3.2

Moment Generating Function

Certain derivations presented in modules 2.4, 2.5 and 2.6 have been somewhat heavy on algebra. For example, determining the mean and variance of the **Binomial distribution** turned out to be fairly tiresome. Another example of hard work was determining the set of probabilities associated with a sum , $P(X+Y=t)$. Many of these tasks are greatly simplified by using **probability generating functions.**

Moment Generating Function: The moment generating function (m.g.f) of a random variable X is dentoed by $M_X(t)$ and it is defined as

$$\boldsymbol{M_X(t) = E(e^{tX})}$$

$$\therefore\ M_X(t) = E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots + \infty$$

$$\therefore\ M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots + \infty \qquad \ldots\ldots \quad (1)$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

which gives the m.g.f in terms of moments.

Therefore the coeffceint of $\frac{t^r}{r!}$ in $M_X(t)$ is μ_r' , where $r = 1, 2, 3, \ldots$ and $\mu_r' = E(X^r)$, moment about origin.

The m.g.f of X about mean $\mu = \mu_1' = E(X)$ is defined as

$$M_{X-\mu}(t) = E\left[e^{t(X-\mu)}\right] = E\left[1 + \frac{t}{1!}(X-\mu) + \frac{t^2}{2!}(X-\mu)^2 + \frac{t^3}{3!}(X-\mu)^3 + \cdots\right]$$

$$= 1 + \frac{t}{1!}E(X-\mu) + \frac{t^2}{2!}E(X-\mu)^2 + \frac{t^3}{3!}E(X-\mu)^3 + \cdots$$

$$= 1 + \frac{t}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \cdots$$

where $E(x-\mu)^r = \mu_r$ is known as the r^{th} central moment for $r = 1, 2, \ldots$

Note that $\mu_1 = E(X-\mu) = E(X) - \mu = \mu - \mu = 0$

Since $M_X(t)$ generates moments, it is called **moment generating function**.

If X is a discrete random variable with p.m.f. $p(x)$ then

$$M_X(t) = E\left(e^{tX}\right) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable with p.d.f. $f(x)$, then

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

Moments Using Moment Generating Function:

Differentiating equation (1) with respect to t and then putting $t = 0$, gives

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

In general,

$$\mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}, \;\; r = 1, 2, 3, \ldots$$

Note: Moment generating function $M_X(t)$ is used to calculate the higher moments.

Theorems on Moment Generating Function:

Theorem 1: $\boldsymbol{M_{aX}(t) = M_X(at)}$, where $\boldsymbol{a}$ is a constant.

Proof: By definition $M_{aX}(t) = E(e^{taX}) = E(e^{atX}) = M_X(at)$

Therefore, $M_{aX}(t) = M_X(at)$

Theorem 2: The moment generating function of the sum of n independent random variables is equal to the product of their respective moment generating functions, $\boldsymbol{i.e.}$, $\boldsymbol{M_{X_1+X_2+X_3+\cdots+X_n}(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t)\ldots M_{X_n}(t)}$

Proof: By definition,

$$M_{X_1+X_2+X_3+\cdots+X_n}(t) = \left[E^{t(X_1+X_2+X_3+\cdots+X_n)}\right]$$

$$= E(e^{tX_1})E(e^{tX_2})E(e^{tX_3})\ldots E(e^{tX_n})$$

(Since $X_1, X_2, \ldots, X_n$ are independent).

Therefore, $M_{X_1+X_2+X_3+\cdots+X_n}(t) = M_{X_1}(t)M_{X_2}(r)\ldots M_{X_n}(t)$
Hence the proof.

Uniqueness Theorem of Moment Generating Function:

The m.g.f. of a distribution, if exists, uniquely determines the distribution. This implies that corresponding to a given probability distribution, there is only one m.g.f (provided it exists) and corresponding to a given m.g.f, there is only one probability distribution. Hence $M_X(t) = M_Y(t) \Longrightarrow X$ and Y are identically distributed.

Effect of Change of Origin and Scale on Moment Generating Function:

Let a random variable X be transformed to a new variable U by changing both the origin and scale in X as $= \frac{X-a}{h}$, where a and h are constants.

