Unit – 2

Probability distributions

2.1

Random Variable

While performing a random experiment we are mainly concerned with the assignment and computation of probabilities of events. In many experiments we are interested in some function of the outcomes of the experiment as opposed to the outcome itself. For instance, in tossing two dice we are interested in the sum of faces of the dice and are not really concerned about the actual outcome. That is, we may be interested in knowing that the sum is seven and not be concerned over whether actual outcome was $(1, 6)$ or $(2, 5)$ or $(3, 4)$ or $(4, 3)$ or $(5, 2)$ or $(6, 1)$. These quantities of interest or more formally these real valued function defined on the sample space are known as **random variables**.

Random variable (r. v): Let S be the sample space associated with a random experiment. Let $\boldsymbol{R}$ be the set of real numbers. If $X: S \longrightarrow \boldsymbol{R}$, $i.e.$, X is a real valued function defined on the sample space, then X is known as a **random variable**. In other words, random variable is a function which takes real values which are determined by the outcomes in the sample space.
The random variables are denoted by capital letters X, Y, Z … etc.

Notation: Let $a, b \in \boldsymbol{R}$. The set of all ω in S such that $X(\omega) = a$ is denoted by $X = a$. That is, $X = a$ denotes the event $\{\omega \in S | X(\omega) = a\}$. Similarly $X \leq a$ denotes the event $\{\omega \in S | X(\omega) \leq a\}$ and $a < X \leq b$ denotes the event $\{\omega \in S | X(\omega) \in (a, b]\}$.

Let us consider a random experiment of three tosses of a coin. Then the sample space S consits of $2^3 = 8$ points as given below.

$$S = \{H, T\} \times \{H, T\} \times \{H, T\}$$

$$= \{HH, HT, TH, TT\} \times \{H, T\}$$

$$= \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$$

For each outcome ω in S define $X(\omega)$ as the number of heads in the outcome ω. Then X may take any one of the values $0, 1, 2$ or 3. For each outcome in S, we have one value of X. Thus,

$$X(HHH) = 3\ ,$$

$$X(HTH) = X(THH) = X(HHT) = 2$$

$$X(TTH) = X(HTT) = X(THT) = 1 \text{ and}$$

$$X(TTT) = 0$$

This shows that X is a random variable.

Note that $X = 0, X = 1, X = 2$ and $X = 3$ respectively denote the events

$\{TTT\}$,$\{TTH, HTT, THT\}$,$\{HTH, THH, HHT\}$ and $\{HHH\}$

Discrete Random Variable (d. r. v): If the random variable assumes only a finite or countably infinite set of values, it is known as **discrete random variable**.
For example, the number of students attending the class, the number of defectives in a lot consisting of manufactured items and the number of accidents taking place on a busy road, etc., are all discrete random variables. In the above example X is a d.r.v.

Continuous Random Variable (c. r. v): If a random variable can assume uncountable set of values, it is said to a **continuous random variable**.
For example, the age, height or weight of the students in a class is all continuous random variables. In case of continuous random variable, we usually talk of the value in a particular interval and not at a point. Generally, discrete random

variable represents *count data* while continuous random variable represent *measured data*.

The probabilistic behavior of a d.r.v.X at each real point is described by a function called **probability mass function** and it is defined below:

Probability Mass Function (p.m.f): Let X be a discrete random variable with distinct values $x_1, x_2, \dots, x_n, \dots$ .The function $p : \boldsymbol{R} \to \boldsymbol{R}$ defined as

$$p(x) = \begin{cases} P(X = x_i) = p_i & if \;\; x = x_i \\ 0 & if \;\; x \neq x_i, i = 1,2, \dots, n, \dots \end{cases}$$

is called the **probability mass function** of r.v.X, if (i) $p(x) \geq 0 \; \forall x \in \boldsymbol{R}$ and
(ii) $\sum_{x \in R} p(x) = 1$

Probability Distribution: The set of all possible ordered pairs $\left(x_i, p(x_i)\right), i = 1,2, \dots, n, \dots$ is called the **probability distribution** of the r.v.X.

In particular, if X takes the values $x_1, x_2, x_3, \dots, x_n$ then the probability of X is usually represented in a tabular form as given below:

Probability Distribution of r. v. x

x	x_1	x_2	x_3	…	x_n
$p(x)$	p_1	p_2	p_3	…	p_n

Note: The concept of probability distribution is analogous to that of frequency distribution. Just as frequency distribution tells us how the total frequency is distributed among different values (or classes) of the variable, similarly a probability distribution tells us how total probability 1 is distributed among the various values which the r. v. can take.

Example 1: Obtain the probability distribution of X, the number of heads in three tosses of a coin (or a simultaneous toss of three coins).

