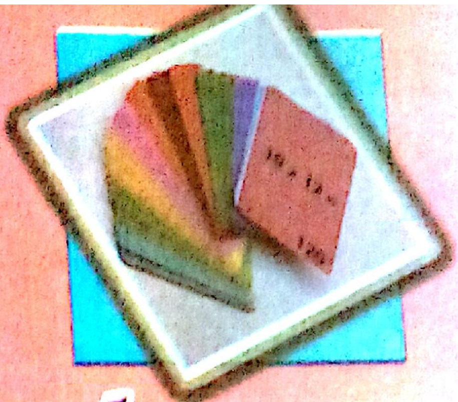


Chapter 5

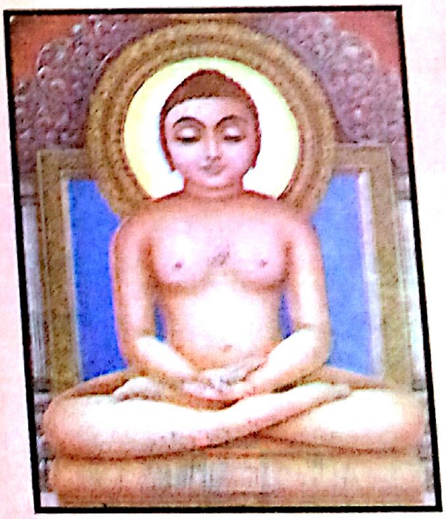
Permutations and Combinations

"Mathematical proofs, like Diamonds, are hard and clear and will be touched with nothing but strict reasoning" - **John Locke**

Introduction

The first known use of permutations and combinations goes back to 6^{th} century B.C. when '*Susruta*' in his medicinal work '*Susruta Samhita*' finds 63 combinations out of 6 different tastes by taking one at a time, two at a time etc. Later in the 3^{rd} century B.C., a Sanskrit scholar '*Pingala*' in his book '*Chandassastra*' used permutations and combinations to determine the number of combinations of a given number of letters by taking one at a time, two at a time etc. The concept of permutations and combinations was treated as a self contained topic in mathematics under the name "*Vikalpa*" by renowned mathematician '*Mahavira*' in 9^{th} century A.D. The credit of stating several important theorems and results on the subject matter of permutations and combinations goes to the renowned scholar "*Bhaskaracharya*". He treated this topic under the name '*Anka Vyastha*' in his famous book '*Leelavathi Ganitham*".

The theory of permutations and combinations in the present sense first appeared in the book 'Ars Conjectandi' written by the renowned mathematician "Jakob Bernoulli" in 17th Century A.D. which was published in 1713 A.D. after his death.



Mahaviracharya
(9th century)

Mahavira *was a 9th century Indian mathematician from Karnataka. He was the author of Ganita Saara Sangraha. He liberated Mathematics from Astronomy*

We must have come across situations like choosing five questions out of eight questions in a question paper or which items to be chosen from the menu card in a hotel etc. We discuss such situations in this chapter. This chapter 'permutations and combinations' is an important chapter in algebra in view of a number of applications in day - to - day life and in the theory of probability. While learning **'permutations and combinations'**, we should be in a position to clearly see whether the concept of a permutation or the concept of a combination is applicable in a given situation. In general, a combination is only a selection while a permutation involves two steps, namely, selection and arrangement. For example, forming a three digit number using the digits 1, 2, 3, 4, 5 is a **'permutation'**. This involves two steps. In the first step we select three digits, say 2, 4, 5. In the second step, we arrange them to form a three digit number such as 245, 452, 542 etc. Forming a set with three elements using the digits 1, 2, 3, 4, 5 is a **'combination'**. This involves only one process, namely, selection of three elements, say 2, 4, 5. Then the element set formed is $\{2, 4, 5\}$ which is same as the sets $\{4, 5, 2\}$ $\{5, 4, 2\}$ etc. Thus, whenever there is importance to the arrangement or order in which the objects are placed, then it is a '**permutation**' and if there is no importance to the arrangement or order, but only selection is required, then it is a '**combination**'. These notions will help us to arrive at the number of arrangements or combinations without actually counting them.

Before going into formal definitions, we introduce **factorial** notation, which is required to calculate the number of permutations or combinations. If n is a positive integer, we define $n!$ (read as n factorial) by mathematical induction as follows.

$$1! = 1$$

and $$n! = n.((n-1)!) \text{ if } n > 1.$$

For example, $$2! = 2(1!) = 2$$

$$3! = 3(2!) = 3.2 = 6$$

$$4! = 4(3!) = 4.6 = 24$$

$$5! = 5(4!) = 5.24 = 120 \text{ etc.}$$

By convention, we define $0! = 1$

Throughout this chapter the letters n, r denote nonnegative integers unless otherwise mentioned.

5.1 Fundamental Principle of Counting - Linear and Circular permutations

Before giving formal definitions of linear and circular permutations, we first learn about the "**Fundamental Principle of Counting**", which plays a very crucial role in the development of the theory of permutations and combinations.

5.1.1 Fundamental principle of Counting

If a work w_1 can be performed in 'm' different ways and a second work w_2 can be performed (after w_1 has been performed in any one of the 'm' ways) in n different ways, then the two works (one after the other) can be performed in 'mn' different ways.

This principle can be easily understood with the help of the following two examples.

5.1.2 Example : If a man has 4 different coloured trousers T_1, T_2, T_3, T_4 and three different coloured shirts S_1, S_2, S_3, then he can select a pair (a trouser and a shirt) in $4 \times 3 = 12$ different ways as explained below.

In this example, we take w_1 as selecting a trouser which can be performed in 4 ways and w_2 as selecting a shirt which can be performed in 3 ways. Hence, by the fundamental principle, he can select a pair in $4 \times 3 = 12$ different ways. They are

$T_1 S_1$	$T_1 S_2$	$T_1 S_3$
$T_2 S_1$	$T_2 S_2$	$T_2 S_3$
$T_3 S_1$	$T_3 S_2$	$T_3 S_3$
$T_4 S_1$	$T_4 S_2$	$T_4 S_3$

5.1.3 Example : If there are four different modes of transport available to travel from Hyderabad (HYD) to Chennai (CH), namely, bus, car, train and aeroplane (we denote these by M_1, M_2, M_3, M_4 respectively) and three different modes of transport from Chennai to Bangalore (BG), namely, bus, train, aeroplane (we denote these by N_1, N_2, N_3 respectively), then how many different modes of transport are available for a person to travel from Hyderabad to Bangalore (via Chennai)?

Solution : Here the work w_1 is to travel from HYD to CH, which can be performed in 4 different ways and the work w_2 is to travel from CH to BG, which can be performed in 3 different ways. Therefore, by the fundamental principle, the two works can be done in $4 \times 3 = 12$ ways. That is, a person can travel from HYD to BG (via CH) in 12 different ways. They are

$M_1 N_1$	$M_1 N_2$	$M_1 N_3$
$M_2 N_1$	$M_2 N_2$	$M_2 N_3$
$M_3 N_1$	$M_3 N_2$	$M_3 N_3$
$M_4 N_1$	$M_4 N_2$	$M_4 N_3$

In this,

M_1N_1 means that the person travels by bus from HYD to CH and again by bus from CH to BG

M_2N_3 means that the person travels by car from HYD to CH and by aeroplane from CH to BG

M_3N_2 means travelling from HYD to CH by train and CH to BG by train again etc.

Now we give the definition of a **linear permutation**.

5.1.4 Definition

From a given finite set of elements (similar or not) selecting some or all of them and arranging them in a line is called a ***'linear permutation'*** *or simply a* ***'permutation'***.

This definition is explained in the following illustrations.

5.1.5 Examples

1. **Example :** From the letters of the word MINT, **two** letter permutations are MI, IM, MN, NM, MT, TM, IN, NI, IT, TI, NT, TN

2. **Example :** From the letters of the word RUNNING

   (i) permutations with **three** letters are RUN, UNN, GUR,GNU, NNN etc.

   (ii) permutations with **Four** letters are RUNN, UNIG, NNIN, GNUN, etc.

3. **Example :** Using the digits 1, 2, 3, 4, 5

   (i) permutations with **two** digits (or two digited numbers) are 12, 13, 32, 52, 53, 45 etc.

   (ii) permutations with **three** digits are 123, 324, 513, 352 etc.

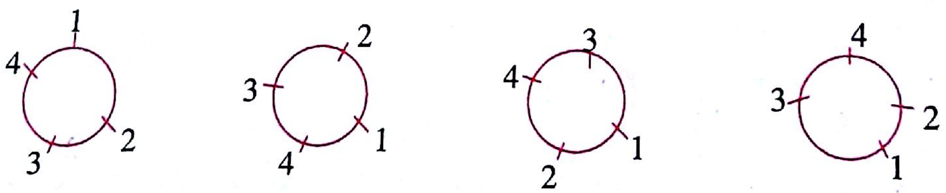
   (iii) permutations with **four** digits are 1234, 4351, 5124 etc.

Next, we define circular permutation in the following.

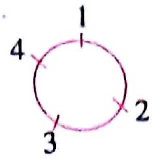
5.1.6 Definition

From a given finite set of things (similar or not) choosing some or all of them and arranging them around a circle is called a ***'circular permutation'***

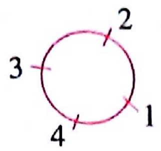
5.1.7 Example : Some of the circular permutations formed using the digits 1, 2, 3, 4 are



The important difference between a circular permutation and a linear permutation is that a linear permutation has a first place (also a last place), where as a circular permutation has no starting place. It can be treated as starting from any one of the elements in it. But how the other elements are arranged relative to this (starting) element is to be taken into consideration. Thus the linear permutations 1234, 2341, 3412, 4123 give rise to only one circular permutation. That is,



Similarly, the linear permutations 1432, 4321, 3214, 2143 give rise to only one circular permutation given below.



Thus in a circular permutation where the first element is placed is not important but how the remaining elements are arranged relative to that element is important.

In some cases like garlands of flowers, chains of beads etc., there is no distinction between the clock-wise and anti clock - wise arrangements of the same circular permutation. They will be treated as a single circular permutation. In such cases, the two circular permutations described above will be treated as a single circular permutation.

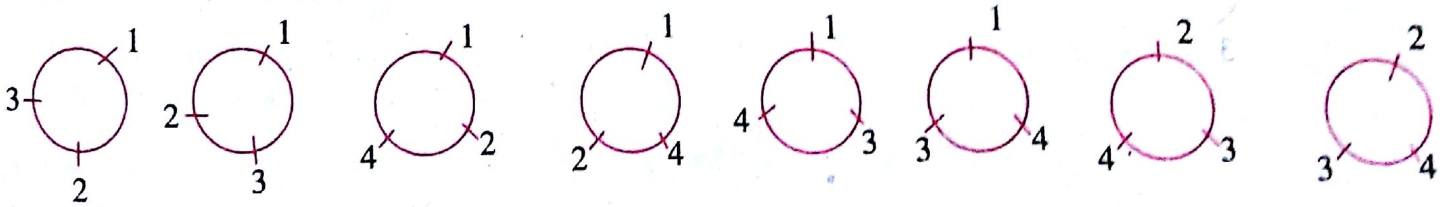
5.1.8 Example : Write all possible (i) linear (ii) circular permutations using the digits 1, 2, 3, 4 taken **three** at a time.

Solution : (i) Required linear permutations are

123	124	134	234
231	241	341	342
312	412	413	423
132	142	143	243
321	421	431	432
213	214	314	324

Thus the number of linear permutations that can be formed using the digits 1, 2, 3, 4 taken **three** at a time is 24.

(ii) Required circular permutations are



Hence, the number of circular permutations that can be formed using the digits 1, 2, 3, 4 taken three at a time is 8.

5.2 Permutations of n dissimilar things taken r at a time

Hereafter by a permutation we mean a linear permutation in which no object is used more than once (that is, a permutation without repetition). If repetition is allowed any where, it will be clearly mentioned.

In example 5.1.8 of the previous section, we have exhibited that the number of permutations of 4 dissimilar things taken 3 at a time 24. But if the number of given things and (or) the number of things to be arranged is large, then it is not easy to enumerate the permutations like in example 5.1.8. Hence we develope a formula to find the number of such permutations in the following.

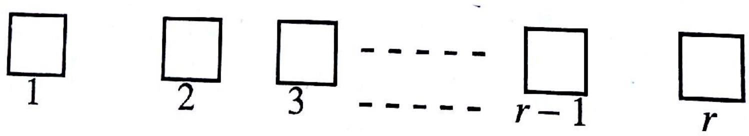
5.2.1 Theorem : *If n, r are positive integers and $r \leq n$, then the number of permutations of n dissimilar things taken r at a time is*

$$n(n-1)(n-2) \ldots (n-r+1). \qquad \textit{That is,} \quad \prod_{k=0}^{r-1} (n-k)$$

Proof : We prove this theorem by using induction on n. If $n = 1$, then $r = 1$. In this case, the theorem is trivial. Assume $n \geq 2$ and suppose that the theorem is true for $n-1$. That is, for any s, $1 \leq s \leq (n-1)$, the number of permutations of $(n-1)$ dissimilar things taken 's' at a time is

$$(n-1)(n-2) \ldots ((n-1)-(s-1)). \text{ That is, } \prod_{k=0}^{s-1} ((n-1)-k).$$

Now suppose n dissimilar things (objects) are given. Observe that the number of the required permutations is equal to the number of ways of filling 'r' blanks using the given n things with one object in each blank. Let us take r blanks.



