Algebra of Sets and Counting Methods

The algebra of sets and counting methods are useful in understanding the basic concepts of *probability*. These concepts are briefly reviewed from the point of view of probability.

Sets and Elements of sets: The fundamental concept in the study of the probability is the set.
A set is a well defined collection of objects and denoted by upper case English letters. The objects in a set are known as **elements** and denoted by lower case letters. A set can be written in two ways. Firstly, if the set has a finite number of elements, we may list the elements, separated by commas and enclosed in brackets. For example, a set A with elements $1, 2, 3, 4, 5$ and 6, it may be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

Secondly, the set may be described by a statement or a rule. Then A may be written as

$$A = \{x \mid x \text{ is a natural number less than or equal to } 6\}$$

If x is an element of the set A, we write $x \in A$. If x is not a element of the set A, then we write $x \notin A$.

Equal Sets: Two sets A and B are said to be ***equal*** or ***identical*** if they have exactly the same elements and we write as $A = B$

Subset: If every element of the set A belong to the set B, $i.e.$, if $x \in A \implies x \in B$, then we say that A is a ***subset*** of B and we write $A \subseteq B$ (A is contained in B) or $B \supseteq A$ (B contians A). If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Null set: A ***null*** or an ***empty set*** is one which does not contain any element at all and denoted by $\emptyset$.

Note:

1. Every set is a subset it self

2. An empty set is a subset of every set.
3. A set containing only one elements is conceptually different from the element itself .
4. In all applications of set theory, especially in probability theory, we shall have a fixed set S (say), given in advance and we shall be concerned only with subsets of S. This set is referred to universal set.

1) Union or sum:

$$A \cup B = \{x|\, x \in A \;\; or \;\; x \in B\}$$

$$\bigcup_{i=1}^{n} A_i = \{x|\;\; x \in A_i \quad for\ at\ least\ one\ i = 1, 2, \dots, n\}$$

2) Intersection or Product:

$$A \cap B = \{x|\;\; x \in A \;\; and \;\; x \in B\}$$

$$\bigcap_{i=1}^{n} A_i = \{x|\, x \in A_i \;\; for\ all\ i = 1, 2, \dots, n\}$$

If $A \cap B = \emptyset$, then we say that A and B are **disjoint sets.**

3) Relative Difference: $A - B = \{x|\;\; x \in A \;\;\; and \;\; x \notin B\}$

4) Complement: $\bar{A} = S - A$

Algebra of Sets:

If A, B and C are subsets of a universal set S, then the following laws hold:

Commutative laws: $A \cup B = B \cup A,\;\; A \cap B = B \cap A$

Associative laws: $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$

Distributive laws:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$

Complementary laws: $A \cup \bar{A} = S, A \cap \bar{A} = \emptyset, A \cup S = S, A \cap S = A$

Difference laws: $A - B = A \cap \bar{B} = A - (A \cap B) = (A \cup B) - B$,

$A - (B - C) = (A - B) \cup (A - C), (A \cup B) - C = (A - C) \cup (B - C)$,

$(A \cap B) \cup (A - B) = A, (A \cap B) \cap (A - B) = \emptyset$

De – Morgan's laws:

$\overline{A \cup B} = \bar{A} \cap \bar{B}, \overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \bar{A}_i$ and $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \bar{A}_i$

Involution law: $\overline{(\bar{A})} = A$

Idempotent law: $A \cup A = A, A \cap A = A$

Class of Sets:
A group of sets will be termed as a class of sets. We shall define some useful types of classes used in probability.

Field: A field $\mathbb{F}$ (or algebra) is a non – empty class of sets which is closed under the formation of finite unions and under complementation. Thus,

(i) $A \in \mathbb{F},\ B \in \mathbb{F} \implies A \cup B \in \mathbb{F}$ and
(ii) $A \in \mathbb{F} \implies \bar{A} \in \mathbb{F}$

***σ* – Field:** A σ – field $\mathbb{B}$ (or σ – algebra) is a non – empty class of sets that is closed under the formation of countable union and complementation. Thus,

(i) $A_i \in \mathbb{B},\ i = 1,2, \dots, \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathbb{B}$
(ii) $A \in \mathbb{B} \Rightarrow \bar{A} = \mathbb{B}$

