1.4

Theorems in Probability

In this module, we shall prove some theorems which help us to evaluate the probabilities of some complicated events in a rather simple way. In proving these theorems, we shall follow the axiomatic approach based on the three axioms given in axiomatic definition of probability in module 1.3 on definitions of probability.

In a problem on probability, we are required to evaluate probability of certain statements. These statements can be expressed in terms of set notation and whose probabilities can be evaluated using theorems in probability. Let A and B be two events in S. Certain statements in set notation are given in the following table.

S. No.	Statement	Set notation
1.	At least one of the events A or B occurs	$A \cup B$
2.	Both the events A and B occur	$A \cap B$
3.	Neither A nor B occurs	$\bar{A} \cap \bar{B}$
4.	Event A occurs and B does not occur	$A \cap \bar{B}$
5.	Exactly one of the events A or B occurs	$(\bar{A} \cap B) \cup (A \cap \bar{B}) = A \Delta B$
6.	Not more than one of the events A or B occurs	$(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B})$
7.	If event A occurs, so does B	$A \subset B$
8.	Events A and B are mutually exclusive	$A \cap B = \phi$
9.	Complement of event A	$\bar{A}$
10.	Sample space	S

Example 1: Let A, B and C are three events in S. Find expression for the events in set notation.

(i) only A occurs
(ii) both A and B, but not C, occur
(iii) all three events occur
(iv) at least one occurs
(v) at least two occur
(vi) one and no more occurs
(Vii) two and no more occur
(viii) none occurs

Solution:

(i) $A \cap \bar{B} \cap \bar{C}$

(ii) $A \cap B \cap \bar{C}$

(iii) $A \cap B \cap C$

(iv) $A \cup B \cup C$

(v) $(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \bar{B} \cap \bar{C}) \cup \left(\bar{A} \cap B \cap \overline{C}\right) \cup (\bar{A} \cap \bar{B} \cap C)$

(vii) $(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$

(viii) $(\bar{A} \cap \bar{B} \cap \bar{C}) = (\overline{A \cup B \cup C})$

Theorems on Probability

Theorem 1: Probability of the impossible event is zero, i.e., $P(\phi) = 0$.

Proof: We know that $S \cup \phi = S \Rightarrow P\ (S) = P(S \cup \phi)$

$$\Rightarrow P(S) = P(S) + P\ (\phi) \text{ (Axiom 3)}$$

$$\Rightarrow P(\phi) = 0$$

Theorem 2: Probability of the complementary event $\bar{A}$ of A is given by $P(\bar{A}) = 1 - P(A)$.

Proof: Since A and $\bar{A}$ are mutually exclusive events in S,

$$A \cup \bar{A} = S \Rightarrow P(A \cup \bar{A}) = P(S) \Rightarrow P(A) + P(\bar{A}) = 1 \text{ (Axioms 2 and 3)}$$

$$\Rightarrow P(\bar{A}) = 1 - P(A)$$

Corollary 1: $\mathbf{0 \leq P(A) \leq 1}$

Proof: We have $P(A) = 1 - P(\bar{A}) \leq 1$ ($\because$ $P(\bar{A}) \geq 0$, by Axiom 1)

Further, $P(A) \geq 0$ (by Axiom 1). Therefore, $0 \leq P(A) \leq 1$

Corollary 2: $\boldsymbol{P(\phi) = 0}$

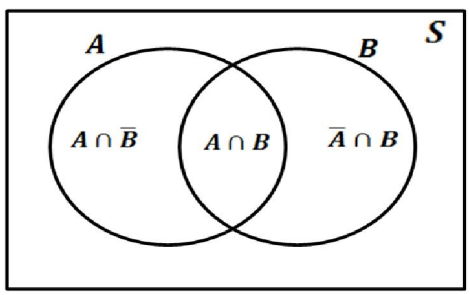
Proof: Since $\phi = \bar{S}$, $P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0$ (by Axiom 2)

$$\Rightarrow P(\phi) = 0$$

Theorem 3: For any two events A and B, we have

(i) $\boldsymbol{P(\bar{A} \cap B) = P(B) - P(A \cap B)}$ **(ii)** $\boldsymbol{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$

Proof:



(i) From the Venn diagram, we have,

$$B = (A \cap B) \cup (\bar{A} \cap B),$$

where $(\bar{A} \cap B)$ and $(A \cap B)$ are mutually exclusive events. Hence by Axiom 3,

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$
$$\boldsymbol{\Rightarrow P\ (\bar{A} \cap B) = P(B) - P(A \cap B)}$$

(ii) Similarly, we have,

$$A = (A \cap B) \cup (A \cap \bar{B}),$$

where $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive events. Hence by Axiom 3

