P1:

Obtain the p.d.f. of $Z = X + Y$, where X and Y are two independent random variables with the following p.d.fs:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & ; \quad a < x < b \\ 0 & ; \quad otherwise \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{d-c} & ; \quad c < y < d\,; \; d-c < b-a \\ 0 & ; \quad otherwise \end{cases}$$

Solution:

The two p.d.fs are shown in the *fig.* 1. To evaluate the limits of integration of the p.d.f. of Z, we consider the following regions represented by the diagram shown in *fig.* 2.

When $z < a + c$, $f_Z(z) = 0$. When $a + c \le z \le a + d$ (see *figure 2(i*), we obtain)

$$f_Z(z) = \frac{1}{(b-a)(d-c)} \int_a^{z-c} dy = \frac{z-c-a}{(b-a)(d-c)}$$

When $a + d \le z \le$ b + c (see *fig.* $2(ii)$), we obtain)

$$f_Z(z) = \frac{1}{(b-a)(d-c)} \int_{z-d}^{z-c} dy = \frac{1}{b-a}$$

When $b + c \le z \le b + d$ (see *fig.* $2(ii)$), we obtain

$$f_Z(z) = \frac{1}{(b-a)(d-c)} \int_{z-d}^{b} dy = \frac{b+d-z}{(b-a)(d-c)}$$

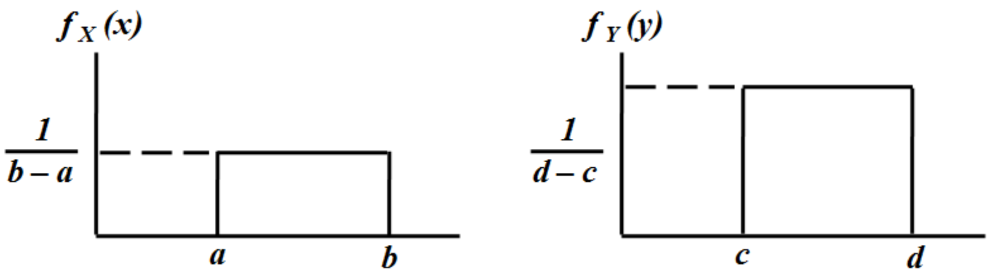

Figure 1: p.d.fs of X and Y

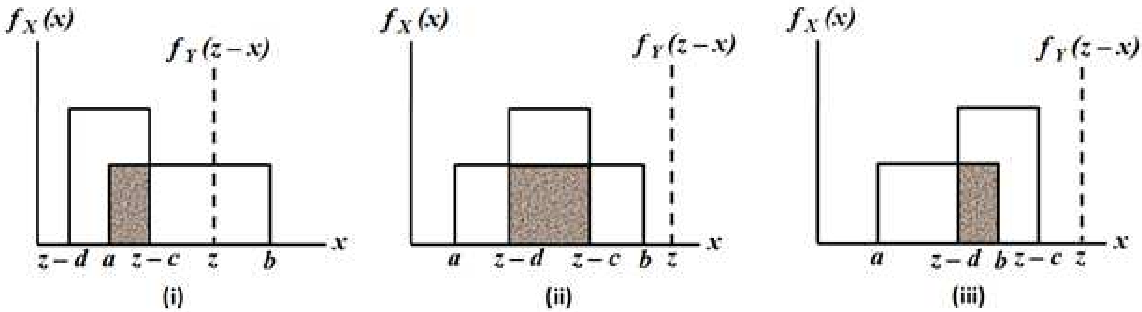

Figure 2: Convolution of the p.d.fs for different values z

Finally, when $z > b + d$, $f_Z(z) = 0$, thus, the p.d.f of Z is given by

$$f_Z(z) = \begin{cases} 0 & , \quad z < a + c \\ \dfrac{z - a - c}{(b-a)(d-c)} & , \quad a + c \leq z \leq a + d \\ \dfrac{1}{b-a} & , \quad a + d \leq z \leq b + c \\ \dfrac{b + d - z}{(b-a)(d-c)} & , \quad b + c \leq z \leq b + d \\ 0 & , \quad z > b + d \end{cases}$$

The p.d.f is graphically illustrated in the following figure which is a **trapezoid**. Note that when $b - a = d - c$, the p.d.f reduces to an **isosceles triangle** centered at $z = \frac{a+c+b+d}{2}$. In the special case when $a = c$ and $b = d$, the isosceles triangle is centered at $z = a + b$.