Lecture: 02 Tues, 21st Aug, 2008

1st Order Diff Eq:-

1st order and 1st degree differential equations are of nine types.

① Separable Eq.
② Exact Eq.
③ Not Exact Eq.
④ Linear Eq.
⑤ Non-Linear Eq.
⑥ Homogeneous Eq.
⑦ Equation Reducible to hom. eq.
⑧ Bernoulli's Eq.
⑨ Orthogonal Eq.

General form of 1st order.

General form of 1st degree / order diff equation is:

$$\boxed{Mdx + Ndy = 0}$$

Exercise 1.1

Solve the following D.E

(i) $y' = x^2$
(ii) $y' = \sin 3x$
(iii) $y' = x^4$
(iv) $y' = xe^{-x^2}$

(i) Solutions:

$\frac{dy}{dx} = x^2 \Rightarrow$ (separable)

$dy = x^2 dx$

$\int dy = \int x^2 dx$

$y = \frac{x^3}{3} + C$

(ii) $y' = \sin 3x$

$\frac{dy}{dx} = \sin 3x$

$dy = \sin 3x \, dx$

$\int dy = \int \sin 3x \, dx$

$y = \frac{-\cos 3x}{3} + C$

$y'' = x^{-4}$

$\frac{d^2y}{dx^2} = x^{-4}$

V.S

$\cancel{\frac{dy}{dx} \frac{dy}{dx} = x}$

$d^2y = x^{-4}\, dx^2$

$\int d^2y = \int x^{-4}\, dx \cdot dx$

$dy = \left(\frac{x^{-3}}{-3} + c_1\right) dx$

$\int dy = -\frac{1}{3} \int (x^{-3} + c_1)\, dx$

$y = -\frac{1}{3}\left(\frac{x^{-2}}{-2}\right) + c_1x + c_2$

$y = \frac{x^{-2}}{6} + c_1x + c_2$

(4) $y' = xe^{-x^2}$

$\frac{dy}{dx} = xe^{-x^2}$

$dy = xe^{-x^2}\, dx$

$\int dy = \int x e^{-x^2}\, dx$

Let,
$-x^2 = t$
$-2x\, dx = dt$
$x\, dx = -\frac{1}{2} dt$

$y = \int -\frac{1}{2} e^t\, dt$

$= -\frac{1}{2} \int e^t\, dt$

$= -\frac{1}{2} e^t + c$

$y = -\frac{1}{2} e^{-x^2} + c$

State the order of D.E & verify that the given function is the solution of corresponding D.E when (a, b, c are constt).

(5) $y' + y = x^2 - 2$, $y = ce^{-x} + x^2 - 2x$

(6) $y'' + y = 0$, $y = a\cos x + b\sin x$

(7) $y''' = e^x$, $y = e^x + ax^2 + bx + c$

(8) $y'' + 2y' + 2y = 0$, $y = e^{-x}(a\cos x + b\sin x)$

(9) $x + yy' = 0$, $x^2 + y^2 = 1$.

<u>Solutions:</u>

(5) <u>1st part</u>.

order $\rightarrow$ first order Eq.

Given that,

$y' + y = x^2 - 2$ — (i)

$y = ce^{-x} + x^2 - 2x$ — (ii)

Diff eq (ii) w.r.t x

$y' = -ce^{-x} + 2x - 2$

Eq (i) $\Rightarrow$ $-\cancel{ce^x} + 2x - 2 + \cancel{ce^x} + x^2 - 2x = x^2 - 2$

$-2 + x^2 = x^2 - 2$

$-2 + x^2 - x^2 + 2 = 0$

$0 = 0$

6 $y'' + y = 0$, $y = a\cos x + b\sin x$.

Solution

Given that,

$y'' + y = 0$ —(i)

$y = a\cos x + b\sin x$ —(ii)

diff eq (ii) w.r.t x

$y' = -a\sin x + b\cos x$

$y'' = -a\cos x - b\sin x$

Eq (i) $\Rightarrow$

$y'' + y = 0$

$-a\cos x - b\sin x + a\cos x + b\sin x = 0$

$\cancel{a\cos x} + \cancel{b\sin x} = 0$

$0 = 0$

Hence the given ~~sol~~ fcn is the sol of the given order "2" DE

7 $y''' = e^x$, $y = e^x + ax^2 + bx + c$

Sol-

Given that,

$y''' = e^x$ --- (i)

$y = e^x + ax^2 + bx + c$ —(ii)

diff eq (ii) w.r.t x:

$y' = e^x + 2ax + b$

$y'' = e^x + 2a$

$y''' = e^x$

put in (i) $\Rightarrow$ $e^x = e^x$

(3) $y'' + 2y' + 2y = 0;\ y = e^{-x}(a\cos x + b\sin x)$

Sol:-

$y = e^{-x}(a\cos x + b\sin x)$ — (i)

$y'' + 2y' + 2y = 0$ — (ii)

diff (i) w.r.t x

$y = e^{-x}(a\cos x + b\sin x)$

$= e^{-x}\frac{d}{dx}(a\cos x + b\sin x) + (a\cos x + b\sin x)\frac{d}{dx}e^{-x}$

