Berger's Classification Theorem on Riemannian Holonomies

MA 333 Final Presentation

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Introduction

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Statement of Berger's Classification

Let (M, g) be a simply-connected, irreducible, nonsymmetric Riemannian Manifold of dim m, then the holonomy group $\mathrm{Hol}(p)$ of M is exactly one of the following:

- $m = n, \quad \mathrm{Hol}(p) = \mathrm{SO}(n)$
- $m = 2n, \ \mathrm{Hol}(p) = \mathrm{U}(n)$
- $m = 2n, \ \mathrm{Hol}(p) = \mathrm{SU}(n)$
- $m = 4n, \ \mathrm{Hol}(p) = \mathrm{Sp}(n)$
- $m = 4n, \ \mathrm{Hol}(p) = \mathrm{Sp}(n)\mathrm{Sp}(1)$
- $m = 7, \quad \mathrm{Hol}(p) = \mathrm{G}_2$
- $m = 8, \quad \mathrm{Hol}(p) = \mathrm{Spin}(7)$

Preliminary Facts and Nomenclature

Lie Group Representations

G be a compact Lie subgroup of $\mathrm{SO}(n)$. Then G has a natural representation/action on $\mathbb{R}^n$. The stabilizer of $v \in \mathbb{R}^n$ is called the *isotropy subgroup* of v, G_v.

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If the slice representation is trivial, we say that v is a *principal vector*. The set of principal vectors is open and dense in $\mathbb{R}^n$.
If M is an embedded submanifold of $\mathbb{R}^n$, then then there is a derived normal holonomy group $\mathrm{Hol}^{\perp}(\nabla^{\perp})$ wrt the normal connection $\nabla^{\perp}$.

Reducible Spaces

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(M, g) is called a *reducible* space if it is isometric to a nontrivial Riemannian product $(M_1, g_1) \times (M_2, g_2)$.
An *irreducible* space is one which cannot be written as a Riemannian product.

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Note, Holonomy representation of $(M_1, g_1) \times (M_2, g_2)$ is $\mathrm{Hol}(M_1, g_1) \times \mathrm{Hol}(M_2, g_2)$. The converse is also true, but locally!
The global converse is achieved if we have completeness.

Reducible Spaces

Theorem 1 (de Rham Decomposition).

Let (M, g) *be simply-connected and complete. Then there is a unique decomposition upto isometry and permutations*

$$(M, g) = \prod_{i=1}^{k} (M_i, g_i)$$

where (M_i, g_i) *are complete, simply-connected and irreducible. Moreover, the holonomy representation of* $\mathrm{Hol}_p(M)$ *over* T_pM *is the product of the representations of* $\mathrm{Hol}_{p_i}(M_i)$ *over* $T_{p_i}M_i$*.*

Symmetric Spaces

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Definition 2.

A Riemannian space (M, g) is said to be *locally symmetric* if for each $x \in M$, there exist isometries s_x defined on some neighbourhood of x such that:

$$s_x(x) = x, \text{ and } s_{x*} = -\mathrm{id}$$

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If these s_x are defined on all of M, then M is said to be a *globally symmetric*.

We have the following result:

Theorem 3.

Let (M, g) be a Riemannian symmetric space. Then

(i) $\nabla R = 0$

(ii) *M is complete.*

(iii) *Involutive isometries s_x are transitive on M.*

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Note, (i) follows because

$$\begin{aligned} \nabla R &= s_x^*(\nabla R) \\ (\nabla R)(X,Y,Z,W) &= s_{x*}(\nabla R)(X,Y,Z,W) \\ &= s_{x*}(\nabla_{s_{x*}X} R)(s_{x*}Y, s_{x*}Z, s_{x*}W) \\ &= (-1)^5 \nabla R(X,Y,Z,W) \end{aligned}$$

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Suppose $x, y \in M, X \in T_pM$ st $y = \exp_x(\epsilon X)$ be joined by a geodesic γ on $[0, \epsilon]$. Then consider the extension of γ to a larger domain,

$$\tilde{\gamma}(t) := \begin{cases} \exp_x(\epsilon X) & t \in [0, \epsilon] \\ s_y(\exp_x((t - \epsilon)X)) & t \in [\epsilon, 2\epsilon] \end{cases}$$

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Now, suppose γ is a geodesic from x to y with $\gamma(2T) = y$. (which exists since M is complete) Then note $s_{\gamma(T)}$ is an isometry taking x to y. This show the involutive isometries are transitive on M.

s-Representations

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The final result on symmetric spaces implies $M = G/H$, where $G = \mathrm{Isom}^0(M)$, and H be the isotropy subgroup of a point $o \in M$. This follows from the Orbit-Stabilizer Theorem.

