4.3

Weak Law of Large Numbers

Let $\{X_n\}$ be a sequence of r.vs and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the mean of first n r.vs. The weak laws deal with *limits of probabilities involving* $\overline{X}_n$. The strong laws deal with *probabilities involving limits of* $\overline{X}_n$.

Definition of Weak Law of Large Numbers

A sequence $\{X_n\}$ of r.vs is said to satisfy the ***Weak Law of Large Numbers (WLLN)*** if

$$\lim_{n\longrightarrow\infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \epsilon\right] = 1$$

for any $\epsilon > 0$, where $S_n = \sum_{i=1}^{n} X_i$, $i.e.,\ \frac{S_n}{n} \xrightarrow{P} E\left(\frac{S_n}{n}\right)$

Theorem1: Let $\{X_n\}$ be a sequence of r.vs and let $S_n = X_1 + \cdots + X_n$ with $B_n = V(S_n) < \infty$. If $\frac{B_n}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, then for any $\epsilon > 0$,

$$\lim_{n\longrightarrow\infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \epsilon\right] = 1$$

i.e., **$\{X_n\}$ satisfies WLLN.**

Proof: On applying Chebychev's inequality to the variable $\frac{S_n}{n}$, we have

$$P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| \geq \epsilon\right] \leq \frac{V\left(\frac{S_n}{n}\right)}{\epsilon^2} = \frac{V(S_n)}{n^2\epsilon^2} = \frac{B_n}{n^2\epsilon^2} \longrightarrow 0$$

as $n \longrightarrow \infty$. Thus,

$$\lim_{n\longrightarrow\infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| \geq \epsilon\right] = 0 \Longrightarrow \lim_{n\longrightarrow\infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \epsilon\right] = 1$$

$\Longrightarrow \{X_n\}$ satisfies WLLN.

Corollary: Let $\{X_n\}$ be a sequence of r.vs, $\overline{X_n} = \frac{S_n}{n}$ and $\mu = E\left(\frac{S_n}{n}\right)$. If $\{X_n\}$ satisfying WLLN. Then

$$\lim_{n\to\infty} P[\overline{X_n} \leq k] = \begin{cases} 0 & ,\text{if } k < \mu \\ 1 & ,\text{if } k > \mu \end{cases}$$

Proof: Since WLLN holds for $\{X_n\}$, we have

$$\lim_{n\to\infty} P[|\overline{X_n} - \mu| < \epsilon] = 1 \Longrightarrow \lim_{n\to\infty} P[|\overline{X_n} - \mu| \geq \epsilon] = 0 \qquad \dots(1)$$

Since $\{\overline{X_n} \leq \mu - \epsilon\} \subset \{|\overline{X_n} - \mu| \geq \epsilon\}$, we have

$$P(\overline{X_n} \leq \mu - \epsilon) \leq P(|\overline{X_n} - \mu| \geq \epsilon)$$

$$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq \mu - \epsilon) \leq \lim_{n\to\infty} P(|\overline{X_n} - \mu| \geq \epsilon)$$

$$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq \mu - \epsilon) = 0$$

$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq k) = 0$, where $k = \mu - \epsilon$, $i.e., k < \mu$ since $\epsilon > 0$

$$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq k) = 0 \ \ if\ k < \mu$$

Further, $P(\overline{X_n} \leq \mu + \epsilon) + P(|\overline{X_n} - \mu| > \epsilon) \geq 1$, since the region is larger than sample space covered.

$$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq \mu + \epsilon) \geq 1 \ \left(\because \lim_{n\to\infty} P(|\overline{X_n} - \mu| > \epsilon) = 0\right)$$

$$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq \mu + \epsilon) = 1$$

$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq k) = 1$ where $k = \mu + \epsilon$ $i.e., k > \mu$ since $\epsilon > 0$

$\Longrightarrow \lim_{n\to\infty} P(\overline{X_n} \leq k) = 1$ if $k > \mu$

Thus, $\lim_{n\to\infty} P(\overline{X_n} \leq k) = \begin{cases} 0\ , & k < \mu \\ 1\ , & k > \mu \end{cases}$

Variations of the WLLN

The following are some special cases of Theorem1 which are stated without proof.

