P2:

Examine if CLT holds for the sequence $\{X_k\}$ with p.m.f. $P(X_k = \pm k^\lambda) = \frac{1}{2}$.

Solution:

We have $\mu_k = E(X_k) = k^\lambda \frac{1}{2} - k^\lambda \frac{1}{2} = 0$,

${\sigma_k}^2 = V(X_k) = k^{2\lambda} \frac{1}{2} + k^{2\lambda} \frac{1}{2} = k^{2\lambda}$ and

$\rho_k^3 = E\{|X_k - 0|^3\} = E(|X_k|^3) = k^{3\lambda} \cdot \frac{1}{2} + k^{3\lambda} \cdot \frac{1}{2} = k^{3\lambda}$

Let $S_n = \sum_{k=1}^{n} X_k$. Then we have $\mu = \sum_{k=1}^{n} \mu_k = 0$, $\sigma^2 = \sum_{k=1}^{n} {\sigma_k}^2 = \sum_{k=1}^{n} k^{2\lambda}$

$\rho^3 = \sum_{k=1}^{n} {\rho_k}^3 = \sum_{k=1}^{n} k^{3\lambda}$ and

$$\frac{\rho^3}{(\sigma^2)^{\frac{3}{2}}} = \frac{\sum_{k=1}^{n} k^{3\lambda}}{\left(\sum_{k=1}^{n} k^{2\lambda}\right)^{\frac{3}{2}}} = \frac{n^{3\lambda+1}}{3\lambda+1} \times \left(\frac{2\lambda+1}{n^{2\lambda+1}}\right)^{\frac{3}{2}} \qquad \left(\begin{array}{l} \because \sum_{k=1}^{n} k^\alpha = \int_0^n x^\alpha dx = \frac{n^{\alpha+1}}{\alpha+1} \\ Euler - maclaurian\ formula \end{array}\right)$$

$$= \frac{(2\lambda+1)^{\frac{3}{2}}}{(3\lambda+1)} n^{(3\lambda+1)-(2\lambda+1)\frac{3}{2}}$$

$$= \frac{(2\lambda+1)^{\frac{3}{2}}}{(3\lambda+1)} n^{-\frac{1}{2}} \to 0\ as\ n \to \infty$$

Since Liapnounoff's condition is satisfied, CLT holds.