Algorithms and Data Structures

COMP261
Minimum Spanning Trees

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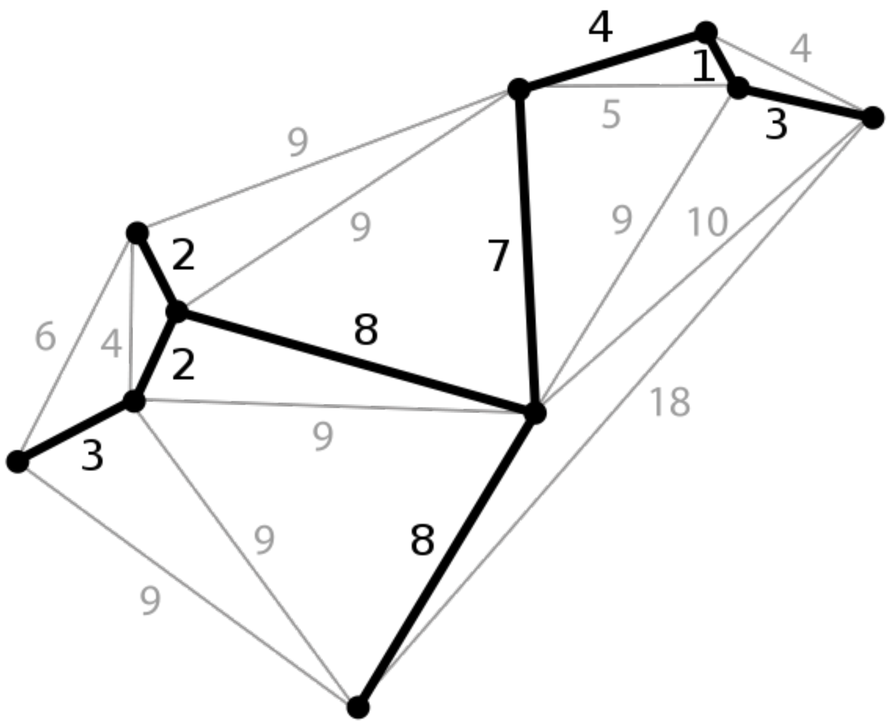
Outline

- Why Minimum Spanning Tree
- What is Minimum Spanning Tree
- Finding Minimum Spanning Tree
  - Prim’s Algorithm
  - Kruskal’s Algorithm

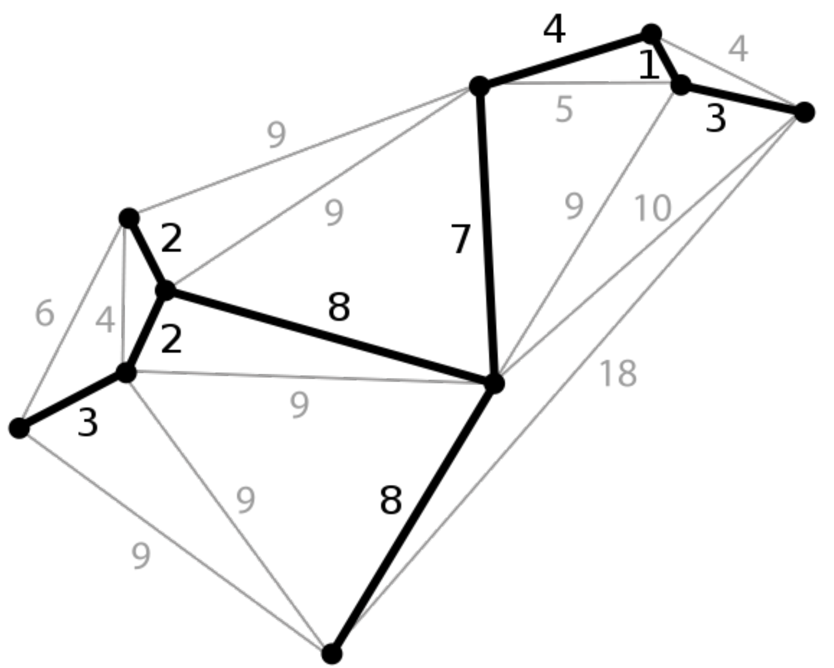
Why Minimum Spanning Tree

- Many applications
  - Telecommunication (cover the network with minimum cost)
  - Cluster analysis (e.g. gene expression clustering)
  - Image segmentation



