

P3:

Find the conditional probability density function of $X_{(i)}$ given $X_{(j)}$ for $1 \leq i < j \leq n$.

Solution:

The c.p.d.f. of $X_{(i)}$ given $X_{(j)}$ is given by

$$\begin{aligned} f_{X_{(i)}|X_{(j)}}(x|y) &= \frac{f_{X_{(i)},X_{(j)}}(x,y)}{f_{X_{(j)}}(y)} \\ &= \frac{\frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(x)]^{i-1} [F(y)-F(x)]^{j-i-1} [1-F(y)]^{n-j} f(x)f(y)}{\frac{n!}{(j-1)!(n-j)!} [F(y)]^{j-1} [1-F(y)]^{n-j} f(y)} \\ &= \frac{(j-1)!}{(i-1)!(j-i-1)!} [F(x)]^{i-1} [F(y) - F(x)]^{j-i-1} [F(y)]^{1-j} f(x) \text{ for } y \geq x \end{aligned}$$