1.3.

Definitions of Probability

The probability of a given event is an expression of likelihood or chance of occurrence of an event. How the number is assigned would depend on the interpretation of the term 'probability'. There is no general agreement about its interpretation. However, broadly speaking, there are four different schools of thought on the concept of probability.

Mathematical (or classical or A priori) definition of probability

Let S be a sample space associated with a random experiment. Let A be an event in S. We make the following assumptions on S:

(i) It is discrete and finite

(ii) The outcomes in it are equally likely

Then the probability of happening (or occurrence) of the event A is defined by

$$P(A) = \frac{Number\ of\ outcomes\ in\ A}{Number\ of\ outcomes\ in\ S} = \frac{n(A)}{n(S)}$$

Note:

i) The probability of non-happening (or non-occurrence) of A is given by

$$P(\bar{A}) = \frac{Number\ of\ outcomes\ in\ \bar{A}}{Number\ of\ outcomes\ in\ S} = \frac{n(\bar{A})}{n(S)} = \frac{n(S)-n(A)}{n(S)} = 1 - \frac{n(A)}{n(S)} = 1 - P(A)$$

That is $P(\bar{A}) = 1 - P(A)$

ii) If $A = \phi$, then $P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$. That is, probability of an impossible event is zero.

iii) If $A = S$, then $P(S) = \frac{n(S)}{n(S)} = 1$. That is, probability of a certain event is one.

iv) For any event A in S, $0 \leq P(A) \leq 1$.

v) The odds in favour of A are given by $n(A) : n(\bar{A}) = P(A) : P(\bar{A})$.

vi) The odds against of A are given by $n(\bar{A}) : n(A) = P(\bar{A}) : P(A)$.

vii) If the odds in favour of A are $a : b$, then $P(A) = \frac{a}{a+b}$.
viii) If the odds against of A are $c : d$, then $P(A) = \frac{d}{c+d}$.
ix) $n(A)$ and $n(S)$ are counted by using methods of counting discussed in **Module 1.1**.

Limitations: The mathematical definition of probability breaks down in the following cases:

(i) The outcomes in the sample space are not equally likely.
(ii) The number of outcomes in the sample space is infinite.

Statistical (or Empirical or Relative Frequency or Von Mises) Definition of Probability

If a random experiment is performed repeatedly under identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.

Symbolically, if in N trials an event A happens a_N times, then the probability of the happening of A is given by

$$P(A) = \lim_{N \to \infty} \frac{a_N}{N} \quad \dots (1.3.1)$$

Note:

i) Since the probability is obtained objectively by repetitive empirical observations, it is known as Empirical Probability.
ii) The empirical probability approaches the classical probability as the number of trials becomes indefinitely large.

Limitations of Empirical Probability

(i) If an experiment is repeated a large number of times, the experimental conditions may not remain identical.
(ii) The limit in (1.3.1) may not attain a unique value, however large N may be.

Subjective definition of probability: In this method, probabilities are assigned to events according to the knowledge, experience and belief about the happening of the events. The main limitation of this definition is, it varies from person to person.

Axiomatic Definition of Probability: Let S be a sample space and let $\mathbb{B}$ be a σ-field associated with S. A probability function (or measure) P is a real valued set function having domain B and which satisfies the following three axioms:

1. $P(A) \geq 0$, for every $A \in \mathbb{B}$ (Non-negativity)
2. $P(S) = 1, i.e, P$ is normed (Normality)
3. If $A_1, A_2, \dots, A_n, \dots$ are mutually exclusive events in S, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ (σ - additive or countably additive)

Thus, the probability function is a normed measure on (the measurable space) $(\mathrm{S}, \mathbb{B}, \mathrm{P})$ is called a **Probability space**. This definition is useful in proving theorems on probability.

Note: The elements of $\mathbb{B}$ are events in S.

Solved Examples using Mathematical Definition of Probability

In this section, we use mathematical definition of probability for computing probabilities. Also we use methods of counting for counting the number of outcomes in an event and sample space.

Example 1: A uniform die is thrown at random. Find the probability that the number on it is (i) even (ii) odd (iii) even or multiple of 3 (iv) even and multiple of 3 (v) greater than 4

Solution:

(i) The number of favourable cases to the event of getting an even number is 3, viz., 2,4,6.

$\therefore$ Required probability $= \frac{3}{6} = \frac{1}{2}$

(ii) The number of favourable cases to the event of getting an odd number is 3, viz., 1, 3, 5.

$\therefore$ Required probability $= \frac{3}{6} = \frac{1}{2}$

(iii) The number of favourable cases to the event of getting even or multiple of 3 is 4, viz., 2, 3, 4, 6.