Minimum Spanning Tree

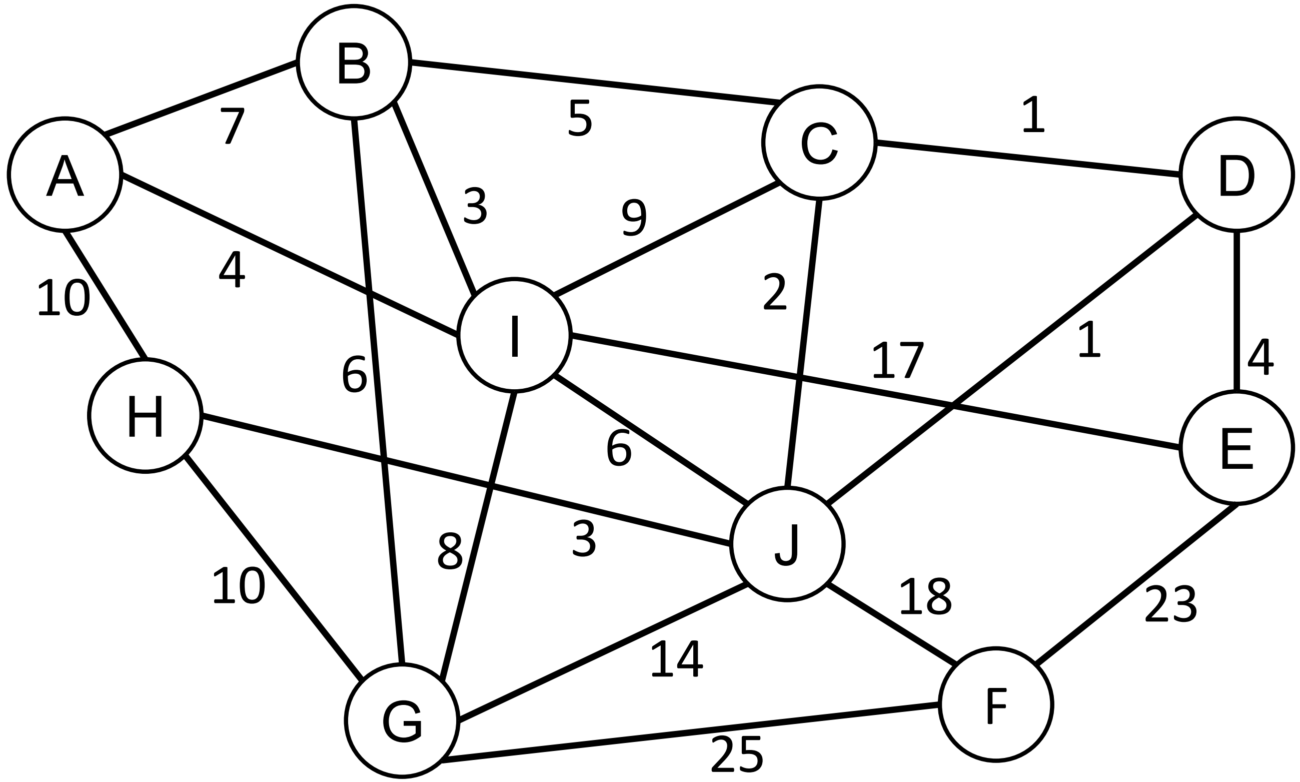
- Given a connected, undirected, weighted graph
- A **spanning tree** is a subgraph that contains all the nodes but is a tree (no cycle)
  - A spanning tree does not require a weighted graph
- A **minimum spanning tree (MST)** is a spanning tree with the minimum total weight among all the spanning trees, i.e. its total weight is no greater than the total weight of any other spanning tree



