2.4

Discrete Probability Distributions

Modules 2.1 and 2.2 deal with general properties of random variables**.** Random variables with special probability distributions are encountered in different fields of *science* and *engineering*. Some specific **discrete probability distributions** are discussed in this module and some specific **continuous probability distributions** are discussed in the next module 2.5.

Discrete Uniform Distribution: A r.v.X is said to have a **discrete uniform distribution** over the range $[1, n]$, if its p.m.f. is given by

$$p(x) = P(X = x) = \begin{cases} \dfrac{1}{n} & , \quad x = 1,2,\ldots,n \\ 0 & , \quad \text{otherwise} \end{cases}$$

Notation: $X \sim U(n)$,read as X follows discrete uniform distribution with parameter n.

Note: If all possible values of a r.v. are equally likely, then this distribution is used.

Example 1: If an unbiased coin is tossed once and X is equal to number of heads, then $X = 0,1$ and

$$P(X = 0) = P(X = 1) = \tfrac{1}{2} \text{ and } X \sim U(2).$$

Example 2: If an unbiased die is thrown once and X is equal to number on the die, then $x = 1,2,3,4,5,6$ and $P(X = i) = \frac{1}{6}$ for $i = 1,2,3,4,5,6$ and $X \sim U(6)$.

Mean and Variance: We have $E(X) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}$

and $E\left(X^2\right) = \frac{1}{n}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6}$

Thus $V(X) = E\left(X^2\right) - \left(E(X)\right)^2 = \frac{(n+1)(n-1)}{12}$

Bernoulli Experiment: A random experiment whose outcomes are of two types, **success (S)** and **failure (F),** occurring with probabilities p and $q(= 1 - p)$ respectively, is called a **Bernoulli experiment**.

Conducting a Bernoulli experiment once is known as **Bernoulli trial**. Note that p and q are same in each trial and outcomes of different trials are independent.

Bernoulli distribution: In a Bernoulli experiment, if a r.v.X is defined such that it takes value 1 with probability p when S occurs and 0 with probability q when F occurs, then we say that X follows Bernoulli distribution and its p.m.f. is given by

$$p(x) = P(X = x) = \begin{cases} p^x q^{1-x} & , \quad x = 0,1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Examples:

1) Tossing of a coin (results a head or tail)
2) Performance of a student in an examination (results pass or failure)
3) Sex of an unborn child (results female or male)

Mean and Variance:

Mean $= \mu = E(X) = 0 \times q + 1 \times p = p$

and $E(X^2) = 0^2 \times q + 1^2 \times p = p$

$\therefore$ Variance $= \sigma^2 = E(X^2) - \left(E(X)\right)^2 = p - p^2 = p(1 - p) = pq$

Binomial Distribution: Suppose we conduct n independent Bernoulli trials and we define

$$X = \text{number of successes in } n \text{ trials.}$$

Then X is a discrete random variable and it takes the values 0,1,2,…,n.

Derivation of $P(X = x)$: Note that $X = x$ means that there are x successes and $(n - x)$ failures in n trials in a specified order (say) SSFSFFFS … FSF.

Since outcomes of different trials are independent, by Multiplication Theorem, we have

$$P(SSFSFFFS \ldots FSF) = P(S) \cdot P(S) \cdot P(F) \cdot P(S) \cdot P(F) \cdot P(F) \cdot P(F) \cdot P(S) \cdot \ldots P(F) \cdot P(S) \cdot P(F)$$

$$= p\ p\ q\ p\ q\ q\ q\ p \ldots q\ p\ q$$

$$= \underbrace{p \cdot p \cdot \ldots \cdot p}_{(x\ times)} \cdot \underbrace{q \cdot q \cdot \ldots \cdot q}_{(n-x\ times)} = p^x q^{n-x}$$

But x successes in n trials can occur in $\binom{n}{x}$ orders and the probability for each of these orders is same, viz., $p^x q^{n-x}$. Hence by addition theorem of probability

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Definition: A r.v. X is said to follow a **binomial distribution** with parameters n and p if its p.m.f. is given by

$$p(x) = P(X = x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & , \quad x = 0,1,2, \ldots, n, 0 < p < 1, q = 1 - p \\ 0 & , \quad \text{otherwise} \end{cases}$$

Notation: $X \sim B(n, p)$. Read as X follows binomial distribution with parameters n and p.

