P2:

Find the p.d.f of U, which is the sum of X and Y that are independent random variables with the following p.d.fs:

$$\boldsymbol{f_X(x) = \lambda e^{-\lambda x} \quad , \quad x \geq 0}$$

$$\boldsymbol{f_Y(y) = \lambda^2 y e^{-\lambda y} \quad , \quad y \geq 0}$$

Solution:

Since X and Y are independent random variables and the p.d.f of U is given by

$$f_U(u) = \int_0^u f_X(x) f_Y(u-x)\, dx$$

$$= \int_0^u \lambda e^{-\lambda x} \lambda^2 (u-x) e^{-\lambda(u-x)} dx$$

$$= \lambda^3 e^{-\lambda u} \int_0^u (u-x)\, dx$$

$$= \lambda^3 e^{-\lambda u} \left[ux - \frac{x^2}{2}\right]_0^u = \lambda^3 e^{-\lambda u} \left[u^2 - \frac{u^2}{2}\right]$$

$$= \frac{\lambda^3 u^2 e^{-\lambda u}}{2}$$

$$= \frac{\lambda^3 u^2 e^{-\lambda u}}{2!} \quad u \geq 0$$

This is known as **Erlang – 3 distribution**.