1.2

Basic Concepts in Probability

Introduction to uncertainty

Every day we have been coming across statements like the ones mentioned below:

1. Probably it will rain tonight.
2. It is quiet likely that there will be a good yield of paddy this year.
3. Probably I will get a first class in the examination.
4. India might win the cricket series against Australia
   and so on.

In all the above statements some element of uncertainty or chance is involved. A numerical measure of uncertainty is provided by a very important branch of statistics known as **Theory of Probability**. In the words of *Prof. Ya-Lin-Chou*: *Statistics is the science of decision making with calculated risks in the face of uncertainty*.

History of Probability

The history of probability suggests that its theory developed with the study of *games of chance*, such as *rolling of dice*, *drawing a card from a pack of cards*, etc. Two French gamblers had once decided that any one person who will first get a 'particular point' will win the game. If the game is stopped before reaching that point, the question is how to share the stake. This and similar other problems were then posed by the great French mathematician *Blaise Pascal*, who after consulting another great French mathematician *Pierre de Fermat*, gave the solution of the problems and then laid down a strong foundation of probability. Later on, another French mathematician, *Laplace*, improved the definition of probability.

Coins, Dice and Playing Cards: The basic concepts in probability are better explained using *coins*, *dice* and *playing cards*. The knowledge of these is very much useful in solving problems in probability.

Coin: A coin is round in shape and it has two sides. One side is known as ***head (H)*** and the other is known as ***tail (T).*** When a coin is tossed, the side on the top is known as the result of the toss.

Die: A die is cube in shape in which length, breadth and height are equal. It has six faces which have same area and numbered from 1 to 6. The plural of die is dice. When a die is thrown, the number on the top face is the result of the throw.

Pack of Cards: A pack of cards 52 cards. It is divided into four suits called *spades*, *clubs*, *hearts* and *diamonds*. Spades and clubs are black; hearts and diamonds are red in colour. Each suit consists of 13 cards, of which *nine* cards are numbered from 2 to 10, an ace, jack, queen and king. We shuffle the cards and then take a card from the top which is the result of selecting a card.

Basic Concepts in Probability

The following basic concepts are very important in understanding the definitions of the probability:

Experiment: The process of making an observation or measurement and observation about a phenomenon is known as an ***experiment***.

Example1: Sitting in the balcony of the house and watching the movement of clouds in the sky is an experiment.

Example2: For given values of pressure (P), measuring the corresponding values of volume (V) of a gas and observing that $P \cdot V = k$(constant) is an experiment. The experiments are of two types:

Deterministic experiment: If an experiment produces the same result when it is conducted several times under identical conditions, then the experiment is known as ***determinant experiment***.
All the experiments in physical and engineering sciences are deterministic.

Random Experiment: If an experiment produces different results even though it is conducted several times under identical conditions, then the experiment is known as ***random experiment***. All the experiments in social sciences are random.

Trial: Conducting a random experiment once is known as a ***trial***.

Outcome: A result of a random experiment in a trial is known as an ***outcome***. Outcomes are denoted by lowercase letters $a, b, c, d, e, \dots$.

Equally Likely Outcomes: Outcomes of a random experiment are said to be ***equally likely*** if all have the same chance of occurrence. Getting a H and T in a balanced coin are equally likely. The outcomes $1,2,3,4,5$ and 6 are equally likely if the die is a cube.

Sample space: The set of all possible outcomes of a random experiment is known as a ***sample space*** and denoted by **S**.

Event: A subset of the sample space is known as an ***event***.
The events are denoted by uppercase letters $\mathrm{A}, \mathrm{B}, \mathrm{C}$ etc.

Happening of an event: We say that an event happens (or occurs) if any one outcome in it happens (or occurs).

Elementary Event: A singleton set consisting an outcome of a random experiment is known as an ***elementary event***.

Favorable outcomes: The outcomes in an event are known as ***favorable outcomes*** or ***cases*** of that event.

Impossible Event: An event with no outcome in it is known as ***impossible event*** and is denoted by $\boldsymbol{\phi}$.