Solution:

The sample space S consists of $2^3 = 8$ sample points, as give below:

$$S = \{H,T\} \times \{H,T\} \times \{H,T\} = \{HH, HT, TH, TT\} \times \{H,T\}$$

$$= \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$$

Obviously, X is a random variable which can take the values 0, 1, 2 or 3.

The probability distribution of X is computed as given below.

No. of heads $\boldsymbol{a}$	$\boldsymbol{X = a}$ $\{\boldsymbol{\omega \in S \mid X(\omega) = a}\}$	**No. of favourable cases**	$\boldsymbol{p(x) = P(X = a)}$
0	$\{TTT\}$	1	$\frac{1}{8}$
1	$\{TTH, HTT, THT\}$	3	$\frac{3}{8}$
2	$\{HTH, THH, HHT\}$	3	$\frac{3}{8}$
3	$\{HHH\}$	1	$\frac{1}{8}$

Hence, the probability distribution of X is given by:

x :	0	1	2	3
$p(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Probability Density Function (p. d. f): Let X be a continuous random variable defined on the sample space S. Let $f(x)$ be a real valued function defined on $\boldsymbol{R}$ such that, for any real numbers a and $b (a < b)$, $P(a \leq X \leq b) = \int_a^b f(x)dx$.

If the function $f(x)$ satisfies (i) $f(x) \geq 0 \quad \forall x \in \boldsymbol{R}$ and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$ then $f(x)$ is known as probability density function (p.d.f) of X

Note:

1. If X is a c. r. v., then $P(X = a) = 0$ where a is some real number.
2. Unlike discrete probability distribution, a continuous probability distribution can't be presented in a tabular form.

Cumulative Distribution Function (c. d. f): The cumulative distribution function of a r. v. X is defined by

$$F(x) = P(X \leq x) = = \begin{cases} \sum_{t \leq x} p(t) & \text{if } X \text{ is a d.r.v. with p.m.f } p(x) \\ \int_{-\infty}^{x} f(t)dt & \text{if } X \text{ is a c.r.v. with p.d.f } f(x) \end{cases}$$

Note: If X is a continuous random variable, then $\frac{d}{dx}F(x) = f(x)$

Properties of c.d.f.

1. If $a < b$, $P(a < X \leq b) = F(b) - F(a)$
2. $0 \leq F(x) \leq 1$ and $F(x) \leq F(y)$ if $x < y$
3. $F(-\infty) = 0$ and $F(\infty) = 1$
4. Discontinuities of $F(x)$ are atmost countable.

Note: The c.d.f. is used to find the cumulative probabilities in a probability distribution.

Example 2:

(i) **Find the constant $\boldsymbol{k}$ such that**

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{kx^2} & , \quad \mathbf{0} < x < 3 \\ \mathbf{0} & , \quad \textbf{otherwise} \end{cases}$$

is a p.d.f.

(ii) **Compute $\boldsymbol{P}(1 < x < 2)$**

(iii) Find the c.d.f and use it to compute $P(1 < x \leq 2)$

Solution:

(i) $f(x)$ is a p.d.f if

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow k\int_0^3 x^2 dx \Rightarrow k\left[\frac{x^3}{3}\right]_0^3 = 1 \Rightarrow k = \frac{1}{9}$$

$$f(x) = \begin{cases} \frac{1}{9}x^2 & , \quad 0 < x < 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(ii)

$$P(1 < x < 2) = \int_1^2 f(x)dx$$

$$= \int_1^2 \frac{1}{2}x^2 dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2 = \frac{7}{27}$$

(iii) We have,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

If $x < 0$, then $F(x) = 0$. If $0 \leq x < 3$, then

$$F(x) = \int_{-\infty}^{x} f(u)du = \frac{1}{9}\int_0^x u^2 du = \frac{x^3}{27}$$

If $x \geq 3$, then

$$F(x) = \int_0^3 f(u)du + \int_3^x f(u)du = \frac{1}{9}\int_0^3 u^2 du + \int_3^x du = \frac{1}{9} \times 9 + 0 = 1$$

Thus, required c.d.f is

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{x^3}{27} & , \quad 0 \leq x < 3 \\ 1 & , \quad x \geq 3 \end{cases}$$

Hence $P(1 < x \leq 2) = P(x \leq 2) - P(x \leq 1)$
$$= F(2) - F(1)$$
$$= \frac{2^3}{27} - \frac{1^3}{27} = \frac{8}{27} - \frac{1}{27} = \frac{7}{12}$$

Example 3: **A die is tossed twice. Getting *an odd number* is termed as a success. Find the probability distribution and c.d.f of the number of successes.**

Solution: Since the cases favorable to getting an odd number in a throw of a die are1,3,5, i.e., 3 in all.