To fill the first blank, we can use any one of the given n things. Thus, the first blank can be filled in 'n' different ways.

After filling up the first blank, we are left with $(n-1)$ things and $(r-1)$ blanks. These $(r-1)$ blanks are to be filled with these $(n-1)$ things. By induction hypothesis (taking $s = r-1$), the number of such permutations is equal to $\prod_{k=0}^{r-2} ((n-1)-k)$.

Here we have performed two works w_1 and w_2. The work w_1 is filling up the first blank and the work w_2 is filling up the remaining $(r-1)$ blanks using the remaining $(n-1)$ things. Hence, by the fundamental principle (5.1.1) the number of ways in which the two works w_1 and w_2 can be performed, (that is, the r blanks can be filled using the given n things) is

$$n.\prod_{k=0}^{r-2}((n-1)-k)$$

$$= n(n-1)(n-2)\ldots((n-1)-(r-2))$$

$$= n(n-1)(n-2)\ldots(n-r+1)$$

$$= \prod_{k=0}^{r-1}(n-k).$$

Hence, by mathematical induction, the theorem follows.

Notation : The number of permutations of n dissimilar things taken r at a time is denoted by ${}^nP_r$ or $P(n, r)$. However, we use the notation ${}^nP_r$ only. Thus, for $1 \le r \le n$,

$\boxed{{}^nP_r = n(n-1)\ldots(n-r+1)}$ and we write ${}^nP_0 = 1$ by convention.

5.2.2 Formula : If $n \ge 1$ and $0 \le r \le n$, then $\boxed{{}^nP_r = \dfrac{n!}{(n-r)!}}$

For $1 \le r \le n$, from Theorem 5.2.1, we get

$${}^nP_r = n(n-1)\ldots(n-r+1)$$

$$= \frac{[n(n-1)\ldots(n-r+1)][(n-r)(n-r-1)\ldots2.1]}{(n-r).(n-r-1)\ldots2.1}$$

$$= \frac{n!}{(n-r)!}$$

By convention, ${}^nP_0 = 1 = \dfrac{n!}{(n-0)!}$

5.2.3 Note : $\boxed{{}^nP_n = n! \text{ and } {}^nP_0 = 1}$

5.2.4 Theorem : For $1 \le r \le n$, ${}^nP_r = n\,.\,{}^{(n-1)}P_{(r-1)}$

First method : We know that the number of ways of filling up r blanks with n dissimilar things is ${}^nP_r$. The first blank can be filled with any one of the given n things in n ways.

Now we can fill the remaining $(r-1)$ blanks with the remaining $(n-1)$ things in ${}^{(n-1)}P_{(r-1)}$ ways. Thus the number of ways of filling up r blanks with n things is $n\,.\,{}^{(n-1)}P_{(r-1)}$

That is, $${}^{n}P_{r} = n \,.\, {}^{(n-1)}P_{(r-1)}$$

Second Method : $$n \,.\, {}^{(n-1)}P_{(r-1)} = n \cdot \frac{(n-1)!}{((n-1)-(r-1))!} = \frac{n((n-1)!)}{(n-r)!} = \frac{n!}{(n-r)!} = {}^{n}P_{r}.$$

5.2.5 Note : From the above Formula we also get that

$$\begin{aligned} {}^{n}P_{r} &= n \,.\, {}^{(n-1)}P_{(r-1)} \\ &= n \,.\, (n-1) \,.\, {}^{(n-2)}P_{(r-2)} \\ &= n \,.\, (n-1) \,.\, (n-2) \,.\, {}^{(n-3)}P_{(r-3)} \text{ etc.} \end{aligned}$$

5.2.6 Example : Find the number of permutations of 4 dissimilar things taken 3 at a time.

Solution : From Formula 5.2.2, the required number of permutations is

$${}^{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24.$$

5.2.7 Note

In example 5.1.8, we exhibited these permutations and then counted them as 24. Here we have used the formula.

5.2.8 Example : Find the numb of all 4 letter words that can be formed using the letters of the word EQUATION. How many of thes rds begin with E? How many end with N? How many begin with E and end with N?

Solution : The word EQUATION has 8 distinct letters. We have to fill up 4 places using these 8 letters.

$$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$$

This can be done in ${}^{8}P_{4} = 8.7.6.5 = 1680$ ways. Hence the number of 4 letter words that can be formed using the letters of the word EQUATION is 1680.

Words beginning with E : Fill the first place with E as shown below.

$$\boxed{E}\ \boxed{}\ \boxed{}\ \boxed{}$$

Now we are left with 7 letters and 3 places. They can be filled in

$${}^{7}P_{3} = 7.6.5 = 210 \text{ ways.}$$

Thus the number of 4 letter words that begin with E is 210.

Words ending with N : This can be done in the same way as above. First fill the last place with N as shown below

$\square \quad \square \quad \square \quad \boxed{N}$

Then the remaining 3 places with the remaining 7 letters can be filled in ${}^7P_3 = 210$ ways. Hence the number of 4 letter words ending with N is 210.

Words beginning with E and ending with N : Fill the first place with E and the last with N as shown below

$\boxed{E} \quad \square \quad \square \quad \boxed{N}$

Now the remaining 2 places with the remaining 6 letters can be filled in

${}^6P_2 = 6 \times 5 = 30$ ways. Thus, the number of 4 letter words that begin with E and end with N is 30.

5.2.9 To prove ${}^nP_r = {}^{(n-1)}P_r + r \, . \, {}^{(n-1)}P_{(r-1)}$

In this section, we prove the following important result using the definition of ${}^nP_r$.

5.2.10 Theorem : *Let n, r be positive integers and $1 \le r < n$. Then*

$${}^nP_r = {}^{(n-1)}P_r + r \, . \, {}^{(n-1)}P_{(r-1)}.$$

First proof (From fundamentals) : We know that ${}^nP_r$ is the number of ways of filling up r places using n things. Let us take one thing among the given n things and name it as T. Let

m_1 = number of permutations containing T

m_2 = number of permutations not containing T.

Thus, ${}^nP_r = m_1 + m_2$.

To find m_1, we first put 'T' in one of the r places. This can be done in r ways. Now we can fill the remaining $(r-1)$ places with the remaining $(n-1)$ things in ${}^{(n-1)}P_{(r-1)}$ ways. Therefore,

$$m_1 = r \, . \, {}^{(n-1)}P_{(r-1)}.$$

To find m_2, leave T aside and fill the r places with the remaining $(n-1)$ things in ${}^{(n-1)}P_r$ ways. Thus

$$m_2 = {}^{(n-1)}P_r.$$

Therefore,

$$^{n}P_r = m_1 + m_2 = {}^{(n-1)}P_r + r \cdot {}^{(n-1)}P_{(r-1)}.$$

Second Proof (Using the formula of $^{n}P_r$)

$$^{(n-1)}P_r + r \cdot {}^{(n-1)}P_{(r-1)} = \frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!(n-r) + r.(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!.\, n}{(n-r)!} = \frac{n!}{(n-r)!} = {}^{n}P_r.$$

5.2.11 Examples

1. Example : Find the number of 4 letter words that can be formed using the letters of the word MIXTURE which

(i) contain the letter X

(ii) do not contain the letter X

Solution

We have to fill up 4 places using the 7 letters of the word MIXTURE. Take 4 places.

(i) First we put X in one of the 4 places. This can be done in 4 ways. Now we can fill the remaining 3 places with the remaining 6 letters in $^{6}P_3$ ways. Thus the number of 4 letter words containing the letter X are

$$4 \times {}^{6}P_3 = 4 \times 120 = 480$$

(ii) Leaving the letter X, we fill the 4 places with the remaining 6 letters in $^{6}P_4$ ways. Thus, the number of 4 letter words that do not contain the letter X is

$$^{6}P_4 = 360.$$

. **Example :** Find the number of ways of arranging 5 boys and 4 girls in a row so that the row (i) begins and nds with boys (ii) begins with a boy and ends with a girl.

lution : (i) The total number of persons is 9 (5 boys + 4 girls). Let us take 9 places.

B								B

First we fill the first and the last places with boys. This can be done in ${}^5P_2$ ways. Now, we have to fill up the remaining 7 places with the remaining 7 persons (3 boys + 4 girls) in 7! ways. Hence the required number of arrangements is

$$ {}^5P_2 \times 7! = 20 \times 5040 = 100800. $$

(ii)

B								G

We fill the first place with one of the boys in 5 ways and last place with one of the girls in 4 ways. The remaining 7 places can be filled with the remaining 7 persons (4 boys + 3 girls) in 7! ways. Hence the number of required arrangements is

$$ 5 \times 4 \times 7! = 20 \times 5040 = 100800. $$

5.2.12 Solved Problems

1. Problem : *If* ${}^nP_4 = 1680$, *find* n.

Solution : We know that ${}^nP_4$ is the product of 4 consecutive integers of which n is the largest. That is

$$ {}^nP_4 = n(n-1)(n-2)(n-3) $$

and

$$ 1680 = 8 \times 7 \times 6 \times 5 $$

on comparing the largest integers, we get $n = 8$.

2. Problem : *If* ${}^{12}P_r = 1320$, *find* r.

Solution : $1320 = 12 \times 11 \times 10 = {}^{12}P_3$. Thus $r = 3$.

3. Problem : If ${}^{(n+1)}P_5 : {}^nP_5 = 3 : 2$, find n.

Solution : ${}^{(n+1)}P_5 : {}^nP_5 = 3 : 2$

$$ \Rightarrow \frac{(n+1)!}{(n-4)!} \times \frac{(n-5)!}{n!} = \frac{3}{2} $$

$$ \Rightarrow \frac{n+1}{n-4} = \frac{3}{2} $$

$$ \Rightarrow 2n + 2 = 3n - 12 \Rightarrow n = 14. $$

4. Problem : *If* ${}^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$, *find* r.

Solution : ${}^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$

$$ \Rightarrow \frac{(56)!}{(56-(r+6))!} \times \frac{(54-(r+3))!}{(54)!} = \frac{30800}{1} $$

$$\Rightarrow \frac{(56)!}{(50-r)!} \times \frac{(51-r)!}{(54)!} = \frac{30800}{1}$$

$$\Rightarrow 56 \times 55 \times (51-r) = 30800$$

$$\Rightarrow (51-r) = \frac{30800}{56\times55} = 10$$

$$\Rightarrow r = 41.$$

5. Problem : *In how many ways 9 mathematics papers can be arranged so that the best and the worst (i) may come together (ii) may not come together ?*

Solution

(i) If the best and worst papers are treated as one unit, then we have $9-2+1=7+1=8$ papers. Now these can be arranged in $(7+1)!$ ways and the best and worst papers between themselves can be permuted in $2!$ ways. Therefore the number of arrangements in which best and worst papers come together is $8!\,2!$.

(ii) Total number of ways of arranging 9 mathematics papers is $9!$. The best and worst papers come together in $8!\,2!$ ways. Therefore the number of ways they may not come together is $9!-8!\,2!$

$$= 8!(9-2) = 8! \times 7.$$

6. Problem : *Find the number of ways of arranging 6 boys and 6 girls in a row. In how many of these arrangements*

(i) all the girls are together

(ii) no two girls are together

(iii) boys and girls come alternately ?

Solution: 6 boys + 6 girls = 12 persons. They can be arranged in a row in $(12)!$ ways.

(i) Treat the 6 girls as one unit. Then we have

6 boys + 1 unit of girls.

They can be arranged in $7!$ ways. Now, the 6 girls among themselves can be permuted in $6!$ ways. Hence, by the fundamental principle, the number of arrangements in which all 6 girls are together is $7! \times 6!$.

(ii) First we arrange 6 boys in a row in 6! ways. The girls can be arranged in the 7 gaps between the boys (including the gap in the beginning and the gap in the ending). These gaps are shown below by the letter X.

X	B	X	B	X	B	X	B	X	B	X	B	X
1		2		3		4		5		6		7

Now, the 6 girls can be arranged in these 7 gaps in $^7P_6$ ways. Hence, by the fundamental principle, the number of arrangements in which no two girls come together is $6! \times {}^7P_6 = 6! \times 7! = 7 \times 6! \times 6!$

(iii) Let us take 12 places. The row may begin with either a boy or a girl. That is, 2 ways. If it begins with a boy, then all odd places (1, 3, 5, 7, 9, 11) will be occupied by boys and the even places (2, 4, 6, 8, 10, 12) by girls. The 6 boys can be arranged in the 6 odd places in 6! ways and the 6 girls can be arranged in the 6 even places in 6! ways. Thus the number of arrangements in which boys and girls come alternately is $2 \times 6! \times 6!$.

Note

In the above, one may think that questions (ii) and (iii) are same. But they are not (as evident from the answers). In Question (ii), after arranging 6 boys, we found 7 gaps and 6 girls are arranged in these 7 gaps. Hence one place remains vacant. It can be any one of the 7 gaps. But in Question (iii), the vacant place should either be at the beginning or at the ending but not in between. Thus, only 2 choices for the vacant place

7. Problem. *Find the number of 4-letter words that can be formed using the letters of the word MIRACLE. How many of them*

(i) begin with an vowel

(ii) begin and end with vowels

(iii) end with a consonant?

