P2:

For geometric distribution $p(x) = 2^{-x}\,; x = 1, 2, 3, \ldots$, prove that Chebychev's inequality gives $P\{|X-2| \le 2\} > \frac{1}{2}$, while the actual probability is $\frac{15}{16}$.

Solution:

$$E(X) = \sum x\,p(x) = \sum_{x=1}^{\infty} \frac{x}{2^x} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots$$

$$= \frac{1}{2}\left(1 + 2A + 3A^2 + 4A^3 + \ldots\right) = \frac{1}{2}(1-A)^{-2} = 2, \left(A = \frac{1}{2}\right)$$

$$E(X^2) = \sum x^2\,p(x) = \sum_{x=1}^{\infty} \frac{x^2}{2^x} = \frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \ldots$$

$$= \frac{1}{2}\left\{1 + 4A + 9A^2 + \ldots\right\}, where\ A = \frac{1}{2}$$

$$= \frac{1}{2}(1+A)(1-A)^{-3} = 6$$

$\therefore Var(X) = \sigma^2 = E(X^2) - \{E(X)\}^2 = 6 - 4 = 2 \Longrightarrow \sigma = \sqrt{2}$

Using Chebychev's inequality, we get $P\{|X - E(X)| > k\sigma|\} \le \frac{1}{k^2}$

With $k = \sqrt{2}$, we get $P\left\{|X-2| > \sqrt{2}\cdot\sqrt{2}\right\} \le \frac{1}{2} \Longrightarrow P[|X-2| \le 2] > 1 - \frac{1}{2} = \frac{1}{2}$

The actual probability is given by

$P\{|X-2| \le 2\} = P\{0 \le X \le 4\} = P\{X = 1,2,3\ or\ 4\}$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$