Certain or Sure Event: An event consisting of all possible outcomes of a random experiment is known as ***certain*** or ***sure event*** and it is same as the sample space.

Exhaustive Events: The events in a sample space are said to be ***exhaustive*** if their union is equal to the sample space. The events $A_1, A_2, \dots, A_n$ in S are said to be exhaustive if

$$\bigcup_{i=1}^{n} A_i = S$$

Mutually Exclusive Events: Two or more events in the sample space are said to be ***mutually exclusive*** if the happening of one of them precludes the happening of the others. Mathematically two events A and B in S are said to be mutually exclusive if $A \cap B = \phi$.

Example3: Consider a random experiment of tossing a coin. The possible outcomes are H and T. Thus, the sample space is given by $S = \{H, T\}$ and $n(S) = 2$ where $n(S)$ is the total number of outcomes in S.

Example 4: Consider a random experiment of tossing two coins (or two tosses of a coin). The sample space is given by $S = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$ and $n(S) = 2^2 = 4$.

Example 5: Consider a random experiment of tossing three coins (or three tosses of a coin). The sample space is given by

$$S = \{H, T\} \times \{H, T\} \times \{H, T\} = \{H, T\} \times \{HH, HT, TH, TT\}$$

$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

and $n(S) = 2^3 = 8$.

Let us define some events in the sample space as below:

E_1: Three heads

E_2: Three tails

E_3: Exactly one head

E_4: Exactly two heads

E_5: At least one head

E_6: At least two heads

Then these events are represented by the following subsets of S:

$$E_1 = \{HHH\};$$

$$E_2 = \{TTT\}$$

$$E_3 = \{HTT, THT, TTH\};$$

$$E_4 = \{HHT, HTH, THH\};$$

$$E_5 = \{HHH, HHT, HTH, HTT, THH, THT, TTH\} \text{ and}$$
$$E_6 = \{HHH, HHT, HTH, THH\}.$$

Note that $E_1 \cup E_2 \cup E_3 \cup E_4 = S$ and hence E_1, E_2 ,E_3 andE_4 are exhaustive events in S. Further, $E_i \cap E_j = \phi$, where $i \neq j$. Hence, E_1, E_2 ,E_3 and E_4 are mutually exclusive events in S.

Note: In general, if a random experiment consists of tossing N coins (or N tosses of a coin), then $n(S) = 2^N$.

Example 6: Let us consider a random experiment of throwing a die. Since we can obtain any one of the six faces 1,2,3,4,5 and 6, the sample space is given by $S = \{1,2,3,4,5,6\}$ and $n(S) = 6$.

Now define $E_1 = \{1,3,5\}, E_2 = \{2,4,6\}$ and $E_3 = \{3,6\}$. We say that E_1 happens or occurs if we get the outcome $1,3 \ or \ 5$. In otherwords, we say that E_1 happens if we get an odd number. Similarly, we say that E_2 happens if we get an even number and E_3 happens if we get a multiple of 3.

Since E_1, E_2 and E_3 are subsets of S; E_1, E_2 and E_3 are events in S. Since $E_1 \cup E_2 = S$, E_1 and E_2 are exhaustive events in S. Since $E_1 \cup E_3 = \{1,3,5,6\} \neq S$, E_1 and E_3 are not exhaustive events in S. Since $E_1 \cap E_2 = \phi$, E_1 and E_2 are mutually exclusive events in S. Since $E_1 \cap E_3 = \{3\}$, E_1 and E_3 are not mutually exclusive events in S. Similarly E_2 and E_3 are not mutually exclusive events in S.

Example 7: In a random experiment of throwing two dice (or two throws of a die), the sample space is given by

$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

$$\{(1,1), (1,2), \dots, (1,6),$$

$$(2,1), (2,2), \dots, (2,6),$$

$$(3,1), (3,2), \dots, (3,6)$$

$$(4,1), (4,2), \dots, (4,6)$$

$$(5,1), (5,2), \dots, (5,6)$$

$$(6,1), (6,2), \dots, (6,6)\}$$

where in the outcome (a, b) , a represents the number obtained on the first die and b represents the number on the second die. Obviously $(a, b) \neq (b, a)$ unless $a = b$. The number of outcomes in S is given by $S = 6^2 = 36$.