Probability of success$(S) = \frac{3}{6} = \frac{1}{2}$; Probability of failure $(F) = 1 - \frac{1}{2} = \frac{1}{2}$.

If X dentoes the number of successes in two throws of a die, then X is a random variable which takes the values 0,1,2.

$$P(X = 0) = P[F \text{ in } 1^{\text{st}} \text{ throw and } F \text{ in } 2^{\text{nd}} \text{ throw}]$$
$$= P(FF) = P(F) \times P(F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$
$$P(X = 1) = P(S \text{ and } F) + P(F \text{ and } S)$$
$$= P(S)P(F) + P(F)P(S)$$
$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}.$$
$$P(X = 2) = P(S \text{ and } S) = P(S)P(S) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \quad .$$

Hence the probability distribution of X is given by :

x	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The c.d.f is given by

$$F(x)=\begin{cases} 0 & if \quad x<0 \\ \frac{1}{4} & if \quad 0\le x<1 \\ \frac{3}{4} & if \quad 1\le x<2 \\ 1 & if \quad x\ge 2 \end{cases}$$

Example 4: Two cards are drawn

(a) successively with replacement
(b) simultaneously (successively without replacement),

from a well shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Solution: Let X denote the number of aces obtained in a draw of two cards. Obviously, X is a random variable which can take the values 0, 1 or 2.

(a) Probability of drawing an ace is $\frac{4}{52}=\frac{1}{13}$

$\Longrightarrow$ Probability of drawing a non-ace is $1-\frac{1}{13}=\frac{12}{13}$.

Since the cards are drawn with replacement, all the draws are independent.

$$P(X=2)=P(\text{Ace and Ace})=P(\text{Ace})\times P(\text{Ace})=\frac{1}{13}\times\frac{1}{13}=\frac{1}{169}$$

$$P(X=1)=P(\text{Ace and Non-ace})+P(\text{Non-ace and Ace})$$

$$=P(\text{Ace})\times P(\text{Non-ace})+P(\text{Non-ace})\times P(\text{Ace})$$

$$=\frac{1}{13}\times\frac{12}{13}+\frac{12}{13}\times\frac{1}{13}=\frac{24}{169}.$$

$$P(X=0)=P(\text{Non-ace and Non-ace})$$

$$=P(\text{Non-ace})\times P(\text{Non-ace})=\frac{12}{13}\times\frac{12}{13}=\frac{144}{169}.$$

Hence, the probability distribution of X is given by:

x :	0	1	2
$p(x)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

(b) If cards are drawn without replacement, then exhaustive number of cases of drawing 2 cards out of 52 cards is ${}^{52}C_2$.

$\therefore P(X=0) = P(\text{No ace}) = P(\text{Both cards are non-aces})$

$$= \frac{{}^{48}C_2}{{}^{52}C_2} = \frac{48 \times 47}{52 \times 51} = \frac{188}{221}$$

$P(X=1) = P(\text{one ace}) = P(\text{one ace and one non-ace})$

$$= \frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{4 \times 48 \times 2}{52 \times 51} = \frac{32}{221}$$

$P(X=2) = P(\text{both aces}) = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{4 \times 3}{52 \times 51} = \frac{1}{221}$

Hence, the probability distribution of X is given by :

x :	0	1	2
$p(x)$:	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Example 5: **If X is a continuous random variable with p.d.f**

$$f(x) = \begin{cases} kx, & 0 \le x < 1 \\ k, & 1 \le x < 2 \\ -k(x-3), & 2 \le x < 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) **Determine k.**

(ii) **Compute $P(x \leq 1.5)$**

Solution:

(i) Since $f(x)$ is the p.d.f, so we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} kx\,dx + \int_{1}^{2} k\,dx + \int_{2}^{3} -k(x-3)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^1 + k\,[x]_1^2 - k\left[\frac{x^2}{2} - 3x\right]_2^3 = 1$$

$$\Rightarrow \frac{k}{2} + 2k - k - k\left[\left(\frac{9}{2} - 9\right) - (2-6)\right] = 1$$

$$\Rightarrow k\left[\frac{1}{2} + 2 - 1 - \frac{9}{2} + 9 + 2 - 6\right] = 1$$

$$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

(ii)

$$P(x \leq 1.5) = \int_{-\infty}^{1.5} f(x)\,dx = \int_{0}^{1} f(x)\,dx + \int_{-\infty}^{1.5} f(x)\,dx$$

$$= k\int_{0}^{1} x\,dx + \int_{1}^{1.5} k\,dx = k\left[\frac{x^2}{2}\right]_0^1 + k\,[x]_1^{1.5} = k\left[\frac{1}{2} + \frac{1}{2}\right] = k = \frac{1}{2}$$