The m.g.f of U (about origin) is given by

$$M_U(t) = E(e^{tU}) = E\left[e^{t\left(\frac{X-a}{h}\right)}\right] = E\left(e^{\left(\frac{tX}{h}-\frac{ta}{h}\right)}\right) = e^{-\frac{at}{h}}E\left(e^{\left(\frac{tX}{h}\right)}\right)$$

$$\therefore\ \boldsymbol{M_{\frac{X-a}{h}}(t) = e^{-\frac{at}{h}}M_X\left(\frac{t}{h}\right)}$$

Note: If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$

Example 1: If X represents the outcome when a fair die is tossed, find the m.g.f. of X and hence, find $E(X)$ and $Var(X)$.

Solution: When a fair die is tossed

$$P(X = x) = \frac{1}{6}, \quad x = 1, 2, 3, 4, 5, 6$$

$$\therefore \; M_X(t) = \sum_{X=1}^{6} e^{tx} P(X = x) = \frac{1}{6} \sum_{x=1}^{6} e^{tx}$$

$$= \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

$\therefore$ Mean $= E(X) = \frac{7}{2}$

Now, $E[X^2]_{t=0} = \left\{\frac{d^2}{dt^2}[M_X(t)]\right\}_{t=0}$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0}$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

Example 2: Find the m.g.f. of the random variable X whose probability function $\boldsymbol{P(X = x) = \frac{1}{2^x}, x = 1, 2, 3, \ldots}$ and hence find its mean.

Solution: By definition,

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x=0}^{\infty} e^{tx} P(X = x) = \sum_{x=1}^{\infty} e^{tX}\left(\frac{1}{2^x}\right) = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$=\left[\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\cdots\right]$$

$$=\frac{e^t}{2}\left[1+\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\cdots\right]=\frac{e^t}{2}\left(1-\frac{e^t}{2}\right)^{-1}$$

$$=\frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1}=\frac{e^t}{2}\left(\frac{2}{2-e^t}\right)=\frac{e^t}{2-e^t}$$

Therefore, $M_X(t)=\frac{e^t}{2-e^t}$

$$\mu_1'=\left[\frac{d}{dt}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}=\left[\frac{(2-e^t)e^t-e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}=\frac{(2-1)1+1}{(2-1)^2}=2$$

Thus, $E(X)$ =mean= 2

Example 3: If the moments of a random variable X are defined by $E(X^r)=0.6, r=1,2,\ldots$ .Show that $P(X=0)=0.4$, $P(X=1)=0.6$, and $P(X\geq 2)=0$.

Solution: We know that

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

where $\mu_r'=E(X^r)=0.6$

$$\therefore\ M_X(t)=1+\sum_{r=1}^{\infty}\frac{t^r}{r!}\mu_r'=1+\sum_{r=1}^{\infty}\frac{t^r}{r!}(0.6)=1+(0.6)\left(\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)$$

$$=1+(0.6)(e^t-1)=1-0.6+0.6e^t=0.4+0.6e^t \qquad \ldots\ldots.(1)$$

But by definition,

$$M_X(t)=E\left(e^{tx}\right)=\sum_{r=0}^{\infty}e^{tx}P(X=x)$$

$M_X(t)=P(X=0)+e^tP(X=1)+e^{2t}P(X=2)+e^{3t}P(X=3)+$...+... ...... (2)

From equations (1) and (2), we have

$$0.4 + 0.6e^t = P(X=0) + e^t P(X=1) + e^{2t} P(X=2) + e^{3t} P(X=3) + \cdots$$

Equating the coefficients of like terms on both sides,

$$P(X=0) = 0.4,\ P(X=1) = 0.6$$

$$P(X=2) = P(X=3) = \cdots = 0 \quad \Rightarrow P(X \geq 2) = 0$$

Example 4: Find the m.g.f. of a random variable whose moments are $\boldsymbol{\mu_r = (r+1)!\ 2^r}$.

Solution: By definition, we have $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\ \mu_r'$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\ (r+1)!\, 2^r = \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$= 1 + 2(2t) + 3(2t)^2 + \cdots = (1-2t)^{-2} = \frac{1}{(1-2t)^2}$$

$$\therefore \quad M_X(t) = \frac{1}{(1-2t)^2}$$

Example 5: If $X \sim B(n,p)$, find the m.g.f of X and hence find its mean and variance.

Solution: Since $X \sim B(n,p)$, its p.m.f. is given by

$p(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, 2, \dots, n$ and $q = 1 - p$.