Real life examples:

1) Number of heads in n tosses of a coin
2) Number of boys in a family of n children
3) Number of times hitting a target in n attempts

Note:

1.

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q + p)^n = 1$$

2. The c.d.f. of X is given by

$$F(x) = P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k q^{n-k} \quad, x = 0,1,2, \dots, n$$

Example 1: Four fair coins are tossed. If the outcomes are assumed to be independent, then find the p.m.f. and c.d.f. of the number of heads obtained.

Solution: Let X be the no. of heads in tossing 4 coins.

Then $X \sim B\left(4, \frac{1}{2}\right)$ where $p = P(head) = \frac{1}{2}$.

Thus $p(x) = P(X = x) = \binom{4}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

$$= \binom{4}{x}\left(\frac{1}{2}\right)^4 = \binom{4}{x}\left(\frac{1}{16}\right) \text{ for } x = 0,1,2,3,4.$$

Then $p(0) = \binom{4}{0}\left(\frac{1}{16}\right) = \frac{1}{16}$

$p(1) = \binom{4}{1}\left(\frac{1}{16}\right) = \frac{4}{16}$

$p(2) = \binom{4}{2}\left(\frac{1}{16}\right) = \frac{6}{16}$

$p(3) = \binom{4}{3}\left(\frac{1}{16}\right) = \frac{4}{16}$

$p(4) = \binom{4}{4}\left(\frac{1}{16}\right) = \frac{1}{16}$

The p.m.f $p(x)$ and c.d.f $F(x)$ are given in the following table.

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

Example 2: A and B play a game in which their chances of winning are in the ratio 3 : 2. Find A's chance of winning at least three games out of the five games played.

Solution:

Define $X =$ No. of games A winning out of 5.

Here $p = P(A\ winning) = \frac{3}{5}$ and $n = 5$ and $X \sim B\left(5, \frac{3}{5}\right)$. Thus,

$$p(x) = P(X = x) = \binom{5}{x}\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{5-x} \text{ for } x = 0,1,\dots,5.$$

Required to find:

$$P(A \text{ winning at least 3 out of 5 games}) = P(X \geq 3)$$

$$= P(X = 3) + P(X = 4) + P(X = 5)$$

$$= \binom{5}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 + \binom{5}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^1 + \binom{5}{5}\left(\frac{3}{5}\right)^5$$

$$= \frac{3^3}{5^5}[10 \times 4 + 5 \times 3 \times 2 + 1 \times 9]$$

$$= \frac{27 \times (40+30+9)}{3125} = 0.68$$

Example 3: The probability of a man hitting a target is $\frac{1}{4}$.

(i) **If he fires 7 times, what is the probability of his hitting the target at least twice?**

(ii) **How many times must he fire so that the probability of his hitting the target at least once is greater than $\frac{2}{3}$?**

Solution: Let X be the no. of times a man hitting the target in 7 fires. Here $p = P(man\ hitting\ the\ target) = \frac{1}{4}$ and $n = 7$. Then $X \sim B\left(7, \frac{1}{4}\right)$ and

$$p(x) = \binom{7}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{7-x} \text{ for } x = 0,1,2,\dots,7.$$

(i) $P(at\ least\ two\ hits) = P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - [p(0) + p(1)]$$

$$= 1 - \left\{\binom{7}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^7 + \binom{7}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6\right\} = \frac{4547}{8192} = 0.55$$

(ii) Find n such that $P(X \geq 1) > \frac{2}{3}$

$$\Longrightarrow 1 - P(X = 0) > \frac{2}{3}$$

$$\Longrightarrow -1 + P(X = 0) < -\frac{2}{3}$$

$$\Longrightarrow P(X = 0) < \frac{1}{3}$$

$$\Longrightarrow \binom{n}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n < \frac{1}{3}$$

$$\Longrightarrow \left(\frac{3}{4}\right)^n < \frac{1}{3}$$

$$\Longrightarrow n[\log 3 - \log 4] < \log 1 - \log 3$$

$$\Longrightarrow n[\log 4 - \log 3] > \log 3$$

$\Longrightarrow n > \frac{\log 3}{\log 4 - \log 3} = 3.8$,since n cannot be fractional, the required number of shots is 4.