$\therefore$ Required probability $= \frac{4}{6} = \frac{2}{3}$

(iv) The number of favourable cases to the event of getting even and multiple of 3 is 1, viz., 6.

$\therefore$ Required probability $= \frac{1}{6}$

(v) The number of favourable cases to the event of getting greater than 4 is 2, viz., 5 and 6.

$\therefore$ Required probability $= \frac{2}{6} = \frac{1}{3}$

Example 2: Four cards are drawn at random from a pack of 52 cards. Find the probability that

(i) They are a king, a queen, a jack and an ace.
(ii) Two are kings and two are aces.
(iii) All are diamonds.
(iv) Two are red and two are black.
(v) There is one card of each suit.
(vi) There are two cards of clubs and two cards of diamonds.

Solution: Four cards can be drawn from a well shuffled pack of 52 cards in ${}^{52}C_4$ ways, which gives the exhaustive number of cases.

(i) 1 king can be drawn out of the 4 kings is ${}^{4}C_1 = 4$ ways. Similarly, 1 queen, 1 jack and an ace can each be drawn in ${}^{4}C_1 = 4$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable number of cases are ${}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1$.

Hence, required probability $= \frac{{}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_4} = \frac{256}{{}^{52}C_4}$

(ii) Required probability $=\frac{{}^{4}C_2 \times {}^{4}C_2}{{}^{52}C_4}$

(iii) Since 4 cards can be drawn out of 13 cards (since there are 13 cards of diamond in a pack of cards) in ${}^{13}C_4$ ways,

Required probability $=\frac{{}^{13}C_4}{{}^{52}C_4}$

(iv) Since there are 26 red cards (of diamonds and hearts) and 26 black cards (of spades and clubs) in a pack of cards,

Required probability $=\frac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$

(v) Since, in a pack of cards there are 13 cards of each suit,

Required probability $=\frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4}$

(vi) Required probability $=\frac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}$

Example 3: What is the chance that a non-leap year should have fifty-three Sundays?

Solution: A non-leap year consists of 365 days, $i.e.$, 52 full weeks and one over-day. A non-leap year will consist of 53 Sundays if this over-day is Sunday. This over-day can be anyone of the possible outcomes:

(i) Sunday (ii) Monday (iii) Tuesday (iv) Wednesday (v) Thursday (vi) Friday (vii) Saturday, $i.e.$, 7 outcomes in all. Of these, the number of ways favourable to the required event viz., the over-day being Sunday is 1.

$\therefore$ Required probability $=\frac{1}{7}$

Example 4: Find the probability that in 5 tossings, a perfect coin turns up head at least 3 times in succession.

Solution: In 5 tossings of a coin, the sample space is:

$$S = \{H, T\} \times \{H, T\} \times \{H, T\} \times \{H, T\} \times \{H, T\}, (H : \text{head}\,; T : \text{tail})$$

$\therefore$ Exhaustive number of cases $= 2^5 = 32.$

The favourable cases for getting at least three heads in succession are :

Starting with 1st toss: $HHHTH,\ \ HHHTT,\ \ HHHHT,\ \ \ HHHHH$

Starting with 2nd toss: $THHHT,\ \ THHHH$

Starting with 3rd toss: $TTHHH,\ \ HTHHH$

Hence, the total number of favourable cases for getting at least 3 heads in succession are 8.

$\therefore$ Required probability $= \frac{Number\ of\ favourable\ cases}{Exhaustive\ number\ of\ cases} = \frac{8}{32} = \frac{1}{4} = 0.25$

Example 5: A bag contains 20 tickets marked with numbers 1 to 20. One ticket is drawn at random. Find the probability that it will be a multiple of (i)2 or 5, (ii)3 or 5

Solution: One ticket can be drawn out of 20 tickets in ${}^{20}C_1 = 20$ ways, which determine the exhaustive number of cases.

(i) The number of cases favourable to getting the ticket number which is:

(a) a multiple of 2 are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, $i.e.$, 10 cases.

(b) a multiple of 5 are 5, 10, 15, 20 i.e., 4 cases

Of these, two cases viz., 10 and 20 are duplicated.

Hence the number of distinct cases favourable to getting a number which is a multiple of 2 or 5 are: $10 + 4 - 2 = 12.$

$\therefore$ Required probability $= \frac{12}{20} = \frac{3}{5} = 0.6$

(ii) The cases favourable to getting a multiple of 3 are 3, 6, 9, 12, 15, 18 i.e., 6 cases in all and getting a multiple of 5 are 5, 10, 15, 20 i.e., 4 cases in all. Of these, one case viz., 15 is duplicated.