Then the m.g.f. of X is given by

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} \left(pe^t\right)^x q^{n-x}$$

$$\Rightarrow M_X(t) = (q + pe^t)^n$$

$\frac{d}{dt} M_X(t) = n(q + pe^t)^{n-1} pe^t \Rightarrow$Mean$= \mu_1' = \left[\frac{dM_X(t)}{dt}\right]_{t=0} = np$

Next, $\frac{d^2}{dt^2}\left(M_X(t)\right) = np[(n-1)(q+pe^t)^{n-2}pe^{2t} + (q+pe^t)^{n-1}e^t]$

$\Longrightarrow\ \mu_2' = \left[\frac{d^2}{dt^2}\left(M_X(t)\right)\right]_{t=0} = np[(n-1)p+1] = np[np-p+1] = np(np+q)$

$\Longrightarrow\ \mu_2' = n^2p^2 + npq$

Now, variance $= \sigma^2 = \mu_2' - (\mu_1')^2 = n^2p^2 + npq - n^2p^2 = npq$

Thus, $\mu = np$ and $\sigma^2 = npq$

Note that $\sigma^2 = npq = \mu q$ where $(0 < q < 1)$ .Thus, $\mu > \sigma^2$.

Note: *For binomial distribution, mean is always greater than variance.*

Example 6 : If $X \sim P(\lambda)$, find its m.g.f. and hence find its mean and variance.

Solution: Since $X \sim P(\lambda)$, then its p.m.f. is given by

$$p(x) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{\lambda!}\ , x = 0, 1, \ldots \text{ and } \lambda > 0$$

The m.g.f. of X is given by

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

$\Longrightarrow\ M_X(t) = e^{\lambda(e^t-1)}$

Since $\frac{d}{dt}\left(M_X(t)\right) = e^{\lambda(e^t-1)}\lambda e^t$; Mean$= \mu = \mu' = \left[\frac{d}{dt}\left(M_X(t)\right)\right]_{t=0} = \lambda.$

Now, $\frac{d^2}{dt^2}\left(M_X(t)\right) = \lambda\left[e^{\lambda(e^t-1)}e^t + e^{\lambda(e^t-1)}\lambda e^{2t}\right]$

Then $\mu_2' = \left[\frac{d^2 M_X(t)}{dt^2}\right]_{t=0} = \lambda(1+\lambda) = \lambda + \lambda^2$

Thus, variance $= \sigma^2 = \mu_2' - (\mu_1')^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$

Therefore, $\mu = \sigma^2 = \lambda$

Note: *Mean and variance are same for Poisson distribution.*

Example 7: If $X \sim NB(r, p)$, find its m.g.f. and hence find its mean and variance.

Solution*:* Since $X \sim NB(r, p)$, its p.m.f. is given by

$$p(x) = \binom{-r}{x} p^r (-q)^x, \qquad x = 0, 1, 2, \ldots$$

The m.g.f of X is given by

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \binom{-r}{x} p^r (-q)^x$$

$$= \sum_{x=0}^{\infty} \binom{-r}{x} p^r \left(-qe^t\right)^x = p^r \sum_{x=0}^{\infty} \binom{-r}{x} \left(-qe^t\right)^x$$

$\Longrightarrow \quad M_X(t) = p^r (1 - qe^t)^{-r}.$

Now, $\frac{d}{dt}\left(M_X(t)\right) = p^r(-r)(1 - qe^t)^{-(r+1)}(-qe^t) = qrp^r(1 - qe^t)^{-(r+1)} e^t$

Mean$= \mu = \mu_1' = \left[\frac{d}{dt}\left(M_X(t)\right)\right]_{t=0} = qrp^r(1 - q)^{-(r+1)} = qrp^r(p)^{-(r+1)} = \frac{rq}{p}$

Further, $\frac{d^2}{dt^2}\left(M_X(t)\right) = rqp^r \frac{d}{dt}\left\{(1 - qe^t)^{-(r+1)} e^t\right\}$

$$= rqp^r\left\{-(r+1)(1 - qe^t)^{-(r+2)}(-qe^{2t}) + (1 - qe^t)^{-(r+1)} e^t\right\}$$