Fundamental Principle of Addition (Principle of inclusion- exclusion)

Let $A_1, A_2, \dots A_m$ be m sets and the elements in each sets are different. Then the number of ways of selecting an element from A_1 or A_2 or …A_m is given by

$$n\left(\bigcup_{i=1}^{m} A_i\right)=\sum_{i=1}^{m} n(A_i)-\sum_{\substack{i=1\\ i<j}}^{m}\sum_{j=1}^{m} n\left(A_i \cap A_j\right)+\sum_{\substack{i=1\\ i<j<k}}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} n\left(A_i \cap A_j \cap A_k\right)- \dots$$

$$\dots.+(-1)^{m-1}\, n\left(\bigcap_{i=1}^{m} A_i\right)$$

where $n(A)$ represents the number of elements in A.

Note:

1. $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$
2. $n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_1 \cap A_3)$

$$-n(A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$$

Example 1: Find the number of ways of selecting

(i) A diamond or heart

(ii) An ace or a spade

from a pack of 52 cards

Solution: Let A_1 be the set of diamonds, A_2 be the set of hearts, A_3 be the set of aces and A_4 set of spades.

(i) Here $(A_1) = 13$, $n(A_2) = 13$ and A_1, A_2 are disjoint.
Hence $n(A_1 \cap A_2) = 0$ and
$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2) = 13 + 13 - 0 = 26$

(ii) Here $n(A_3) = 4$ and $n(A_4) = 13$. Note that A_3 and A_4 are not disjoint and $n(A_3 \cap A_4) = 1$. Hence $n(A_3 \cup A_4) = 4 + 13 - 1 = 16$

Note: If $A_1, A_2, \dots, A_n$ are pair-wise disjoint sets, then there will be no common elements to these sets and hence

$$n\left(\bigcup_{i=1}^{m} A_i\right)=\sum_{i=1}^{m} n\left(A_i\right)$$

Fundamental Principle of Multiplication (Product rule)

Let $A_1, A_2, .., A_m$ be m sets and the elements in each set are distinct. Then the number of ways of selecting first object from A_1, second object from A_2, ..., m^{th} object from A_m in succession is given by

$$n(A_1 \times A_2 \times \ldots \times A_m) = n(A_1).n(A_2)\ldots n(A_m)$$

Example 2: A man has 8 different shirts and 6 different pants. In how many different ways, he can be dressed?

Solution: Choosing a dress means selection of one shirt and one pant. The total number of ways of choosing a dress is $8 \times 6 = 48$

Example 3: Two dice are thrown.

(i) How many different outcomes are there?
(ii) How many different outcomes with distinct values (no doubles)?

Solution: On each die, we may get the number 1 or 2 or 3 or 4 or 5 or 6.
One outcome means one number on first die and another number on second die.

(i) Number of different out comes $= 6 \times 6 = 36$
(ii) Number of different out comes $= 6 \times 5 = 30$

Permutations

A permutation is an arrangement or an ordered selection of objects. There is importance to the order of objects in a permutation.

1) The number of permutations of n different objects taken $r(\leq n)$ at a time is

$^{n}P_{r} = n(n-1)\ldots(n-(r-1))$ when repetition of objects is not allowed.

The number of permutations of n different objects taken $r(\leq n)$ at a time is n^r when repetition of objects is allowed any number of times.

2) The number of permutations of n different objects taken all at a time when repetition of objects is not allowed is $n!$
3) If there are n objects, n_1 of type 1, n_2 of type 2,..., n_k of type k , where $n_1 + n_2 + n_3 \ldots + n_k = n$, then the number of permutations of these n objects taken all at a time is

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_k!}$$

4) The number of permutations of n different objects taken r at a time without repetitions in which
   (i) k particular objects will always occur is $^{n-k}P_{r-k}\ ^{r}P_{k}$
   (ii) s particular objects will never occur is $^{(n-s)}P_{r}$
   (iii) k particular objects will always occur and s particular objects will never occur is $^{(n-k-s)}P_{r-k}\ ^{r}P_{k}$

Combinations

A combination is an unordered selection or subset of the objects. There is no importance to the order of the objects in a combination.

1) The number of combinations of n different objects taken $r(\leq n)$ at a time is denoted by $^{n}C_{r}$ and $^{n}C_{r} = \frac{^{n}P_{r}}{r!}$ when repetition of objects is not allowed.

