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1) Investment Opportunity Set

I have selected Sec 3 and Sec 7 for the investment opportunity set, and their respective expected returns(r), Standard deviations (σ), and Correlation coefficient (ρ) are as below.

Ticker	Expected Return E(r)	Standard Deviation (σ)	Correlation Coefficient (ρ)
Sec 3	4.64%	7.65%	0.09
Sec 7	3.89%	6.20%	

Let's assume the investment weights in Sec 3 and Sec 7 are w_3 and w_7 respectively.

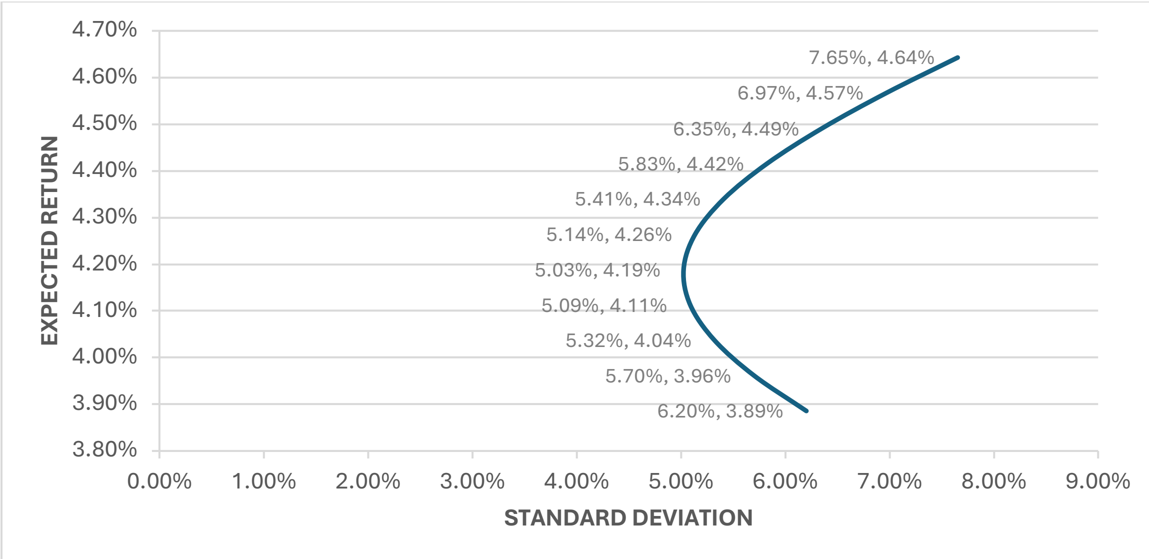
Such that, total weight (w_3 + w_7) = 1

Then, the Expected Return of portfolio $r_p = w_3 r_3 + w_7 r_7$,

and the Standard Deviation of the Portfolio $\sigma_p = \sqrt{w_3^2\sigma_3^2 + w_7^2\sigma_7^2 + 2w_3 w_7 \sigma_3 \sigma_7 \rho_{3,7}}$

To obtain the investment opportunity set, we assume different investment weights between Sec 3 and Sec 7 and calculate the expected return and standard deviations for each weight combination, as shown below.

Investment in Sec 3 (w_3)	Investment in Sec 7 (w_7)	The expected return of portfolio (r_p)	Standard Deviation of the Portfolio (σ_p)
100%	0%	4.64%	7.65%
90%	10%	4.57%	6.97%
80%	20%	4.49%	6.35%
70%	30%	4.42%	5.83%
60%	40%	4.34%	5.41%
50%	50%	4.26%	5.14%
40%	60%	4.19%	5.03%
30%	70%	4.11%	5.09%
20%	80%	4.04%	5.32%
10%	90%	3.96%	5.70%
0%	100%	3.89%	6.20%



2) Minimum Variance portfolio

A minimum variance portfolio is nothing but a portfolio in which the investment weight's combination of the risky assets has the lowest risk out of all the combinations of investments opportunity set.

If we observe the graph of the investment opportunity set for Sec 3 and Sec 7 above, the portfolio risk starts to decrease as we diversify the portfolio from Sec 7 to Sec 3; however, it starts increasing again when the weight of investment in Sec 3 keeps rising. Along the curve, the leftmost point is the investment combination with the minimum possible variance, and that portfolio is called the **Minimum variance portfolio**.

