Berger's Classification Theorem on Riemannian Holonomies

MA 333 Final Presentation

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Introduction

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Let (M, g) be a simply-connected, irreducible, nonsymmetric Riemannian Manifold of dim m, then the holonomy group $\mathrm{Hol}(p)$ of M is exactly one of the following:

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Let (M, g) be a simply-connected, irreducible, nonsymmetric Riemannian Manifold of dim m, then the holonomy group $\mathrm{Hol}(p)$ of M is exactly one of the following:

- $m = n, \quad \mathrm{Hol}(p) = \mathrm{SO}(n)$
- $m = 2n, \ \mathrm{Hol}(p) = \mathrm{U}(n)$
- $m = 2n, \ \mathrm{Hol}(p) = \mathrm{SU}(n)$
- $m = 4n, \ \mathrm{Hol}(p) = \mathrm{Sp}(n)$
- $m = 4n, \ \mathrm{Hol}(p) = \mathrm{Sp}(n)\mathrm{Sp}(1)$
- $m = 7, \quad \mathrm{Hol}(p) = \mathrm{G}_2$
- $m = 8, \quad \mathrm{Hol}(p) = \mathrm{Spin}(7)$

Preliminary Facts and Nomenclature

Lie Group Representations

G be a compact Lie subgroup of $\mathrm{SO}(n)$. Then G has a natural representation/action on $\mathbb{R}^n$. The stabilizer of $v \in \mathbb{R}^n$ is called the *isotropy subgroup* of v, G_v.

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If M is an embedded submanifold of $\mathbb{R}^n$, then then there is a derived normal holonomy group $\mathrm{Hol}^{\perp}(\nabla^{\perp})$ wrt the normal connection $\nabla^{\perp}$.

Reducible Spaces

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(M, g) is called a *reducible* space if it is isometric to a nontrivial Riemannian product $(M_1, g_1) \times (M_2, g_2)$.
An *irreducible* space is one which cannot be written as a Riemannian product.

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An *irreducible* space is one which cannot be written as a Riemannian product.
Note, Holonomy representation of $(M_1, g_1) \times (M_2, g_2)$ is $\mathrm{Hol}(M_1, g_1) \times \mathrm{Hol}(M_2, g_2)$. The converse is also true, but locally!
The global converse is achieved if we have completeness.

Reducible Spaces

Theorem 1 (de Rham Decomposition).

Let (M, g) be simply-connected and complete. Then there is a unique decomposition upto isometry and permutations

$$(M, g) = \prod_{i=1}^{k} (M_i, g_i)$$

where (M_i, g_i) are complete, simply-connected and irreducible. Moreover, the holonomy representation of $\mathrm{Hol}_p(M)$ over T_pM is the product of the representations of $\mathrm{Hol}_{p_i}(M_i)$ over $T_{p_i}M_i$.

Proof is via distributions.

Symmetric Spaces

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Definition 2.

A Riemannian space (M, g) is said to be *locally symmetric* if for each $x \in M$, there exist isometries s_x defined on some neighbourhood of x such that:

$$s_x(x) = x, \text{ and } s_{x*} = -\mathrm{id}$$

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If these s_x are defined on all of M, then M is said to be a *globally symmetric*.

We have the following result:

Theorem 3.

Let (M, g) be a Riemannian symmetric space. Then

(i) $\nabla R = 0$

(ii) *M is complete.*

(iii) *Involutive isometries s_x are transitive on M.*

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Note, (i) follows because

$$\begin{aligned} \nabla R &= s_x^*(\nabla R) \\ (\nabla R)(X, Y, Z, W) &= s_{x*}(\nabla R)(X, Y, Z, W) \\ &= s_{x*}(\nabla_{s_{x*}X} R)(s_{x*}Y, s_{x*}Z, s_{x*}W) \\ &= (-1)^5 \nabla R(X, Y, Z, W) \end{aligned}$$

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Suppose $x, y \in M, X \in T_pM$ st $y = \exp_x(\epsilon X)$ be joined by a geodesic γ on $[0, \epsilon]$. Then consider the extension of γ to a larger domain,

$$\tilde{\gamma}(t) := \begin{cases} \exp_x(\epsilon X) & t \in [0, \epsilon] \\ s_y(\exp_x((t - \epsilon)X)) & t \in [\epsilon, 2\epsilon] \end{cases}$$

This shows M is complete.

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Now, suppose γ is a geodesic from x to y with $\gamma(2T) = y$. (which exists since M is complete) Then note $s_{\gamma(T)}$ is an isometry taking x to y. This show the involutive isometries are transitive on M.

s-Representations

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Definition 4.

The isotropy representation of a semisimple simply-connected symmetric space is called an *s-Representation*.

A semisimple symmetric space is $M = G/H$ as before such that G is semisimple.

Holonomy and Curvature

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Theorem 5 (Ambrose-Singer).

Let ∇ *be a connection on* (M, g)*. Then*

$$\mathfrak{hol}_p(\nabla) = \mathrm{span}_{\mathbb{R}}\{P_\gamma^{-1}(F_\nabla \cdot v \wedge w)_x P_\gamma : \gamma \textit{ from } p \textit{ to } x\}$$

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For Levi-Civita connection,

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And for symmetric spaces,

$$\mathfrak{hol}(p) = \mathrm{span}_{\mathbb{R}}\{R_p(X, Y) : X, Y \in T_pM\}$$

because $\nabla R = 0$.

Berger's Holonomy Theorem

Lemma 6.

