P2:

Let $\{X_n\}$ be a sequence of independent r.vs such that

$$P(X_n = \pm 1) = \frac{1}{2}(1 - 2^{-n}), \;\; P(X_n = \pm 2^{-n}) = 2^{-n-1}$$

Does the $WLLN$ hold for this sequence?

Solution:

Here $E(X_n) = \frac{1}{2}(1 - 2^{-n}) - \frac{1}{2}(1 - 2^{-n}) + 2^{-n}.2^{-n-1} - 2^{-n}.2^{-n-1} = 0$

and $V(X_n) = E(X_n^2) = \frac{1}{2}(1 - 2^{-n}) + \frac{1}{2}(1 - 2^{-n}) + 2^{-2n}.2^{-n-1} + 2^{-2n}.2^{-n-1}$

$$= (1 - 2^{-n}) + 2^{-3n}$$

$\Longrightarrow V(X_n) = 2^{-3n} - 2^{-n} + 1$

Let $S_n = \sum_{i=1}^{n} X_i$. Then

$V(S_n) = \sum_{i=1}^{n} V(X_i)$ $\qquad\qquad$ ($\because$ $X_i\, s$ are independent)

$$= \sum_{i=1}^{n}\left(2^{-3i} - 2^{-i} + 1\right) = \sum_{i=1}^{n} 2^{-3i} - \sum_{i=1}^{n} 2^{-i} + n$$

$\Longrightarrow B_n = V(S_n) = \frac{1}{7}(1 - 8^{-n}) - (1 - 2^{-n}) + n = -\frac{1}{78^n} + \frac{1}{2^n} - \frac{6}{7} + n$

$\Longrightarrow \frac{B_n}{n^2} = -\frac{1}{7.n^2.8^n} + \frac{1}{n^2.2^n} - \frac{6}{7.n^2} + \frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$

Since $\frac{B_n}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, $WLLN$ holds for the given sequence.