P4:

Find the m.g.f. of exponential $E(\lambda)$ distribution and hence find its mean and variance.

Solution:

Since $X \sim E(\lambda)$, its p.d.f. is given by $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $\lambda > 0$

The m.g.f. of X is given by

$$M_X(t) = E\left[e^{tX}\right] = \lambda \int_0^\infty e^{tx} e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$

$$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r \quad for\ \lambda > t$$

$$\Rightarrow M_X(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r \quad for\ \lambda > t$$

But $\mu_r' =$ coefficient of $\frac{t^r}{r!}$ in $M_X(t) = \frac{r!}{\lambda^r}$ for $r = 1, 2, \ldots$

Thus, $\mu_1' = \frac{1}{\lambda}$ and $\mu_2' = \frac{2}{\lambda^2}$ and hence $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Thus, $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$