P3:

If $(\boldsymbol{X}, \boldsymbol{Y}) \sim \boldsymbol{BN}\left(\mathbf{3}, \mathbf{1}, \mathbf{16}, \mathbf{25}, \frac{3}{5}\right)$, then find

(i) $\boldsymbol{P}(\mathbf{3} < Y < 8 | X = 7)$

(ii) $\boldsymbol{P}(-\mathbf{3} < X < 3 | Y = -4)$**.**

Solution:

If $(X, Y) \sim BN(\mu_1, \mu_2, {\sigma_1}^2, {\sigma_2}^2, \rho)$, then

$$(Y|X = x) \sim N\left[\mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x - \mu_1), {\sigma_2}^2(1 - \rho^2)\right]$$

and $$(X|Y = y) \sim N\left[\mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(y - \mu_2), {\sigma_1}^2(1 - \rho^2)\right]$$

(i) $E(Y|X = 7) = \mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x - \mu_1) = 1 + \frac{3}{5} \cdot \frac{5}{4}(7 - 3) = 4$ and

$V(Y|X = 7) = {\sigma_2}^2(1 - \rho^2) = 25\left(1 - \frac{9}{25}\right) = 16$

Thus $(Y|X = 7) \sim N(4,16)$.

Consider $P(3 < Y < 8|X = 7) = P\left(\frac{3-4}{4} < Z < \frac{8-4}{4}\right)$

$$= P\left(-\frac{1}{4} < Z < 1\right)$$

$$= P\left(-\frac{1}{4} < Z < 0\right) + P(0 < Z < 1)$$

$$= P(0 < Z < 0.25) + P(0 < Z < 1)$$

$$= 0.0987 + 0.3413 \quad \text{(use table)}$$

$$= 0.44$$

(ii) $E(X|Y = -4) = \mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(y - \mu_2)$

$$= 3 + \frac{3}{5}\left(\frac{4}{5}\right)(-4 - 1) = \frac{3}{5}$$

and $V(X|Y = -4) = \sigma_1^2(1 - \rho^2) = 16\left(1 - \frac{9}{25}\right) = \left(\frac{16}{5}\right)^2$

Thus $V(X|Y=-4) \sim N\left[\frac{3}{5}, \left(\frac{16}{5}\right)^2\right]$

$\therefore \; P(-3 < X < 3|Y=-4) = P\left(\frac{-3-\frac{3}{5}}{\frac{16}{5}} < Z < \frac{3-\frac{3}{5}}{\frac{16}{5}}\right) = P(-1.125 < Z < 0.75)$

$= P(-1.125 < Z < 0) + P(0 < Z < 0.75)$

$= P(0 < Z < 1.125) + P(0 < Z < 0.75)$ (use table)

$= 0.3708 + 0.2734 = 0.6442$