Let us define the following events in S.

E_1: Sum of points on two dice is 5

E_2: Sum of points on two dice is 6

E_3: Sum of points on two dice is even

E_4: Sum of points on two dice is odd

E_5: Sum of points on two dice is greater than 12

E_6: Sum of points on two dice is divisible by 3

E_7: Sum is greater than or equal to 2 and is less than or equal to 12

Then the events E_1 to E_7 as subsets of S are given below.

$E_1 = \{(1,4),(2,3),(3,2),(4,1)\}$ and $n(E_1) = 4$

$E_2 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ and $n(E_2) = 5$

The sum of the points on the two dice is even if the points obtained on each die is (i) even or (ii) odd. Thus

$E_3 = (\{2,4,6\} \times \{2,4,6\}) \cup (\{1,3,5\} \times \{1,3,5\})$

$\{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6),(1,1),(1,3),(1,5),$

$(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$

and $n(E_3) = (3 \times 3) + (3 \times 3) = 9 + 9 = 18.$

Similarly,

$E_4 = (\{2,4,6\} \times \{1,3,5\}) \cup (\{1,3,5\} \times \{2,4,6\})$

$\{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5),(1,2),(1,4),(1,6),$

$(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$

and $n(E_4) = (3 \times 3) + (3 \times 3) = 18.$

Further, $E_5 = \phi, i.e.$, E_5 is an impossible event and $E_7 = S, i.e.$, E_7 is a certain event. Hence $n(E_5) = 0$ and $n(E_7) = 36$.

The sum of the points on the two dice is divisible by 3 if their sum is 3,6,9 or 12. Thus

$E_6 = \{(1,2),(2,1),(1,5),(2,4),(3,3),(4,2),(5,1),(3,6),(4,5),(5,4),(6,3),(6,6)\}$ and $n(E_6) = 12$.

Note: In general, if the random experiment consists of throwing of N dice (or N throws of a die), the number of outcomes in S is given by $n(S) = 6^N$.

Example 8: Let us consider the random experiment of tossing a coin and a die together. Then the sample space is given by

$S = \{H, T\} \times \{1,2,3,4,5,6\}$

$= \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

and $n(S) = 2 \times 6 = 12$.

Note: In the above examples 3 to 8, if the coins and dice are unbiased, the outcomes in the sample spaces are equally likely. Normally, the coins are balanced and hence are unbiased. If a die is a cube, then all the surfaces have the same area and also it is unbiased.

Example 9: Let us consider the random experiment of selecting two balls simultaneously from and urn containing 4 balls of different colours red(R), blue(B), yellow(Y) and white(W). Then the sample space is given by

$$S = \{RB, RY, RW, BY, BW, YW\} \text{ and } n(S) = 4_{C_2} = 6$$

Example 10: If the random experiment consists of selecting two balls one after the other with replacement in Example 9, the sample space is given by

$S = \{R, B, Y, W\} \times \{R, B, Y, W\} =$
$\{RR, RB, RY, RW, BR, BB, BY, BW, YR, YB, YY, YW, WR, WB, WY, WW\}$ and $n(S) = 4 \times 4 = 16$.

Example 11: If the random experiment consists of selecting two balls one after the other without replacement in Example7, the sample space is given by

$S = \{RB, RY, RW, BR, BY, BW, YR, YB, YW, WR, WB, WY\}$ and
$n(S) = 4 \times 3 = 12$.

Example12: Consider a random experiment of tossing a coin until head appears. Its sample space is given by

$$S = \{H, TH, TTH, TTTH, \dots\}$$

where TTH represents tail in first, tail in second and head in third tosses and so on. Obviously, $n(S)$ is infinite.

Example13: Consider a random experiment of tossing a coin repeatedly until head or tail appears twice in succession. Thus the sample space is given by

$$S = \{HH, TT, THH, HTT, HTHH, THTT, \dots\}$$

and $n(S)$ is infinite.