Theorem 2: (Bernoulli's WLLN)

Let $\{X_n\}$ be a sequence of Bernoulli trials with probability of success equal to p. If S_n is the number of successes in n trials, then

$$\lim_{n\longrightarrow\infty} P\left[\left|\frac{S_n - np}{n}\right| < \epsilon\right] = 1, \ \ \forall\, \epsilon > 0$$

Theorem 3: (Khinchine's WLLN)

Let $\{X_n\}$ be a sequence of i.i.d.r.vs with $E(X_i) = \mu < \infty$, $i = 1,2, \ldots$, then the WLLNs holds i.e.,

$$\lim_{n\longrightarrow\infty} P\left[\left|\frac{S_n}{n} - \mu\right| > \epsilon\right] = 0$$

Theorem 4: (Bernstein's WLLN)

Let $\{X_n\}$ be a sequence of random variables for which $var(X_n) = {\sigma_n}^2 < k,\ \forall\, i,$ where k is independent of n. If $\sigma_{ij} = cov(X_i, X_j) \longrightarrow 0$ as $|i - j| \longrightarrow \infty$ (*Asymptotic uncorrelatedness*) then the WLLN holds.

Example1: Let $\{X_n\}$ be i.i.d.r.vs with mean μ and variance σ^2, if

$$\frac{{X_1}^2 + {X_2}^2 + \cdots + {X_n}^2}{n} \xrightarrow{P} c$$

as $n \longrightarrow \infty$ for some constant $c(0 \leq c < \infty)$, then find c.

Solution: Here $E(X_i) = \mu$ and $V(X_i) = \sigma^2\ \forall\, i$.

Let $S_n = {X_1}^2 + {X_2}^2 + \cdots + {X_n}^2$. Then

$E(S_n) = nE({X_1}^2)$ (∵ Xs are i.i.d.r.vs)

$$= n\left[V(X_1) + \left(E(X_1)\right)^2\right]$$

$$\Rightarrow E(S_n) = n(\sigma^2 + \mu^2)$$

Since $E(X^2) = V(X) + \left(E(X)\right)^2 = \sigma^2 + \mu^2$ exists for each X^2 in S_n, by Khinchine's WLLN, we have

$$\frac{S_n}{n} = \frac{{X_1}^2 + {X_2}^2 + \cdots + {X_n}^2}{n} \quad E\left({X_1}^2\right) = \mu^2 + \sigma^2$$

Thus, $c = \mu^2 + \sigma^2$.

Example 2: If the i.i.d. r.vs $X_k (k = 1, 2, \ldots)$ assume the value $2^{r-2\ln r}$ with probability $\frac{1}{2^r}$, examine if the WLLN holds for the sequence $\{X_k\}$.

Solution:

$$E(X_k) = \sum_{r=1}^{\infty} 2^{r-2\ln r} \cdot \frac{1}{2^r} = \sum_{r=1}^{\infty} \left(2^{-2}\right)^{\ln r} = \sum_{r=1}^{\infty} \left(\frac{1}{4}\right)^{\ln r}$$

$$= \sum_{r=1}^{\infty} (r)^{\ln\left(\frac{1}{4}\right)} \left(\because a^{\ln n} = n^{\ln a}\right)$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{r}\right)^{\ln 4} \textit{ converges since } \ln 4 = 1.39 > 1 \quad \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \textit{ converges if } p > 1\right)$$

$$\textit{Thus } E(X_k) < \infty$$

Since $\{X_k\}$ are i.i.d.r.vs with $E(X_k) < \infty$, the WLLN holds for the sequence, by Khinchine's theorem.

Example 3: Let $\{X_n\}$ be a sequence of i.i.d $U(0, 1)$ r.vs. For the geometric mean $G_n = (X_1 . X_2 . \ldots . X_n)^{\frac{1}{n}}$, show that $G_n \xrightarrow{P} c$ where c is some constant. Find c.

Solution: Let $Y = -\ln X$ where $X \sim U(0,1)$. The c.d.f. of Y is given by

$$F_Y(y) = P(Y \le y) = P(-\ln X \le y) = P(X \ge e^{-y}) = \int_{e^{-y}}^{1} 1\, dx = 1 - e^{-y}$$

$\Longrightarrow F_Y(y) = 1 - e^{-y}$ and the p.d.f of Y is given by

$f_Y(y) = \frac{d}{dx}\left(F_Y(y)\right) = e^{-y}$ for $y > 0$.

Then $E(Y) = V(Y) = 1$.