Solution : The word MIRACLE has 7 letters. Hence the number of 4 letter words that can be formed using these letters is $^7P_4 = 7 \times 6 \times 5 \times 4 = 840$

Let us take 4 blanks.

$\square \quad \square \quad \square \quad \square$

(i) We can fill the first place with one of the 3 vowels (I, A, E) in $^3P_1 = 3$ ways. Now, the remaining 3 places can be filled using the remaining 6 letters in

$$^6P_3 = 120 \text{ ways.}$$

Thus the number of 4 letter words that begin with an vowel is $3 \times 120 = 360$.

(ii) Fill the first and last places with 2 vowels in ${}^3P_2 = 6$ ways.

The remaining 2 places can be filled with the remaining 5 letters in ${}^5P_2 = 20$ ways.

Thus the number of 4 letter words that begin and end with vowels is

$$6 \times 20 = 120.$$

(iii) We can fill the last place with one of the 4 consonants (M, R, C, L) in ${}^4P_1 = 4$ ways.

The remaining 3 places can be filled with the remaining 6 letters in ${}^6P_3$ ways.

Thus the number of 4 letter words that end with an vowel is

$$4 \times {}^6P_3 = 4 \times 120 = 480.$$

8. Problem : *Find the number of ways of permuting the letters of the word PICTURE so that*

(i) *all vowels come together*

(ii) *no two vowels come together.*

(iii) *the relative positions of vowels and consonants are not distrurbed.*

Solution : The word PICTURE has 3 vowels (I, U, E) and 4 consonants (P, C, T, R)

(i) Treat the 3 vowels as one unit. Then we can arrange 4 consonants + 1 unit of vowels in 5! ways. Now the 3 vowels among themselves can be permuted in 3! ways. Hence the number of permutations in which the 3 vowels come together is $5! \times 3! = 720$.

(ii) First arrange the 4 consonants in 4! ways. Then in between the vowels, in the beginning and in the ending, there are 5 gaps as shown below by the letter X

X	□	X	□	X	□	X	□	X
1		2		3		4		5

In these 5 places we can arrange the 3 vowels ${}^5P_3$ ways. Thus the number of words in which no two vowels come together is $4! \times {}^5P_3 = 24 \times 60 = 1440$.

(iii) The three vowels can be arranged in their relative positions in 3! ways and the 4 consonants can be arranged in their relative positions in 4! ways.

V	C	C	V	C	V	C

The required number of arrangements is $3!\,4! = 144$.

Note

In the above problem, from (i) we get that the number of permutations in which the vowels do not come together is

= Total number of permutations – number of permutations in which 3 vowels come together.

$= 7! - 5!\,.\,3! = 5040 - 720 = 4320.$

But the number of permutations in which no two vowels come together is only 1440. In the remaining 2880 permutations two vowels come together and third appears away from these.

9. Problem : *If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order, find the rank of the word PRISON.*

Solution : The letters of the given word in dictionary order is

I N O P R S

In the dictionary order, first we write all words that begin with I. If we fill the first place with I, then the remaining 5 places can be filled with the remaining 5 letters in 5! ways. That is, there are 5! words that begin with I. Proceeding like this, after writing all words that begin with I, N, O, we have to write the words that begin with P. Among them first come the words with first two letters P, I. As above there are 4! such words. On proceeding like this, we get

I	–	–	–	–	–	→	5! words
N	–	–	–	–	–	→	5! words
O	–	–	–	–	–	→	5! words
P	I	–	–	–	–	→	4! words
P	N	–	–	–	–	→	4! words
P	O	–	–	–	–	→	4! words
P	R	I	N	–	–	→	2! words
P	R	I	O	–	–	→	2! words
P	R	I	S	N	–	→	1! words
P	R	I	S	O	N	→	1 word

Hence the rank of the word PRISON is

$$3 \times 5! + 3 \times 4! + 2 \times 2! + 1! + 1 = 360 + 72 + 4 + 1 + 1 = 438.$$

10. Problem : *Find the number of 4 digit numbers that can ... the digits 2, 3, 5, 6, 8 (without repetition). How many of them are divisible by*

(i) 2 (ii) 3 (iii) 4 (iv) 5 (v) 25

Solution

The number of 4 digit numbers that can be formed using the 5 digits 2, 3, 5, 6, 8 is ${}^5P_4 = 120$.

(i) Divisible by 2 : For a number to be divisible by 2, the units place should be filled with an even digit. This can be done in 3 ways (2 or 6 or 8).

			X

Now, the remaining 3 places can be filled with the remaining 4 digits in ${}^4P_3 = 24$ ways. Hence, the number of 4-digit numbers divisible by 2 is

$$3 \times 24 = 72.$$

(ii) Divisible by 3 : A number is divisible by 3 if the sum of the digits in it is a multiple of 3. Since the sum of the given 5 digits is 24, we have to leave either 3 or 6 and use the digits 2, 5, 6, 8 or 2, 3, 5, 8. In each case, we can permute them in 4! ways. Thus the number of 4-digit numbers divisible by 3 is

$$2 \times 4! = 48.$$

(iii) Divisible by 4 : A number is divisible by 4 if the number formed by the digits in the last two places (tens and units places) is a multiple of 4.

		X	X

Thus we fill the last two places (as shown in the figure) with one of

28, 32, 36, 52, 56, 68

That is done in 6 ways. After filling the last two places, we can fill the remaining two places with the remaining 3 digits in ${}^3P_2 = 6$ ways.

Thus, the number of 4-digit numbers divisible by 4 is $6 \times 6 = 36$.

(iv) Divisible by 5 : After filling the units place with 5 (one way), the remaining 3 places can be filled with the remaining 4 digits in ${}^4P_3 = 24$ ways. Hence the number of 4-digit numbers divisible by 5 is 24.

(v) Divisible by 25 : Here also we have to fill the last two places (that is, units and tens place) with 25 (one way) as shown below.

		2	5

Now the remaining 2 places can be filled with the remaining 3 digits in ${}^3P_2 = 6$ ways. Hence the number of 4-digit numbers divisible by 25 is 6.

11. Problem : *Find the sum of all 4-digit numbers that can be formed using the digits 1, 3, 5, 7, 9.*

Solution : We know that the number of 4-digit numbers that can be formed using the given 5 digits is $^5P_4 = 120$. Now we find their sum.

We first find the sum of the digits in the unit place of all these 120 numbers. If we fill the units place with 1 as shown below,

			1

then the remaining 3 places can be filled with the remaining 4 digits in $^4P_3$ ways. This means, the number of 4 digit numbers having 1 in units place is $^4P_3$. Similarly, each of the digits 3, 5, 7, 9 appear 24 times in units place. By adding all these digits we get the sum of the digits in units place of all 120 numbers as

$$^4P_3 \times 1 + {}^4P_3 \times 3 + {}^4P_3 \times 5 + {}^4P_3 \times 7 + {}^4P_3 \times 9 = {}^4P_3 \times 25.$$

Similarly, we get the sum of the digits in Tens place as $^4P_3 \times 25$. Since it is in 10's place, its value is $^4P_3 \times 25 \times 10$.

Similarly, the values of the sum of the digits in 100's place and 1000's place are

$$^4P_3 \times 25 \times 100 \text{ and } {}^4P_3 \times 25 \times 1000$$

respectively. Hence the sum of all the 4-digit numbers formed by using the digits 1, 3, 5, 7, 9 is

$$^4P_3 \times 25 \times 1 + {}^4P_3 \times 25 \times 10 + {}^4P_3 \times 25 \times 100 + {}^4P_3 \times 25 \times 1000$$

$$= {}^4P_3 \times 25 \times 1111 \qquad \ldots (*)$$

$$= 24 \times 25 \times 1111 = 6{,}66{,}600.$$

Note

1. From (*) in the above example, we can derive that the sum of all r-digit numbers that can be formed using the given 'n' non - zero digits ($1 \le r \le n \le 9$) is

$$^{(n-1)}P_{(r-1)} \times \text{sum of the given digits} \times 111..1 \ (r \text{ times})$$

2. In the above, if '0' is one digit among the given n digits, then we get that the sum of the r-digit numbers that can be formed using the given n digits (including '0')

$$= \{^{(n-1)}P_{(r-1)} \times \text{sum of the given digits} \times 111..1 \ (r \text{ times})\}$$

$$- \{^{(n-2)}P_{(r-2)} \times \text{sum of the given digits} \times 111..1 \ ((r-1) \text{ times})\}.$$

12. Problem : *How many four digited numbers can be formed using the digits 1, 2, 5, 7, 8, 9? How many of them begin with 9 and end with 2 ?*

Solution : The number of four digited numbers that can be formed using the given digits 1, 2, 5, 7, 8, 9 is $^6P_4 = 360$. Now, the first place and last place can be filled with 9 and 2 in one way.

9			2

The remaining 2 places can be filled by the remaining 4 digits 1, 5, 7, 8. Therefore these two places can be filled in $^4P_2$ ways. Hence, the required number of ways $= 1 \,.\, ^4P_2 = 12$.

13. Problem : *Find the number of injections of a set A with 5 elements to a set B with 7 elements.*

Solution : The first element of A can be mapped to any one of the 7 elements in 7 ways. The second element of A can be mapped to any one of the remaining 6 elements in 6 ways. Proceeding like this we get the number of injections from A to B as $^7P_5 = 2520$.

Note

If a set A has m elements and the set B has n elements, then the number of injections from A into B is ^nP_m if $m \le n$ and 0 if $m > n$.

14. Problem : *Find the number of ways in which 4 letters can be put in 4 addressed envelopes so that no letter goes into the envelope meant for it.*

Solution : Required number of ways is $4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 12-4+1 = 9$.

Note

If there are n things in a row, a permutation of these n things such that none of them occupies its original position is called a derangement of n things.

The number of derangements of n distinct things is $n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^n\frac{1}{n!}\right) = 9$.

Exercise 5(a)

I. 1. If $^nP_3 = 1320$, find n.

2. If $^nP_7 = 42 \,.\, ^nP_5$, find n.

3. If $^{(n+1)}P_5 : {}^nP_6 = 2 : 7$, find n.

4. If $^{12}P_5 + 5 \,.\, ^{12}P_4 = {}^{13}P_r$, find r.

5. If $^{18}P_{(r-1)} : {}^{17}P_{(r-1)} = 9 : 7$, find r.

6. A man has 4 sons and there are 5 schools within his reach. In how many ways can he admit his sons in the schools so that no two of them will be in the same school.

II. 1. If there are 25 railway stations on a railway line, how many types of single second class tickets must be printed, so as to enable a passenger to travel from one station to another.

2. In a class there are 30 students. On the New year day, every student posts a greeting card to all his / her classmates. Find the total number of greeting cards posted by them.

3. Find the number of ways of arranging the letters of the word TRIANGLE so that the relative positions of the vowels and consonants are not disturbed.

4. Find the sum of all 4 digited numbers that can be formed using the digits 0, 2, 4, 7, 8, without repetition.

5. Find the number of numbers that are greater than 4000 which can be formed using the digits 0, 2, 4, 6, 8 without repetition.

6. Find the number of ways of arranging the letters of the word MONDAY so that no vowel occupies even place.

7. Find the number of ways of arranging 5 different mathematics books, 4 different physics books and 3 different chemistry books such that the books of the same subject are together.

III. 1. Find the number of 5 letter words that can be formed using the letters of the word CONSIDER. How many of them begin with "C", how many of them end with "R", and how many of them begin with "C" and end with "R" ?

2. Find the number of ways of seating 10 students $A_1, A_2, ..., A_{10}$ in a row such that

(i) A_1, A_2, A_3 sit together

(ii) A_1, A_2, A_3 sit in a specified order.

(iii) A_1, A_2, A_3 sit together in a specified order.

3. Find the number of ways in which 5 red balls, 4 black balls of different sizes can be arranged in a row so that (i) no two balls of the same colour come together (ii) the balls of the same colour come together.

4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 5, 6, 7, if no two digits are the same. How many of them are divisible by

(i) 2 (ii) 3 (iii) 4 (iv) 5 (v) 25

5. If the letters of the word MASTER are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the ranks of the words (i) REMAST (ii) MASTER.

6. If the letters of the word BRING are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the 59th word.

7. Find the sum of all 4 digited numbers that can be formed using the digits 1, 2, 4, 5, 6 without repetition.

8. There are 9 objects and 9 boxes. Out of 9 objects, 5 cannot fit in three small boxes. How many arrangements can be made such that each object can be put in one box only.

5.3 Permutations when repetitions are allowed

In the previous sections we have learnt about the number of permutations of 'n' dissimilar things taken 'r' at a time, when the repetition of the things is not allowed. In this section, we learn about the number of permutations of n dissimilar things taken r at a time when each thing can be repeated any number of times. (That is, when repetition is allowed).

5.3.1 Theorem : Let n and r be positive integers. If the repetition of things is allowed, then the number of permutations of 'n' dissimilar things taken 'r' at a time is n^r.