Hence, the number of distinct cases favourable to getting a multiple of 3 or 5 is $6 + 4 - 1 = 9$.

$\therefore$ Required probability $= \frac{9}{20} = 0.45$

Example 6: An urn contains 8 white and 3 red balls. If two balls are drawn at random, find the probability that

(i) both are white, (ii) both are red, (iii) one is of each color.

Solution: Total number of balls in the urn is $8 + 3 = 11$. Since 2 balls can be drawn out of 11 balls in ${}^{11}C_2$ ways,

Exhaustive number of cases $= {}^{11}C_2 = \frac{11 \times 10}{2} = 55$

(i) If both the drawn balls are white, they must be selected out of the 8 white balls and this can be done in ${}^{8}C_2 = \frac{8 \times 7}{2} = 28$ ways.

$\therefore$ Probability that both the balls are white $= \frac{28}{55}$

(ii) If both the drawn balls are red, they must be drawn out of the 3 red balls and this can be done in ${}^{3}C_2 = 3$ ways. Hence, the probability that both the drawn balls are red $= \frac{3}{55}$.

(iii) The number of favourable cases for drawing one white ball and one red ball is ${}^{8}C_1 \times {}^{3}C_1 = 8 \times 3 = 24$

$\therefore$ Probability that one ball is white and other is red $= \frac{24}{55}$

Example 7: The letters of the word 'article' are arranged at random. Find the probability that the vowels may occupy the even places.

Solution: The word 'article' contains 7 distinct letters which can be arranged among themselves in $7!$ ways. Hence exhaustive number of cases is $7!$.

In the word 'article' there are 3 vowels, viz., a, i and e and these are to be placed in, three even places, viz., 2nd, 4th and 6th place. This can be done in $3!$, ways. For each arrangement, the remaining 4 consonants can be arranged in $4!$ ways. Hence, associating these two operations, the number of favourable cases for the vowels to occupy even places is $3! \times 4!$.

$\therefore$ Required probability $= \frac{3!4!}{7!} = \frac{3!}{7\times6\times5} = \frac{1}{35}$

Example 8: Twenty books are placed at random in a shelf. Find the probability that a particular pair of books shall be:

(i) Always together **(ii) Never together**

Solution: Since 20 books can be arranged among themselves in $20!$ ways, the exhaustive number of cases is $20!$.

(i) Let us now regard that the two particular books are tagged together so that we shall regard them as a single book. Thus, now we have $(20-1) = 19$ books which can be arranged among themselves in $19!$ ways. But the two books which are fastened together can be arranged among themselves in $2!$ ways.

Hence, associating these two operations, the number of favourable cases for getting a particular pair of books always together is $19! \times 2!$.

$\therefore$ Required probability is $\frac{19!\times2!}{20!} = \frac{2}{20} = \frac{1}{10}$.

(ii) Total number of arrangement of 20 books among themselves is $20!$ and the total number of arrangements that a particular pair of books will always be together is $19!\,2!$, [See part (i)]. Hence, the number of arrangements in which a particular pair of books is never together is

$$20! - 2 \times 19! = (20 - 2) \times 19! = 18 \times 19!$$

$\therefore$ Required probability $= \frac{18 \times 19!}{20!} = \frac{18}{20} = \frac{9}{10}$

Aliter: P [A particular pair of books shall never be together]

$= 1 - P$[A particular pair of books is always together]$= 1 - \frac{1}{10} = \frac{9}{10}$.

Example 9: n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.

Solution: The n persons can be seated in n chairs at a round table in $(n-1)!$ ways, which gives the exhaustive number of cases.

If two specified persons, say, A and B sit together, then regarding A and B fixed together, we get $(n-1)$ persons in all, who can be seated at a round table in $(n-2)!$ ways. Further, since A and B can interchange their positions in 2! ways, total number of favourable cases of getting A and B together is $(n-2)! \times 2!$.

Hence, the required probability is: $\frac{(n-2)! \times 2!}{(n-1)!} = \frac{2}{n-1}$

Aliter: Let us suppose that of the n persons, two persons, say, A and B are to be seated together at a round table. After one of these two persons, say A occupies the chair, the other person B can occupy any one of the remaining $(n-1)$ chairs. Out of these $(n-1)$ seats, the number of seats favourable to making B sit next to A is 2 (since B can sit on either side of A). Hence the required probability is $\frac{2}{n-1}$.