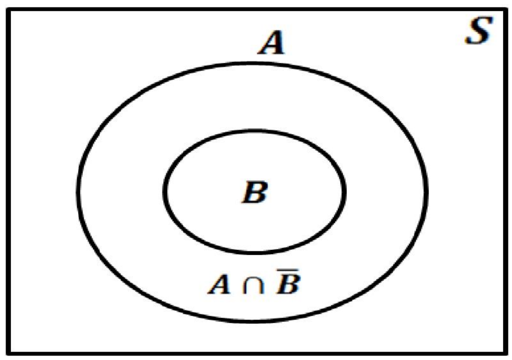
$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$\Longrightarrow \boldsymbol{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$$

Theorem 4: If $\boldsymbol{B \subset A}$, then

(i) $\boldsymbol{P(A \cap \bar{B}) = P(A) - P(B)}$ **(ii)** $\boldsymbol{P(B) \leq P(A)}$

Proof:



(i) If $B \subset A$, then B and $A \cap \bar{B}$ are mutually exclusive events and

$$A = B \cup (A \cap \bar{B})$$

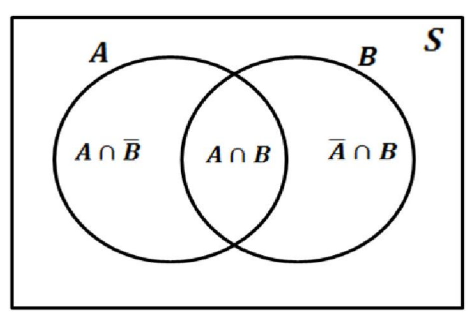
$$\Longrightarrow P(A) = P(B) + P(A \cap \bar{B}) \text{ (Axiom 3)}$$

$$\Longrightarrow P(A \cap \bar{B}) = P(A) - P(B)$$

(ii) We have $P(A \cap \bar{B}) \geq 0$ (Axiom 1). Hence $P(A) - P(B) \geq 0 \Rightarrow P(B) \leq P(A)$.
Thus, $B \subset A \Rightarrow P(B) \leq P(A)$.

Theorem 5: Addition Theorem of Probability for Two Events:
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any two events in $\boldsymbol{S}$. Then
$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Proof: From Venn diagram, we have



$$A \cup B = A \cup (\bar{A} \cap B)$$

where A and $\bar{A} \cap B$ are mutually exclusive events in S.

$\therefore\ P(A \cup B) = P(A) + P(\bar{A} \cap B)$ (Axiom 3)

$= P(A) + P(B) - P(A \cap B)$ (From Theorem 3)

Thus, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note:

1. If A and B are mutually exclusive events then $A \cap B = \phi$ and hence $P(A \cap B) = P(\phi) = 0$. Thus, if A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.
2. The addition theorem of probability for three events is given by

$$\boldsymbol{P(A \cup B \cup C) =}$$

$$\boldsymbol{P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$$

This can be proved first by taking $A \cup B$ as one event and C as second event and repeated application of Theorem 5

$$P(A \cup B \cup C) = P\big((A \cup B) \cup C\big) = P(A \cup B) + P(C) - P\big((A \cup B) \cap C\big)$$

$$= P(A \cup B) + P(C) - P\big((A \cap C) \cup (B \cap C)\big)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

3. Addition Theorem of Probability for $\boldsymbol{n}$-Events

Let $\boldsymbol{A_1, A_2, .., A_n}$ be $\boldsymbol{n}$ events in $\boldsymbol{S}$. Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} P(A_i \cap A_j) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j<k}}^{n} \sum_{k=1}^{n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Example 2: If two dice are thrown, what is the probability that the sum is (i) greater than 8, (ii) neither 7 nor 11 (iii) an even number on the first die or a total of 8?

Solution:

(i) If two dice are thrown, then $n(S) = 6^2 = 36$. Let T be the event getting the sum of the numbers greater than 8 on the two dice. Then $T = A \cup B \cup C \cup D$, where A, B, C and D repectively the events of getting the sum of 9,10,11 and 12. Note that A, B, C and D are pair wise mutually exclusive events. Therefore

$$P(T) = P(A) + P(B) + P(C) + P(D)$$

Note that $A = \{(3,6),(4,5),(5,4),(6,3)\}$ and $P(A) = \frac{4}{36}$

$B = \{(4,6),(5,5),(6,4)\}$ and $P(B) = \frac{3}{36}$

$C = \{(5,6),(6,5)\}$ and $P(C) = \frac{2}{36}$

$D = \{(6,6)\}$ and $P(D) = \frac{1}{36}$

$\therefore P(T) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$

(ii) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11.Then

$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ and $P(A) = \frac{6}{36}$

$B = \{(5,6),(6,5)\}$ and $P(B) = \frac{2}{36}$

$\therefore$ Required probability $=$ P (neither 7 nor 11)