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Definition 4.

The isotropy representation of a semisimple simply-connected symmetric space is called an *s-Representation*.

A semisimple symmetric space is $M = G/H$ as before such that G is semisimple.

Holonomy and Curvature

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Theorem 5 (Ambrose-Singer).

Let ∇ be a connection on (M, g). Then

$$\mathfrak{hol}_p(\nabla) = \mathrm{span}_{\mathbb{R}}\{P_\gamma^{-1}(F_\nabla \cdot v \wedge w)_x P_\gamma : \gamma \text{ from } p \text{ to } x\}$$

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And for symmetric spaces,

$$\mathfrak{hol}(p) = \operatorname{span}_{\mathbb{R}}\{R_p(X, Y) : X, Y \in T_pM\}$$

because $\nabla R = 0$.

Berger's Holonomy Theorem

Lemma 6.

Let G be a compact subgroup of $\mathrm{SO}(n)$ acting on $\mathbb{R}^n$, not transitive on the sphere and $v \in \mathbb{R}^n$ be a principal vector. Then $\mathbb{R}^n$ is spanned by the family of tangent spaces $\nu_\gamma(t)(G.\gamma(t))$ for some $\xi \in \nu_v(G.v)$, $\gamma(t) = v + t\xi, t \in \mathbb{R}$.

Some More Results

Theorem 7.

If G, a subgroup of $\mathrm{SO}(n)$ acts on $\mathbb{R}^n$ as an s-representation, then the connected component of the normalizer of G in $\mathrm{SO}(n)$, $N_o(G) = G$.

Normal Holonomy Theorem

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Theorem 8 (Normal Holonomy Theorem).

Let M be a submanifold of $\mathbb{R}^n$ with $p \in M$. Then $\mathrm{Hol}_0^{\perp}(p)$ acts as an s-representation on $\nu_p M$ (upto the subspace of fixed points), i.e. there exist decompositions

$$\nu_p M = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$
$$\mathrm{Hol}_0^{\perp}(p) = G_1 \times \ldots \times G_k$$

such that G_i acts on V_i irreducibly as an isotropy representation of a simple symmetric space and trivially on all other V_j.

Full submanifolds

Definition 9.

A submanifold of $\mathbb{R}^n$ is called *full* if it is not contained in any proper affine subspace of $\mathbb{R}^n$.

Note, an orbit $G.v$ is full iff G acts irreducibly on $\mathbb{R}^n$.

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Theorem 10.

Let G be an orthogonal group acting on $\mathbb{R}^n$ such that the orbit $G.v$ is full. Then the connected component of the slice representation $(G_v)_o$ is contained in the normal holonomy group $\mathrm{Hol}_v^{\perp}(G.v)$.

Cartan's Theorem on Existence Totally Geodesic Manifolds

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Theorem 11 (Existence of Totally Geodesic Manifolds).

Let $\tilde{M}$ be a Riemannian manifold, $p \in \tilde{M}$, and V a linear subspace of $T_p\tilde{M}$. There exists a totally geodesic submanifold M of $\tilde{M}$ with $p \in M$ and $T_pM = V$ if and only if there exists $\epsilon > 0$ s.t. for every geodesic γ in $\tilde{M}$ with $\gamma(0) = p$, $\dot{\gamma}(0) \in V \cap B_\epsilon(0)$, the Riemannian curvature tensor of $\tilde{M}$ preserves the parallel translate of V along γ from p to $\gamma(1)$.

Another Lemma

We will also use the following lemma eventually:

Lemma 12.

Let M be a Riemannian manifold s.t. every holonomy transformation of T_pM extends via the expoenential map tio a local isometry. Then M is locally symmetric.

The Glueing Lemma

The following lemma will be useful:
First for any $v \in T_pM$, define $\mathcal{F}_v$ to be the family of subspaces of T_pM all containing v, which under the exponential map within injectivity radii form totally geodesic, locally symmetric manifolds.

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Lemma 13.

Let M be a Riemannian manifold. Assume that for any given v in some dense subset of the Euclean ball of injectivity radius at p, the family $\mathcal{F}_v$ spans T_pM. Then the involution s_p is an isometry of the ball.

