P3:

Let $\{X_n\}$ be a sequence of independent r.vs such that

$$\boldsymbol{P(X_n = \pm 1) = \frac{1}{2}(1 - 2^{-n}), P(X_n = \pm 2^{-n}) = 2^{-n-1}}$$

Does the $SLLN$ hold for this sequence?

Solution:

Here $E(X_n) = 1\frac{1}{2}(1 - 2^{-n}) - 1\,\frac{1}{2}(1 - 2^{-n}) + 2^{-n}2^{-n-1} - 2^{-n}2^{-n-1} = 0$

and $\sigma_n^2 = V(X_n) = E(X_n^2) = 1^2\frac{1}{2}(1 - 2^{-n}) + 1^2\frac{1}{2}(1 - 2^{-n}) + 2^{-2n}2^{-2n-1}$

$+2^{-2n}\,2^{-n-1}$

$$= 2\,\frac{1}{2}(1 - 2^{-n}) + 2.\,2^{-2n}2^{-n-1}$$

$\Longrightarrow \sigma_n^2 = 1 - 2^{-n} + 2^{-3n}$

Further, we have $\sum_{n=1}^{\infty}\frac{\sigma_n^2}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{2^n n^2} + \sum_{n=1}^{\infty}\frac{1}{2^{3n} n^2}$ converges.

Thus $\{X_n\}$ obeys $SLLN$.