Proof : The number of required permutations is equal to the number of ways of filling up 'r' blank places with the given n things (repetitions allowed). We prove this by using induction on r. If $r = 1$, then the number of ways of filling up one blank using the given n things is $n = n^1$. Therefore the result is true for $r = 1$. Assume that $r > 1$ and that the result is true for $(r-1)$. That is the number of ways of filling up $(r-1)$ blank places with the given n things is $n^{(r-1)}$. Now suppose 'n' dissimilar things are given. Now we take r blank places as shown below.

$$\begin{array}{ccccc} \square & \square & \square & ------ & \square \\ 1 & 2 & 3 & \cdots\cdots & r \end{array}$$

The blank 1 can be filled with any one of the given n things in 'n' ways. Now, we are left with $(r-1)$ blanks and n things (because the object used in the first place can be used again). By induction hypothesis the remaining $(r-1)$ places can be filled with the given n things in $n^{(r-1)}$ ways. Therefore, by the fundamental principle, the number of ways of filling up 'r' blanks with the given 'n' things is

$$n \times n^{(r-1)} = n^r.$$

Hence the theorem follows by mathematical induction.

Note : The number of permutations of n dissimilar things taken 'r' things at a time with atleast one repetition is $n^r - {}^nP_r$.

5.3.2 Definition

A number or a word which reads same either from left to right or from right to left is called a Palindrome. Some examples of palindromes are ATTA, ROTOR, 12321, 120021 etc.

Note : The number of palindromes with r distinct letters that can be formed using given n distinct letters is

(i) $n^{r/2}$ if r is even

(ii) $n^{\frac{r+1}{2}}$ if r is odd.

5.3.3 Examples

1. Example : Find the number of permutations of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 when repetition is allowed.

Solution : From Theorem 5.3.1, the number of 4-digit numbers is

$$6^4 = 1296.$$

2. Example : Find the number of 4 letter words that can be formed using the letters of the word PISTON in which atleast one letter is repeated.

Solution : The given word has 6 letters. The number of 4 letter words that can be formed using these 6 letters

(i) when repetition is allowed is 6^4

(ii) When repetition is not allowed is ${}^6P_4$

Hence, the number 4 letter words in which atleast one letter is repeated is

$$6^4 - {}^6P_4 = 1296 - 360 = 936.$$

3. Example : A number lock has 3 rings and each ring has 9 digits 1, 2, 3, ..., 9. Find the maximum numb of unsuccessful attempts that can be made by a person who tries to open the lock without knowing the k code.

Solution : Each ring can be rotated in 9 different ways. Thus, the total number of different ways in whic rings can be rotated is 9^3. Out of these attempts, only one attempt is a successful attempt. Hence, maximum number of unsuccessful attempts is

$$9^3 - 1 = 729 - 1 = 728.$$

4. Example : Find the number of (i) 6 (ii) 7 letter palindromes that can be formed using the letters of the word EQUATION.

Solution : (i) 8^3 (ii) 8^4 (using the note below 5.3.2 definition)

5. Example : Find the number of seven digited palindromes that can be formed using 0, 1, 2, 3, 4.

Solution : First place can be filled in 4 ways (using only non-zero digits). Remaining three places can be filled in 5 ways each.

$\therefore$ Number of palindromes $= 4 \times 5^3$.

5.3.4 Solved Problems

1. Problem : *Find the number of 5-letter words that can be formed using the letters of the word MIXTURE which begin with an vowel when repetitions are allowed.*

Solution : We have to fill up 5 blanks using the letters of the word MIXTURE having 7 letters among which there are 3 vowels. Fill the first place with one of the vowels (I or U or E) in 3 ways as shown below

				X

Each of the remaining 4 places can be filled in 7 ways(since we can use all 7 letters each time). Thus the number of 5-letter words is $3 \times 7 \times 7 \times 7 \times 7 = 3 \times 7^4$.

2. Problem: *Find the number of functions from a set A with m elements to a set B with n elements.*

Solution : Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$

To define the image of a_1 we have n choices (any element of B). Then we can define the image of a_2 again in n ways (since a_1, a_2 can have same image). Thus we can define the image of each of the m elements in n ways. Therefore the number of functions from A to B is

$$n \times n \times ... \times n \text{ } (m \text{ times}) = n^m.$$

3. Problem : *Find the number of surjections from a set A with n elements to a set B with 2 elements when n>1.*

Solution : Let $A = \{a_1 a_2 ... a_n\}$ and $B = \{x, y\}$. From the above problem, the total number of functions from A onto B is 2^n. For a function to be a surjection its range should contain both x, y. Observe that the number of functions which are not surjections that is, the functions which contain x or y alone in the range is 2. Hence the number of surjections from A to B is $2^n - 2$.

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Note : In the above problem, even if B has more than 2 elements also we can derive a formula to find the number of surjections from A to B. But this result is beyond the scope of this book and hence it is not included here.

4. Problem : *Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 that are divisible by*

(i) 2 (ii) 3

when repetition is allowed.

Solution

(i) Numbers divisible by 2

Take 4 blanks. For a number to be divisible by 2, the units place should be filled with an even digit. This can be done in 3 ways (2 or 4 or 6).

			X

Now, each of the remaining 3 places can be filled in 6 ways. Hence the number of 4-digit numbers that are divisible by 2 is

$$3 \times 6^3 = 3 \times 216 = 648.$$

(ii) Numbers divisible by 3

Fill the first 3 places with the given 6 digits in 6^3 ways.

X	X	X	

Now, after filling up the first 3 places with three digits, if we fill up the units place in 6 ways, we get 6 consecutive positive integers. Out of any six consecutive integers exactly **two** are divisible by 3. Hence the units place can be filled in 2 ways. Hence the number of 4-digit numbers divisible by 3 is $2 \times 216 = 432$

5. Problem : *Find the number of 5-letter words that can be formed using the letters of the wor EXPLAIN that begin and end with a vowel when repetitions are allowed.*

Solution : We can fill the first and last places with vowels each in 3 ways (E or A or I).

X				X

Now each of the remaining 3 places can be filled in 7 ways (using any letter of giv 7 letters). Hence the number of 5 letter words which begin and end with vowels is

$$3^2 \times 7^3 = 9 \times 343 = 3087.$$

Exercise 5(b)

I. 1. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 4, 5, 7, 8 when repetition is allowed.

2. Find the number of 5 letter words that can be formed using the letters of the word RHYME if each letter can be used any number of times.

3. Find the number of functions from a set A containing 5 elements into a set B containing 4 elements.

II. 1. Find the number of palindromes with 6 digits that can be formed using the digits

(i) 0, 2, 4, 6, 8 (ii) 1, 3, 5, 7, 9

2. Find the number of 4-digit telephone numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 with atleast one digit repeated.

3. Find the number of bijections from a set A containing 7 elements onto itself.

4. Find the number of ways of arranging 'r' things in a line using the given 'n' different things in which atleast one thing is repeated.

5. Find the number of 5 letter words that can be formed using the letters of the word NATURE that begin with N when repetition is allowed.

6. Find the number of 5-digit numbers divisible by 5 that can be formed using the digits 0, 1, 2, 3, 4, 5, when repetition is allowed.

7. Find the number of nur ers less than 2000 that can be formed using the digits, 1, 2, 3, 4 if repetition is allowed.

III. 1. 9 different letters of an alphabet are given. Find the number of 4 letter words that can be formed using these 9 letters which have

(i) no letter is repeated (ii) atleast one letter is repeated.

2. Find the number of 4-digit numbers which can be formed using the digits 0, 2, 5, 7, 8 that are divisible by (i) 2 (ii) 4 when repetition is allowed.

3. Find the number of 4-digit numbers that can be formed using the digits 0, 1, 2, 3, 4, 5 which are divisible by 6 when repetition of the digits is allowed.

5.4 Circular Permutations

In all the previous sections in this chapter, we have studied about linear permutations (That is, permutations rranged linearly) with or without repetition. In this section, we learn about the arrangement of given objects ound a circle. These permutations are called **Circular permutations** (Definition 5.1.6).

In circular permutations, there are two types of arrangements. One is clock-wise and the other is anti clock-wise as shown in the Figures 5.1 and 5.2

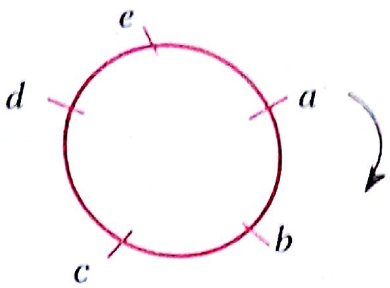

Fig. 5.1
Clock - wise arrangement

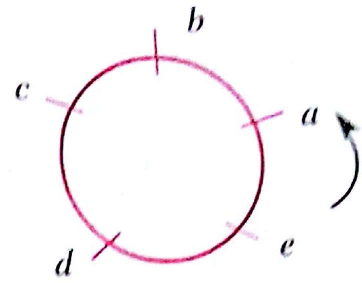

Fig. 5.2
Anti-clock-wise arrangement

The above two circular permutations are same but for the direction. In circular permutations, in general, the direction is also important and hence we regard the two permutations described in the Fig. 5.1 and Fig. 5.2 as two different circular permutations. However, in some special cases we treat the clock - wise and anti clock-wise arrangements of the same circular permutations as identical. We will discuss such cases later.

5.4.1 Theorem

The number of circular permutations of 'n' dissimilar things (taken all at a time) is (n – 1)!

Proof : First method

In a circular permutation, there is no first place or beginning place. Hence which thing we use first or which place we fill first does not matter. But how we arrange the remaining things relative to the first object already placed is to be calculated. Take n places around a circle as shown in the Fig. 5.3.

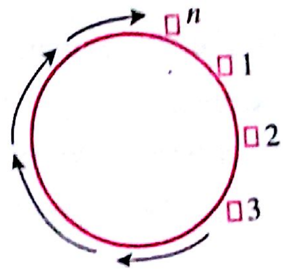

Fig. 5.3

Put any one of the given n things in any one of the n places. Now the remaining $(n-1)$ things can be arranged in the remaining $(n-1)$ places in $(n-1)!$ ways. Therefore, the number of circular permutations of n things taken all at a time is $(n-1)!$

Second Method

Let M be the number of circular permutations of n things taken all at a time. If we take one such permutation it looks as in the Fig. 5.4.

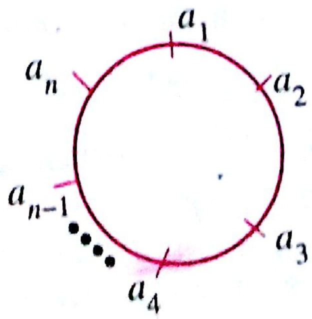

Fig. 5.4

This circular permutation gives rise to n linear permutations (read in either clock-wise or anti clock-wise direction, but not both) as exhibited below.

$$a_1\, a_2\, a_3 \ldots\, a_{n-1}\, a_n$$

$$a_2\, a_3\, a_4 \ldots\, a_n\, a_1$$

$$a_3\, a_4\, a_5 \ldots\, a_1\, a_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_n\, a_1\, a_2 \ldots\, a_{n-2}\, a_{n-1}$$

Thus, one circular permutation gives rise to n linear permutations and hence M circular permutations give us $M \times n$ linear permutations. But we know that the number of linear permutations of n dissimilar things (taken all at a time) is ${}^nP_n = n!$

Hence, we get $M \times n = n!$ and $M = \dfrac{n!}{n} = (n-1)!$.

5.4.2 Note

In case of (hanging type of circular permutations like) the garlands of flowers, chains of beads etc. a circular permutation looks like clock-wise arrangement when seen from one side and anti clock-wise arrangement from the other side. Hence, they will be treated as identical. Therefore, the number of circular permutations, of n things in these cases is $\dfrac{(n-1)!}{2}$ (half of the number of the actual circular permutations).

5.4.3 Example : Find the number of ways of arranging 5 boys and 5 girls around a circle.

Solution : Total number of persons $n = 10$ (5 boys + 5 girls).

Therefore, the number of circular permutations is $(n-1)! = 9!$

5.4.4 Example : Find the number of ways of arranging 8 persons around a circular table if two particular persons were to sit together.

Solution : Treat the two particular persons as one unit. Then we have $6 + 1 = 7$ entities. They can be arranged around a circular table in $6!$ ways. Now, the two particular persons can be permuted among themselves in $2!$ ways. Therefore, the number of required arrangements is

$$6! \times 2! = 1440.$$

5.4.5 Solved Problems

1. Problem : *Find the number of ways of arranging 8 men and 4 women around a circular table. In how many of them*

(i) *all the women come together*

(ii) *no two women come together*

Solution : Total number of persons = 12 (8 men + 4 women)

Therefore, the number of circular permutations is (11)!

(i) Treat the 4 women as single unit. Then we have

$$8 \text{ men} + 1 \text{ unit of women} = 9 \text{ entities.}$$

They can be arranged around a circular table in 8! ways. Now, the 4 women among themselves can be arranged in 4! ways. Hence by the Fundamental principle, the required number of arrangements is $8! \times 4!$

(ii) First we arrange 8 men around the circular table in 7! ways. There are 8 places in between them as shown in Fig. 5.5 by the symbol x. (one place in between any two consecutive men)

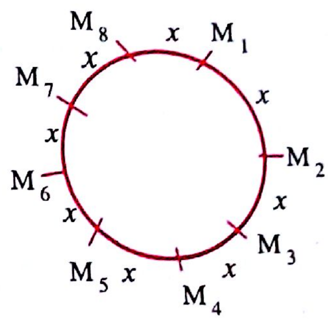

Fig. 5.5

Now we can arrange the 4 women in these 8 places in ${}^8P_4$ ways. Thus, the number of circular permutations in which no two women come together is

$$7! \times {}^8P_4.$$

2. Problem : *Find the number of ways of seating 5 Indians, 4 Americans and 3 Russians at a round table so that*

(i) *all Indians sit together*

(ii) *no two Russians sit together*

(iii) *persons of same nationality sit together.*

Solution

(i) Treat the 5 Indians as one unit. Then we have 4 Americans + 3 Russians + 1 unit of Indians = 8 entities.

