P2:

Two random variables X and Y have j.p.d.f.

$$f(x,y) = \begin{cases} \frac{xy}{96}, & 0 < x < 4, 1 < y < 5 \\ 0, & otherwise \end{cases}$$

Find the correlation coefficient between X and Y.

Solution:

The marginal p.d.fs. of X and Y are given by

$$f_1(x) = \int_1^5 f(x,y)\,dy = \frac{1}{96} x \int_1^5 y\,dy = \frac{x}{8} \quad for\ 0 < x < 4$$

$$and\ \ f_2(y) = \int_0^4 f(x,y)\,dx = \frac{1}{96} y \int_0^4 x\,dx = \frac{y}{12} \quad for\ 1 < y < 5$$

$$\text{Then } E(X) = \int_0^4 x f(x)\,dx = \int_0^4 x \cdot \frac{x}{8}\,dx = \frac{1}{8}\int_0^4 x^2\,dx = \frac{8}{3}$$

$$and\ \ E(Y) = \int_1^5 y f_2(y)\,dy = \int_1^5 y \cdot \frac{y}{12}\,dy = \frac{1}{12}\int_1^5 y^2\,dy = \frac{31}{9}$$

$$E(XY) = \int_1^5\int_0^4 xy f(x,y)\,dx\,dy = \frac{1}{96}\int_1^5\int_0^4 x^2y^2\,dx\,dy = \frac{1}{96}\left(\int_1^5 y^2\,dy\right)\left(\int_0^4 x^2\,dx\right) = \frac{248}{27}$$

$$E(X^2) = \int_0^4 x^2 f_1(x)\,dx = \frac{1}{8}\int_0^4 x^3\,dx = 8$$

$$E(Y^2) = \int_1^5 y^2 f_2(y)\,dy = \frac{1}{12}\int_1^5 y^3\,dy = 13$$

$$\sigma_X{}^2 = V(X) = E(X^2) - \left(E(X)\right)^2 = 8 - \left(\frac{8}{3}\right)^2 = \frac{8}{9}$$

$$\sigma_Y{}^2 = V(Y) = E(Y^2) - \left(E(Y)\right)^2 = 13 - \left(\frac{31}{9}\right)^2 = \frac{92}{81}$$

$$\therefore \rho = \rho(X,Y) = \frac{cov(X,Y)}{\sigma_X\,\sigma_Y} = E(XY) - \frac{E(X)E(Y)}{\sigma_X\,\sigma_Y} = 0 \implies \rho = 0$$