P4:

Find $f_{UW}(u,w)$ if $U = X^2 + Y^2$ and $W = X^2$

Solution:

From the second equation we have that $x = \pm\sqrt{w}$. Substitute this value of x in the first equation, we obtain $y = \pm\sqrt{u-w}$, which is real only when $u \geq w$. Also ,

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & 0 \end{vmatrix} = -4xy$$

Thus,

$$f_{UW}(u,w) = \frac{f_{XY}(\sqrt{w},\sqrt{u-w})}{4\left|\sqrt{w(u-w)}\right|} + \frac{f_{XY}(\sqrt{w},-\sqrt{u-w})}{4\left|-\sqrt{w(u-w)}\right|} + \frac{f_{XY}(-\sqrt{w},\sqrt{u-w})}{4\left|-\sqrt{w(u-w)}\right|} + \frac{f_{XY}(-\sqrt{w},-\sqrt{u-w})}{4\left|\sqrt{w(u-w)}\right|}$$

$$= \frac{f_{XY}(\sqrt{w},\sqrt{u-w}) + f_{XY}(\sqrt{w},-\sqrt{u-w})}{4\left|\sqrt{w(u-w)}\right|} + \frac{f_{XY}(-\sqrt{w},\sqrt{u-w}) + f_{XY}(-\sqrt{w},-\sqrt{u-w})}{4\left|\sqrt{w(u-w)}\right|}$$

where $u > w > 0$