P1:

Investigate the a.s. convergence of $\{\overline{X_n}\}$ to $\mathbf{0}$ where X_ns are independent and

$$\boldsymbol{P(X_n = 0) = 1 - \frac{1}{n}, P(X_n = \pm 1) = \frac{1}{2n}}$$

Solution:

Here $E(X_n) = 0\left(1 - \frac{1}{n}\right) + \frac{1}{2n} - \frac{1}{2n} = 0$ and

$\sigma_n^2 = V(X_n) = E(X_n^2) = 0^2\left(1 - \frac{1}{n}\right) + 1^2.\frac{1}{2n} + 1^2.\frac{1}{2n} = \frac{1}{n}$

Thus, we have

$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent and by $SLLN$ $\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right)$

But $E\left(\frac{S_n}{n}\right) = \frac{1}{n}E\left(\sum_{k=1}^{n} X_i\right) = \frac{1}{n}\sum_{k=1}^{n}(X_i) = 0$ and $\frac{S_n}{n} = \bar{X}_n$

Therefore, $\bar{X}_n \xrightarrow{a.s.} 0$