P4:

Let $X_1, X_2, \ldots$ be i.i.d.r.vs with common p.d.f. $N(0,1)$. Determine the limiting distribution of the r.v.

$$\boldsymbol{W_n = \sqrt{n}\left(\frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2}\right)}$$

Solution:

Let $U_n = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ and $V_n = \frac{X_1^2+X_2^2+\cdots+X_n^2}{n}$. Then $W_n = \frac{U_n}{V_n}$

Since each $X \sim N(0,1)$, the m.g.f. of X is given by $M_X(t) = e^{\frac{t^2}{2}}$

Then the m.g.f. of U_n is given by

$$M_{U_n}(t) = E\left[e^{tU_n}\right] = E\left[e^{\left(\frac{t}{\sqrt{n}}\right)\sum_{i=1}^{n} X_i}\right]$$

$$= \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{\sqrt{n}}\right) \qquad (\because X\ s \text{ are independent})$$

$$= \prod_{i=1}^{n} e^{t^2/2n} \qquad (\because X\ s \text{ are identically distributed})$$

$$= e^{\frac{t^2}{2n}} . e^{\frac{t^2}{2n}} . e^{\frac{t^2}{2n}} . e^{\frac{t^2}{2n}} \ldots . e^{\frac{t^2}{2n}}\ (n\ times) = e^{\frac{t^2}{2}}$$

$\Longrightarrow M_{U_n}(t) = e^{\frac{t^2}{2}}$ which is the m.g.f. of $N(0,1)$ r.v. By uniqueness of m.g.f. ,

$U_n \sim N(0,1)$ $i.e.,$ $U_n \xrightarrow{L} Z$, where $Z \sim N(0,1)$.

Next, we find the m.g.f. of V_n and identify its probability distribution.

First, the m.g.f. of X^2 is given by

$$M_{X^2}(t)=E\left[e^{tx^2}\right]=\int_{-\infty}^{\infty}e^{tx^2}f(x)dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx^2}e^{-\frac{1}{2}x^2}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2(1-2t)}dx$$

Let $\left(\sqrt{1-2t}\right)x = y \Longrightarrow dx = (1-2t)^{-\frac{1}{2}}\,dy$. Then

$$M_{X^2}(t)=\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y^2}dy\right\}(1-2t)^{-1/2}=(1-2t)^{-1/2}$$

($\because$ total probability of $N(0,1)$ is 1)

$\Longrightarrow M_{X^2}(t) = (1-2t)^{-\frac{1}{2}}$

Next, $M_{V_n}(t) = E[e^{tV_n}] = E\left[e^{\left(\frac{t}{n}\right)(X_1^2+\cdots+X_n^2)}\right]$

$$=\prod_{i=1}^{n}E\left[e^{\left(\frac{t}{n}\right)X_i^2}\right] \qquad (\because X\ s \text{ are independent})$$

$$=\prod_{i=1}^{n}M_{X^2}\left(\frac{t}{n}\right)$$

$$=\prod_{i=1}^{n}\left(1-\frac{2t}{n}\right)^{-\frac{1}{2}}=\left(1-\frac{2t}{n}\right)^{-\frac{n}{2}}$$

$\Longrightarrow\ M_{V_n}(t) = \left(1-\frac{2t}{n}\right)^{-\frac{n}{2}},\ t < \frac{n}{2}$,which is the m.g.f. of a Gamma distribution with two parameters $\alpha = \frac{n}{2}$ and $\beta = \frac{2}{n}$

The $p.d.f.$ of Gamma variate with two parameters (α,β) is defined by

$$f(x) = \frac{1}{\sqrt{\alpha}}\frac{1}{\beta^{\alpha}}e^{-\frac{1}{\beta}x}x^{\alpha-1} \text{ for } x > 0\,,\ \ \alpha > 0, \beta > 0$$

Notation: $X\sim G(\alpha,\beta)$

If $X\sim G(\alpha,\beta)$ then its m.g.f. is given by $M_X(t) = (1-pt)^{-\alpha}$

Mean= $E(X) = \alpha\beta$ and variance $= V(X) = \beta^2\alpha$

The variance of V_n is given by $V(V_n) = \beta^2\alpha = \left(\frac{2}{n}\right)^2 \frac{n}{2} = \frac{2}{n}$

We have for any $\epsilon > 0$,

$$P\{|V_n - 1| > \epsilon\} \leq \frac{V(V_n)}{\epsilon^2} \qquad \text{(By chebychev's inequality)}$$

$$= \frac{2}{n\epsilon^2} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Thus, $V_n \xrightarrow{\boldsymbol{P}} 1$. We have thus shown that $U_n \xrightarrow{\boldsymbol{L}} Z$ and $V_n \xrightarrow{\boldsymbol{P}} 1$

If follows by Slutsky's theorem (iii)

$$W_n = \frac{U_n}{V_n} \xrightarrow{\boldsymbol{L}} \frac{Z}{1} = Z, \text{ where } Z \text{ is } N(0,1)$$

Hence, $W_n \sim N(0,1)$.