Thus, the sequence $\{Y_n\}$ is iid with finite mean $E(Y_n) = 1$. Hence, by Khinchine's WLLN

$$\sum_{i=1}^{n} \frac{Y_i}{n} \xrightarrow{P} E(Y_1) = 1 \qquad \dots (1)$$

$$But \;\; \ln G_n = \sum_{i=1}^{n} \ln \frac{X_i}{n} = -\sum_{i=1}^{n} \frac{Y_i}{n}$$

$$\Longrightarrow \sum_{i=1}^{n} \frac{Y_i}{n} = -\, ln\, G_n \qquad \dots (2)$$

From (1) and (2), we have

$$-\, ln\, G_n \xrightarrow{P} 1 \quad i.e., \;\; G_n \xrightarrow{P} e^{-1}$$

Thus, $c = \frac{1}{e}$.

Example 4: Show that the sequence $P\left(X_k = \pm 2^k\right) = \frac{1}{2}$ of independent r.vs does not obey the WLLN.

Solution: Here $E(X_k) = 2^k \frac{1}{2} - 2^k \frac{1}{2} = 0$ and

$$V(X_k) = E\left({X_k}^2\right) = 2^{2k} \frac{1}{2} + 2^{2k} \frac{1}{2} = 2^{2k} = 4^k.$$

Let $S_n = X_1 + \cdots + X_n$. Then $E(S_n) = 0$ and $V(S_n) = \sum_{k=1}^{n} V(X_k)$

($\because$ Xs are independent)

$$\Rightarrow B_n = V(S_n) = \sum_{k=1}^{n} 4^k = (4 + 4^2 + \cdots + 4^n) = \frac{4(4^n - 1)}{4 - 1} = \frac{4}{3}(4^n - 1)$$

Now, $\lim\limits_{n\to\infty} \frac{B_n}{n^2} = \frac{4}{3} \lim\limits_{n\to\infty} \left(\frac{4^n - 1}{n^2}\right) \qquad \left(\frac{\infty}{\infty}\ form\right)$

$$= \frac{4}{3} \lim_{n\to\infty} \frac{4^n \ln 4}{2n} = \frac{4}{3} (\ln 4)^2 \frac{1}{2} \lim_{n\to\infty} 4^n \longrightarrow \infty$$

Since $\frac{B_n}{n^2} \longrightarrow \infty$ as $n \longrightarrow \infty$, WLLN does not hold.

Example 5: Show that the following sequence of independent r.vs does not obey WLLN:

$$\boldsymbol{P\left[X_k = \pm(2k - 1)^{\frac{1}{2}}\right] = \frac{1}{2}}$$

Solution: $E(X_k) = (2k - 1)^{\frac{1}{2}} \cdot \frac{1}{2} - (2k - 1)^{\frac{1}{2}} \cdot \frac{1}{2} = 0$

$$V(X_k) = E\left({X_k}^2\right) = (2k - 1) \cdot \frac{1}{2} + (2k - 1) \cdot \frac{1}{2} = 2k - 1$$

Let $S_n = \sum\limits_{k=1}^{n} X_k$. Then

$$B_n = V(S_n) = \sum_{k=1}^{n} V(X_k) \qquad (\because X\text{s are independent})$$

$$= \sum_{k=1}^{n} (2k - 1) = n^2$$

$$\Longrightarrow B_n = n^2 \Longrightarrow \frac{B_n}{n^2} = 1$$

$$\Longrightarrow \lim_{n\to\infty} \frac{B_n}{n^2} = 1$$

Thus, it follows that $\{X_k\}$ does not hold WLLN.

Example 6: Let X_i can have only two values i^α and $-i^\alpha$ with equal probabilities. If $\{X_i\}$ is a sequence of independent r.vs, then show that WLLN holds if $\alpha < \frac{1}{2}$.

Solution: Here $E(X_i) = i^\alpha \frac{1}{2} - i^\alpha \frac{1}{2} = 0$ and

$$V(X_i) = E\left(X_i^{\,2}\right) = i^{2\alpha}\frac{1}{2} + i^{2\alpha}\frac{1}{2} = i^{2\alpha}$$

Let $S_n = \sum_{k=1}^{n} X_k$. Then

$$B_n = V(S_n) = \sum_{i=1}^{n} V(X_i) \qquad (\because X_i\text{s are independent})$$

$$= \sum_{i=1}^{n} i^{2\alpha} = 1^{2\alpha} + 2^{2\alpha} + \cdots + n^{2\alpha}$$

$$= \int_0^n x^{2\alpha}dx \quad (\text{Euler} - \text{Maclaurion formula})$$

$\Longrightarrow B_n = \frac{n^{2\alpha+1}}{2\alpha+1} \Longrightarrow \frac{B_n}{n^2} = \frac{n^{2\alpha+1}}{2\alpha+1} \longrightarrow 0$ as $n \longrightarrow \infty$ if $\alpha < \frac{1}{2}$

Thus, $\frac{B_n}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$ when $\alpha < \frac{1}{2}$

Therefore, $\{X_n\}$ holds WLLN when $\alpha < \frac{1}{2}$.

