P2:

Let $\{X_n\}$ be a sequence of r.vs defined by $P(X_n = 0) = 1 - \frac{1}{n^r}$ and $P(X_n = n) = \frac{1}{n^r}$, $r > 0, n = 1, 2, \dots$.

Show that (i) $X_n \xrightarrow{P} 0$ and (ii) $X_n \overset{r}{\not\to} 0$

Solution:

(i) We have

$$P(|X_n - 0| > \epsilon) = P(X_n = n) = \frac{1}{n^r}$$

Therefore $P(|X_n - 0| > \epsilon) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $X_n \xrightarrow{P} 0$

(ii) We have $E(|X_n - 0|^r) = E(|X_n|^r) = 0^r \left(1 - \frac{1}{n^r}\right) + n^r \left(\frac{1}{n^r}\right) = 1$

$$\Rightarrow E(|X_n - 0|^r) = 1$$
$$\Rightarrow \lim_{n\to\infty} E(|X_n - 0|^r) = 1$$
$$\Rightarrow X_n \overset{r}{\not\to} 0$$

Thus $X_n \xrightarrow{P} 0$ but $X_n \overset{r}{\not\longrightarrow} 0$