4.4

Strong Law of Large Numbers

Definition: A sequence of r.vs $\{X_n\}$ is said to satisfy the **strong law of large numbers** (**SLLN**) if

$$\left[\frac{S_n - E(S_n)}{n}\right] \xrightarrow{a.s} 0 \text{ as } n \longrightarrow \infty$$

We state the following theorems without proof which are useful in checking whether a given *sequence satisfies SLLN or not.*

Theorem1: (Kolmogorov's SLLN)

This theorem is helpful when the r.vs in the sequence are *independent but not identically distributed.*

Statement: Let $\{X_n\}$ be a sequence of independent r.vs with $E(X_i) = \mu$ and $V(X_i) = \sigma_i^2 < \infty$ for $= 1, 2, \dots$. If $\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$, then the SLLN holds for the sequence $\{X_n\}$.

Theorem 2:

This theorem is helpful when the r.vs in the sequence are independent and identically distributed (i.i.d).

Statement: The sequence $\{X_n\}$ of i.i.d.r.vs holds SLLN iff $E(X_n)$ exists.

Theorem 3: (Borel's SLLN):

This theorem is helpful when the sequence consists of *Bernoulli trials*.

Statement: For a sequence of Bernoulli trials with constant probability of success, the SLLN holds.

Example 1: Let $\{X_n\}$ be a sequence of independent random variables with p.m.f. given by

$$\boldsymbol{P(X_n = \pm 2^n) = \frac{1}{2^{(2n+1)}}\ , P(X_n = 0) = 1 - \frac{1}{2^{2n}}}$$

Does the SLLN hold for $\{X_n\}$?

Solution: We have $E(X_n) = 2^n \frac{1}{2^{2n+1}} - 2^n \frac{1}{2^{2n+1}} = 0$ and

$\sigma_n{}^2 = V(X_n) = E\left(X_n{}^2\right) = 2^{2n} \frac{1}{2^{2n+1}} + 2^{2n} \frac{1}{2^{2n+1}} = 1$

Further, $\sum_{n=1}^{\infty} \frac{\sigma_n{}^2}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($\because \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$).

Hence, the SLLN holds for $\{X_n\}$.

Example 2: For what value of α does the SLLN hold for the sequence

$$\boldsymbol{P(X_k = \pm k^\alpha) = \frac{1}{2}}$$

Solution: We have $E(X_k) = k^\alpha \frac{1}{2} - k^\alpha \frac{1}{2} = 0$ and

$\sigma_k{}^2 = V(X_k) = E\left(X_k{}^2\right) = k^{2\alpha} \frac{1}{2} + k^{2\alpha} \frac{1}{2} = k^{2\alpha}$

Further, $\sum_{k=1}^{\infty} \frac{\sigma_k{}^2}{k^2} = \sum_{k=1}^{\infty} \frac{k^{2\alpha}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^{2-2\alpha}}$ converges if $2 - 2\alpha > 1$

($\because \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$).

$$\Rightarrow 2\alpha < 1 \Rightarrow \alpha < \frac{1}{2}$$

Thus, SLLN holds if $\alpha < \frac{1}{2}$.

Example 3: Let $\{X_n\}$ be a sequence of independent r.vs with p.m.f. given by

$$\boldsymbol{P\left(X_n = \pm\frac{1}{n}\right) = \frac{1}{2}}$$

Check whether SLLN holds for $\{X_n\}$ or not.

Solution: We have $E(X_n) = \frac{1}{n}\frac{1}{2} - \frac{1}{n}\frac{1}{2} = 0$ and

$\sigma_n{}^2 = V(X_n) = E(X_n{}^2) = \frac{1}{n^2}\frac{1}{2} + \frac{1}{n^2}\frac{1}{2} = \frac{1}{n^2}$

Further, $\sum_{n=1}^{\infty}\frac{\sigma_n{}^2}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ converges ($\because \sum_{k=1}^{\infty}\frac{1}{k^p}$ converges if $p > 1$).

Therefore, $\{X_n\}$ obeys SLLN.

Example 4: Let $\{X_n\}$ be a sequence of independent r.vs with p.m.f. given by

$$\boldsymbol{P(X_k = \pm 2^{-k}) = \frac{1}{2}}$$

Check whether SLLN holds or not.

Solution: Here $E(X_k) = 2^{-k}\frac{1}{2} - 2^{-k}\frac{1}{2} = 0$ and

$\sigma_k{}^2 = V(X_k) = E(X_k{}^2) = 2^{-2k}\frac{1}{2} + 2^{-2k}\frac{1}{2} = 2^{-2k}$

Further $\sum_{k=1}^{\infty}\frac{\sigma_k{}^2}{k^2} = \sum_{k=1}^{\infty}2^{-2k}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k^2 2^{2k}}$ converges. Therefore, $\{X_n\}$ obeys the SLLN.

Example 5: Let $\{X_n\}$ be i.i.d.r.vs with mean μ and variance σ^2 and as $n \longrightarrow \infty$,

$$\frac{{X_1}^2 + \cdots + {X_n}^2}{n} \xrightarrow{a.s} c$$

for some constant c $(0 \leq c < \infty)$, then find c.

Solution: Here $E(X_i) = \mu$ and $V(X_i) = \sigma^2 \; \forall \; i$.

Let $S_n = {X_1}^2 + \cdots + {X_n}^2$. Then

$$E(S_n) = nE({X_1}^2) = n\left[V(X_1) + \left(E(X_1)\right)^2\right] = n(\sigma^2 + \mu^2)$$

$$\Longrightarrow E(S_n) = n(\sigma^2 + \mu^2)$$

$$\Longrightarrow E\left(\frac{S_n}{n}\right) = \sigma^2 + \mu^2$$

By Theorem 2,

$$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right) = (\sigma^2 + \mu^2)$$

$$\Longrightarrow \frac{{X_1}^2 + \cdots + {X_n}^2}{n} \xrightarrow{a.s} c, \text{ where } c = \sigma^2 + \mu^2 .$$

Example 6: If the i.i.d.r.vs $\{X_n\}$ assume the value $2^{r-2\ln r}$ with probability $\frac{1}{2^r}$ for $r = 1, 2, \ldots$, examine if the SLLN holds for the sequence $\{X_n\}$.

Solution: By Theorem 2, SLLN holds for i.i.d.r.vs $\{X_n\}$ if $E(X_k)$ exists $\forall \; k$.

Here we have to verify whether $E(X_k)$ is finite or not.

We have

$$E(X_k) = \sum_{r=1}^{\infty} 2^{r-2\ln r} \frac{1}{2^r} = \sum_{r=1}^{\infty} 2^{-2\ln r} = \sum_{r=1}^{\infty} \left(\frac{1}{4}\right)^{\ln r}$$

$$= \sum_{r=1}^{\infty} r^{\ln\left(\frac{1}{4}\right)} \; \left(\because a^{\ln n} = n^{\ln a}\right)$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{r}\right)^{\ln 4} = \sum_{r=1}^{\infty} \frac{1}{r^{\ln 4}} \text{ where } ln\, 4 = 1.39 > 1$$

which converges.

Thus, $E(X)$ is finite and hence the SLLN holds for $\{X_n\}$.