To calculate the expected return and standard deviation of the minimum variance portfolio, we first need to calculate the weight of the investment at which the portfolio has a minimum variance, as shown below.

$$w_{Min}(D) = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2\,Cov(r_D, r_E)} \quad (\because Cov(r_D, r_E) = \sigma_D \sigma_E \rho_{DE})$$

Substituting the values in the above equation

$$w_{Min}(Sec\ 3) = \frac{\sigma_7^2 - Cov(r_3, r_7)}{\sigma_3^2 + \sigma_7^2 - 2\,Cov(r_3, r_7)}$$

Wherein, σ_7^2 is the variance of sec 7, σ_3^2 is the variance of Sec 3 and $Cov(r_3, r_7)$ is covariance between sec 3 and Sec 7

$$w_3 = \frac{0.062^2 - 0.062 * 0.0765 * 0.09}{0.0765^2 + 0.062^2 - 2 * 0.062 * 0.0765 * 0.09}$$

$$w_3 = 0.3864$$

$$= 38.64\%$$

$$w_7 = 1 - 0.38.64 \quad (\because w_7 + w_3 = 1)$$

$$w_7 = 61.36\%$$

So, the weights Sec 3 and Sec 7 in the minimum variance portfolio are 38.64% and 61.36% respectively.

Expected Return of portfolio $r_p = w_3 r_3 + w_7 r_7$ (from question 1)

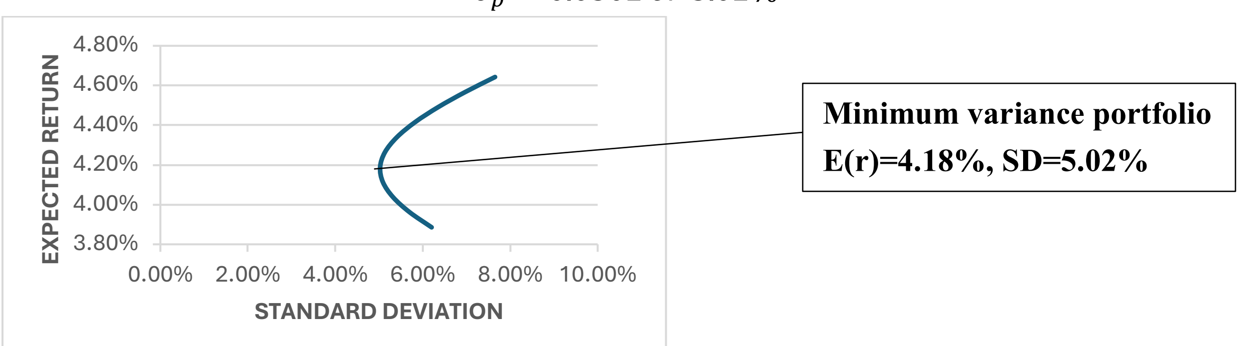
$$r_p = 0.3864 * 0.0464 + 0.6136 * 0.0389$$

$$r_p = 0.0418 \; or \; 4.18\%$$

Standard Deviation of the Portfolio $\sigma_p = \sqrt{w_3^2\sigma_3^2 + w_7^2\sigma_7^2 + 2w_3w_7\sigma_3\sigma_7\rho_{3,7}}$ (from question 1)

$$\sigma_p = \sqrt{0.3864^2 * 0.0765^2 + 0.6136^2 * 0.062^2 + 2 * 0.3864 * 0.6136 * 0.0765 * 0.062 * 0.09}$$

$$\sigma_p = 0.0502 \; or \; 5.02\%$$



Hence, the expected return E(r) of the **Minimum variance Portfolio** is **4.18%** approximately , and the Standard deviationσ_p of the **Minimum variance Portfolio** is **5.02%** approximately.

3) Efficiency frontier

I have selected Sec 1, Sec 2, and Sec 7, and the respective returns(r), Standard deviations (σ), and Correlation coefficients (ρ) matrix are as below.

Ticker	**Expected Return E(r)**	**Standard Deviation (σ)**
Sec 1	6.43%	9.35%
Sec 2	9.46%	15.38%
Sec 6	2.89%	2.06%

Correlation Matrix (ρ)			
	Sec 1	Sec 2	Sec 6
Sec 1	1	0.12	0.18
Sec 2	0.12	1	0.02
Sec 6	0.18	0.02	1

Let's assume the investment weights in Sec 1, Sec 2, and Sec 6 are w_1, w_2 and w_6 respectively.

