P3:

Examine if CLT holds for the sequence $\{X_k\}$ with p.m.f. $P(X_k = \pm 2^{-k}) = \frac{1}{2}$.

Solution:

We have $\mu_k = E(X_k) = 2^{-k} \cdot \frac{1}{2} - 2^{-k} \cdot \frac{1}{2} = 0$,

$\sigma_k{}^2 = V(X_k) = E(X_k{}^2) = 2^{-2k} \cdot \frac{1}{2} + 2^{-2k} \cdot \frac{1}{2} = 2^{-2k}$ and

$\rho_k{}^3 = E\{|X_k - 0|^3\} = E(|X_k|^3) = 2^{-3k} \cdot \frac{1}{2} + 2^{-3k} \cdot \frac{1}{2} = 2^{-3k}$

Let $S_n = \sum_{k=1}^{n} X_k$. Then we have $\mu = \sum_{k=1}^{n} \mu_k = 0$

$$\sigma^2 = \sum_{k=1}^{n} \sigma_k{}^2 = \sum_{k=1}^{n} 2^{-2k} = \sum_{k=1}^{n} \frac{1}{2^{2k}} = \sum_{k=1}^{n} \frac{1}{4^k} = \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^n} = \frac{1}{3}\left(1 - 4^{-n}\right) \text{ and}$$

$$\rho^3 = \sum_{k=1}^{n} \rho_k{}^3 = \sum_{k=1}^{n} 2^{-3k} = \sum_{k=1}^{n} \frac{1}{8^k} = \frac{1}{8} + \frac{1}{8^2} + \ldots + \frac{1}{8^n} = \frac{1}{7}\left(1 - 8^{-n}\right) \text{ and}$$

$$\frac{\rho^3}{\left(\sigma^2\right)^{\frac{3}{2}}} = \frac{\frac{1}{7}\left(1 - 8^{-n}\right)}{\left[\frac{1}{3}\left(1 - 4^{-n}\right)\right]^{\frac{3}{2}}} \to \frac{1}{7}\left(\frac{1}{3}\right)^{-\frac{3}{2}} \quad as\ n \to \infty$$

Since Liapounoff's condition is not satisfied, we can't say CLT holds or not.