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That is, glueing together locally symmetric spaces gives locally symmetric spaces.
Proof follows by constructing a parallel frame in which Jacobi operator diagonalizes, and then using density.

Auxillary Theorem

Theorem 14.

Let M be a Riemannian manifold, $p \in M$ and let ρ be the injectivity radius at p. Assume $\Phi = \mathrm{Hol}(p)$ acts irreducibly over T_pM. Denote $N^v = \exp_p(\nu_v(\Phi.v) \cap B^E_\rho(0))$. Then for all $v \in T_pM, v \neq 0$,

(i) *N^v is totally geodesic. Further, N^v splits off the geodesic γ_v. And $Hol_p(N^v) = \Phi^v \subseteq (\Phi_v)_o$.*

(ii) *The normal holonomy group $\Phi^\perp$ of $\Phi.v$ at v acts by isometries on N^v. Moreover, $\Phi^\perp \supseteq \Phi^v$, holonomy group of N^v at p.*

(iii) *N^v is locally symmetric.*

Auxillary Theorem

(i) The proof for totally geodesic follows from the Cartan's criterion for Existence of totally geodesic submanifolds, after showing that the curavture endomorphisms $R(\xi, \eta)$ belong to the isotropy algebra at v, and that the isotropy algebra acts invariantly on the normal space $\nu_v(\Phi.v)$. That is, the normal space is preserved by the curvature tensor.

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(iii) Let $v \in T_pM$, and c be a curve in N^v from p to q. Then parallel transport maps $\Phi(p)$ to $\Phi(q)$.

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(iii) Let $v \in T_pM$, and c be a curve in N^v from p to q. Then parallel transport maps $\Phi(p)$ to $\Phi(q)$.
So it maps $\Phi.v$ isometrically to $\Phi(q).(P_c(v))$. Then $T_qN^{P_c(v)} = T_qN^v$. So $N^{P_c(v)}$ and N^v coincide in a neigbourhood of q. Thus holonomy transformations of N^v extend to local isometries by the exponential map.

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So it maps $\Phi.v$ isometrically to $\Phi(q).(P_c(v))$. Then $T_qN^{P_c(v)} = T_qN^v$. So $N^{P_c(v)}$ and N^v coincide in a neigbourhood of q. Thus holonomy transformations of N^v extend to local isometries by the exponential map. It follows from Lemma 12, that N^v is locally symmetric.

Berger's Theorem (Finally!)

Theorem 15.

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Let $p \in M$. The set of principal vectors of T_pM is open and dense. And $\Phi = \mathrm{Hol}(p)$ is not transitive on the sphere.

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So earlier lemma gives $\gamma_\xi(t) = v + t\xi$ in $\nu_v(\Phi.v)$ with $\nu_{\gamma_\xi(t)}(\Phi.\gamma(t))$ spanning T_pM.

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From Theorem 14, and the Glueing Lemma 21, we conclude that M is locally symmetric at p.

Berger's List

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Group G	Transitive Action on Sphere
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$\mathrm{SU}(n)$	$\mathbb{S}^{2n-1}$
$\mathrm{U}(n)$	$\mathbb{S}^{2n-1}$
$\mathrm{Sp}(n)$	$\mathbb{S}^{4n-1}$
$\mathrm{Sp}(n)\mathrm{U}(1)$	$\mathbb{S}^{4n-1}$
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathbb{S}^{4n-1}$
G_2	$\mathbb{S}^{6}$
$\mathrm{Spin}(7)$	$\mathbb{S}^{7}$
$\mathrm{Spin}(9)$	$\mathbb{S}^{15}$

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Holonomy groups of all possible simply-connected irreducible nonsymmetric manifolds are classified as follows:

Holonomy Group	$\dim(M)$	Corresponding Geometry
$\mathrm{SO}(n)$	$m = n$	Riemannian manifold
$\mathrm{U}(n)$	$m = 2n$	Kähler manifold
$\mathrm{SU}(n)$	$m = 2n$	Calabi-Yau manifold
$\mathrm{Sp}(n)$	$m = 4n$	Hyperkähler manifold
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$m = 4n$	Quaternion-Kähler manifold
G_2	$m = 7$	G_2 manifold
$\mathrm{Spin}(7)$	$m = 8$	$\mathrm{Spin}(7)$ manifold

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Thanks!