P3:

Show that $\boldsymbol{WLLN}$ holds to the mean of a sequence of independent r.v s $\boldsymbol{X_k}$ specified by

$$\boldsymbol{P\left(X_k = \pm\sqrt{\ln k}\right) = \frac{1}{2}}$$

Solution:

Here $E(X_k) = \sqrt{ln\, k}\,.\frac{1}{2} - \sqrt{ln\, k}\,,\frac{1}{2} = 0$ and

$$V(X_k) = E(X_k^2) = \frac{1}{2}ln\, k + \frac{1}{2}ln\, k = ln\, k$$

Let $S_n = \sum_{k=1}^{n} S_k$. Then

$V(S_n) = \sum_{k=1}^{n} V(X_k)$ $\quad\quad$ ($\because X_K$ $'s$ are independent)

$$= \sum_{k=1}^{n} ln\, k = ln\, 1 + ln\, 2 + \cdots + ln\, n = ln(n!)$$

$\Longrightarrow V(S_n) = B_n = ln(n!)$ ... (1)

By Stirling's approximation

$$n! = \sqrt{2\pi}e^{-n}\, n^{n+\frac{1}{2}} \quad \text{... (2)}$$

We have $\frac{B_n}{n^2} = \frac{ln\left(\sqrt{2\pi}e^{-n}.n^{n+\frac{1}{2}}\right)}{n^2} = \frac{-n+\left(n+\frac{1}{2}\right)\ln n + ln\sqrt{2\pi}}{n^2}$

$$= \frac{\left(ln\sqrt{2\pi} - n\right)}{n^2} + \frac{1}{n}\left(1 + \frac{1}{2n}\right)\ln n$$

$$= \frac{\ln\sqrt{2\pi} - n}{n^2} + \left(1 + \frac{1}{2n}\right)\ln\left(n^{1/n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad \left(\because \lim_{n\to\infty} n^{1/n} = 1\right)$$

Hence $\lim_{n\to\infty} \frac{B_n}{n^2} = 0$. Thus, the sequence $\{X_n\}$ holds $WLLN$.