Mean of Binomial Distribution:

$$\mu = E(X) = \sum_{x=0}^{n} x\, p(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \left(\frac{n}{x}\right)\binom{n-1}{x-1} p^x q^{n-x} = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

$$= np(q + p)^{n-1} = np$$

$$\Longrightarrow \mu = np$$

Variance of Binomial Distribution:

$\sigma^2 = V(X) = npq$ **(See $\boldsymbol{P_1}$ for proof)**

Example 4: One hundred balls are tossed into 50 boxes. What is the expected number of balls in the tenth box.

Solution: If we think of the balls tossed as Bernoulli trials in which a success is defined as getting a ball in the tenth box, then $p = \frac{1}{50}$. If X denotes the number of balls that go into the tenth box.

Then $X \sim B\left(100, \frac{1}{50}\right)$ and $E(X) = np = 100 \times \frac{1}{50} = 2$.

Example 5: The mean and variance of binomial distribution are $\mathbf{4}$ and $\frac{\mathbf{4}}{\mathbf{3}}$ respectively. Find $\boldsymbol{P}(\boldsymbol{X} \geq \mathbf{1})$.

Solution: Here $X \sim B(n, p)$. But $np = 4$ and $np(1-p) = \frac{4}{3}$.

Hence $4(1-p) = \frac{4}{3} \Longrightarrow 1 - p = \frac{1}{3} \Longrightarrow p = \frac{2}{3}$ and $n = \frac{4}{p} = 4 \times \frac{3}{2} = 6$.

Thus, $X \sim B\left(6, \frac{1}{3}\right)$ and hence $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - \binom{6}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^6 = 1 - \left(\frac{1}{3}\right)^6$$

Poisson Distribution:

If $n \to \infty$ and $p \to 0$ such that $\lambda = np$ fixed, then $\binom{n}{x} p^x (1-p)^{n-x} \longrightarrow \frac{e^{-\lambda}\lambda^x}{x!}$ which is the p.m.f. of Poisson distribution (**See $\boldsymbol{P_2}$**). Thus the p.m.f. of **Poisson distribution** is obtained as the limit of p.m.f. of **binomial distribution.**

Definition: A r.v.X is said to follow a **Poisson distribution** with parameter λ if its p.m.f. is given by

$$p(x) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & , \quad x = 0,1,2,\dots; \lambda > 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Notation: Read $X \sim P(\lambda)$ as: X follows poisson distribution with parameter λ.

Real life examples

1) Number of defectives in a packet of 100 blades.
2) Number of telephone calls received at a particular telephone exchange in some unit of time.
3) Number of print mistakes in a page of a book.
4) The number of fragments received by a surface area 'A' from a fragment atom bomb.
5) The emission of radio active (alpha) particles.
6) Number of air accidents in some unit of time.

Note :

1. $$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1\left(\because e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!}\right)$$

2. The c.d.f. of X is given by

$$F(x) = P(X \le x) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}$$

Example 6: Messages arrive at a switchboard in a Poisson manner at an average rate of six per hour. Find the probability for each of the following events:

a) Exactly two messages arrive within one hour.

b) No message arrives within one hour.

c) At least three messages arrive within one hour.

