P4:

For hyper geometric distribution, show that mean and variance are given by

$$\boldsymbol{\mu = E(X) = \frac{nM}{n}} \quad \textbf{and}$$

$$\boldsymbol{\sigma^2 = V(X) = \frac{NM(N-M)(N-n)}{N^2(N-1)}}$$

when $\boldsymbol{\min(n, M) = n}$.

Proof:

We have $p(x) = \dfrac{\dbinom{M}{x}\dbinom{N-M}{n-x}}{\dbinom{N}{n}}$

$$E(X) = \sum_{x=0}^{n} x\,p(x) = \sum_{x=0}^{n} \frac{x\dbinom{M}{x}\dbinom{N-M}{n-x}}{\dbinom{N}{n}}$$

$$= \sum_{x=1}^{n} \frac{x\left(\dfrac{M}{x}\right)\dbinom{M-1}{x-1}\dbinom{N-M}{n-x}}{\dbinom{N}{n}}$$

$$= \frac{M}{\dbinom{N}{n}} \sum_{x=1}^{n} \binom{M-1}{x-1}\binom{N-M}{n-x} = \frac{M}{\dbinom{N}{n}} \binom{N-1}{n-1}$$

$$= \frac{M}{\left(\dfrac{N}{n}\right)\dbinom{N-1}{n-1}} \binom{N-1}{n-1} \quad \left(\because \sum_{x=1}^{n} \binom{M-1}{x-1}\binom{N-M}{n-x} = \binom{N-1}{n-1}\right)$$

$$\Rightarrow \mu = \frac{nM}{N}$$

$$Now, E\left(X^2\right) = \sum_{x=0}^{n} x^2 \ p\left(x\right) = \sum_{x=0}^{n}\left[x\left(x-1\right)+x\right]p\left(x\right) = \sum_{x=2}^{n} x\left(x-1\right)px + \sum_{x=0}^{n} xp\left(x\right)$$

$$= \sum_{x=2}^{n} \frac{x\left(x-1\right)\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} + \mu$$

$$= \sum_{x=2}^{n} \frac{x\left(x-1\right)\left(\frac{M}{x}\right)\left(\frac{M-1}{x-1}\right)\binom{M-2}{x-2}\binom{N-M}{n-x}}{\binom{N}{n}} + \mu$$

$$= \frac{M\left(M-1\right)}{\binom{N}{n}} \sum_{x=2}^{n} \binom{M-2}{x-2}\binom{N-M}{n-x} + \mu$$

$$= \frac{M\left(M-1\right)}{\binom{N}{n}} \binom{N-2}{n-2} + \mu \quad \left(\because \sum_{x=2}^{n} \binom{M-2}{x-2}\binom{N-M}{n-x} = \binom{N-2}{n-2}\right)$$

$$= \frac{M\left(M-1\right)}{\left(\frac{N}{n}\right)\left(\frac{N-1}{n-1}\right)\binom{N-2}{n-2}} \binom{N-2}{n-2} + \mu$$

$$\Rightarrow E\left(X^2\right) = \frac{n\left(n-1\right)M\left(M-1\right)}{N\left(N-1\right)} + \frac{nM}{N}$$

$$Thus,\ V(X) = E(X^2) - (E(X))^2$$

$$= \frac{n(n-1)M(M-1)}{N(N-1)} + \frac{nM}{N} - \frac{n^2M^2}{N^2}$$

$$= \frac{NM(N-M)(N-n)}{N^2(N-1)}\ (on\ simplification)$$