transit (M) or continue driving their automobile (A). The following transition matrix has been obtained:

$$\begin{array}{cc} & \text{This year} \\ & \begin{array}{cc} \quad M & \quad A \end{array} \\ \text{Next year} \begin{array}{c} M \\ A \end{array} & \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}.$$

Suppose that the population of the area remains constant, and that initially 30% of the commuters use mass transit and 70% use their automobiles.

(a) What percentage of the commuters will be using the mass transit system after 1 year? After 2 years?

(b) What percentage of the commuters will be using the mass transit system in the long run?

Theoretical Exercise

T.1. Is the transpose of a transition matrix of a Markov chain also a transition matrix of a Markov chain? Explain.

MATLAB Exercises

The computation of the sequence of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ *as in Examples 3 and 4 can easily be done using* MATLAB *commands. Once the transition matrix T and initial state vector* $\mathbf{x}^{(0)}$ *are entered into* MATLAB, *the state vector for the kth observation period is obtained from the* MATLAB *command*

T^k * x

ML.1. Use MATLAB to verify the computations of state vectors in Example 3 for periods 1 through 5.

ML.2. In Example 4, if the initial state is changed to

$$\begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix},$$

determine $\mathbf{x}^{(5)}$.

ML.3. In MATLAB, enter **help sum** and determine the action of command **sum** on an $m \times n$ matrix. Use command **sum** to determine which of the following are Markov matrices.

(a) $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

(b) $\begin{bmatrix} 0.5 & 0.6 & 0.7 \\ 0.3 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.0 \end{bmatrix}$

(c) $\begin{bmatrix} 0.66 & 0.25 & 0.125 \\ 0.33 & 0.25 & 0.625 \\ 0.00 & 0.50 & 0.250 \end{bmatrix}$

Ex 2.5

The transition Probabilities arranged in $n \times n$ matrix $T=[t_{ij}]$ is called transition matrix of markov chain

where

$$t_{1j}+t_{2j}+t_{3j}+\cdots+t_{nj}=1 \quad \text{—— (1)}$$

entries in each column of T are non-negative & add upto 1 as in (1)

(a) not transition b/c $a_{11}+a_{21}=0.3+0.4 \neq 1$

(b) transition as $0.2+0.8+0.0=1$
$0.3+0.5+0.2=1$
$0.1+0.7+0.2=1.$

(c) transition because $0.55+0.45=1$
$0.33+0.67=1.$

(d) Not transition because All each column are not equal to 1.

Probability vector

The vector $u=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ is called Probability vector if

$u_1+u_2+\cdots u_n=1$

Q2 (a) $\frac{1}{2}+\frac{1}{3}+\frac{2}{3} = \frac{3+2+4}{6} = 9/6 \neq$ not P.V

(b) $0+1+0=1$ Prob: vector

(c) $\frac{1}{4}+\frac{1}{6}+\frac{1}{3}+1/4 = \frac{3+2+4+3}{12} = \cancel{12}/\cancel{12} = 1$ Prob:V.

(d) $\frac{1}{5}+\frac{2}{5}+\frac{1}{10}+\frac{2}{10} = \frac{2+4+1+2}{10} = \frac{9}{10} \neq 1$ not P.V.

Q3

$$\begin{bmatrix} \square & 0.4 & 0.3 \\ 0.3 & \square & 0.5 \\ \square & 0.2 & \square \end{bmatrix}$$

For $j=1$

$a_{11}+a_{21}+a_{31}=1$

$\square + 0.3 + \square = 1$

$a_{11}+a_{31} = 1-0.3 = 0.7$

$a_{11}+a_{31} = 0.7$ So there are more than Possible values. ((0.5+0.2) etc).

For $j=2$ $a_{12}+a_{22}+a_{31}=1$

$0.4 + \boxed{a_{22}} + 0.2 = 1$

$a_{22} = 1-0.2-0.4 = 0.4$ =, $\boxed{a_{22}=0.4}$

Ex 2.5

For $j=3$

$$a_{13}+a_{23}+a_{33}=1$$

$$0.3+0.5+a_{33}=1$$

$$a_{33}=1-0.3-0.5=0.2$$

$$\boxed{a_{33}=0.2}$$

Q4 $j=1$

$$0.2+0.3+a_{13}=1$$

$$a_{13}=1-0.2-0.3=0.5$$

$$\boxed{a_{13}=0.5}$$

$j=2$ $\quad 0.1+a_{22}+a_{32}=1$

$a_{22}+a_{32}=1-0.1=0.9$ (so there are more

them possible values: $\boxed{a_{22}+a_{32}=0.9}$

$j=3$

$$0.3+0.5+a_{33}=1$$

$$a_{33}=1-0.3-0.5=0.2$$

$$\boxed{a_{33}=0.2}$$

Q5 $T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$

(a) If $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x^{(1)} = Tx^{(0)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}.$$

