P4:

Let $X_1, \dots, X_n$ be i.i.d. standardized variates with $E(X_i^{\,n}) < \infty$. Find the limiting distribution of

$$\boldsymbol{Z_n = \frac{\sqrt{n}(X_1X_2 + X_3X_4 + \cdots + X_{2n-1}X_{2n})}{\left(X_1^{\,2} + X_2^{\,2} + \cdots + X_{2n}^{\,2}\right)}}$$

Solution:

Since X_is are i.i.d standardized variates, we have

$$E(X_i) = 0\,, V(X_i) = E\left(X_i^{\,2}\right) = 1\,, i = 1,2,\dots,n$$

Let $Y_i = X_{2i-1}X_{2i}\ \ , i = 1,2,\dots,n$

$\Longrightarrow E(Y_i) = E(X_{2i-1})E(X_{2i}) = 0$ (∵ X_is are independent)

and $V(Y_i) = E\left(Y_i^{\,2}\right) = E\left[X_{2i-1}^{\,2}\, X_{2i}^{\,2}\right] = E\left[X_{2i-1}^{\,2}\right] E\left[X_{2i}^{\,2}\right] = 1 \cdot 1 = 1$

Hence $Y_i\,, i = 1,2,\dots,n$ are also i.i.d.r.vs. Hence CLT holds for $S_n = \sum_{i=1}^{n} Y_i$. Further,

$$E(S_n) = \sum_{i=1}^{n} E(Y_i) = 0 \text{ and}$$

$$V(S_n) = \sum_{i=1}^{n} V(Y_i) = n$$

Then by CLT

$$U_n = \frac{S_n - E(S_n)}{\sqrt{V(S_n)}} = \frac{X_1X_2 + \cdots + X_{2n-1}X_{2n}}{\sqrt{n}} \xrightarrow{L} Z \text{ where } Z \sim N(0,1) \text{ as } n \longrightarrow \infty.$$

Also $E\left(X_i^{\,2}\right) = 1$ (finite), $i = 1,2,\dots,n$.

Hence, by Khinchine's theorem, WLLN applies to the sequence $\{X_i^{\,2}\}$, $i = 1,2,\dots,2n$ so that

$$V_n = \frac{X_1^{\,2} + X_2^{\,2} + \cdots + X_{2n}^{\,2}}{2n} \xrightarrow{P} E\left(X_i^{\,2}\right) = 1 \text{ as } n \longrightarrow \infty.$$

Hence, by Slutsky's theorem, we have

$$\lim_{n\to\infty}(U_n \cdot V_n) = \frac{2\sqrt{n}(X_1X_2 + \cdots + X_{2n-1}X_{2n})}{{X_1}^2 + \cdots + {X_n}^2} \xrightarrow{L} \frac{Z}{1} = Z{\sim}N(0{,}1)$$

Thus, $\frac{\sqrt{n}(X_1X_2+\cdots+X_{2n-1}X_{2n})}{{X_1}^2+\cdots+{X_n}^2} \xrightarrow{L} \frac{Z}{2}{\sim}N\left(0,\frac{1}{4}\right)$ (If $Z{\sim}N(0{,}1)$, then $cZ{\sim}N(0,c^2)$).