They can be arranged at a round table in $(8-1)! = 7!$ ways.

Now, the 5 Indians among themselves can be arranged in 5! ways. Hence, the required number arrangements is $7! \times 5!$.

Now, arrange the 5 Indians + 4 Amercians around the table in $(9-1)! = 8!$ ways.

(ii) First we arrange the 5 Indians + 4 Amercians around the table in $(9-1)! = 8!$ ways. Now, there are 9 gaps in between these 9 persons (one gap between any two consecutive persons). The 3 Russians can be arranged in these 9 gaps in ${}^9P_3$ ways. Hence, the required number of arrangements is

$$8! \times {}^9P_3$$

(iii) Treat the 5 Indians as one unit, the 4 Amercians as one unit and the 3 Russians as one unit. These 3 units can be arranged at round table in $(3-1)! = 2!$ ways.

Now, the 5 Indians among themselves can be permuted in 5! ways. Similarly, the 4 Amercians in 4! ways and 3 Russians in 3! ways. Hence, the required number of arrangements is

$$2! \times 5! \times 4! \times 3!$$

3. Problem : *Find the number of different chains that can be prepared using 7 different coloured beads.*

Solution : We know that the number of circular permutations of hanging type that can be formed using n things is $\frac{1}{2}\{(n-1)!\}$. Hence the number of different ways of preparing the chains $= \frac{1}{2}\{(7-1)!\} = \frac{6!}{2} = 360$.

4. Problem : *Find the number of different ways of preparing a garland using 7 distinct red roses and 4 distinct yellow roses such that no two yellow roses come together.*

Solution : First we arrange 7 red roses in a circular form (garland form) in $(7-1)! = 6!$ ways Now, there are 7 gaps in between the red roses and we can arrange the 4 yellow roses in these 7 gaps in ${}^7P_4$ ways. Thus the total number of circular permutations is $6! \times {}^7P_4$.

But, this being the case of garlands, clock-wise and anti clock-wise arrangements look a like. Hence, the required number of ways is $\frac{1}{2}\left(6! \times {}^7P_4\right)$

Exercise 5(c)

I. 1. Find the number of ways of arranging 7 persons around a circle.

2. Find the number of ways of arranging the chief minister and 10 cabinet ministers at a circular table so that the chief minister always sits in a particular seat.

3. Find the number of ways of preparing a chain with 6 different coloured beads.

II. 1. Find the number of ways of arranging 4 boys and 3 girls around a circle so that all the girls sit together.

2. Find the number of ways of arranging 7 gents and 4 ladies around a circular table if no two ladies wish to sit together.

3. Find the number of ways of arranging 7 guests and a host around a circle if 2 particular guests wish to sit on either side of the host.

4. Find the number of ways of preparing a garland with 3 yellow, 4 white and 2 red roses of different sizes such that the two red roses come together.

III. 1. Find the number of ways of arranging 6 boys and 6 girls around a circular table so that

(i) all the girls sit together (ii) no two girls sit together

(iii) boys and girls sit alternately

2. Find the number of ways of arranging 6 red roses and 3 yellow roses of different sizes into a garland. In how many of them

(i) all the yellow roses are together (ii) no two yellow roses are together

3. A round table conference is attended by 3 Indians, 3 Chinese, 3 Canadians and 2 Americans. Find the number of ways of arranging them at the round table so that the delegates belonging to same country sit together.

4. A chain of beads is to be prepared using 6 different red coloured beads and 3 different blue coloured beads. In how many ways can this be done so that no two blue coloured beads come together.

5. A family consists of father, mother, 2 daughters and 2 sons. In how many different ways can they sit at a round table if the 2 daughters wish to sit on either side of the father?

5.5 Permutations with Constraint repetitions

In section 5.3, we have learnt about the permutations in which repetition of things is allowed. That is, each of the given n dissimilar things can be used any number of times. But in the literature we come across many words whose spelling contains certain repeated letters like MATHEMATICS, COFFEE, ASSOCIATION, EAMCET etc. and many numbers with repeated digits like 47436, 3007, 141516 etc. In this section, we find the number of permutations of such words or numbers.

5.5.1 Theorem

The number of linear permutations of 'n' things in which 'p' things are alike and the rest are different is $\frac{n!}{p!}$.

Proof : Let M be the number of permutations of 'n' things in which 'p' things are alike and the rest are different. If we take one such permutation, it contains p things which are alike. If we replace these p like things by p dissimilar things, then we can arange among themselves (without disturbing the relative positions of the other things) in $p!$ ways. In other words, one permutation when p things are alike gives rise to $p!$ permutations when all are different. Therefore, from the M such permutations, we get $M \times (p!)$ permutations when all n things are different. But, we know that the number of permutations of n dissimilar things (taken all at a time) is $n!$. Hence

$$M \times p! = n! \quad \text{or} \quad M = \frac{n!}{p!}$$

We can extend this result for the case when we have more than one set of alike things in the given n things by applying theorem 5.5.1 repeatedly as given in the following.

5.5.2 Theorem

The number of linear permutations of 'n' things in which there are p like things of one kind, q like things of second kind, r like things of the third kind and the rest are different is

$$\frac{n!}{p!\, q!\, r!}$$

Now, we apply these results in the following examples.

5.5.3 Example : Find the number of ways of arranging the letters of the word SPECIFIC. In how many of them

(i) the two C's come together?

(ii) the two I's do not come together?

Solution : The given word has 8 letters in which there are 2 I's (alike letters) and 2 C's. Hence, they can be arranged in

$$\frac{8!}{2!\, 2!} = 10,080 \text{ ways (using the Theorem 5.5.2)}$$

(i) Treat the 2 C's as one unit. Then we have $6 + 1 = 7$ letters in which two letters (I's) are alike. Hence by theorem 5.5.1 they can be arranged in

$$\frac{7!}{2!} = 2520 \text{ ways}$$

Now, the 2C's among themselves can be arranged in $\frac{2!}{2!} = 1$ way. Thus, the number of required arrangements is 2520.

(ii) Keeping the 2 I's aside, arrange the remaining 6 letters can be arranged in $\frac{6!}{2!} = 360$ ways (since there are two C's among 6 letters). Among these 6 letters we find 7 gaps as shown below

$$- S - P - E - C - F - C -$$

The two I's can be arranged in these 7 gaps in

$$\frac{{}^7P_2}{2!} \text{ ways}$$

Hence, the number of required arrangements is

$$\frac{6!}{2!} \times \frac{{}^7P_2}{2!} = 360 \times 21 = 7560.$$

5.5.4 Solved Problems

1. Problem : *Find the number of ways of arranging the letters of the word SINGING so that*

(i) *they begin and end with I*

(ii) *the two G's come together*

(iii) *relative positions of vowels and consonants are not disturbed.*

Solution

(i) First we fill the first and last places with I's in $\frac{2!}{2!} = 1$ way as shown below

I						I

Now, we fill the remaining 5 places with the remaining 5 letters S, N, G, N, G in

$$\frac{5!}{2!\,2!} = 30 \text{ ways.}$$

Hence, the number of required permutations is 30.

(ii) Treat the two G's as one unit. Then we have 6 letters in which there are 2 I's and 2 N's. Hence they can be arranged in

$$\frac{6!}{2!\,2!} = 180 \text{ ways.}$$

Now, the two G's among themselves can be arranged in $\frac{2!}{2!} = 1$ way. Hence the number o required permutations is 180.

(iii) In the word SINGING, there are 2 vowels which are alike i.e., I, and there are 5 consonan of which 2N's and 2G's are alike and one S is different.

$$\underline{C}\ \ \underline{V}\ \ \underline{C}\ \ \underline{C}\ \ \underline{V}\ \ \underline{C}\ \ \underline{C}$$

The two vowels can be interchanged among themselves in $\frac{2!}{2!} = 1$ way. Now, the 5 consonants can be arranged in the remaining 5 places in $\frac{5!}{2!2!} = 30$ ways.

$\therefore$ Number of required arrangements $= 1 \times 30 = 30.$

2. Problem : *Find the number of ways of arranging the letters of the word $a^4 b^3 c^5$ in its expanded form.*

Solution : The expanded form of $a^4 b^3 c^5$ is

$$aaaa\ \ bbb\ \ ccccc$$

This word has 12 letters in which there are 4 a's, 3 b's and 5c's. By Theorem 5.5.2, they can be arranged in $\frac{12!}{4!\ 3!\ 5!}$ ways.

3. Problem : *Find the number of 5-digit numbers that can be formed using the digits 1, 1, 2, 2, 3. How many of them are even?*

Solution : In the given 5 digits, there are two 1's and two 2's. Hence they can be arranged in

$$\frac{5!}{2!\ 2!} = 30 \text{ ways.}$$

Now, to find even numbers fill the units place by 2. Now the remaining 4 places can be filled using the remaining digits 1, 1, 2, 3, in

$$\frac{4!}{2!} = 12 \text{ ways}$$

Thus the number of 5-digit even numbers that can be formed using the digits 1, 1, 2, 2, 3 is 12.

4. Problem : *There are 4 copies (alike) each of 3 different books. Find the number of ways of arranging these 12 books in a shelf in a single row.*

Solution : We have 12 books in which 4 books are alike of one kind, 4 books are alike of second kind and 4 books are alike of third kind. Hence, by Theorem 5.5.2., they can be arranged in a shelf in a row in

$$\frac{12!}{4!\ 4!\ 4!} \text{ ways}$$

In problem 9 of solved problems 5.2.12, we have calculated the rank of the word PRISION. In the following problem we find the rank of a word when it contains repeated letters.

5. **Problem :** *If the letters of the word EAMCET are permuted in all possible ways and if the words thus formed are arranged in the dictionary order, find the rank of the word EAMCET.*

Solution : The dictionary order of the letters of given word is

A C E E M T

In the dictionary order the words which begin with the letter A come first. If we fill the first place with A, remaining 5 letters can be arranged $\frac{5!}{2!}$ ways (since there are two E's). On proceeding like this (as in problem 9 of 5.2.12) we get

$$A\ -\ -\ -\ -\ -\ \rightarrow \frac{5!}{2!} \text{ words}$$

$$C\ -\ -\ -\ -\ -\ \rightarrow \frac{5!}{2!} \text{ words}$$

$$E\ A\ C\ -\ -\ -\ \rightarrow 3! \text{ words}$$

$$E\ A\ E\ -\ -\ -\ \rightarrow 3! \text{ words}$$

$$E\ A\ M\ C\ E\ T\ \rightarrow 1 \text{ word}$$

Hence the rank of the word EAMCET is

$$2 \times \frac{5!}{2!} + 2 \times 3! + 1 = 120 + 12 + 1 = 133.$$

Exercise 5(d)

I. 1. Find the number of ways of arranging the letters of the words.

(i) INDEPENDENCE (ii) MATHEMATICS

(iii) SINGING (iv) PERMUTATION

(v) COMBINATION (vi) INTERMEDIATE

2. Find the number of 7-digit numbers that can be formed using 2, 2, 2, 3, 3, 4, 4.

1. Find the number of 4-letter words that can be formed using the letters of the word RAMANA.

2. How many numbers can be formed using all the digits 1, 2, 3, 4, 3, 2, 1 such that even digits always occupy even places?

3. In a library, there are 6 copies of one book, 4 copies each of two different books, 5 copies each of three different books and 3 copies each of two different books. Find the number of ways of arranging all these books in a shelf in a single row.

4. A book store has 'm' copies each of 'n' different books. Find the number of ways of arranging these books in a shelf in a single row.

5. Find the number of 5-digit numbers that can be formed using the digits $0, 1, 1, 2, 3$.

6. In how many ways can the letters of the word CHEESE be arranged so that no two E's come together?

III. 1. Find the number of ways of arranging the letters of the word ASSOCIATIONS. In how many of them

   (i) all the three S's come together.

   (ii) the two A's do not come together.

2. Find the number of ways of arranging the letters of the word MISSING so that the two S's are together and the two I's are together.

3. If the letters of the word AJANTA are permuted in all possible ways and the words thus formed are arranged in dictionary order, find the ranks of the words (i) AJANTA (ii) JANATA

5.6 Combinations - Definitions and Certain Theorems

At the beginning of this chapter, we have exhibited the difference between a permutation and a combination. A combination is only a selection. There is no importance to the order or arrangment of things in a combination. Thus a combination of 'n' things taken 'r' at a time can be regarded as a subset with 'r' elements of a set containing 'n' elements. The number of combinations of 'n' dissimilar things taken 'r' at a time is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$ and it is equal to the number of subsets with 'r' elements of set containing 'n' elements. In the succeeding theorem we develop a formula to find ${}^nC_r$.

5.6.1 Theorem

The number of combinations of 'n' dissimilar things taken 'r' at a time is $\frac{^nP_r}{r!}$. That is $^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$.