Example 7: Let X_i be i.i.d. r.vs, $i = 1, 2, \ldots$ with mean μ and variance $\sigma^2 < \infty$ and let $S_n = X_1 + \cdots + X_n$. Show that the WLLN does not hold for the sequence $\{S_n\}$, but it holds for the sequence $\{a_n S_n\}$ provided $na_n \longrightarrow 0$.

Solution: Let $T_n = S_1 + S_2 + \cdots + S_n$, then

$$T_n = X_1 + (X_1 + X_2) + (X_1 + X_2 + X_3) + \cdots + (X_1 + X_2 + \cdots + X_n)$$

$$= nX_1 + (n-1)X_2 + \cdots + 2X_{n-1} + X_n$$

$$\therefore B_n = V(T_n) = n^2\sigma^2 + (n-1)^2\sigma^2 + \cdots + 2^2\sigma^2 + \sigma^2$$

$$= (1^2 + 2^2 + \cdots + (n-1)^2 + n^2)\sigma^2$$

$$\Longrightarrow B_n = \frac{n(n+1)(2n+1)}{6}\sigma^2 \Longrightarrow \frac{B_n}{n^2} \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Since $\frac{B_n}{n^2} \not\to 0$, $WLLN$ does not hold for $\{S_n\}$.

For the sequence $\{a_n S_n\}$, we have

$$T_n = a_1 S_1 + a_2 S_2 + \cdots + a_n S_n$$

$$= a_1 X_1 + a_2(X_1 + X_2) + a_3(X_1 + X_2 + X_3) + \cdots + a_n(X_1 + X_2 + \cdots + X_n)$$

$$= (a_1 + a_2 + \cdots + a_n)X_1 + (a_2 + a_3 + \cdots + a_n)X_2 + \cdots + (a_{n-1} + a_n)X_{n-1} + a_n X_n$$

and $B_n = V(T_n) = (a_1 + a_2 + \cdots + a_n)^2\sigma^2 + (a_2 + \cdots + a_n)^2\sigma^2 + \cdots$

$$+(a_{n-1} + a_n)^2\sigma^2 + a_n^2\sigma^2$$

$$= [{a_1}^2 + 2{a_2}^2 + \cdots + n{a_n}^{22} + 2(a_1a_2 + \cdots + a_1a_n) +$$

$$4(a_2a_3 + \cdots + a_2a_n) + \cdots + 2(n-1)a_{n-1}a_n]\sigma^2 \qquad \ldots(1)$$

Since $na_n \longrightarrow 0$, by Cauchy theorem $\lim\limits_{n\to\infty}\left[\frac{a_1+2a_2+\cdots+na_n}{n}\right] = 0$

Again $na_n \longrightarrow 0 \implies a_n = \frac{\epsilon}{n}$ so that $a_n \longrightarrow 0$ as $n \longrightarrow \infty$ and hence

$\lim\limits_{n\to\infty}\left[\frac{a_1+a_2+\cdots+a_n}{n}\right] = 0$ (by Cauchy theorem)

Let $\theta_n = \frac{(a_1+a_2+\cdots+a_n)(a_1+2a_2+\cdots+na_n)}{n^2}$

$$= \frac{\left[\left(\sum_{i=1}^{n} ia_i^2\right) + 3a_1a_2 + 4a_1a_3 + \ldots + (n+1)a_1a_n + \ldots + (2n-1)a_{n-1}a_n\right]}{n^2} \quad \ldots..(2)$$

By two preceding limits, $\theta_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Hence from (1) and (2), we have $\frac{B_n}{n^2} < \theta_n\sigma^2 \longrightarrow 0$ as $n \longrightarrow \infty$

Therefore, $WLLN$ holds for the sequence $\{a_n S_n\}$ provided $na_n \longrightarrow 0$.