Solution: Let X be the r.v. that denotes the number of messages arriving at the switchboard within a one-hour interval. Then $X \sim P(6)$ and its p.m.f is given by

$$P(X = x) = p(x) = \frac{e^{-6}6^x}{x!} \text{ for } x = 0,1,2, \ldots$$

a) $P(X=2)=\frac{e^{-6}6^2}{2!}=\frac{36}{2}e^{-6}=18e^{-6}$.

b) $P(X=0)=\frac{e^{-6}6^0}{0!}=e^{-6}$.

c) $P(X\geq 3)=1-\{P(X=0)+P(X=1)+P(X=2)\}$

$$=1-\left\{\frac{e^{-6}6^0}{0!}+\frac{e^{-6}6^1}{1!}+\frac{e^{-6}6^2}{2!}\right\}=1-e^{-6}\{1+6+18\}=1-25e^{-6}$$

Example 7: In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

Solution:

$\lambda=$ Average number of typo-graphical errors/page$=\frac{390}{520}=0.75$

Let $X=$ Number of errors per page

Then $X\sim P(0.75)$ and $p(x)=P(X=x)=\frac{e^{-0.75}(0.75)^x}{x!}$

P(No error)$=P(X=0)=p(0)=e^{-0.75}$

P(A random sample of 5 pages contain no error)$=[p(0)]^5=[e^{-0.75}]^5=e^{-3.75}$

Mean of Poisson Distribution : The mean of poisson distribution is given by

$$\mu=E(X)=\sum_{x=0}^{\infty}x\,p(x)=\sum_{x=1}^{\infty}\frac{x\,e^{-\lambda}\lambda^x}{x!}=\lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}=\lambda e^{-\lambda}e^{\lambda}=\lambda$$

$\Rightarrow \mu=\lambda$

Variance of Poisson Distribution:

$$E(X^2)=\sum_{x=0}^{\infty}x^2P(x)=\sum_{x=0}^{\infty}[x(x-1)+x]p(x)$$

$$= \sum_{x=2}^{\infty} x(x-1)p(x) + \sum_{x=1}^{\infty} xp(x) = \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda = e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$\Longrightarrow E(X^2) = \lambda^2 + \lambda$

The variance of Poisson distribution is given by

$$V(X) = E(X^2) - \big(E(X)\big)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$\Longrightarrow V(X) = \lambda$

Note that *for Poisson distribution, mean and variance are equal.*

Example 8: If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent Poisson variates such that $\boldsymbol{P(X=1) = P(X=2)}$ and $\boldsymbol{P(Y=2) = P(Y=3)}$, find the variance of $\boldsymbol{X - 2Y}$.

Solution: Let $X \sim P(\lambda)$ and $Y \sim P(\mu)$. Then

$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0,1,2, \dots;\ \lambda > 0$ and

$P(Y = y) = \frac{e^{-\mu}\mu^y}{y!}$ for $y = 0,1,2, \dots;\ \mu > 0.$

Since $P(X = 1) = P(X = 2)$; $\lambda e^{-\lambda} = \frac{e^{-\lambda}\lambda^2}{2} \Longrightarrow \lambda^2 - 2\lambda = 0$

$\Longrightarrow \lambda(\lambda - 2) = 0 \Longrightarrow \lambda = 0, 2 \Longrightarrow \lambda = 2$ ($\because \lambda = 0$ is not admissible)

Since $P(Y = 2) = P(Y = 3)$, then $\frac{e^{-\mu}\mu^2}{2} = \frac{e^{-\mu}\mu^3}{6} \Longrightarrow \mu^3 - 3\mu^2 = 0$

$\Longrightarrow \mu^2(\mu - 3) = 0 \Longrightarrow \mu = 0{,}3 \Longrightarrow \mu = 3.$

$V(X - 2Y) = 1^2\, V(X) + (-2)^2\, V(Y) = \lambda + 4\mu = 2 + 4 \times 3 = 2 + 12 = 14$

Negative Binomial (or Pascal) Distribution:

Let X denote the number of failures before the r^{th} success in a sequence of Bernoulli trials. Then the number of trials required is $X + r$.

Derivation of $\boldsymbol{P(X = x)}$:

In $x + r$ trials, the last trial must be a success whose probability is p. In the remaining $(x + r - 1)$ trials, we must have $(r - 1)$ successes whose probability is $\binom{x + r - 1}{r - 1} p^{r-1} q^x$ (Using binomial distribution).