Then $\mu_2' = \left[\frac{d^2}{dt^2}\left(M_X(t)\right)\right]_{t=0} = rqp^r\left[(r+1)qp^{-(r+2)} + p^{-(r+1)}\right]$

$$= r(r+1)q^2p^{-2} + rqp^{-1} = \frac{rq}{p}\left(\frac{(r+1)q}{p} + 1\right) = \frac{rq}{p^2}(rq + 1)$$

$$\Longrightarrow \quad \mu_2' = \frac{r^2q^2}{p^2} + \frac{rq}{p^2}$$

Hence, variance $= \sigma^2 = \mu_2' - (\mu_1')^2 = \frac{r^2q^2}{p^2} + \frac{rq}{p^2} - \frac{r^2q^2}{p^2} \Longrightarrow \quad \sigma_2^2 = \frac{rq}{p^2}$

Example 8:Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{3}e^{-\frac{x}{3}} & , \quad x > 0 \\ 0 & , \quad otherwise \end{cases}$$

Find

(i) $P(X > 3)$

(ii) **M.g.f. of X**

(iii) $E(X)$ **and** $Var(X)$

Solution:

(i) $$P(X>3)=\int_3^\infty f(x)\,dx=\int_3^\infty \frac{1}{3}e^{-\frac{x}{3}}dx=\frac{1}{3}\left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right]_3^\infty = -(0-e^{-1}) = e^{-1} = \frac{1}{e}$$

(ii) $$M_X(t)=E\left(e^{tX}\right)=\int_0^\infty e^{tx} f(x)dx$$

$$=\int_0^\infty e^{tx}\frac{1}{3}e^{-\frac{x}{3}}dx=\frac{1}{3}\int_0^\infty e^{\left(t-\frac{1}{3}\right)x}dx=\frac{1}{3}\int_0^\infty e^{-\left(\frac{1}{3}-t\right)x}dx=\frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^\infty$$

$$=\frac{1}{3}\left[0-\frac{1}{-\left(\frac{1}{3}-t\right)}\right]=\frac{1}{3}\left[\frac{1}{\left(\frac{1-3t}{3}\right)}\right]$$

$$M_X(t) = \frac{1}{1-3t} = (1-3t)^{-1}$$

$$\frac{d}{dt}[M_X(t)] = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$$

(iii) $$E(X) = Mean = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = 3$$

$$\frac{d^2}{dt^2}[M_X(t)] = -6(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = 18$$

$$Var(X) = E(X^2) - [E(X)]^2 = 18 - 9 = 9$$

Example 9: Let X be a discrete random variable with p.d.f.

$$\boldsymbol{p(x) = \begin{cases} \frac{1}{x(x+1)} & , \quad x = 1, 2 \ldots \\ 0 & , \quad otherwise \end{cases}}$$

Show that $\boldsymbol{E(X)}$ does not exist even though m.g.f. exist.

Solution:

$$E(X) = \sum_{x=1}^{\infty} x\, p(x) = \sum_{x=1}^{\infty} \frac{1}{x+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{x=1}^{\infty} \frac{1}{x} - 1$$

But $\boldsymbol{\sum_{x=1}^{\infty} \frac{1}{x}}$ is a divergent series.

Therefore, $E(x)$ does not exist and hence, no moment exists.

Now, m.g.f. of X is given by

$$M_X(t) = \sum_{x=1}^{\infty} p(x) e^{tx} = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

Substituting $z = e^t$,

$$M_X(t) = \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)} = \frac{z}{1.2} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \ldots$$

$$= z\left(1 - \frac{1}{2}\right) + z^2\left(\frac{1}{2} - \frac{1}{3}\right) + z^3\left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$= \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right) - \frac{z}{2} - \frac{z^2}{3} - \frac{z^4}{4} \ldots$$

$$= -\log(1 - z) - \frac{1}{z}\left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right)$$

$$= -\,log(1-z) + 1 + \frac{1}{z}\,log(1-z)\,, |z| < 1$$

$$= 1 + \left(\frac{1}{z} - 1\right) log(1-z)\,,\ \ |z| < 1$$

$$M_X(t) = \begin{cases} 1 + (e^{-\,t} - 1)\, log(1 - e^{t}) & , \qquad\qquad t < 0 \\ 1 & , \qquad for \quad t = 0 \end{cases}$$

and $M_X(t)$ does not exist for $t > 0$.