P2:

Find the j.p.d.f. of the mid range $\boldsymbol{M = \frac{1}{2}\left[X_{(1)} + X_{(2)}\right]}$.

Solution:

Let $x = u$ and $y = \frac{1}{2}(u+v)$.Then $u = x$ and $v = 2y - x$ and the jacobian of transformation is given by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \text{ and } |J| = 2$$

$X_{(1)}$ and $X_{(2)}$ given in P_1, the $\mathrm{j.p.d.f.}$ of $X_{(1)}$ and M is given by

$$f_{X_{(1)},M}(x,y) = f_{X_{(1)},X_{(n)}}(x,y)|J|$$

$$= 2n(n-1)[F(2y-x) - F(x)]^{n-2} f(x) f(2y-x)$$

The m.p.d.f. of M is given by

$$f_M(y) = \int_{-\infty}^{\infty} f_{X_{(1),M}}(x,y)dx$$

$$= 2n(n-1)\int_{-\infty}^{\infty}\left[F(2y-x) - F(x)\right]^{n-2} f(x) f(2y-x)dx$$