P1:

Show that WLLN follows from CLT for the sequence of i.i.d.r.vs .

Solution:

Let $\{X_n\}$ be a sequence of i.i.d.r.vs with $E(X_i) = \mu$ and $V(X_i) = \sigma^2\ \forall i.$

Then $\{X_n\}$ obeys WLLN if

$$\lim_{n \longrightarrow \infty} P\left[\left|\frac{S_n}{n} - \mu\right| > \epsilon\right] = 0 \text{ (By Khinchine's WLLN)}$$

for $\epsilon > 0$ and $S_n = \sum_{i=1}^{n} X_i$.

$$\Longrightarrow \lim_{n \longrightarrow \infty} P\left[\left|\frac{S_n}{n} - \mu\right| < \epsilon\right] = 1 \quad \text{.... (1)}$$

Since $\epsilon > 0$ is fixed, for any positive constant k, we have

$$k\sigma\sqrt{n} < \epsilon n \quad \text{.... (2)}$$

for all sufficiently large values of n (because any large multiple of $\sqrt{n}$ is negligible in comparison with any small multiple of n)

From (2), we have $|S_n - n\mu| < k\sigma\sqrt{n} \Longrightarrow \{|S_n - n\mu| < \epsilon n\}$

$i.e.,\ \frac{|S_n - n\mu|}{\sigma\sqrt{n}} < k \Longrightarrow \left|\frac{S_n - n\mu}{n}\right| < \epsilon$

Therefore, $P\left\{\frac{|S_n - n\mu|}{\sigma\sqrt{n}} < k\right\} \leq P\left\{\left|\frac{S_n - n\mu}{n}\right| < \epsilon\right\}$

$$i.e,\ \lim_{n \longrightarrow \infty} P\left\{\frac{|S_n - n\mu|}{\sigma\sqrt{n}} < k\right\} \leq \lim_{n \longrightarrow \infty} P\left\{\left|\frac{S_n - n\mu}{n}\right| < \epsilon\right\} \quad \text{... (3)}$$

As $n \longrightarrow \infty$, by Lindeberg-Levy CLT, $Z = \frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$ $i.e.$, standard normal variate. Then, we can choose large k such that

$$\lim_{n \longrightarrow \infty} P\left\{\frac{|S_n - n\mu|}{\sigma\sqrt{n}} < k\right\} = P(|Z| < k) = 1 \quad \text{...(4)}$$

From (3) and (4), we have

$$\lim_{n \longrightarrow \infty} P\left\{\left|\frac{S_n}{n} - \mu\right| < \epsilon\right\} \geq 1$$

$\Longrightarrow \lim_{n \longrightarrow \infty} P\left\{\left|\frac{S_n}{n} - \mu\right| < \epsilon\right\} = 1$ (∵ Probability can't be more than 1) which is 1.

Therefore, WLLN holds.