   The number of combinations of n different objects taken $r(\leq n)$ at a time is $^{(n+r-1)}C_{r}$ when repetition of objects is allowed.
2) The number of combinations of n different objects taken r at a time without repetitions in which
   (i) k particular objects will always occur is $^{(n-k)}C_{r-k}$

(ii) s particular objects will never occur is $(n-s)_{C_r}$

(iii) k particuarl objects will always occur and s particular objects will never occur is $(n-k-s)_{C_{r-k}}$

3) The number of combinations of n different objects taken any number (one or more) at a time when repetitions are not allowed is

$$n_{C_1} + n_{C_2} + \cdots + n_{C_n} = 2^n - 1$$

4) The total number of combinations of $(n_1 + n_2 + \cdots + n_k)$ objects taken any number at a time when n_1 objects are of type 1, n_2 are of type 2, … , n_k are of type $k = (n_1 + 1)(n_2 + 1) \ldots (n_k + 1) - 1$

5) The total number of combinations of $(n_1 + n_2 + \cdots + n_k + m)$ objects **taken any number at a time** when n_1 objects are of type 1, n_2 are of type 2, … , n_k are of type k =$(n_1 + 1)(n_2 + 1) \ldots (n_k + 1). 2^m - 1$

Circular Permutations

An arrangement of objects arranged in a circle is known as a circular permutation.

1) The number of circular permutations of n different objects taken all at a time is $(n-1)!$
2) The number of circular permutations of n different objects taken all at a time when clockwise and anticlockwise arrangements are considered the same (as in Necklace, Garland) is $\frac{(n-1)!}{2}$
3) The number of circular permutations of n different objects taken r at a time is $n_{C_r}(r-1)! = \frac{n_{P_r}}{r}$
4) The number of circular permutations of n different objects taken r at a time when no distinction is made between clockwise and anticlockwise direction $= \frac{1}{2} . \frac{n_{P_r}}{r}$.

Distribution or Occupancy Problems

The number of ways, r objects can be distributed among n different boxes, depends upon the fact: how many objects are permitted to be in one box and whether the objects are different or not. Problems involving the distribution of objects among boxes are called distribution or occupancy problems.

The distribution of different objects corresponds to permutations and distribution of identical objects corresponds to combinations.

Distribution of Different Objects:

1. The number of ways of distributing r different objects into n different boxes if

(i) no restriction is placed on the number of objects permitted in a box is n^r.

(ii) a particular box contains exactly k objects is $r_{C_k}.(n-1)^{r-k}$.

(iii) at most one object is permitted into a box is $n_{P_r}(n \geq r)$.

2. The number of ways of distributing r_i objects to the i^{th} box for $i = 1, 2, \dots, n$ such that $r_1 + r_2 + \cdots + r_n = r$ is given by

$$\frac{r!}{r_1!r_2!r_3!\dots r_n!}$$

Distribution of Identical objects

1) The number of ways of distributing r identical objects into n different boxes if

   (i) no restriction is placed on the number of objects permitted per box is $(n+r-1)_{C_r}$ (*Bose – Einstein formula*)

   (ii) A particular box contains exactly k objects is

$$((n-1)+(r-k)-1)_{C_{r-k}} = (n-r-k-2)_{C_r}$$

   (iii) atmost one object is permitted per box is $n_{C_r}\ (r \leq n)$

   (*Fermi – Dirac formula*)

Example 4

S.No	Objects	Arrangement	Problem	Answer
1	5 boys and 4 girls	Row	No two girls together	$5! \times 6_{P_4}$
2	5 boys and 4 girls	Circle	No two girls together	$4! \times 5_{P_4}$
3	5 boys and 5 girls	row	No two girls together	$5! \times 6_{P_5}$
4	5 boys and 5 girls	row	Boys and girls alternate	$5! \times 5! \times 2!$
5	5 boys and 5 girls	circle	No two girls together	$4! \times 5_{P_5} = 4! \times 5!$
6	5 boys and 5 girls	circle	Boys and girls alternate	$4! \times 5!$
7	$5 +$ signs and $4 -$ signs	row	No two $-s$ together	$1 \times 6_{C_4}$
8	$5 +$ signs and $4 -$ signs	circle	No two $-s$ together	$1 \times 5_{C_4}$
9	$5 +$ signs and $5 -$ signs	row	No two $-s$ together	$1 \times 6_{C_5}$
10	$5 +$ signs and $5 -$ signs	row	$+$ and $-$ alternate	$1 + 1 = 2$
11	$5 +$ signs and $5 -$ signs	circle	No two $-s$ together	1
12	$5 +$ signs and $5 -$ signs	circle	$+$ and $-$ alternate	1

Example 5:

(i) Find the number of 4 -letter words that can be formed using the letters of the word **EQUATION.**

(ii) How many of these words begin with E?