$= e^{-x}(-a\sin x + b\cos x) + (a\cos x + b\sin x)(-e^{-x})$

$= e^{-x}(-a\sin x + b\cos x - a\cos x - b\sin x)$

Again diff:-

$= e^{-x}\frac{d}{dx}(-a\sin x + b\cos x - a\cos x - b\sin x) + (-a\sin x + b\cos x - a\cos x - b\sin x)\frac{d}{dx}e^{-x}$

$= e^{-x}(-a\cos x - b\sin x + a\sin x - b\cos x) - e^{-x}(-a\sin x + b\cos x - a\cos x - b\sin x)$

$= e^{-x}(-\cancel{a\cos x} - \cancel{b\sin x} + a\sin x - b\cos x + a\sin x - b\cos x + \cancel{a\cos x} + \cancel{b\sin x})$

$= e^{-x}(2a\sin x - 2b\cos x)$

$y'' = 2e^{-x}(a\sin x - 2b\cos x)$

Taking (ii)

$y'' + 2y' + 2y = 0$

$2e^{-x}a\sin x - 2e^{-x}b\cos x + 2(-e^{-x}a\sin x + e^{-x}b\cos x - e^{-x}a\cos x - e^{-x}b\sin x) + 2(e^{-x}a\cos x + e^{-x}b\sin x)$

$\cancel{2e^{-x}a\sin x} - \cancel{2e^{-x}b\cos x} - \cancel{2e^{-x}a\sin x} + \cancel{2e^{-x}b\cos x} - \cancel{2e^{-x}a\cos x} - \cancel{2e^{-x}b\sin x} + \cancel{2e^{-x}a\cos x} + \cancel{2e^{-x}b\sin x} = 0$

$0 = 0$

(a) $x + yy' = 0$, $x^2+y^2=1$

Given that,

$x^2+y^2=1$ —— (i)

$x+yy'=0$ —— (ii)

$\rightarrow$ $x^2+y^2=1$

$y^2 = 1-x^2$

$2y\frac{dy}{dx} = 1-2x$

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Sol- Given that,

diff eq (i),

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = \frac{-2x}{2y} \Rightarrow \boxed{y' = -x/y}$

put in (ii)

$x + yy' = 0$

$x + y\left(-\frac{x}{y}\right) = 0 \Rightarrow 0=0$

(b) what happen with the D-E in problem (a) if.

$x^2 - y^2 = 1$

$2x - 2yy' = 0$

$2x = 2yy'$

$2yy' = 2x$

$y' = \frac{2x}{2y} = \frac{x}{y}$

eq $\Rightarrow$ (ii) $x - y\left(\frac{x}{y}\right) = 0$

$x - x = 0$

$0 = 0$

If d-E is $x^2-y^2=1$ then the other eq becomes $x - yy' = 0$.

If ∞ replace by 2

Sol:
No change.

Initial Value Problems:

Verify that y is a solution of DE find 'c' so that the resulting particular solution the given initial condition. Graph the solution

(12) $x^3 + y^3 y' = 0$, $x^4 + y^4 = c$ $(y>0)$
$y = 1$ when $x = 0$

(13) $y' + 2y = 2.8$, $y = ce^{-2x} + 1.4$,
$y = 1.0$ when $x = 0$.

(14) $xy' = 3y$, $y = cx^3$, $y = 16$, $x = -4$.

(15) $yy' = 2x$, $y^2 - 2x^2 = c$ $(c>0)$, $y(1) = \sqrt{3}$

(16) $y' = y\tan x$, $y = c\sec x$, $y(0) = \pi/2$

(17) $4yy' + x = 0$, $x^2 + 4y^2 = c$ $(y>0)$, $y(2) = 1$.

Solutions:

Given that

(12) $x^3 + y^3 y' = 0$ — (i), $x^4 + y^4 = c$ — (ii) $(y>0)$, $y = 1$, $x = 0$

1st part: diff eq (ii) w.r.t x:

$$4x^3 + 4y^3 y' = 0$$

$$4y^3 y' = -4x^3$$

$$y' = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3} \Rightarrow \boxed{y' = \frac{-x^3}{y^3}}$$

eq (i): $x^3 + y^3 y' = 0$

$$x^3 + y^3\left(\frac{-x^3}{y^3}\right) = 0$$

$$x^3 - x^3 = 0$$

$$0 = 0$$

Hence the given f-n is the solution of corresponding D.E.

2nd part:

To find C:

put $x = 0$ and $y = 1$ in eq (ii)

$x^4 + y^4 = c$

$(0)^4 + (1)^4 = c \Rightarrow \boxed{c = 1}$

eq (ii) $\Rightarrow x^4 + y^4 = 1$

3rd part.

$y = (1 - x^4)^{1/4}$

To plot graph:

$x^4 + y^4 = 1 \quad (y > 0)$

Homework

Ex: 1.1

Ques: 13 → 17

(13) $y' + 2y = 2.8$; $y = ce^{-2x} + 1.4$, $y = 1.0$ when $x = 0$

Solution:

1st part:

$y' + 2y = 2.8 \quad \dots (i)$

$y = ce^{-2x} + 1.4 \quad \dots (ii)$

diff (ii) w.r.t x.

