P1:

Find the m.g.f. of uniform distribution $\boldsymbol{U[a,b]}$ and hence obtain the mean and variance of the distribution.

Solution:

Since $X \sim U[a,b]$, the p.d.f. is given by $f(x) = \frac{1}{b-a}$, $a < x < b$.

The m.g.f. is given by

$$M_X(t) = E\left[e^{tX}\right] = \int_a^b e^{ty} . \frac{1}{b-a} dy = \frac{1}{b-a} \int_a^b e^{ty} dy$$

$$= \frac{1}{b-a} \left[\frac{e^{ty}}{t}\right]_a^b = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

$$\Longrightarrow \quad M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)} \qquad \text{for} \quad t \neq 0$$

$$\Longrightarrow tM_X(t) = \frac{e^{tb} - e^{ta}}{b-a} \qquad \dots (1)$$

On differentiating both sides of (1) w.r.t t, we get

$$t\, M_X'(t) + M_X(t) = \frac{(b\, e^{tb} - a\, e^{ta})}{b-a}$$

On differentiating both sides of (1) twice w.r.t t, we get

$$2M_X'(t) + tM_X''(t) = \frac{b^2 e^{bt} - a^2 e^{at}}{b-a}$$

In general, on differentiating both sides of (1) w.r.t. t, $(k+1)$ times we get

$$(k+1)\, \frac{d^k}{dt^k}\big(M_X(t)\big) + t\, \frac{d^{k+1}}{dt^{k+1}}\big(M_X(t)\big) = \frac{b^{k+1} e^{bt} - a^{k+1}\, e^{at}}{b-a}$$

$$\Longrightarrow \quad \frac{d^k M_X(t)}{dt^k} = \frac{1}{k+1}\left[\frac{b^{k+1} e^{bt} - a^{k+1}}{b-a} - t\, \frac{d^{k+1} M_X(t)}{dt^{k+1}}\right] \qquad \dots (2)$$

When $t = 0$ on both sides of (2), we get

$$\mu_k' = \frac{1}{k+1}\frac{(b^{k+1}-a^{k+1})}{b-a} \qquad \text{......}(3)$$

When $k = 1$, the mean is given by

$$\text{mean}= \mu = \mu_1' = \frac{1}{2}\left(\frac{b^2-a^2}{b-a}\right) = \frac{b+a}{2}$$

When $k = 2$, $\mu_2' = \frac{1}{3}\left(\frac{b^3-a^3}{b-a}\right) = \frac{(b^2+ab+a^2)\,(b-a)}{3\,(b-a)}$

$$= \frac{b^2+ab+a^2}{3}$$

Therefore, the Variance is given by

$$\sigma^2 = \mu_2' - (\mu_1')^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{4(b^2+ab+a^2)-3(b^2+2ab+a^2)}{12} = \frac{(b-a)^2}{12}$$

$$\Longrightarrow \quad \sigma^2 = \frac{(b-a)^2}{12}$$