Example 10: In a village of 21 inhabitants, a person tells a rumour to a second person, who in turn repeats it to a third person, etc. at each step the recipient of the rumour is chosen at random from the 20 people available. Find the probability that the rumour will be told 10 times without:

(i) returning to the originator ; **(ii) being repeated to any person**

Solution: Since any person can tell the rumour to any one of the remaining $21 - 1 = 20$ people in 20 ways, the exhaustive number of cases that the rumour will be told 10 times is 20^{10}.

(i) Let us define the event :

E_1 : The rumour will be told 10 times without returning to the originator.

The originator can tell the rumour to any one of the remaining 20 persons in 20 ways, and each of the $10 - 1 = 9$ recipients of the rumour can tell it to any of the remaining $20 - 1 = 19$ persons (without returning it to the originator) in 19 ways. Hence the favourable number of cases for E_1 are 20×19^9. The required probability is given by :

$$P(E_1) = \frac{20 \times 19^9}{20^{10}} = \left(\frac{19}{20}\right)^9$$

(ii) Let us define the event :

E_2 : The rumour is told 10 times without being repeated to any person.

In this case the first person (narrator) can tell the rumour to any one of the available $21 - 1 = 20$ persons; the second person can tell the rumour to any one of the remaining $20 - 1 = 19$ persons; the third person can tell the rumour to anyone of the remaining $20 - 2 = 18$ persons; …; the 10th person can tell the rumour to any one of the remaining $20 - 9 = 11$ persons.

Hence the favourable number of cases for E_2 are $20 \times 19 \times 18 \times \ldots \times 11$.

$\therefore$ Required probability $= P(E_2) = \frac{20\times19\times18\times\ldots\times11}{20^{10}}$

Example 11: If 10 men, among whom are A and B, stand in a row, what is the probability that there will be exactly 3 men between A and B?

Solution: If 10 men stand in a row, then A can occupy any one of the 10 positions and B can occupy any one of the remaining 9 positions. Hence, the exhaustive number of cases for the positions of two men A and B are $10 \times 9 = 90$.

The cases favourable to the event that there are exactly 3 men between A and B are given below:

(i) A is in the 1st position and B is in the 5th position.

(ii) A is in the 2nd position and B is in the 6th position.

. . . .

. . . .

. . . .

(vi) A is in the 6th position and B is in the 10th position.

Further, since A and B can interchange their positions, the total number of favourable cases $= 2 \times 6 = 12$.

∴ Required probability $= \frac{12}{90} = \frac{2}{15} = 0.1333$

Example12: A five digit number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.

Solution: The total number of ways in which the five digits 0, 1, 2, 3, 4 can be arranged among themselves is $5!$. Out of these, the number of arrangements which begin with 0 (and therefore will give only 4-digited numbers) is $4!$.

Hence the total number of five digited numbers that can be formed from digits 0, 1, 2, 3, 4 is $5! - 4! = 120 - 24 = 96$

The number formed will be divisible by 4 if the number formed by the two digits on extreme right (i.e., the digits in the unit and tens places) is divisible by 4. Such numbers are:

$$04, \quad 12, \quad 20, \quad 24, \quad 32 \text{ and } 40$$

If the numbers end in 04, the remaining three digits viz., 1, 2 and 3 can be arranged among themselves in 3! in each case.

If the numbers end with 12, the remaining three digits 0, 2, 3 can be arranged in 3! ways. Out of these we shall reject those numbers which start with 0 (i.e., have 0 as the first digit). There are $(3-1)! = 2!$ such cases. Hence, the number of five digited numbers ending with 12 is : $3! - 2! = 6 - 2 = 4$

Similarly the number of 5 digited numbers ending with 24 and 32 each is 4. Hence the total number of favourable cases is: $3 \times 3! + 3 \times 4 = 18 + 12 = 30$

Hence, required probability $= \frac{30}{96} = \frac{5}{16}$

Example13: There are four hotels in a certain town. If 3 men check into hotels in a day, what is the probability that each checks into a different hotel?

Solution: Since each man can check into any one of the four hotels in ${}^4C_1 = 4$ ways, the 3 men can check into 4 hotels in $4 \times 4 \times 4 = 64$ ways, which gives the exhaustive number of cases.

If three men are to check into different hotels, then first man can check into any one of the 4 hotels in ${}^4C_1 = 4$ ways; the second man can check into any one of the remaining 3 hotels in ${}^3C_1 = 3$ ways; and the third man can check into any one of the remaining two hotels in ${}^2C_1 = 2$ ways. Hence, favourable number of cases for each man checking into a different hotel is: ${}^4C_1 \times {}^3C_1 \times {}^2C_1 = 4 \times 3 \times 2 = 24$

$\therefore$ Required probability $= \frac{24}{64} = \frac{3}{8} = 0.375$