4.1. Order Statistics

Exercise

1. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the set of order statistics of independent r.v s with common p.d.f.
$$f(x) = \begin{cases} \beta\, e^{-x\beta} &, \quad x \ge 0 \\ 0 &, \quad otherwise \end{cases}$$
Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for any $s > r$.

2. Let $X_1, X_2, \dots, X_n$ be i.i.d .r.vswith p.d.f.
$$f(y) = \begin{cases} y^{\alpha} & if \quad 0 < y < 1 \\ 0 & otherwise, \quad \alpha > 0 \end{cases}$$
Show that $\frac{X_{(i)}}{X_{(n)}}$, $i = 1, 2, \dots, n-1$ and $X_{(n)}$ are independent.

3. Let $X_1, X_2, \dots, X_n$ be i.i.d .r.v s with common p.d.f .
$$f(x) = \alpha \frac{\sigma^{\alpha}}{x^{\alpha+1}}, \; x > \sigma \text{ where } \alpha > 0, \; \sigma > 0$$
Show that $X_{(1)}$ and $\left(\frac{X_{(2)}}{X_{(1)}}, \dots, \frac{X_{(n)}}{X_{(1)}}\right)$ are independent.

4. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the order statistics of n independent r.v.s $X_1, X_2, \dots, X_n$ with common p.d.f.
$$f(x) = \begin{cases} 1 & if \;\; 0 < x < 1 \\ 0 & otherwise \end{cases}$$
Show that $Y_1 = \frac{X_{(1)}}{X_{(2)}}, \quad Y_2 = \frac{X_{(2)}}{X_{(3)}}, \dots, Y_{n-1} = \frac{X_{(n-1)}}{X_{(n)}}$ and $Y_n = X_{(n)}$ are independent.

5. An urn contains N identical marbles numbered 1 through N. Form this urn n marbles are drawn, and let $X_{(n)}$ be the largest number drawn. Show that
$$P\left[X_{(n)} = k\right] = \frac{\binom{k-1}{n-1}}{\binom{N}{n}}, \quad k = n, n+1, \dots, N$$
and $E\left(X_{(n)}\right) = \frac{n(N+1)}{(n+1)}$.