P2:

Find the m.g.f. of Normal $N(\mu, \sigma^2)$ distribution and hence find its mean and variance.

Solution:

Since $X \sim N(\mu, \sigma^2)$, its p.d.f. is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

The m.g.f. of X is given by

$$M_X(t) = E\left[e^{tX}\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx}\, exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Let $z = \frac{x-\mu}{\sigma}$. Thus $x = \mu + \sigma z$ and $dx = \sigma . dz$

Thus, $M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(t(\mu + \sigma z)\right) exp\left(-\frac{z^2}{2}\right) dz$

$$= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(z^2 - 2t\sigma z\right)\right] dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left\{(z-\sigma t)^2 - \sigma^2 t^2\right\}\right] dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{t\mu + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} exp\left[-\frac{1}{2}(z-\sigma t)^2\right] dz$$

Let $u = z - \sigma t \implies du = dz$

$$= e^{t\mu + \frac{1}{2}\sigma^2 t^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

$\Longrightarrow\ M_X(t) = e^{t\mu+\frac{1}{2}\sigma^2t^2} \Longrightarrow\ M_X'(t) = e^{t\mu+\frac{1}{2}\sigma^2t^2}(\mu+\sigma^2 t)$

$\Longrightarrow\ \mu =$Mean$= \mu_1' = M_X'(t)|_{t=0} = \mu$

Further, $M_X''(t) = e^{t\mu+\frac{1}{2}\sigma^2t^2}(\mu+\sigma^2 t)^2 + e^{t\mu+\frac{1}{2}\sigma^2t^2}(\sigma^2)$

$\Longrightarrow\ \mu_2' = M_X''(t)|_{t=0} = \mu^2+\sigma^2 \Longrightarrow\ \mu_2' = \mu^2+\sigma^2$

The variance is given by

Variance $= \mu_2' - (\mu_1')^2 = \mu^2+\sigma^2-\mu^2 = \sigma^2.$