Proof : Let A be the set containing the given n dissimilar things. Then the number of combinations (^nC_r) of these n dissimilar things taken r at a time is equal to the number of subsets of A containing r elements. If we select one such subset of A containing 'r' elements, these 'r' things can be arranged in a line in $r!$ ways. In other words, one combination gives rise to $r!$ permutations (of n things taken r at a time). Thus from these ^nC_r combinations we get $^nC_r \times r!$ permutations. But we know that the number of permutations of 'n' things taken 'r' at a time is ^nP_r. Hence

$$^nC_r \times r! = {}^nP_r$$

Therefore,
$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!.r!}$$

5.6.2 Note : From the above theorem $^nC_r = \frac{^nP_r}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)\,...\,1}$. Thus the numerator is the product of r consecutive integers in decreasing order starting from n while the denominator is the product of r consecutive integers in decreasing order, starting from r. For examples

$$^9C_3 = \frac{9\times8\times7}{3\times2\times1} = 84 \text{ and } {}^{10}C_4 = \frac{10\times9\times8\times7}{1\times2\times3\times4} = 210.$$

The following is a direct consequence of Theorem 5.6.1.

5.6.3 Corollary : The number of different subsets of 'r' elements of a set containing 'n' elements is ^nC_r.

5.6.4 Examples

1. Example : Find the number of ways of selecting 7 members from a contingent of 10 soldiers.

Solution : The number of ways of selecting 7 members out of 10 soldiers is

$$^{10}C_7 = \frac{10!}{3!\,7!} = 120. \text{ (from theorem 5.6.1)}$$

2. Example : If a set A has 8 elements, find the number of subsets of A, containing at least 6 elements.

Solution : We have to fix the number of subsets of A, containing 6 or 7 or 8 elements.

Number of subsets of A, containing exactly 6 elements $= {}^8C_6$

Number of subsets of A, containing exactly 7 elements $= {}^8C_7$

Number of subsets of A, containing exactly 8 elements $= {}^8C_8$

Therefore, the number of subsets of A containing atleast 6 elements

$$= {}^8C_6 + {}^8C_7 + {}^8C_8 = 28 + 8 + 1 = 37.$$

Whenever we select r elements out of n elements, we will be left with $(n-r)$ elements. Thus, the number of ways of selecting 'r' elements from the given n elements is equal to the number of ways of leaving $(n-r)$ elements. This is proved in the following.

5.6.5 Theorem

If n, r are integers with $0 \le r \le n$, the ${}^nC_r = {}^nC_{(n-r)}$

Proof:
$$ {}^nC_{(n-r)} = \frac{n!}{(n-(n-r))!.(n-r)!} \text{ (from theorem 5.6.1)}$$
$$= \frac{n!}{r!.(n-r)!} = {}^nC_r.$$

5.6.6 Corollary : For any positive integer n, ${}^nC_n = {}^nC_0 = 1$.

Proof: From Theorem 5.6.1

$$ {}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!n!} = 1 \quad (\text{since } 0! = 1)$$

Taking $r = n$ in theorem 5.6.5, we get ${}^nC_n = {}^nC_{(n-n)} = {}^nC_0$.

Thus ${}^nC_n = {}^nC_0 = 1$.

5.6.7 Theorem

If m, n are distinct positive integers, then the number of ways of dividing $(m + n)$ things into two groups containing 'm' things and 'n' things is $\frac{(m+n)!}{m!n!}$.

Proof : Whenever we select 'm' things out of the given $(m + n)$ things, we will be left with n things and hence two groups one containing m things and the other containing 'n' things are formed. Therefore, the number of ways of dividing $(m+n)$ dissimilar things into two groups containing 'm' things out of 'n' things.

= The number of ways of selecting 'm' things out of $(m+n)$ things

$$= {}^{(m+n)}C_m = \frac{(m+n)!}{(m+n-m)!.m!} = \frac{(m+n)!}{m!.n!}$$

5.6.8 Corollary : If m, n, p are distinct positive integers, then the number of ways of dividing '$m+n+p$' things into 3 groups containing 'm' things, 'n' things and 'p' things is

$$\frac{(m+n+p)!}{m!.n!.p!}$$

Proof : First we select in 'm' things from the given '$m+n+p$' things to form the first group in ${}^{(m+n+p)}C_m$ ways. Now, from the remaining $(n+p)$ things we select 'n' things to form second group in ${}^{(n+p)}C_n$ ways. Then the remaining 'p' things automatically form the third group. Thus the number of ways of dividing the given $m+n+p$ dissimilar things into 3 groups containing m, n, p things

$$= {}^{m+n+p}C_m \times {}^{n+p}C_n = \frac{(m+n+p)!}{(m+n+p-m)!\,m!} \times \frac{(n+p)!}{(n+p-n)!\,n!}$$

$$= \frac{(m+n+p)!}{(n+p)!\,m!} \times \frac{(n+p)!}{p!\,n!} = \frac{(m+n+p)!}{m!\,n!\,p!}.$$

5.6.9 Corollary : The number of ways of dividing $2n$ dissimilar things into two equal groups containing 'n' things in each is $\frac{(2n)!}{2!\,n!\,n!}$.

Proof : As in Theorem 5.6.7. we can divide '$2n$' things into two groups of n elements each in $\frac{2n!}{n!n!}$ ways. Since the two groups have equal number of elements, we can interchange them in $2!$ ways. Though they have given rise to the same division, they are counted as '2' divisions in the above calculation. Hence we can divide the given '$2n$' things into two equal groups having n elements in each group in

$$\frac{2n!}{2!\,n!\,n!} \text{ ways}$$

This result can be extended to mn things in the following.

5.6.10 Corollary : The number of ways of dividing 'mn' dissimilar things into 'm' equal groups each containing 'n' elements is

$$\frac{(mn)!}{m!(n!)^m}$$

If we have to distribute 'mn' things equally among 'm' persons we use the following.

5.6.11 Corollary : The number of ways of distributing 'mn' dissimilar things equally among 'm' persons is

$$\frac{(mn)!}{(n!)^m}$$

Proof : First we divide 'mn' things into 'm' equal groups (using corollary 5.6.10) in $\frac{(mn)!}{m!(n!)^m}$ ways. Now we have m groups (not identical, they contain only equal number of elements) and m persons. Hence they can be distributed in $m!$ ways. Hence, by the fundamental principle, the number of ways of distributing 'mn' things to 'm' persons equally is

$$\frac{(mn)!}{m!(n!)^m} \times m! = \frac{(mn)!}{(n!)^m}$$

5.6.12 Example : A candidate is required to answer 6 out of 10 questions which are divided into two groups A and B each containing 5 questions. He is not permitted to attempt more than 4 questions from either group. Find the number of different ways in which the candidate can choose six questions.

Solution : The candidate can answer 4 questions from group A and 2 questions from group B or 3 questions from each group or 2 questions from group A and 4 questions from group B. The number of ways of choosing questions by the candidate with

(i) 4 from group A and 2 from group B $= {}^5C_4 \times {}^5C_2 = 50.$

(ii) 3 from each group $= {}^5C_3 \times {}^5C_3 = 100.$

(iii) 2 from group A and 4 from group B $= {}^5C_2 \times {}^5C_4 = 50.$

Thus the number of ways of selecting 6 questions out of 10 questions is $50 + 100 + 50 = 200.$

5.6.13 Solved Problems

1. Problem : *Find the number of ways of selecting 4 boys and 3 girls from a group of 8 boys and 5 girls.*

Solution : 4 boys can be selected from the given 8 boys in ${}^8C_4$ ways and 3 girls can be selected from the given 5 girls in ${}^5C_3$ ways. Hence, by the Fundamental principle, the number of required selections is

$${}^8C_4 \times {}^5C_3 = 70 \times 10 = 700.$$

2. Problem : *Find the number of ways of selecting 4 English, 3 Telugu and 2 Hindi books out of 7 English, 6 Telugu and 5 Hindi books.*

Solution : The number of ways of selecting

4 English books out of 7 books $= {}^7C_4$

3 Telugu books out of 6 books $= {}^6C_3$

2 Hindi books out of 5 books $= {}^5C_2$

Hence, the number of required ways $= {}^7C_4 \times {}^6C_3 \times {}^5C_2 = 35 \times 20 \times 10 = 7000.$

3. **Problem :** *Find the number of ways of forming a committee of 4 members out of 6 boys and 4 girls such that there is atleast one girl in the committee.*

Solution : The number of ways of forming a committee of 4 members out of 10 members (6 boys + 4 girls) is ${}^{10}C_4$. Out of these, the number of ways of forming the committee having no girl is ${}^{6}C_4$ (we select all 4 members from boys). Therefore, the number of ways of forming the committees having atleast one girl is

$$ {}^{10}C_4 - {}^{6}C_4 = 210 - 15 = 195. $$

4. **Problem :** *Find the number of ways of selecting 11 member cricket team from 7 bats men, 6 bowlers and 2 wicket keepers so that the team contains 2 wicket keepers and atleast 4 bowlers.*

Solution : The required cricket team can have the following compositions

Bowlers	Wicket Keepers	Batsmen	Number of ways of selecting team
4	2	5	${}^6C_4 \times {}^2C_2 \times {}^7C_5 = 15 \times 1 \times 21 = 315$
5	2	4	${}^6C_5 \times {}^2C_2 \times {}^7C_4 = 6 \times 1 \times 35 = 210$
6	2	3	${}^6C_6 \times {}^2C_2 \times {}^7C_3 = 1 \times 1 \times 35 = 35$

Therefore, the number of ways of selecting the required cricket team

$$ = 315 + 210 + 35 = 560. $$

5. **Problem :** *If a set of 'm' parallel lines intersect another set of 'n' parallel lines (not parallel to the lines in the first set), then find the number of parallelograms formed in this lattice structure.*

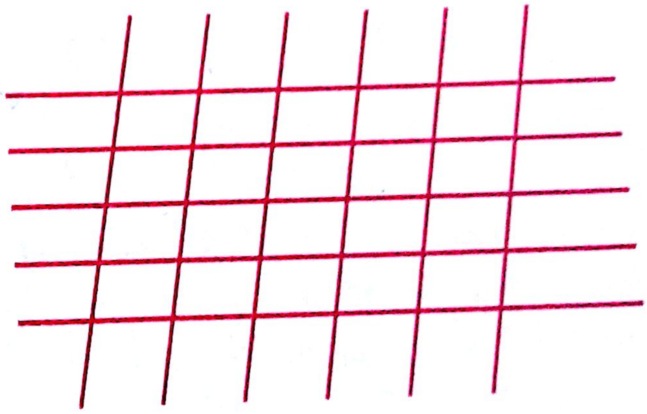

Fig. 5.6

Solution : Whenever we select 2 lines from the first set of m lines and 2 lines from the second set of n lines, one parallelogram is formed as shown in the figure.

Thus, the number of parallelograms formed is ${}^mC_2 \times {}^nC_2$.

6. **Problem :** *There are 'm' points in a plane out of which 'p' points are collinear and no three of the points are collinear unless all the three are from these p points. Find the number of different*

(i) *straight lines passing through pairs of distinct points.*

(ii) *triangles formed by joining these points (by line segments).*

Solution

(i) From the given 'm' points, by drawing straight lines passing through 2 distinct points at a time, we are supposed to get '${}^mC_2$' number of lines. But, since 'p' out of these 'm' points are collinear, by forming lines passing through these 'p' points 2 at a time we get only one line instead of getting ${}^pC_2$. Therefore, the number of different lines as required is

$$ {}^mC_2 - {}^pC_2 + 1. $$

(ii) From the given m points, by joining 3 at a time, we are supposed to get ${}^mC_3$ number of triangles. Since p points out of these m points are collinear, by joining these p points 3 at a time we do not get any triangle (we get only a line) when we are supposed to get ${}^pC_3$ number of triangles. Hence the number of triangles formed by joining the given m points is

$$ {}^mC_3 - {}^pC_3. $$

Note : The number of diagonals in an n-sided polygon $= {}^nC_2 - n = \dfrac{n(n-3)}{2}$.

7. Problem : *A teacher wants to take 10 students to a park. He can take exactly 3 students at a time and will not take the same group of 3 students more than once. Find the number of times*

(i) each student can go to the park (ii) the teacher can go to the park

Solution

(i) To find the number of times a student can go to the park, we have to select 2 more students from the remaining 9 students. This can be done in ${}^9C_2$ ways. Hence, each student can go to park ${}^9C_2 = 36$ times.

(ii) The number of times the teacher can go to park

= The number of different ways of selecting 3 students out of 10.

$= {}^{10}C_3 = 120.$

8. Problem : *A double decker minibus has 8 seats in the lower deck and 10 seats in the upper deck. Find the number of ways of arranging 18 persons in the bus if 3 children want to go to the upper deck and 4 old people can not go to the upper deck.*

Solution

Allowing 3 children to the upper deck and 4 old people to the lower deck, we are left with 11 people and 11 seats (7 in the upper deck and 4 in the lower deck). We can select 7 people for the upper deck out of the 11 people in ${}^{11}C_7$ ways. The remaining 4 persons go to lower deck. Now we can arrange 10 persons (3 children and 7 others) in the upper deck and 8 persons (4 old people and 4 others) in the lower deck in 10! and 8! ways respectively. Hence, the required number of arrangements

$$ = {}^{11}C_7 \times 10!\ 8!. $$

5.6.14 Certain theorems on combinations

In this section, we prove two important theorems about ^nC_r. In theorem 5.6.5, we have proved that $^nC_r = {^nC_{n-r}}$. In this section, we prove the converse of this result. Before that, we prove the following lemma which will be used to prove the theorem.