- Two algorithms to find MST
  - Prim's Algorithm
  - Kruskal's Algorithm
  - Both can prove correctness

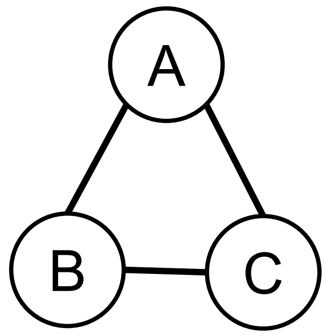
Prim’s Algorithm

- Grow the tree from a root node
  - Randomly select a root node, initialise a single-node tree
  - Repeatedly add one node outside the tree into the tree until all the nodes are in the tree
    - Add a new edge: one node in the tree, the other outside the tree
    - The added edge has the minimum weight



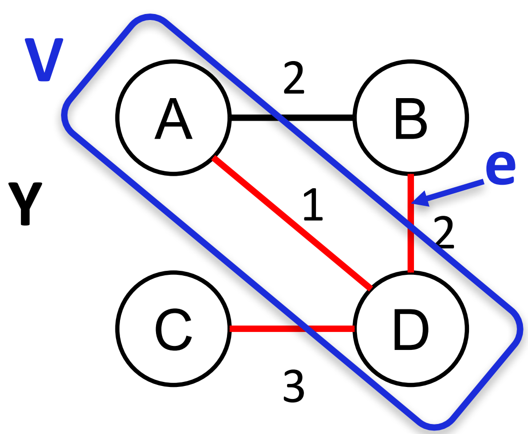
Correctness of Prim's Algorithm

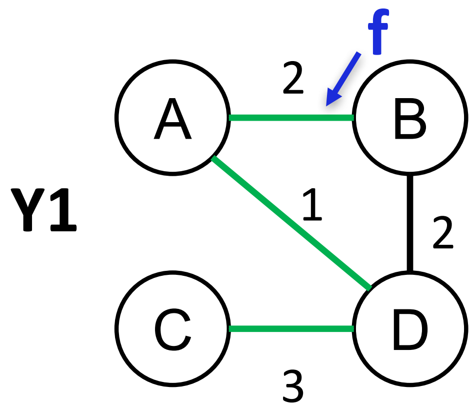
- **Theorem**: the tree obtained by Prim's Algorithm is a MST
- **Proof**:
  - The tree contains all the nodes
    - Algorithm does not stop until all the nodes have been added
  - The tree has no cycle
    - If there is a cycle, then when adding the last edge, the two nodes of the edge are already in the tree (conflicting with the algorithm)



Correctness of Prim's Algorithm

- **Theorem**: the tree obtained by Prim's Algorithm is a MST
- **Proof**:
  - The tree has the minimum total weight
  - Let Y be the tree obtained by Prim's Algorithm
  - Y_1 be the MST, Y_1 not equal to Y
  - Let **e** be the first edge that is not in Y_1 during the algorithm, **V** be the set of nodes in the tree before adding e. We have another edge **f in Y_1**, with one node in V, the other outside V.
  - Since Prim's algorithm selects e rather than f, then $w(e) <= w(f)$
  - If we remove f from Y_1, add e to Y_1, we get Y_2 with $w(Y_2) <= w(Y_1)$
  - Repeat this, … Y_n = Y, and $w(Y_n) <= w(Y_1)$. So Y is also a MST





Prim’s Algorithm

Given: a connected undirected weight graph

Initialise *fringe* to have a root node with *costToTree* = 0 and a dummy edge, all nodes are unvisited;