Thus, by multiplication theorem, we have

$$p(x) = P(X = x) = \binom{x + r - 1}{r - 1} p^{r-1} q^x \cdot p = \binom{x + r - 1}{r - 1} p^r q^x$$

Definition: A random variable X is said to follow a **Negative binomial distribution**(NBD) with parameters r and p if its p.m.f is given by

$$p(x) = P(X = x) = \begin{cases} \binom{x + r - 1}{r - 1} p^r q^x & , \quad x = 0,1,2, \ldots \\ \qquad 0 & , \quad \text{otherwise} \end{cases}$$

Notation: $X \sim NB(r, p)$.

Note:

1. $\binom{x + r - 1}{r - 1} = \binom{x + r - 1}{x} \quad \left(\because \binom{n}{r} = \binom{n}{n - r}\right)$

$$= \frac{(x+r-1)(x+r-2)\ldots(r+1)r}{x!}$$

$$= (-1)^r \frac{(-r)(-r-1)\ldots(-r-x+2)(-r-x+1)}{x!} = (-1)^x \binom{-r}{x}$$

Thus, the p.m.f. of NBD can be written as

$$p(x) = \begin{cases} \binom{-r}{x} p^r (-q)^x & , \quad x = 0,1,2, \ldots \\ \qquad 0 & , \quad \text{otherwise} \end{cases}$$

Further , it is the $(x+1)^{th}$ term in the expansion of $p^r(1-q)^{-r}$, a binomial expansion with negative index. Therefore, the distribution is known as negative binomial distribution.

2. $\sum_{x=0}^{\infty} p(x) = p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x = p^r (1-q)^{-r} = p^r p^{-r} = 1$

Mean of NBD: The mean of NBD is given by

$$\mu = E(x) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=1}^{\infty} x\, p(x) = \sum_{x=1}^{\infty} x \binom{-r}{x} p^r (-q)^x$$

$$= p^r \sum_{x=1}^{\infty} x \left(\frac{-r}{x}\right) \binom{\overline{-r+1}}{x-1} (-q)^x = (-r)(p^r)(-q) \sum_{x=1}^{\infty} \binom{\overline{-r+1}}{x-1} (-q)^{x-1}$$

$$= (-r)(-q)p^r(1-q)^{-(r+1)}$$

$$= rqp^r p^{-(r+1)} = \frac{rq}{p}$$

Variance of NBD: $\sigma^2 = \frac{rq}{p^2}$ **(See P_3 for proof)**

3. Notice that $\frac{\mu}{\sigma^2} = p < 1$ and this implies that mean is smaller than variance in NBD.
4. If $Y =$ Number of trials required to get r^{th} success, then $Y = X + r$ and

$$P(Y = y) = P(X + r = y) = P(X = y - r) = \binom{y-1}{r-1} p^r q^{y-r}$$

for $y = r, r+1, \ldots$ and $E(y) = E(X) + r = \frac{rq}{p} + 1$ and $V(Y) = V(X) = \frac{rq}{p^2}$

Real life examples

1) Number of tails before the third head.
2) Number of girls before the second son.
3) Number of non-defectives before the third defective.

Example 9: Find the probability that there are two daughters before the second son in a family when probability of a son is 0.5.

Solution: Let $X =$ the number of daughters before second son

Then $P(X = x) = \binom{x+2-1}{2-1}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^x = \binom{x+1}{1}\left(\frac{1}{2}\right)^{x+2}$

and $P(X = 2) = \binom{3}{1}\left(\frac{1}{2}\right)^4 = \frac{3}{16}$

Example 10: An item is produced in large numbers. The machine is known to produce 5% defectives. A quality control inspector is examining the items by taking them at random. What is the probability that at least 4 items are to be examined in order to get 2 defectives?

Solution: Let $Y =$ No. of items to be examined in order to get 2 defectives. Here $p =$(defective)$= \frac{5}{100} = 0.05$.

Then $P(Y = y) = \binom{y-1}{2-1}(0.05)^2(0.95)^{y-2}$

$\Rightarrow P(Y = y) = (y-1)(0.05)^2(0.95)^{y-2}$

We want to find

$$P(Y \geq 4) = 1 - \sum_{y=2}^{3} P(Y = y) = 1 - \{P(Y = 2) + P(Y = 3)\}$$

$$= 1 - \{(0.05)^2 + 2(0.05)^2(0.95)\} = 0.9928$$

Geometric distribution:

Let X denotes the number of failures before the first success in a sequence of Bernoulli trials. Then the required number of trials is $X + 1$.

