P3:

For negative binomial distribution, show that $V(X) = \frac{rq}{p^2}$.

Proof:

We already derived that $p(x) = \binom{-r}{x} p^r (-q)^x$

$$E(X) = \sum_{x=0}^{\infty} x\, p(x) = \frac{rq}{p}$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} \left[x(x-1) + x\right] p(x)$$

$$= \sum_{x=2}^{\infty} x(x-1)\, p(x) + \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=2}^{\infty} x(x-1) \binom{-r}{x} p^r (-q)^x + \frac{rq}{p}$$

$$= \sum_{x=2}^{\infty} x(x-1) \left(\frac{-r}{x}\right)\left(\frac{-r-1}{x-1}\right)\binom{-r+2}{x-2} p^r (-q)^x + \frac{nq}{p}$$

$$= r(r+1)\, p^r (-q)^2 \sum_{x=2}^{\infty} \binom{-r+2}{x-2} (-q)^{x-2} + \frac{nq}{p}$$

$$= r(r+1)\, p^r q^2 (1-q)^{-(r+2)} + \frac{nq}{p} \quad \left(\because \sum_{x=2}^{\infty} \binom{-\overline{r+2}}{x-2} (-q)^{x-2} = (1-q)^{-(r+2)} \right)$$

$$= r(r+1)\, p^r q^2 p^{-(r+2)} + \frac{nq}{p}$$

$$\Rightarrow E(X^2) = r(r+1) \frac{q^2}{p^2} + \frac{nq}{p}$$

$$\begin{aligned} Thus,\ V(X) = E\left(X^2\right) - \left(E(X)\right)^2 &= r(r+1)\frac{q^2}{p^2} + \frac{nq}{p} - \frac{r^2q^2}{p^2} \\ &= \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{nq}{p} - \frac{r^2q^2}{p^2} \\ &= \frac{rq}{p^2}(q+p) \end{aligned}$$

$$\Rightarrow V(X) = \frac{rq}{p^2}$$