Repeat until all nodes are visited {

  Choose from fringe the unvisited node (*n**) with minimum *costToTree*;

  Add the corresponding edge to the spanning tree, set *n* as* visited

  for each (edge (*n*, n’*) outgoing from node *n**) {

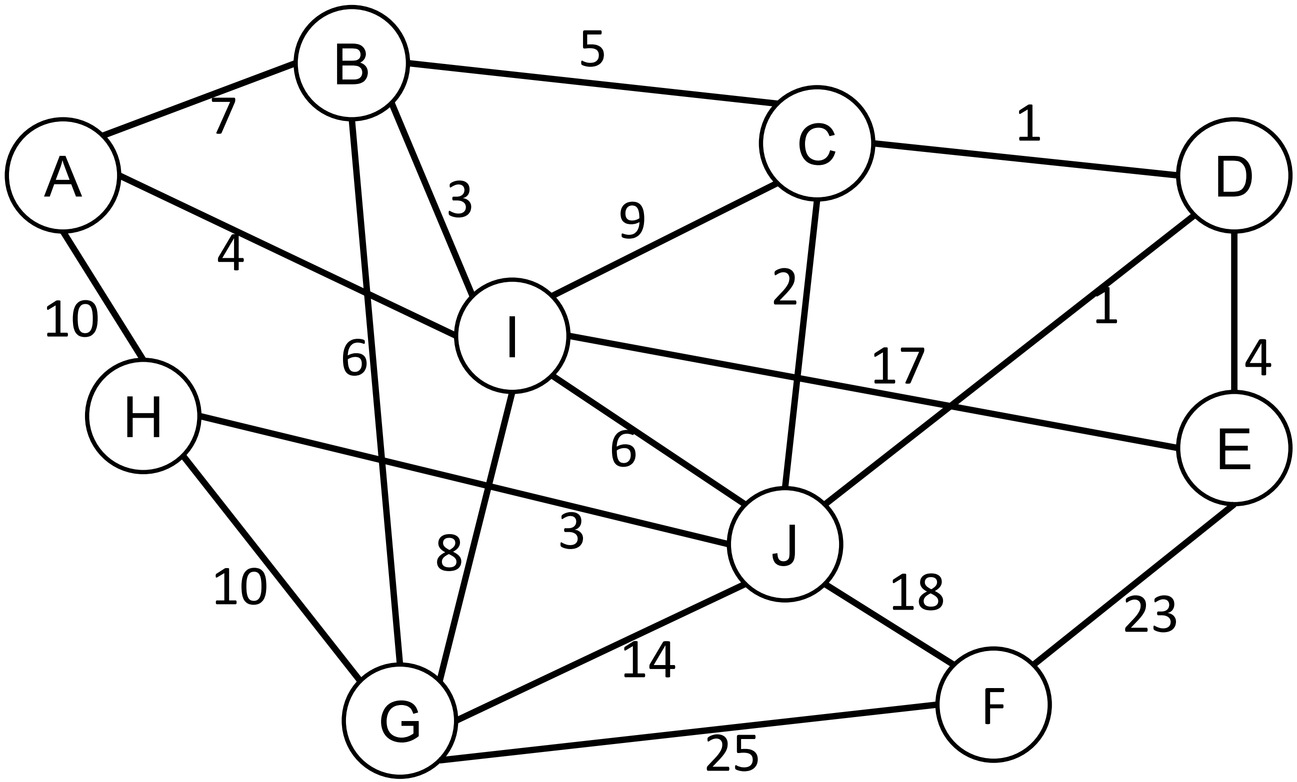
    if (*n’* is not unvisited) **then** add *<n’, (n*,n’), cost(n*,n’)>* into the fringe;

  }

}

Kruskal's Algorithm

- Merge trees
  - Initially, each node is a single-node tree
  - At each step, merge two trees into one
  - The merge cost is the minimum (min-cost edge)