$\frac{dy}{dx} = \frac{d}{dx}(ce^{-2x}) + \frac{d}{dx}(1.4)$

$y' = -2ce^{-2x}$

put y' in (i)

$y' + 2y = 2.8$

$-2ce^{-2x} + 2y = 2.8$

$-2ce^{-2x} + 2(ce^{-2x} + 1.4) = 2.8$

$-\cancel{2ce^{-2x}} + \cancel{2ce^{-2x}} + 2.8 = 2.8$

Hence the given f-n is the solution of corresponding diff eq.

2nd part:

put $y = 1.0$ and $x = 0$ in (ii)

$y = ce^{-x} + 1.4$

$1.0 = ce^{-(0)} + 1.4 \Rightarrow 1 = ce^{0} + 1.4$

$1 = c + 1.4 \Rightarrow 1 - 1.4 = c \Rightarrow \boxed{c = -0.4}$

Hence,

$y = -0.4e^{-x} + 1.4$

(14) $xy' = 3y$, $y = cx^3$, $y = 16$ when $x = -4$

Solution:

$xy' = 3y$ ...(i) $\quad y = cx^3$ ...(ii)

diff eq (ii) w.r.t x.

$y = cx^3$

$y' = 3cx^2$

put y' and y in (i)

$x(3cx^2) = 3(cx^3) \Rightarrow 3cx^3 = 3cx^3$

Hence the given fn is the solution of corresponding D.E.

2nd part:

$y = cx^3$

put $x = -4$ & $y = 16$

$16 = c(-4)^3$

$16 = c(-64) \Rightarrow c = \frac{-16}{64} = -0.25$

Hence $y = -0.25x^3$

(15) $yy' = 2x$, $y^2 - 2x^2 = c\ (y > 0)$; $y(1) = \sqrt{3}$

Solution:

1st part:

$yy' = 2x$ ...(i) $\quad y^2 - 2x^2 = c$ ...(ii)

$y^2 - 2x^2 = c$

$2yy' - 4x = 0$

$2yy' = 4x \Rightarrow y' = \frac{4x}{2y} \Rightarrow y' = \frac{2x}{y}$

Now (i) ⇒ $yy' = 2x$.

$y\left(\frac{2x}{y}\right) = 2x \Rightarrow 2x = 2x$.

Hence, the given fn is the solution of corresponding D.E.

2nd part:

put $x=1$ & $y=\sqrt{3}$.

$y^2 - 2x^2 = c$

$(\sqrt{3})^2 - 2(1)^2 = c \Rightarrow 3 - 2 = c \Rightarrow \boxed{c=1}$

(16) $y' = y\tan x$, $y = c\sec x$, $y(0) = \pi/2$

Sol.

(1st-part)

$y' = y\tan x$ ...(i)

$y = c\sec x$ ...(ii)

(ii) ⇒ $y = c\sec x$.

$y' = c\tan x\sec x$.

(i) ⇒ $y' = y\tan x$

$c\tan x\sec x = c\sec x\tan x$

Hence, the given fn is the solution of corresponding D.E.

(2nd-part)

put $x=0$ & $y=\pi/2$

$y = c\sec x$

$\pi/2 = c\sec(0)$

$\pi/2 = c\frac{1}{\cos 0} \Rightarrow \boxed{c = \pi/2}$

(17) $4yy' + x = 0$, $x^2 + 4y^2 = c$ $(y>0)$, $y(2) = 1$

Solution.

(1st-part):

$4yy' + x = 0$ ...(i) $x^2 + 4y^2 = c$ ...(ii)

(ii) ⇒ $x^2 + 4y^2 = c$

$2x + 8yy' = 0$

$8yy' = -2x \Rightarrow y' = -2x/8y = -x/4y$.

$4yy' + x = 0$

$4y\left(\frac{-x}{4y}\right) + x = 0 \Rightarrow -x + x = 0 \Rightarrow 0 = 0$

Hence, the given fun is the solution of corresponding D.E.

(2nd part)

put $x = 2$ and $y = 1$.

$x^2 + 4y^2 = c$

$(2)^2 + 4(1)^2 = c \Rightarrow 4 + 4 = c \Rightarrow \boxed{c = 8}$

Mon, 12th Sept, 2008

Lecture : 03

Exercise 1.3

Ques — Ques → solvable and Ques 8, separable

Ques — Ques → Reduce to variable sep.

Ques) $yy' + 25x = 0$

Solution:

$yy' = -25x$

$y\frac{dy}{dx} = -25x$ (variable sep)

$y\,dy = -25x\,dx$

Integration,

$\int y\,dy = -25\int x\,dx$

$\frac{y^2}{2} + c_1 = -25\frac{x^2}{2} + c_2$

$\frac{y^2}{2} = -25x^2/2 + c_3$

$y^2 = -25x^2 + 2c_3$

$\boxed{y^2 = -25x^2 + c}$ $2c_3 = c$