P4:

The time one has to spend in point of a bank counter is observed to be a random variable x with the p.d.f.

$$f(x)=\begin{cases} 0 & , \quad x<0 \\ \frac{1}{9}(x+1) & , \quad 0\le x<1 \\ \frac{4}{9}\left(x-\frac{1}{2}\right) & , \quad 1\le x<\frac{3}{2} \\ \frac{4}{9}\left(\frac{5}{2}-x\right) & , \quad \frac{3}{2}\le x<2 \\ \frac{1}{9}(4-x) & , \quad 2\le x<3 \\ \frac{1}{9} & , \quad 3\le x<6 \\ 0 & , \quad x\le 6 \end{cases}$$

Let A, B be the events defined as

> A **: One waits between 0 and 2 minutes inclusive**
>
> B **: One waits between 1 and 3 minutes inclusive**

i. **Show that $P(B|A)=\frac{2}{3}$**

ii. **Show that $P(\overline{A}\cap\overline{B})=\frac{1}{3}$**

Solution:

i.

$$P(A)=\int_0^2 f(x)\,dx=\int_0^1 \frac{1}{9}(x+1)\,dx+\int_1^{\frac{3}{2}}\frac{4}{9}\left(x-\frac{1}{2}\right)dx+\int_{\frac{3}{2}}^{2}\frac{4}{9}\left(\frac{5}{2}-x\right)dx$$

$$=\frac{1}{9}\left[\frac{x^2}{2}+x\right]_0^1+\frac{4}{9}\left[\frac{x^2}{2}-\frac{x}{2}\right]_1^{\frac{3}{2}}+\frac{4}{9}\left[\frac{5x}{2}-\frac{x^2}{2}\right]_{\frac{3}{2}}^{2}$$

$$=\frac{1}{9}\left[\left(\frac{1}{2}+1\right)\right]+\frac{4}{9}\left[\left(\frac{9}{8}-\frac{3}{4}\right)\right]+\frac{4}{9}\left[(5-2)-\left(\frac{15}{4}-\frac{9}{8}\right)\right]=\frac{1}{2}$$

$$P(A\cap B)=P(1\le x\le 2)=\int_1^2 f(x)dx$$

$$=\int_1^{\frac{3}{2}}\frac{4}{9}\left(x-\frac{1}{2}\right)dx+\int_{\frac{3}{2}}^{2}\frac{4}{9}\left(\frac{5}{2}-x\right)dx$$

$$=\frac{4}{9}\left[\frac{x^2}{2}-\frac{x}{2}\right]_1^{\frac{3}{2}}+\frac{4}{9}\left[\frac{5x}{2}-\frac{x^2}{2}\right]_{\frac{3}{2}}^{2}$$

$$=\frac{4}{9}\left[\left(\frac{9}{8}-\frac{3}{4}\right)-\left(\frac{1}{2}-\frac{1}{2}\right)\right]+\frac{4}{9}\left[(5-2)-\left(\frac{15}{4}-\frac{9}{8}\right)\right]=\frac{1}{3}$$

$$P(B)=\int_1^3 f(x)dx$$

$$=\int_1^{\frac{3}{2}}\frac{4}{9}\left(x-\frac{1}{2}\right)dx+\int_{\frac{3}{2}}^{2}\frac{4}{9}\left(\frac{5}{2}-x\right)dx+\int_2^3\frac{1}{9}(4-x)dx$$

$$=\frac{4}{9}\left[\frac{x^2}{2}-\frac{x}{2}\right]_1^{\frac{3}{2}}+\frac{4}{9}\left[\frac{5x}{2}-\frac{x^2}{2}\right]_{\frac{3}{2}}^{2}+\frac{1}{9}\left[4x-\frac{x^2}{2}\right]_2^3$$

$$=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$$

Thus $P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}$

ii. $P\left(\overline{A}\cap\overline{B}\right)=P\left(\overline{A\cup B}\right)=1-P(A\cup B)$

$$=1-[P(A)+P(B)-P(A\cap B)]=1-\frac{2}{3}=\frac{1}{3}$$

Alternatively, $\overline{A}\cap\overline{B}$ means the waiting time more than 3 minutes. So,

$$P\left(\overline{\mathrm{A}}\cap\overline{\mathrm{B}}\right)=P(x>3)=\int_3^{\infty} f(x)dx=\int_3^6 f(x)dx+\int_3^{\infty} f(x)dx$$

$$=\frac{1}{9}\int_3^6 dx+0=\frac{1}{9}[6-3]=\frac{3}{9}=\frac{1}{3}$$