5.6.15 Lemma : If a, b are positive real numbers and k is a positive integer such that

$$(a+1)(a+2)\ldots(a+k) = (b+1)(b+2)\ldots(b+k),$$

then $a = b$

Proof : Suppose $a \neq b$, Then either $a < b$ or $a > b$. Without loss of generality assume $a < b$. Then, for $1 \leq i \leq k$,

$$(b+i)-(a+i) = b-a>0$$

that is, $b+i > a+i$

on multiplying these in equalities, we get

$$(b+1)(b+2)\ldots(b+k) > (a+1)(a+2)\ldots(a+k)$$

which is a contradiction to the hypothesis. Hence $a = b$.

Now, we prove the following.

5.6.16 Theorem : *For $0 \leq r, s \leq n$, if $^nC_r = {^nC_s}$, then either $r = s$ or $r+s = n$. (that is, either $s = r$ or $s = n-r$)*

Proof : Suppose $^nC_r = {^nC_s}$. If $r = s$, the theorem is proved. Assume $r \neq s$. Without loss of generality assume $r < s$. Then $(n-s) < (n-r)$. Now

$$^nC_r = {^nC_s} \Rightarrow \frac{n!}{(n-r)!r!} = \frac{n!}{(n-s)!s!}$$

$$\Rightarrow (n-r)!\,r! = (n-s)!\,s!$$

$$\Rightarrow [(n-r)(n-r-1)\ldots.(n-s+1)]\,(n-s)!\,r! = (n-s)!\,[s(s-1)\ldots(r+1)]\,r!$$

(since $r < s$ and $(n-s) < (n-r)$)

$$\Rightarrow (n-r)(n-r-1)\ldots(n-s+1) = s(s-1)\ldots(r+1)$$

Hence by taking $a = n-s+1$, $b = r+1$ and $k = s-r$ in Lemma 5.6.15, we get

$n-s+1 = r+1$. Hence $r = n-s$ or $n = r+s$.

Now, we prove another important property of ^nC_r which is analogous to Theorem 5.2.10

5.6.17 Theorem : *If $1 \le r \le n$, then* $\boxed{{}^nC_{r-1} + {}^nC_r = {}^{(n+1)}C_r}$.

Proof

First method : We know that ${}^{(n+1)}C_r$ is the number of r–element subsets (That is, subsets having r–elements) of a set A containing $(n+1)$ elements. Fix $a \in$ A. If we select an r–element subset of A, it may or may not contain the fixed element a. We calculate the number of such subsets now.

(i) Number of r–element subsets of A containing the element 'a'

= Number of ways of selecting $(r-1)$ elements from the remaining n elements of A(since already one element, that is, a is selected)

$= {}^nC_{(r-1)}$

(ii) Number of r–element subsets of A not containing the element 'a'

= Number of ways of selecting r–elements from the remaining n elements of A (leaving the element a)

$= {}^nC_r$

Thus, from (i) and (ii), we get

$$ {}^{(n+1)}C_r = {}^nC_{(r-1)} + {}^nC_r $$

Second Method : (Using the formula for ${}^nC_r$)

$$ {}^nC_{(r-1)} + {}^nC_r = \frac{n!}{(n-(r-1))!(r-1)!} + \frac{n!}{(n-r)!r!} $$

$$ = n!\left\{\frac{1}{(n-r+1)!(r-1)!} + \frac{1}{(n-r)!r!}\right\} $$

$$ = n!\left\{\frac{r+(n-r+1)}{(n-r+1)!r!}\right\} = \frac{n!(n+1)}{(n-r+1)!r!} $$

$$ = \frac{(n+1)!}{((n+1)-r)!r!} = {}^{(n+1)}C_r $$

5.6.18 Corollary : If $2 \le r \le n$, then

$$ {}^nC_{(r-2)} + 2.{}^nC_{(r-1)} + {}^nC_r = {}^{(n+2)}C_r $$

Proof : L.H.S. $= {}^nC_{(r-2)} + 2.{}^nC_{(r-1)} + {}^nC_r$

$= ({}^nC_{(r-2)} + {}^nC_{(r-1)}) + ({}^nC_{(r-1)} + {}^nC_r)$

$= {}^{(n+1)}C_{(r-1)} + {}^{(n+1)}C_r$ (by Theorem 5.6.17)

$= {}^{(n+2)}C_r$ (again by Theorem 5.6.17)

= R.H.S.

5.6.19 Theorem : *If p things are alike of one kind, q things are alike of second kind and r things are alike of third kind, then the number of ways of selecting any number of things (one or more) out of these (p + q + r) things is (p + 1) (q + 1) (r + 1) – 1.*

Proof : From the first group of p things, we can select 0 or 1 or 2 or ... or p things. Since all the p things in this group are alike, we have to decide only the number of things to be selected. This can be done in $(p+1)$ ways. Similarly, we can select any number of things from second and third groups (up to a maximum of q and r things respectively) in $(q+1)$ and $(r+1)$ ways respectively. Hence, by the fundamental principle, we can select any number of things from the 3 groups in

$$(p+1) \,.\, (q+1) \,.\, (r+1)$$

ways. But this includes one selecting of '0' from each group. Since we have to select one or more things, the number of required ways is

$$(p+1)(q+1)(r+1) - 1.$$

5.6.20 Corollary : The number of ways of selecting one or more things out of 'n' dissimilar things is $2^n - 1$.

Proof : The given n dissimilar things can be regarded as n groups having 1 alike thing in each group. Hence, by theorem 5.6.19, the number of ways of selecting one or more things out of the given n things is

$$\underbrace{(1+1)(1+1)\ldots(1+1)}_{(n \text{ times})} - 1 = 2^n - 1$$

5.6.21 Corollary : If $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, then the number of positive divisors of

$$n = p_1^{\alpha_1} \,.\, p_2^{\alpha_2} \ldots p_k^{\alpha_k} \text{ is } (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) \text{ (this includes 1 and } n\text{)}.$$

Proof : If $n = p_1^{\alpha_1} \,.\, p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, then any positive divisor of n is of the form $p_1^{\beta_1} \,.\, p_2^{\beta_2} \ldots p_k^{\beta_k}$ where $0 \le \beta_i \le \alpha_i$ for $1 \le i \le k$. Thus β_i can be 0 or 1 or 2 or ... or α_i, that is β_i can take $\alpha_i + 1$ values for $1 \le i \le k$. Hence, by Theorem 5.6.19, the number of positive divisors of n is $(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$. (Since when all β_i's are zeroes we get the divisor 1).

5.6.22 Note

The positive divisors of n, other than 1 and n itself are called proper divisors of n and 1, n are called improper (or trivial) divisors of n. Thus, the number of proper divisors of a positive integer $n = p_1^{\alpha_1} . p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, (where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers) is $(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) - 2$.

5.6.23 Definition

Exponent of a prime in n! ($n \in Z^+$) : *Exponent of a prime p in n! is the largest integer k such that p^k divides n!.*

The exponent of p in $n!$ is given by

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots.$$

(proof of this result is beyond the scope of this book)

5.6.24 Example : Find the number of zeros in 100!.

Solution : $100! = 2^{\alpha} 3^{\beta} 5^{\gamma} 7^{\delta} \ldots.$

$$\text{where } \alpha = \left[\frac{100}{2}\right]+\left[\frac{100}{2^2}\right]+\left[\frac{100}{2^3}\right]+\left[\frac{100}{2^4}\right]+\ldots.$$

$$= 50 + 25 + 12 + 6 + 3 + 1$$

$$= 97$$

$$\gamma = \left[\frac{100}{5}\right]+\left[\frac{100}{5^2}\right]$$

$$= 20 + 4 = 24.$$

Now, the number of zeros in 100! is the power of 10 in 100! which is 24, since $10 = 2 \times 5$.

5.6.25 Example : If there are 5 alike pens, 6 alike pencils and 7 alike erasers, find the number of ways of selecting any number of (one or more) things out of them.

Solution : By Theorem 5.6.19, the required number of ways is

$$(5 + 1)(6 + 1)(7 + 1) - 1 = 335.$$

5.6.26 Example : Find the number of positive divisors of 1080.

Solution : $1080 = 2^3 \times 3^3 \times 5^1$

Hence, by corollary 5.6.21, the number of postive divisors of $1080 = (3+1)(3+1)(1+1) = 32$.

5.6.27 Example : To pass an examination a student has to pass in each of the three papers. In how many ways can a student fail in the examination?

Solution : For each of the three papers there are two choices P or F. There are $2^3 = 8$ choices. But a student passes only if he/she passes in all papers.

$\therefore$ Required no. of ways $= 2^3 - 1 = 7$.

5.6.28 Example : Out of 3 different books on Economics, 4 different books on political science and 5 different books on Geography, how many collections can be made, if each collection consists of (i) exactly one book of each subject (ii) atleast one book of each subject.

Solution

(i) Out of 3 books on Economics exactly one book is chosen in ${}^3C_1$ ways.

Out of 4 books on political science one book can be chosen in ${}^4C_1$ ways, and a Geography book out of 5 books on it can be chosen in ${}^5C_1$ ways.

$\therefore$ Required number of ways $= {}^3C_1 \cdot {}^4C_1 \cdot {}^5C_1 = 3\times4\times5 = 60$.

(ii) As in Example 5.6.27, the number of collections having atleast one book of each subject is

$$(2^3-1)(2^4-1)(2^5-1) = 7\times15\times31 = 3255.$$

Note: In the above problem, if the books of each subject are alike then the required number of ways are (i) 1, (ii) $3\times4\times5 = 60$ respectively.

5.6.29 Solved Problems

1. Problem : *Prove that*

(i) $${}^{10}C_3 + {}^{10}C_6 = {}^{11}C_4$$

(ii) $${}^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3 = {}^{30}C_4$$

Solution

(i) $${}^{10}C_3 + {}^{10}C_6 = {}^{10}C_3 + {}^{10}C_4 \quad (\text{since } {}^nC_r = {}^nC_{(n-r)})$$

$$= {}^{11}C_4 \quad (\text{by Theorem 5.6.17})$$

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(ii) ${}^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3$

$= {}^{25}C_4 + \{{}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3\}$

$= {}^{26}C_4 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$ (since ${}^{25}C_3 + {}^{25}C_4 = {}^{26}C_4$)

$= {}^{27}C_4 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{28}C_4 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{29}C_4 + {}^{29}C_3$

$= {}^{30}C_4$.

2. Problem : (i) *If* ${}^{12}C_{(s+1)} = {}^{12}C_{(2s-5)}$, *find* s

(ii) *If* ${}^{n}C_{21} = {}^{n}C_{27}$, *find* ${}^{50}C_{n}$.

Solution

(i) By Theorem 5.6.16,

$${}^{12}C_{(s+1)} = {}^{12}C_{(2s-5)} \Rightarrow \text{either } s+1 = 2s-5 \text{ or } (s+1)+(2s-5) = 12$$

$$\Rightarrow s = 6 \text{ or } s = \frac{16}{3}.$$

$$\Rightarrow s = 6 \quad \text{(since } s \text{ is a non negative integer)}$$

(ii) By Theorem 5.6.16,

$${}^{n}C_{21} = {}^{n}C_{27} \Rightarrow n = 21 + 27 = 48$$

Therefore, $${}^{50}C_{n} = {}^{50}C_{48} = {}^{50}C_{2} = \frac{50\times 49}{1\times 2} = 1225.$$

3. Problem : *14 persons are seated at a round table. Find the number of ways of selecting two persons out of them who are not seated adjacent to each other.*

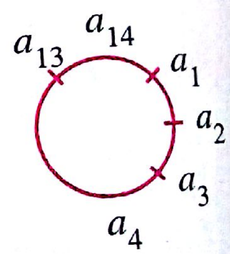

Fig. 5.7

Solution : Let the seating arrangement of given 14 persons at the round table be as shown in Fig. 5.7.

Number of ways of selecting 2 persons out of 14 persons

${}^{14}C_2 = 91$.

In the above arrangement two persons sitting adjacent to each other can be selected in 14 ways (th are $a_1\,a_2, a_2\,a_3, \ldots, a_{13}\,a_{14}, a_{14}\,a_1$).

Therefore, the required number of ways $= 91 - 14 = 77$.

Exercise 5(e)

I. 1. If ${}^{n}C_4 = 210$, find n.

2. If ${}^{12}C_r = 495$, find the possible values of 'r'.

3. If $10.{}^{n}C_2 = 3.{}^{n+1}C_3$, find n.

4. If ${}^{n}P_r = 5040$ and ${}^{n}C_r = 210$, find n and r.

5. If ${}^{n}C_4 = {}^{n}C_6$, find n.

6. If ${}^{15}C_{2r-1} = {}^{15}C_{2r+4}$, find r.

7. If ${}^{17}C_{2t+1} = {}^{17}C_{3t-5}$, find t.

8. If ${}^{12}C_{r+1} = {}^{12}C_{3r-5}$, find r.

9. If ${}^{9}C_3 + {}^{9}C_5 = {}^{10}C_r$, then find r.

10. Find the number of ways of forming a committee of 5 members from 6 men and 3 ladies.

11. In question no. 10, how many committees contain atleast two ladies.

12. If ${}^{n}C_5 = {}^{n}C_6$, then find ${}^{13}C_n$.

II. 1. Prove that for $3 \leq r \leq n$,

$$^{(n-3)}C_r + 3.^{(n-3)}C_{(r-1)} + 3.^{(n-3)}C_{(r-2)} + {}^{(n-3)}C_{(r-3)} = {}^{n}C_r.$$

2. Find the value of ${}^{10}C_5 + 2.{}^{10}C_4 + {}^{10}C_3$.

3. Simplify ${}^{34}C_5 + \sum_{r=0}^{4} {}^{(38-r)}C_4$.

4. In a class there are 30 students. If each student plays a chess game with each of the other student, then find the total number of chess games played by them.

