P3:

Find the m.g.f. of geometric $G(p)$ distribution and hence obtain its mean and variance.

Solution:

If $X \sim G(p)$, its p.m.f is given by $p(x) = q^x p$ for $x = 0, 1, 2, \dots, \quad 0 < p < 1$.

Then the m.g.f. is given by

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx} q^x p = \sum_{x=0}^{\infty} \left(qe^t\right)^x = \frac{p}{1 - qe^t}$$

$$\Longrightarrow \; M_X(t) = \frac{p}{1 - qe^t}$$

Then $\mu_1' = M_X'(t)|_{t=0} = pq(1-q)^{-2} = \frac{q}{p}$ and $\mu_2' = M_X''(t)|_{t=0} = \frac{q}{p} + \frac{2q^2}{p^2}$

and $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{q}{p} + \frac{2q^2}{p^2} - \left(\frac{q}{p}\right)^2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{qp+q^2}{p^2} = \frac{q(p+q)}{p^2} = \frac{q}{p^2}$

Thus, Mean$= \mu = \mu_1' = \frac{q}{p}$ and variance $= \sigma^2 = \frac{q}{p^2}$