P1:

The $\mathbf{j.p.m.f}$ of $\boldsymbol{X}$ and $\boldsymbol{X_n}$ is given by

$X \backslash X_n$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{P(X=x)}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$
$\mathbf{1}$	$\mathbf{\frac{1}{2}}$	$\mathbf{0}$	$\mathbf{\frac{1}{2}}$
$\boldsymbol{P(X_n=x)}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{1}$

Show that $\boldsymbol{X_n \xrightarrow{d} X}$ but $\boldsymbol{X_n \overset{P}{\not\longrightarrow} 0}$

Solution:

The p.m.f of X_n is $P(X_n = 0) = \frac{1}{2} = P(X_n = 1)$ and its c.d.f is given by

$$F_n(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{1}{2} & , \quad 0 \le x < 1 \\ 1 & , \quad x \ge 1 \end{cases}$$

The p.m.f of X is $P(X = 0) = \frac{1}{2} = P(X = 1)$ and its c.d.f is given by

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{1}{2} & , \quad 0 \le x < 1 \\ 1 & , \quad x \ge 1 \end{cases}$$

Thus, $F_n(x) \xrightarrow{w} F(x)$ as $n \longrightarrow \infty$. That is $X_n \xrightarrow{d} X$

But $P\left[|X_n - X| > \frac{1}{2}\right] = P[|X_n - X| = 1]$

$$= P[X_n = 0, X = 1] + P[X_n = 1, X = 0]$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \not\longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and hence $X_n \overset{P}{\not\longrightarrow} 0$. Therefore, $X_n \xrightarrow{d} X$ but $X_n \overset{P}{\not\longrightarrow} 0$