P4:

The j.p.d.f. of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{1}{8}\,(x+y)\,, & 0 \le x \le 2\,, 0 \le y \le 2 \\ 0 \quad , & otherwise \end{cases}$$

Compute $\rho(X, Y)$

Solution:

The m.p.d.f. of X is given by

$$f_1(x) = \int_0^2 f(x,y)dy = \frac{1}{8}\int_0^2 (x+y)\,dy = \frac{1}{8}\left(xy + \frac{y^2}{2}\right)_0^2 = \frac{1}{8}(2x+2) = \frac{x+1}{4}$$

$\Rightarrow f_1(x) = \frac{x+1}{4}$ for $0 \le x \le 2$

Since $f(x,y)$ is symmetric in x and y,

$f_1(x) = f_2(y), E(X) = E(Y)$ and $V(X) = V(Y)$

Now, $E(X) = \int_0^2 x\,f(x)dx = \int_0^2 x\frac{(x+1)}{4}dx$

$$= \frac{1}{4}\int_0^2 (x^2 + x)\,dx = \frac{1}{4}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2 = \frac{1}{4}\left(\frac{8}{3} + 2\right) = \frac{7}{6} \text{ and}$$

$$E(X^2) = \int_0^2 \int_0^2 x^2\left(\frac{x+1}{4}\right)dx = \frac{1}{4}\int_0^2 (x^3 + x^2)dx$$

$$= \frac{1}{4}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^2 = \frac{1}{4}\left[\frac{16}{4} + \frac{8}{3}\right] = \frac{5}{3}$$

$$\therefore V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{5}{3} - \frac{49}{36} = \frac{60 - 49}{36} = \frac{11}{36}$$

Hence $E(X) = E(Y) = \frac{7}{6}$ and $V(X) = V(Y) = \frac{11}{36}$

Further,

$$E(XY) = \int_0^2 \int_0^2 xyf(x,y)dxdy = \frac{1}{8}\int_0^2 \int_0^2 xy(x+y)dxdy --- (1)$$

Now,

$$\int_0^2 xy\,(x+y)dx = y\int_0^2 (x^2 + xy)dx = y\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^2 = y\left[\frac{8}{3} + 2y\right] --(2)$$

From (1) & (2),

$$E(XY) = \frac{1}{8}\int_0^2 \left(\frac{8}{3}y + 2y^2\right)dy = \frac{1}{8}\left[\frac{8}{3}.\frac{y^2}{2} + 2.\frac{y^3}{3}\right]_0^2 = \frac{1}{8}\left[\frac{8}{3} \times 2 + 2x \times \frac{8}{3}\right] = \frac{4}{3}$$

Thus $cov\ (X,Y) = E(XY) - E(X).E(Y) = \frac{4}{3} - \frac{7}{6} \times \frac{7}{6} = \frac{48-49}{36} = -\frac{1}{36}$

and hence $\rho(X,Y) = \frac{cov\ (X,Y)}{\sqrt{V_{(X)}}\sqrt{V_{(Y)}}} = \frac{cov\ (X,Y)}{V(X)} = -\frac{\frac{1}{36}}{\frac{11}{36}} = -\frac{1}{11}$