$= P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B)]$ ($\because$ A and B are mutually exclusive events)

$= 1 - \left[\frac{6}{36} + \frac{2}{36}\right] = 1 - \frac{8}{36} = 1 - \frac{2}{9} = \frac{7}{9}$

(iii) Let A be the event of getting an even number on the first die and B be the event of getting the sum of 8. Therefore,

$A = \{2,4,6\} \times \{1,2,3,4,5,6\} \Rightarrow n(A) = 3 \times 6 = 18,$

$B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\} \Rightarrow n(B) = 5,$

$A \cap B = \{(2,6), (4,4), (6,2)\} \Rightarrow n(A \cap B) = 3$ and
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}$

Example 3: A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution: Let us define the following events:

A : The card drawn is a king

B : The card drawn is a heart

C : The card drawn is a red card

Then, A, B and C are not mutually exclusive.

$n(A) = 4$, $n(B) = 13$, $n(C) = 26$, $n(A \cap B) = 1$, $n(A \cap C) = 2$,
$n(B \cap C) = 13$, $n(A \cap B \cap C) = 1$.

$$P(A \cup B \cup C)$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{2}{52} - \frac{13}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

Compound event: The simultaneous occurrence of two or more events is termed as compound event.

Compound probability: The probability of a compound event is known as compound probability.

Conditional probability: The probability of an event A occuring when it is known that some event B has occurred, is called a conditional probability of the event A, given that B has occurred and denoted by $\boldsymbol{P(A|\ B)}$.

Definition: The conditional probability of the event A, given that B has occurred, denoted by $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \text{ if } P(B) > 0$$

If $P(B) = 0$, $P(A|B)$ is not defined.

Example 4: Consider a family with two children. Assume that each child is likely to be a boy as it is to be a girl. What is the conditional probability that both children are boys, given that (i) the older child is a boy (ii) at least one of the child is a boy?

Solution: We have the sample space $S = \{(b,b),(b,g),(gg\ ),(gg\ )\}$.Define the events:

A : Older child is a boy

B : Younger child is a boy

Therefore, $A = \{(b,b),(b,g)\}$, $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$, $B = \{(b,b),(gb\ )\}$

Then $A \cap B$: both are boys, $A \cap B = \{(b,b)\}$ and $P(A \cap B) = \frac{n(A\cap B)}{n(s)} = \frac{1}{4}$

$A \cup B$: At least one is a boy

and $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

(i) $P\left((A \cap B)|A\right) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

(ii) $P\left((A \cap B)|(A \cup B)\right) = \frac{P[(A\cap B)\cap(A\cup B)]}{P(A\cup B)} = \frac{P(A\cap B)}{P(A\cup B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Independent events: Two events A and B are said to be independent if the happening or non-happening of A is not affected by the happening or non-happening of B. Thus, A and B are independent if and only if the conditional probability of the event A given that B has happened is equal to the probability of A. That is,

$$P(A|B) = P(A) \text{ if } P(B) > 0$$

Similarly $P(B|A) = P(B)$ if $P(A) > 0$

By the definition of conditional probability, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Thus, A and B are independent events, if and only if

$$P(A \cap B) = P(A|B) \cdot P(B) = P(A) \cdot P(B)$$

In general, $A_1, A_2, \dots, A_n$ are independent events, if and only if

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_n)$$

Pair wise Independent Events: A set of events $A_1, A_2, \dots, A_n$ are said to be pairwise independent if every pair of different events are independent.

That is, $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ for all i and j, $i \neq j$.

Mutual Independent Events: A set of events $A_1, A_2, \dots, A_n$ are said to be mutually independent, if $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \dots \cdot P(A_{i_k})$ for every subset $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ of $\{A_1, A_2, \dots, A_n\}$.

Note: Pair wise independence does not imply mutual independence.

Theorem 6: Multiplication Theorem for Two events

Let A and B be any two events, then

$$P(A \cap B) = \begin{cases} P(A).P(B|A) & \text{if } P(A) > 0 \\ P(B).P(A|B) & \text{if } P(B) > 0 \\ P(A).P(B) & \text{if } A \text{ and } B \text{ are independent} \end{cases}$$

The proof follows from definition of conditional probability.

Note: Multiplication Theorem for n-Events $A_1, A_2, A_3, \ldots, A_n$

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = \begin{cases} P(A_1).P(A_2 \mid A_1).P(A_3 \mid (A_1 \cap A_2))\ldots P(A_n \mid (A_1 \cap A_2 \cap \ldots \cap A_{n-1})) \\ P(A_1).P(A_2).P(A_3)\ldots P(A_n), \text{if } A_1, A_2, \ldots, A_n \text{ are independent} \end{cases}$$

Theorem 7: If A_1 and A_2 are independent events, then A_1 and $\overline{A_2}$ are also independent.