Geometric distribution is a particular case of negative binomial distribution with $r = 1$.

Definition: A random variable X is said to follow a **Geometric distribution** (GD) with parameter p if its p.m.f. is given by

$$p(x) = P(X = x) = \begin{cases} p \cdot q^x & for\ \ x = 0,1,2, \dots \\ 0\ , & otherwise \end{cases}$$

Notation: $X \sim GD(r)$

Mean and variance of GD

$\mu = \frac{q}{p}$ and $\sigma^2 = \frac{q}{p^2}$ (take $r = 1$ in μ and σ^2 of NBD)

Note: Let $Y =$ Number of trials required to get first success, then $Y = X + 1$ and

$$P(Y = y) = P(X + 1 = y) = P(X = y - 1) = \begin{cases} p\, q^{y-1} & for\ \ y = 1,2,3, \dots \\ 0\ , & otherwise \end{cases}$$

Further, $E(Y) = E(X) + 1 = \frac{1}{p}$ and $V(Y) = V(X) = \frac{q}{p^2}$.

Real life examples

1) Number of tails before the third head
2) Number of girls before the second son
3) Number of non-defectives before the first defective

Example 11: find the probability that there are two daughters before the first son in a family where probability of a son is 0.5 .

Solution: Let $X =$ Number of daughters before the first son.

Then $P(X = x) = \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{x+1}$ for $x = 0,1,2, \dots$

and $P(X = 2) = \left(\frac{1}{2}\right)^{2+1} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Hyper geometric Distribution:

Consider an urn with N balls, M of which are white and $N - M$ are red. Suppose we draw a sample of n balls at random with replacement. Let X denote the number of white balls in the sample. Then $X \sim B(n,p)$ where $p = \frac{M}{N}$ which remains same for all trials and outcomes of different trials are independent. The p.m.f of X is given by $P(X = x) = \binom{n}{x}\left(\frac{M}{N}\right)^x\left(1 - \frac{M}{N}\right)^{n-x}$ for $i = 1,2,\dots,n$.

If the sample is selected without replacement, p is not same for all trials and outcomes of different trials are not independent and hence binomial distribution cannot be applied.

Derivation of $P(X = x)$: The number of all possible samples without replacement $= \binom{N}{x}$.

The number of samples in which there are x white balls and $(n - x)$ red balls $= \binom{M}{x}\binom{N-M}{n-x}$.

Thus $p(x) = P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$.

Definition: A random variable X is said to follow the **hyper geometric distribution** if its p.m.f is given by

$$p(x) = P(X = x) = \begin{cases} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} & , \quad x = 0,1,\dots \min(n,M) \\ 0 & , \quad otherwise \end{cases}$$

Example 12: A bag contains 4 white balls and 3 green balls. Three balls are drawn. What is the probability that 2 are white.

Solution: $N = 4 + 3 = 7,\ M = 4, n = 3$

$X =$ Number of white balls and

$$P(X = 2) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{4\times3\times3\times6}{2\times7\times6\times5} = \frac{18}{35}$$

Note: $$\sum_{x=0}^{\min(n,M)} p(x) = \begin{cases} \sum_{x=0}^{n} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \dfrac{\binom{N}{n}}{\binom{N}{n}} = 1 \ if\ \min(n,M) = n \\ \sum_{x=0}^{n} \dfrac{\binom{M}{x}\binom{N-M}{M-x}}{\binom{N}{M}} = \dfrac{\binom{N}{M}}{\binom{N}{M}} = 1 \ if\ \min(n,M) = M \end{cases}$$

Mean and variance of Hyper geometric distribution:

The mean is given by $\mu = \frac{nM}{N}$ and variance is given by $\sigma^2 = \frac{NM(N-M)(N-n)}{N^2(N-1)}$ if $\min(n, M) = n$ **(For proof, see $\boldsymbol{P_4}$**).