(iii) How many end with N?

(iv) How many begin with E and end with N?

Solution: The word EQUATION has 8 distinct letters.

(i) Number of 4 letter words is 8_{P_4}

(ii) The first letter E is fixed ($E - --$). The remaining three letters are to be filled with 7 letters. Thus, the number of $4 -$ letter words begin with E is 7_{P_3}

(iii) $- - -N$. No of words ending with N is 7_{P_3}

(iv) $E - -N$. No of words begin with E and end with N is 6_{P_2}

Example 6: Find the number of 4 letter words that can be formed using the letters of the word **MIXTURE** which

(i) contain the letter X

(ii) do not contain the letter X

Solution: Take 4 blanks $- - - -$. We have to fill up 4 blanks using the 7 letters of the word.

(i) First we put X in one of the 4 blanks. This can be done in 4 ways. Now we can fill the remaining 3 palces with the remaining 6 letters in 6_{P_3} ways. Thus, the number of 4 letter words containing the letter X are $4 \times 6_{P_3} = 4 \times 120 = 480$

(ii) Leaving the letter X, we fill the 4 blanks with the remaining 6 letters in 6_{P_4} ways. Thus, the number of 4 letter words that do not contain the letter X is

$$6_{P_4} = 360$$

Example 7: Find all 4 $-$digit numbers that can be formed using the digits $1, 2, 3, 4, 5, 6$ when repetition is allowed.

Solution: The number of 4 $-$digit number with repetitions is 6^4

Example 8: Find the number of ways of arranging the letters of the word **SPECIFIC**. In how many of them

(i) the two Cs come together?
(ii) the two Is do not come together?

Solution: The word **SPECIFIC** has 8 letters in which there are 2 I's and 2 C's. Hence, they can be arranged in

$$\frac{8!}{2!2!} \text{ ways}$$

(i) Treat two C's as one unit. Then we have $6+1=7$ letters in which two letters (I's) are alike.

Thus, the no. of arrangement $=\frac{7!}{2!}$

(ii) Keeping the two I's aside, arrange the remaining 6 letters can be arragned in $\frac{6!}{2!}$ ways. Among these 6 letters we find 7 gaps as shown below.

$$-S-P-E-C-F-C-$$

The two I's can be arranged in these 7 gaps in $\frac{{}^{7}\boldsymbol{P}_{2}}{2!}$

Hence, the number of required arrangements is $\frac{6!}{2!}\times\frac{{}^{7}\boldsymbol{P}_{2}}{2!}$

Example 9: Find the number of ways of selecting 4 boys and 3 girls from a group of 8 boys and 5 girls is

Solution: $8_{C_4} \times 5_{C_3}$

Example 10: Find the number of ways of forming a committee of 4 members out of 6 boys and 4 girls such that there is at least one girl in the committee.

Solution: $10_{C_4} - 6_{C_4}$

Derangements and Matches

If n objects numbered $1, 2, 3, \dots, n$ are distributed at random in n places also numbered $1, 2, \dots, n$ a match is said to occur. If an object occupies the place corresponding to its number, the number of permutations in which no match occurs is

$$\boldsymbol{D_n} = \boldsymbol{n!}\left\{1 - \frac{1}{1\boldsymbol{!}} + \frac{1}{2\boldsymbol{!}} - \frac{1}{3\boldsymbol{!}} + \boldsymbol{\dots} + (-1)^{\boldsymbol{n}} \frac{1}{\boldsymbol{n!}}\right\}$$

This is also known as derangement.

The number of permutations of n objects in which exactly r matches occur is

$$n_{C_r}.D_{n-r} = \frac{n!}{r!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-r}}{(n-r)!}\right\}$$