Proof: **(See P3)**

Theorem 8: If A_1 and A_2 are independent events, then $\overline{A_1}$ and $\overline{A_2}$ are also independent.

Proof: **(See P4)**

Example 5: A fair dice is thrown twice. Let A, B and C denote the following events:

A : First toss is odd; B : Second toss is even; C : Sum of numbers is 7

- **(i) Find $P(A), P(B)$ and $P(C)$.**
- **(ii) Show that A, B and C are pair wise independent**
- **(iii) Show that A, B and C are not independent**

Solution:

(i) The number of outcomes in the sample space S is given by $n(S) = 6^2 = 36$.
We have,

$$A = \{1,3,5\} \times \{1,2,3,4,5,6\} \text{ and } n(A) = 3 \times 6 = 18$$

$$B = \{1,2,3,4,5,6\} \times \{2,4,6\} \text{ and } n(B) = 6 \times 3 = 18$$

$$C = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \text{ and } n(C) = 6$$

Therefore, $P(A) = \frac{18}{36} = \frac{1}{2}$, $P(B) = \frac{18}{36} = \frac{1}{2}$ and $P(C) = \frac{6}{36} = \frac{1}{6}$.

(ii) $A \cap B = \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$

$\therefore n(A \cap B) = 9$ and $P(A \cap B) = \frac{9}{36} = \frac{1}{4}$

But $P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Thus, $P(A \cap B) = P(A) \cdot P(B) \Rightarrow A$ and B are independent.

Next consider $A \cap C = \{(1,6),(3,4),(5,2)\}$

$\therefore n(A \cap C) = 3$ and $P(A \cap C) = \frac{3}{36} = \frac{1}{12}$.

But $P(B) \cdot P(C) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.

Thus, $P(B \cap C) = P(B) \cdot P(C) \Rightarrow B$ and C are independent

(iii) Consider $A \cap B \cap C = \{(1,6),(3,4),(5,2)\}$

$\therefore n(A \cap B \cap C) = 3$ and $P(A \cap B \cap C) = \frac{3}{36} = \frac{1}{12}$

But $P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24}$

Thus, $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

$\Rightarrow A, B$ and C are not independent.

Theorem 9: If A_1 and A_2 are independent events, then

$$\boldsymbol{P(A_1 \cup A_2) = 1 - P(\overline{A_1}) \cdot P(\overline{A_2})}$$

Proof: Consider $RHS = 1 - P(\overline{A_1}) \cdot P(\overline{A_2})$

$$= 1 - \left[\left(1 - P(A_1)\right) \cdot \left(1 - P(A_2)\right)\right]$$

$$= 1 - \left(1 - P(A_1) - P(A_2) + P(A_1) \cdot P(A_2)\right)$$

$$= 1 - 1 + P(A_1) + P(A_2) - P(A_1) \cdot P(A_2)$$

$$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= P(A_1 \cup A_2)$$

Thus, $P(A_1 \cup A_2) = 1 - P(\overline{A_1}) \cdot P(\overline{A_2})$.

Generalization: If $\boldsymbol{A_1, A_2, \dots, A_n}$ are $\boldsymbol{n}$ independent events, then

$$\boldsymbol{P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(\overline{A_1}) \cdot P(\overline{A_2}) \cdot \dots \cdot P(\overline{A_n})}$$

Example 6: A problem in probability is given to three students $\boldsymbol{A, B}$ and $\boldsymbol{C}$ whose chances of solving it are $\boldsymbol{\frac{1}{3}}$, $\boldsymbol{\frac{1}{4}}$ and $\boldsymbol{\frac{1}{5}}$ respectively. Find the probability that the problem will be solved if they all try independently.

Solution: Let E_1, E_2 and E_3 denote the events that the problem is solved by A, B and C respectively. Then, we have

$$P(E_1) = \frac{1}{3} \Rightarrow P(\overline{E_1}) = \frac{2}{3}$$

$$P(E_2) = \frac{1}{4} \Rightarrow P(\overline{E_2}) = \frac{3}{4}$$

$$P(E_3) = \frac{1}{5} \Rightarrow P(\overline{E_3}) = \frac{4}{5}$$

The problem is solved if atleast one of them is able to solve it.

Thus, $P(E_1 \cup E_2 \cup A_3) = 1 - P(\overline{E_1}) \cdot P(\overline{E_2}) \cdot P(\overline{E_3}) = 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = 1 - \frac{2}{5} = \frac{3}{5}$