5. Find the number of ways of selecting 3 girls and 3 boys out of 7 girls and 6 boys.

6. Find the number of ways of selecting a committee of 6 members out of 10 members always including a specified member.

7. Find the number of ways of selecting 5 books from 9 different mathematics books such that a particular book is not included.

8. Find the number of ways of selecting 3 vowels and 2 consonants from the letters of the word EQUATION.

9. Find the number of diagonals of a polygon with 12 sides.

10. If n persons are sitting in a row, find the number of ways of selecting two persons, who are sitting adjacent to each other.

11. Find the number of ways of giving away 4 similar coins to 5 boys if each boy can be given any number (less than or equal to 4) of coins.

III. 1. Prove that $\dfrac{^{4n}C_{2n}}{^{2n}C_n} = \dfrac{1.3.5 \ldots (4n-1)}{\{1.3.5 \ldots (2n-1)\}^2}$

2. If a set A has 12 elements, find the number of subsets of A having

   (i) 4 elements (ii) Atleast 3 elements (iii) Atmost 3 elements.

3. Find the numbers of ways of selecting a cricket team of 11 players from 7 batsmen and 6 bowlers such that there will be atleast 5 bowlers in the team.

4. If 5 vowels and 6 consonants are given, then how many 6 letter words can be formed with 3 vowels and 3 consonants.

5. There are 8 railway stations along a railway line. In how many ways can a train be stopped at 3 of these stations such that no two of them are consecutive?

6. Find the number of ways of forming a committee of 5 members out of 6 Indians and 5 Americans so that always the Indians will be in majority in the committee.

7. A question paper is divided into 3 sections A, B, C containing 3, 4, 5 questions respectively. Find the number of ways of attempting 6 questions choosing atleast one from each section.

8. Find the number of ways in which 12 things be (i) divided into 4 equal groups (ii) distributed to 4 persons equally.

9. A class contains 4 boys and g girls. Every sunday, five students with atleast 3 boys go for a picnic. A different group is being sent every week. During the picnic, the class teacher gives each girl in the group a doll. If the total number of dolls distributed is 85, find g.

Key Concepts

Here we give a brief summary of the results and important concepts of this chapter.

❖ Fundamental principle : If a work W_1 can be performed in m different ways and another work W_2 can be performed in n different ways, then the two works simultaneously can be performed in mn different ways.

❖ If n is a positive integer, then $n! = n\{(n-1)!\}$ and $1! = 1$.

❖ We define $0! = 1$.

❖ The number of permutations of n dissimilar things taken 'r' at a time is deonted by ${}^nP_r$ and ${}^nP_r = \dfrac{n!}{(n-r)!}$ for $0 \le r \le n$.

❖ If n, r are positive integers and $r \le n$, then

(i) ${}^nP_r = n\,.\,{}^{(n-1)}P_{(r-1)}$ (if $r \ge 1$) (ii) ${}^nP_r = n.(n-1).{}^{(n-2)}P_{(r-2)}$. (If $r \ge 2$)

❖ The number of permutations of n dissimilar things taken 'r' at a time

(i) containing a particular thing is $r\,.\,{}^{(n-1)}P_{(r-1)}$.

(ii) not containing a particular thing is ${}^{(n-1)}P_r$.

(iii) containing a particular thing in a particular place is ${}^{(n-1)}P_{(r-1)}$.

❖ If n, r are positive integers and $r \le n$, then ${}^nP_r = {}^{(n-1)}P_r + r\,.\,{}^{(n-1)}P_{(r-1)}$.

❖ The sum of the r-digit numbers that can be formed using the given 'n' distinct non-zero digits $(r \le n \le 9)$ is ${}^{(n-1)}P_{(r-1)} \times$ (sum of all n digits) $\times$ $(111 \ldots . 1)_{(r\text{ times})}$).

❖ In the above, if '0' is one among the given 'n' digits, then the sum is ${}^{(n-1)}P_{(r-1)} \times$ (sum of the digits) $\times\ 111\ldots 1_{(r\text{ times})} - ({}^{(n-2)}P_{(r-2)} \times$ (sum of the digits) $\times$ $(111 \ldots 1)_{(r-1)\text{ times}}$).

❖ The number of permutations of n dissimilar things taken 'r' at a time when repetitions are allowed [i.e., each thing can be used any number of times) is n^r.

❖ The number of circular permutations of n dissimilar things is $(n-1)!$.

- In the case of hanging type circular permutations like garlands of flowers, chains of beads etc., the number of circular permutations of n things is $\frac{1}{2}\{(n-1)!\}$.
- If in the given n things, p like things are of one kind, q alike things are of the second kind, r alike things are of the third kind and the rest are dissimilar, then the number of permutations (of these n things) is $\dfrac{n!}{(p!)(q!)(r!)}$.
- The number of combinations of n things taken 'r' at a time is denoted by ^nC_r and $^nC_r = \dfrac{n!}{(n-r)!r!}$ for $0 \le r \le n$.
- If n, r are integers and $0 \le r \le n$, then $^nC_r = {}^nC_{(n-r)}$.
- $^nC_0 = {}^nC_n$; $^nC_1 = {}^nC_{(n-1)}$.
- The number of ways of dividing '$m + n$' things $(m \neq n)$ into two groups containing m, n things is $^{(m+n)}C_m = {}^{(m+n)}C_n = \dfrac{(m+n)!}{m!n!}$.
- The number of ways of dividing $(m + n + p)$ things $(m, n, p$ are distinct) into 3 groups of m, n, p things is $\dfrac{(m+n+p)!}{(m!)(n!)(p!)}$.
- The number of ways of dividing mn things into m equal groups is $\dfrac{(mn)!}{(n!)^m\,(m!)}$.
- The number of ways if distributing mn things equally to m persons is $\dfrac{(mn)!}{(n!)^m}$.
- If p alike things are of one kind, q alike things are of the second kind, and r alike things are of the third kind, then the number of ways of selecting one or more things out of them is $(p+1)(q+1)(r+1)-1$.
- If m is a positive integer and

$$m = p_1^{\alpha_1}\cdot p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, then the number of divisors of m is $(\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_k + 1)$ (This includes 1 and m).

Historical Note

Ancient Indian Pandit *Pingala's Chandassastra* (ca–200 B.C.) contains material on the theory of permutations and combinations. *Mahaviracharya* and *Bhaskaracharya* (9th Century A.D., 12th Century A.D.) contributed substantially to this topic.

A Hebrew writer *Rabbi Ben Ezra* determined the number of combinations of the then known planets by taking one at a time, two at a time etc., as 1140 approximately. Though he did not know the formula for ${}^nC_r$, he observed that the number of combinations of n planets, taken r at a time is the same as the number of combinations taken $(n-r)$ at a time for certain values of $r\,(\le n)$. The formulae for ${}^nP_r$, ${}^nC_r$ have been given by another Hebrew writer '*Levi Ben Gerson*'.

Answers

Exercise 5(a)

I. 1. 12 2. 12 3. 11 4. 5 5. 5

6. ${}^5P_4 = 120$

II. 1. ${}^{25}P_2 = 600$ 2. ${}^{30}P_2 = 870$ 3. $5! \times 3! = 720$

4. ${}^4P_3 \times 21 \times 1111 - {}^3P_2 \times 21 \times 111 = 5{,}45{,}958$ 5. $3 \times {}^4P_3 + 4 \times 4! = 168$

6. ${}^3P_2 \times 4! = 144$ 7. $3! \times 5! \times 4! \times 3! = 1{,}03{,}680$

III. 1. ${}^8P_5 = 6720$; ${}^7P_4 = 840$; ${}^7P_4 = 840$; ${}^6P_3 = 120$

2. (i) $(8!)\,(3!)$ (ii) $\dfrac{10!}{3!}$ (iii) $8!$

3. (i) $4!\ 5!$ (ii) $2!\ 5!\ 4!$

4. ${}^5P_4 = 120$

(i) $2! \times {}^4P_3 = 48$ (ii) $4! = 24$ (iii) $6 \times {}^3P_2 = 36$

(iv) ${}^4P_3 = 24$ (v) $2 \times {}^3P_2 = 12$

5. (i) 391 (ii) 257

6. IGRBN

7. ${}^4P_3 \times 18 \times 1111 = 4,79,952$

8. ${}^6P_5\, 4! = 17280$

Exercise 5(b)

I. 1. $6^4 = 1296$ 2. $5^5 = 3125$ 3. $4^5 = 1024$

II. 1 (i) 4×5^2 (ii) 5^3 2. $6^4 - {}^6P_4 = 936$

3. ${}^7P_7 = 5040$ 4. $n^r - {}^nP_r$ 5. $6^4 = 1296$

6. $2 \times 5 \times 6^3 = 2160$ 7. $4 + 4^2 + 4^3 + 4^3.1 = 148.$

III. 1. (i) ${}^9P_4 = 3024$ (ii) $9^4 - {}^9P_4 = 3537$

2. (i) $3 \times 4 \times 5^2 = 300$ (ii) 160 3. $5 \times 6^2 \times 1 = 180$

Exercise 5(c)

I. 1. $(7-1)! = 720$ 2. $(10)!$ 3. $\frac{1}{2}[(6-1)!] = 60$

II. 1. $4! \times 3! = 144$ 2. $6! \times {}^7P_4$ 3. $5! \times 2! = 240$

4. $\frac{1}{2}(7! \times 2!) = 5040$

III. 1. (i) $6! \times 6!$ (ii) $5! \times 6!$ (iii) $5! \times 6!$

2. $\frac{1}{2}[(9-1)!] = 20,160$ (i) $\frac{6! \times 3!}{2} = 2160$ (ii) $\frac{5! \times {}^6P_3}{2} = 7200$

3. $3! \times 3! \times 3! \times 3! \times 2! = 2,592$ 4. $\frac{1}{2}(5! \times {}^6P_3) = 7,200$

5. $3! \times 2! = 12$

Exercise 5(d)

I. 1. (i) $\frac{(12)!}{4!.3!.2!}$ (ii) $\frac{(11)!}{2!.2!.2!}$ (iii) $\frac{7!}{2!.2!.2!}$

(iv) $\frac{(11)!}{2!}$ (v) $\frac{(11)!}{2!.2!.2!}$ (vi) $\frac{(12)!}{3!.2!.2!}$

2. $\frac{7!}{3!.2!.2!}$

II. 1. $4!+3\times\frac{4!}{2!}+3\times\frac{4!}{3!} = 72$

2. $\frac{3!}{2!}\times\frac{4!}{2!\times 2!} = 18$

3. $\frac{(35)!}{(3!)^2(4!)^2(5!)^3.6!}$

4. $\frac{(mn)!}{(m!)^n}$

5. $4!+2\cdot\frac{4!}{2!} = 48$

6. $3!\times\frac{{}^4P_3}{3!} = 24$

III. 1. $\frac{(12)!}{3!\ 2!\ 2!\ 2!}$ (i) $\frac{(10)!}{2!\ 2!\ 2!}$ (ii) $\frac{(10)!}{3!\ 2!\ 2!}\times\frac{{}^{11}P_2}{2!}$

2. $5! = 120$

3. (i) 28 (ii) 68

Exercise 5(e)

I. 1. 10

2. 4 or 8

3. 9

4. $n = 10,\ r = 4$

5. 10

6. 3

6. 6

8. 3 or 4

9. 4 or 6

10. ${}^9C_5 = 126$

11. ${}^3C_2 \times {}^6C_3 + {}^3C_3 \times {}^6C_2 = 75$

12. ${}^{13}C_{11} = 78$

II. 2. ${}^{12}C_5 = 792$

3. ${}^{39}C_5$

4. ${}^{30}C_2 = 435$

5. ${}^7C_3 \times {}^6C_3 = 700$

6. ${}^9C_5 = 126$

7. ${}^8C_5 = 56$

8. ${}^5C_3 \times {}^3C_2 = 30$ 9. ${}^{12}C_2 - 12$ 10. $n = 1$

11. ${}^5C_1 + {}^5C_2 \times {}^2C_1 + {}^5C_2 + {}^5C_3 \times {}^3C_1 + {}^5C_4 = 70$

III. 2. (i) ${}^{12}C_4 = 495$

(ii) $2^{12} - \{{}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2\} = 4017$

(iii) ${}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 = 299$

3. ${}^6C_5 \times {}^7C_6 + {}^6C_6 \times {}^7C_5 = 63$

4. ${}^5C_3 \times {}^6C_3 \times 6!$

5. ${}^8C_3 - 6 - 30 = 20$

6. ${}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1 + {}^6C_5 = 281$

7. ${}^{12}C_6 - {}^7C_6 - {}^9C_6 - {}^8C_6 = 805$

8. (i) $\dfrac{(12)!}{(3!)^4 4!}$ (ii) $\dfrac{(12)!}{(3!)^4}$

9. $g = 5.$