Such that, total weight ($w_1 + w_2$ + w_6) = 1

Then, the Expected Return of the portfolio $r_p = w_1 r_1 + w_2 r_2 + w_6 r_6$

To calculate the Standard Deviation of the Portfolioσ_p, we need to prepare a covariance matrix as below

Covariance Matrix				
		w_1	w_2	w_6
		Sec 1	Sec 2	Sec 6
w_1	Sec 1	$Cov(r_1, r_1)$	$Cov(r_1, r_2)$	$Cov(r_1, r_6)$
w_2	Sec 2	$Cov(r_2, r_1)$	$Cov(r_2, r_2)$	$Cov(r_2, r_6)$
w_6	Sec 6	$Cov(r_6, r_1)$	$Cov(r_6, r_2)$	$Cov(r_6, r_6)$

$$\begin{aligned}\sigma_p^2 = w_1^2 Cov(r_1, r_1) + w_1 w_2 Cov(r_1, r_2) + w_1 w_3 Cov(r_1, r_6) + w_2^2 Cov(r_2, r_2) \\ + w_2 w_1 Cov(r_2, r_1) + w_2 w_3 Cov(r_2, r_6) + w_6^2 Cov(r_6, r_6) + w_3 w_1 Cov(r_6, r_1) \\ + w_3 w_2 Cov(r_6, r_2)\end{aligned}$$

Summing up all the weighted covariances, we get the variance of the total portfolio.

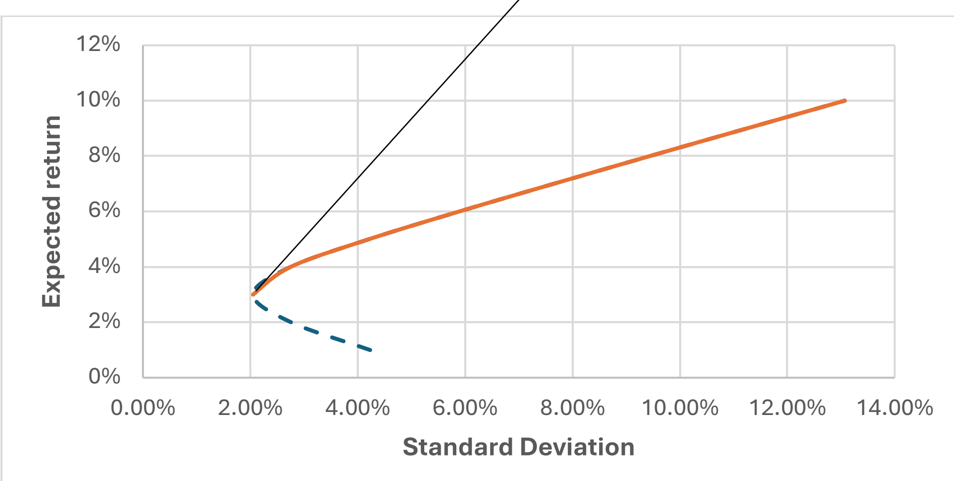
To obtain the efficiency frontier, we first need to calculate the lowest possible portfolio risk for multiple random target returns.

i.e., $\sigma^2_{p\,Min} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(r_i, r_j)$ at any target return r_t.

The lowest possible portfolio risks for some target returns calculated using MS Excel solver are as follows.

Target return(r_t)	**Standard deviation(σ_p)**	**Standard deviation(σ_p)**
1%	4.23%	
2%	2.76%	
3%	2.05%	2.05%
4%	2.75%	2.75%
5%	4.21%	4.21%
6%	5.89%	5.89%
7%	7.65%	7.65%
8%	9.44%	9.44%
9%	11.25%	11.25%
10%	13.07%	13.07%

→ Global Minimum Variance portfolio ($r_t = 3\%, \sigma_p = 2.05\%$)



If we observe the graph above, the portfolio risk (standard deviation) first decreases and then increases again as the target expected return increases. The blue dotted line and the orange solid line combined show the Investment opportunity set with different combinations of investment weights in Sec 1, Sec 2, and Sec 3. However, for every portfolio on the blue dotted line, there is a portfolio on the orange solid line providing a higher return for the same level of risk. So, any rational investor would want to invest in a portfolio with higher returns given the same level of portfolio risk. So, to obtain the highest possible return at the lowest possible portfolio risk for the respective return, one would invest in a portfolio lying on the orange solid line, making it the optimal choice. In a nutshell, all portfolios of Sec 1, Sec 2, and Sec 3 that lie above and upward from the global minimum variance portfolio give the best possible risk and return combinations. Hence, the name efficiency frontier.

