P3:

For each positive integer n there exist integers m and k (uniquely determined) such that

$$n = 2^k + m\,,\ 0 \le m < 2^k\,,\ k = 0, 1, 2, \dots$$

Thus, for $n = 1$, we have $k = 0$ and $m = 0$; for $n = 5$, we have $k = 2$ and $m = 1$; and so on. Define r.vs X_n for $n = 1, 2, \dots n$ on $\Omega = [0, 1]$ by

$$X_n(w) = \begin{cases} 2^k\,, & \frac{m}{2^k} \le w < \frac{m+1}{2^k} \\ 0\,, & \text{otherwise} \end{cases}$$

Show that $X_n \xrightarrow{P} 0$ but $X_n \overset{a.s}{\not\longrightarrow} 0$.

Solution:

Let the probability distribution of X_n be given by

$$P(I) = \text{length of the interval } I \subseteq \Omega.$$

Thus, $P(X_n = 2^k) = \frac{1}{2^k}$ and $P(X_n = 0) = 1 - \frac{1}{2^k}$

The limit $\lim\limits_{n\to\infty} X_n(w) = \lim\limits_{k\to\infty} X_n(w)$ does not exist for any $w \in \Omega$, so that X_n does not converge almost surely. But

$$P(|X_n - 0| > \epsilon) = P(|X_n| > \epsilon) = P(X_n > \epsilon)$$

$$= \begin{cases} 0\,, & \epsilon \ge 2^k \\ \frac{1}{2^k}\,, & 0 < \epsilon < 2^k \end{cases}$$

and we see that

$$P(|X_n - 0| > \epsilon) \longrightarrow 0 \text{ as } n \text{ (and hence } k) \longrightarrow \infty.$$

Thus, $X_n \xrightarrow{P} 0$ but $X_n \overset{a.s}{\not\longrightarrow} 0$.