Let G be a compact subgroup of $\mathrm{SO}(n)$ acting on $\mathbb{R}^n$, not transitive on the sphere and $v \in \mathbb{R}^n$ be a principal vector. Then $\mathbb{R}^n$ is spanned by the family of tangent spaces $\nu_\gamma(t)(G.\gamma(t))$ for some $\xi \in \nu_v(G.v)$,
$\gamma(t) = v + t\xi, t \in \mathbb{R}$.

Some More Results

Theorem 7.

If G, a subgroup of $\mathrm{SO}(n)$ acts on $\mathbb{R}^n$ as an s-representation, then the connected component of the normalizer of G in $\mathrm{SO}(n)$, $N_o(G) = G$.

This is a standard result which

Normal Holonomy Theorem

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Theorem 8 (Normal Holonomy Theorem).

Let M be a submanifold of $\mathbb{R}^n$ with $p \in M$. Then $\mathrm{Hol}_0^{\perp}(p)$ acts as an s-representation on $\nu_p M$ (upto the subspace of fixed points), i.e. there exist decompositions

$$\nu_p M = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$
$$\mathrm{Hol}_0^{\perp}(p) = G_1 \times \ldots \times G_k$$

such that G_i acts on V_i irreducibly as an isotropy representation of a simple symmetric space and trivially on all other V_j.

Full submanifolds

Definition 9.

A submanifold of $\mathbb{R}^n$ is called *full* if it is not contained in any proper affine subspace of $\mathbb{R}^n$.

Note, an orbit $G.v$ is full iff G acts irreducibly on $\mathbb{R}^n$.

Theorem 10.

Let G be an orthogonal group acting on $\mathbb{R}^n$ such that the orbit $G.v$ is full. Then

Cartan gave a criterion for submanifolds of arbitrary Riemannian manifolds being totally geodesic.

Theorem 11 (Existence of Totally Geodesic Manifolds).

Let $\tilde{M}$ be a Riemannian manifold, $p \in \tilde{M}$, and V a linear subspace of $T_p\tilde{M}$. There exists a totally geodesic submanifold M of $\tilde{M}$ with $p \in M$ and $T_pM = V$ if and only if there exists $\epsilon > 0$ s.t. for every geodesic γ in $\tilde{M}$ with $\gamma(0) = p$, $\dot{\gamma}(0) \in V \cap B_\epsilon(0)$, the Riemannian curvature tensor of $\tilde{M}$ preserves the parallel translate of V along γ from p to $\gamma(1)$.

The Glueing Lemma

The following lemma will be useful:
First for any $v \in T_pM$, define $\mathcal{F}_v$ to be the family of subspaces of T_pM all containing v, which under the exponential map within injectivity radii form totally geodesic, locally symmetric manifolds.

Lemma 12.

Let M be a Riemannian manifold. Assume that for any given v in some dense subset of the Euclean ball of injectivity radius at p, the family $\mathcal{F}_v$ spans T_pM. Then the involution s_p is an isometry of the ball.

That is, glueing together locally symmetric spaces gives locally symmetric spaces.

Theorem 13.

Let M be a Riemannian manifold, $p \in M$ and let ρ be the injectivity radius at p. Assume $\Phi = \mathrm{Hol}(p)$ acts irreducibly over T_pM. Denote $N^v = \exp_p(\nu_v(\Phi.v) \cap B_\rho^E(0))$. Then for all $v \in T_pM, v \neq 0$,

(i) *N^v is totally geodesic. Further, N^v splits off the geodesic γ_v. And $Hol_p(N^v) \subseteq \Phi^v$*

(ii) *fd*

(iii) *N^v is locally symmetric.*

Berger's Theorem (Finally!)

Theorem 14.

If the holonomy group of an irreducible Riemannian manifold M is not transitive on the sphere. Then M is locally symmetric.

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Let $p \in M$. The set of principal vectors of T_pM is open and dense. And $\Phi = \mathrm{Hol}(p)$ is not transitive on the sphere.

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So earlier lemma gives $\gamma_\xi(t) = v + t\xi$ in $\nu_v(\Phi.v)$ with $\nu_{\gamma_\xi(t)}(\Phi.\gamma(t))$ spanning T_pM.

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From Theorem 13, and the Glueing Lemma 20, we conclude that M is locally symmetric at p.

Berger's List

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Group G	Transitive Action on Sphere
$\mathrm{SO}(n)$	$\mathbb{S}^{n-1}$
$\mathrm{SU}(n)$	$\mathbb{S}^{2n-1}$
$\mathrm{U}(n)$	$\mathbb{S}^{2n-1}$
$\mathrm{Sp}(n)$	$\mathbb{S}^{4n-1}$
$\mathrm{Sp}(n)\mathrm{U}(1)$	$\mathbb{S}^{4n-1}$
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathbb{S}^{4n-1}$
G_2	$\mathbb{S}^{6}$
$\mathrm{Spin}(7)$	$\mathbb{S}^{7}$
$\mathrm{Spin}(9)$	$\mathbb{S}^{15}$

Some of these cases were later shown to not required to be considered.

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Berger's List

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Holonomy groups of all possible simply-connected irreducible nonsymmetric manifolds are classified as follows:

Holonomy Group	$\dim(M)$	Corresponding Geometry
$\mathrm{SO}(n)$	$m = n$	Riemannian manifold
$\mathrm{U}(n)$	$m = 2n$	Kähler manifold
$\mathrm{SU}(n)$	$m = 2n$	Calabi-Yau manifold
$\mathrm{Sp}(n)$	$m = 4n$	Hyperkähler manifold
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$m = 4n$	Quaternion-Kähler manifold
G_2	$m = 7$	G_2 manifold
$\mathrm{Spin}(7)$	$m = 8$	$\mathrm{Spin}(7)$ manifold

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Thanks!