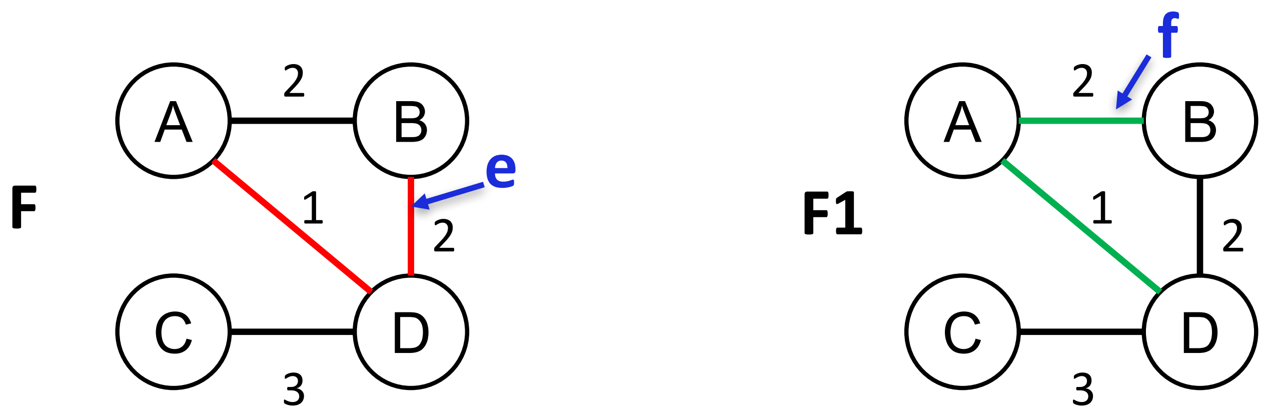
Correctness of Kruskal's Algorithm

- **Theorem**: the tree obtained by Kruskal's Algorithm is a MST
- **Proof**:
  - The tree cannot have cycle (always merge two tree into one)
  - The tree has all the nodes
  - Therefore, the tree is a spanning tree

  - If F is the set of edges chosen at any stage of the algorithm, then there is some minimum spanning tree that contains F.

Correctness of Kruskal's Algorithm

- If F is the set of edges chosen at any stage of the algorithm, then there is some minimum spanning tree that contains F.
  - At the beginning, F is empty, so MST contains F
  - If F is contained in a MST, then F + e is also contained in a MST, where e is the new edge added by Kruskal's algorithm
    - If not, then there is another tree F1 with the same nodes, and is contained in a MST.
    - There exists an edge f in F1 but not in F.
    - F1 – f + e is another tree, and cost(F1 – f + e) = cost(F1) – cost(f) + cost(e) <= cost(F1).
    - F = F1 – f + e is contained in a MST



Kruskal's Algorithm

Given: a connected undirected weight graph (*N* nodes, *M* edges)

Set *forest* as *N* node sets, each containing a node;

Set *fringe* as a priority queue of all the edges ⟨n1, n2, length⟩;

Set *tree* as an empty set of edges;

Repeat until *forest* contains only one tree or edges is empty {

  Get and remove *⟨ n1*, n2*, length*⟩* as the edge with minimum length from fringe;

  If (*n1** and *n2** are in different sets in forest) {

    Merge the two sets in *forest*;

    Add the edge to *tree*;

  }

}

return *tree*;

Complexity of Prim's and Kruskal's Alg

- Depends on data structure
  - If using adjacency matrix for graph?
  - If using priority queue?

- Can we do better in Kruskal's algorithm?
  - Efficiently check two nodes are in different sets in forest
  - Efficiently merge two trees
  - A new data structure: disjoint sets

Summary

- Minimum Spanning Tree (MST)

- Two algorithms for finding MST
  - Prim’s algorithm
  - Kruskal’s algorithm

- Correctness
- Complexity

- Next lecture: disjoint set for efficient Kruskal’s algorithm