4) Diversification Benefits

Diversification is an investment strategy where capital available for investing is divided and invested among various assets to reduce risk.

For example, consider Sec 1, Sec 2, and Sec 6 with expected returns(r) and standard deviations (σ), as shown below.

Ticker	**Expected Return E(r)**	**Standard Deviation (σ)**
Sec 1	6.43%	9.35%
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Sec 6	2.89%	2.06%

Correlation Matrix (ρ)			
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Sec 6	0.18	0.02	1

→ Scenario 1

Let's say we have invested **100%** of the capital available in only **Sec 1**. The expected return shall be **6.43%,** and portfolio risk shall be **9.35%**

→ Scenario 2

Now, let's say we have invested **75%** of capital in **Sec 1** and the remaining **25%** in **Sec 2**

the expected return $E(r) = w_1 r_1 + w_1 r_2$ and

Portfolio risk $\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2}}$

Substituting the values in the above equations

The expected return of the portfolio is **7.18%,** and portfolio risk is **8.38%**

→ **Scenario 3**

Now, let's say we have invested **53%** of capital in **Sec 1**, **37%** of capital in **Sec 2,** and the remaining **10%** of capital in **Sec 6**

the expected return $E(r) = w_1r_1 + w_2r_2 + w_6r_6$ and

Portfolio risk (from Question 3)

$$\begin{aligned}\sigma_p^2 = w_1^2Cov(r_1,r_1) + w_1w_2Cov(r_1,r_2) + w_1w_3Cov(r_1,r_6) + w_2^2Cov(r_2,r_2) \\ + w_2w_1Cov(r_2,r_1) + w_2w_3Cov(r_2,r_6) + w_6^2Cov(r_6,r_6) + w_3w_1Cov(r_6,r_1) \\ + w_3w_2Cov(r_6,r_2)\end{aligned}$$

Substituting the values in the above equations

The expected return of the portfolio is **7.20%,** and portfolio risk is **8%.**

When we compare the above three scenarios, in scenario 2, the expected return increased, and portfolio risk decreased compared to scenario 1. In scenario 3, the expected return further increased, and risk further reduced compared to scenario 2. By observation, it's evident that risk is reduced when investments are diversified.

Rationale

Before understanding the rationale behind the risk reduction due to diversification, let's observe the formulae of the Expected return of a portfolio and variance of a portfolio for a moment.

$$E(r) = w_1r_1 + w_2r_2$$

$$\begin{aligned}\sigma_p^2 &= w_1^2Cov(r_1,r_1) + w_2^2Cov(r_2,r_2) + w_2w_1Cov(r_2,r_1) + w_1w_2Cov(r_1,r_2) \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_2w_1Cov(r_2,r_1)\end{aligned}$$

The expected return of a portfolio is nothing but the weighted average of individual asset's expected return. However, the variance of a portfolio is, unlike the expected return, the weighted sum of covariances, and each weight is the product of the proportions of the pair of assets. The covariance matrix we used in question 3 explains the same even more clearly.

When we observe the portfolio variance formula, we can see that, with weights being constant, the variance of the portfolio changes with the change in covariances of the assets, and the covariance between two assets is calculated as below