$$x^{(2)} = Tx^{(1)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.61 \\ 0.39 \end{bmatrix}.$$

$$x^{(3)} = Tx^{(2)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.61 \\ 0.39 \end{bmatrix} = \begin{bmatrix} 0.583 \\ 0.417 \end{bmatrix}.$$

(b) Show that T is regular.

$$\text{Now } T^2 = T.T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{bmatrix}.$$

As all enteries are non-zero so it's regular.

T is regular if all enteries in some Power of T are Positive.

$0.575 + 0.425 = 1.$

<u>Steady state vector</u>

$$x^{(4)} = Tx^{(3)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.583 \\ 0.417 \end{bmatrix} = \begin{bmatrix} 0.575 \\ 0.425 \end{bmatrix}$$

Ex 2.5

$$x^{(5)} = Tx^{(4)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.575 \\ 0.425 \end{bmatrix} = \begin{bmatrix} 0.573 \\ 0.428 \end{bmatrix}.$$

$$x^{(6)} = Tx^{(5)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.573 \\ 0.428 \end{bmatrix} = \begin{bmatrix} 0.572 \\ 0.429 \end{bmatrix}.$$

$$x^{(7)} = Tx^{(6)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.572 \\ 0.429 \end{bmatrix} = \begin{bmatrix} 0.572 \\ 0.429 \end{bmatrix}$$

Same.

It's steady state vector.

Q6 is similarly to Q5

Q7 (a) $T^2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}.$

So no ~~entry~~ entry is 0 in T^2 so it's regular.

(b) $T^2 = T.T = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} 1/4 & 0 & 0 \\ 1/4 & 1 & 3/4 \\ 1/2 & 0 & 1/4 \end{bmatrix}$$

So all entries in T^2 are not non-zero so its not regular.

* Q (c) + d Part are similarly to a + b Part *

Q9 (a)

$$T^2 = T.T = \begin{bmatrix} 1/2 & 0 \\ 1/2 & \phi \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 3/4 & 1 \end{bmatrix}.$$

it's not regular because $t_{22} = 0$.

(b) let $x^{(0)} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. (standard)

$$x^{(1)} = Tx^{(0)} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}.$$

$$x^{(2)} = Tx^{(1)} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.95 \end{bmatrix}$$

$$x^{(3)} = Tx^{(2)} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.05 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 0.025 \\ 0.975 \end{bmatrix}$$

$$x^{(4)} = Tx^{(3)} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.025 \\ 0.975 \end{bmatrix} = \begin{bmatrix} 0.0125 \\ 0.9875 \end{bmatrix}$$

$$x^{(5)} = Tx^{(4)} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.013 \\ 0.99 \end{bmatrix} \to \begin{bmatrix} 0.00 \\ 1 \end{bmatrix}$$

Ex 2.5

Q10 (a) $T = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$.

$T^2 = \begin{bmatrix} 0.333 & 0.5 \\ 0.6667 & 0.5 \end{bmatrix}$

$T^3 = \begin{bmatrix} 0.444 & 0.4166 \\ 0.5555 & 0.5833 \end{bmatrix}$

$T^4 = \begin{bmatrix} 0.4259 & 0.4305 \\ 0.574 & 0.5694 \end{bmatrix}$.

$T^5 = \begin{bmatrix} 0.429 & 0.4282 \\ 0.5709 & 0.5717 \end{bmatrix}$.

$T^6 = \begin{bmatrix} 0.4284 & 0.4286 \\ 0.5715 & 0.5719 \end{bmatrix}$

$T^7 = \begin{bmatrix} 0.4285 & 0.4285 \\ 0.5714 & 0.5714 \end{bmatrix}$

same.

$T^8 = \begin{bmatrix} 0.4285 & 0.4285 \\ 0.5714 & 0.5714 \end{bmatrix}$

Steady state.