$$Cov(r_i, r_j) = \sigma_i\sigma_j\rho_{ij}$$

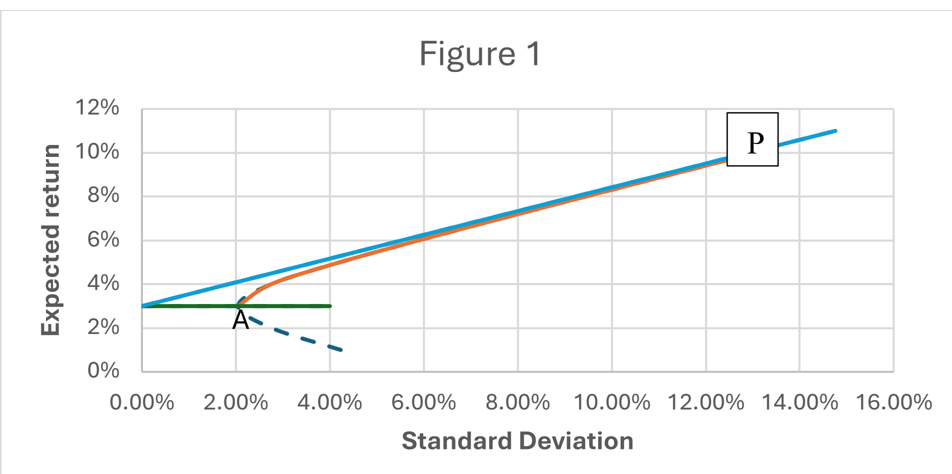
So, other terms being constant, portfolio variance moves proportional to the correlation coefficient (ρ). If we see scenario 1, we can see that the correlation coefficient (ρ_{12}) between Sec 1 and Sec 2 is 0.12, which is less than 1(a perfect correlation). So, dividing our investment among them has reduced the "$2w_2w_1Cov(r_2, r_1)$" term, ultimately reducing the overall risk. A lesser or a negative correlation coefficient or diversifying even further, in turn, reduces the weighted sum of covariances even further, eventually resulting in reduced portfolio risk. So, by diversifying the capital available among different assets, total portfolio risk can be reduced while obtaining better returns.

5) Risky asset portfolio + Risk-free asset

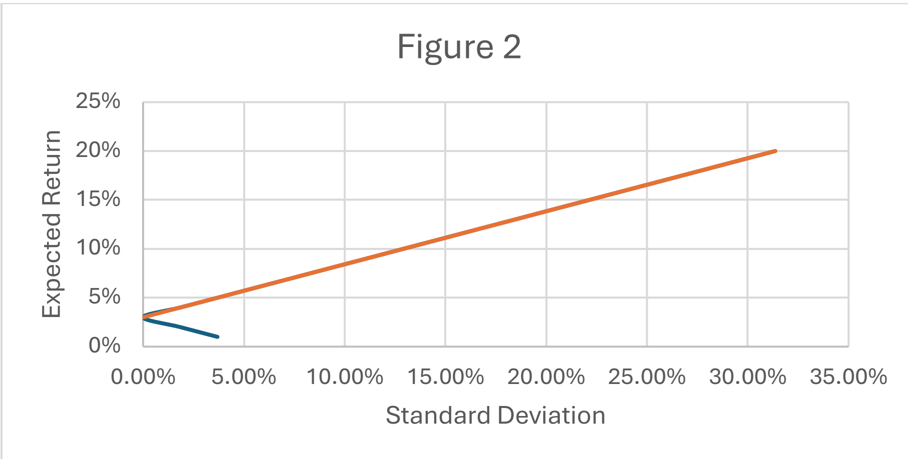
Including a risk-free asset can improve the performance of a portfolio by enhancing its risk-return profile. Adding a risk-free asset to a risky portfolio will transform the efficiency frontier into a capital allocation line (CAL), which will result in higher expected returns for the same level of risk compared to a portfolio composed solely of risky assets.

Let's consider the Investment opportunity set of Sec 1, Sec 2, and Sec 6 from question 3 and draw a Capital Allocation line (CAL) with a risk-free rate of 3% passing through the minimum variance portfolio(A). The risk-return ratio (Sharpe ratio) of this portfolio will be $S = 0.0021$.

Now, let's draw a CAL with the steepest slope, i.e., the highest Sharpe ratio. We can see that it forms a tangency with the efficiency frontier. So, any CAL steeper than the tangent CAL is not feasible because it does not form part of the opportunity set of the portfolio, and any CAL with a lesser slope than the tangent CAL gives a lesser return for any given risk. So, the point(P) at which the steepest CAL touches the efficiency frontier is the portfolio that offers the highest return to risk ratio (Sharpe ratio $(S) = 0.54$) than any other point on the efficiency frontier.



Now, let's draw the efficiency frontier of a portfolio comprising the risk-free asset with a 3% rate, Sec 1, Sec 2, and Sec 3.



If we observe the red solid line (efficiency frontier), we can see that it formed a straight line and overlaps with tangent CAL in Figure 1. In other words, both are the same.

Now, if we compare the efficiency frontiers of portfolios with and without risk-free asset, we can see that for any given portfolio risk, the portfolio with risk-free asset provides more return than the portfolio without risk-free asset. So, any rational investor would choose to invest in portfolio P. Though, depending on the risk aversion, the investor may choose to invest more in the risk free asset, the ratio of investment weights in the risky assets will remain same.