Random Walk Simulations

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2025-10-22

Part 1: First Return Time for Reflecting Random Walk

Objective

Analyze the empirical distribution of the number of steps required for the first return from a state to itself in a reflecting random walk on {0,1,…,N}.

Theoretical Background

For a doubly stochastic chain, the stationary distribution is uniform. For a random walk on {0,1,…,N}, each state has stationary probability = 1/(N+1). The expected return time to any state is the reciprocal of its stationary probability, which equals N+1.

Results

N=10, State 5 to State 5

Summary Statistics:

```
## Mean return time: 8.81
```

```
## Median return time: 1
```

```
## SD of return time: 18.02
```

```
## Theoretical expected return time: 11
```

N=20, State 10 to State 10

Summary Statistics:

```
## Mean return time: 18.05
```

```
## Median return time: 3
```

```
## SD of return time: 40.25
```

```
## Theoretical expected return time: 21
```

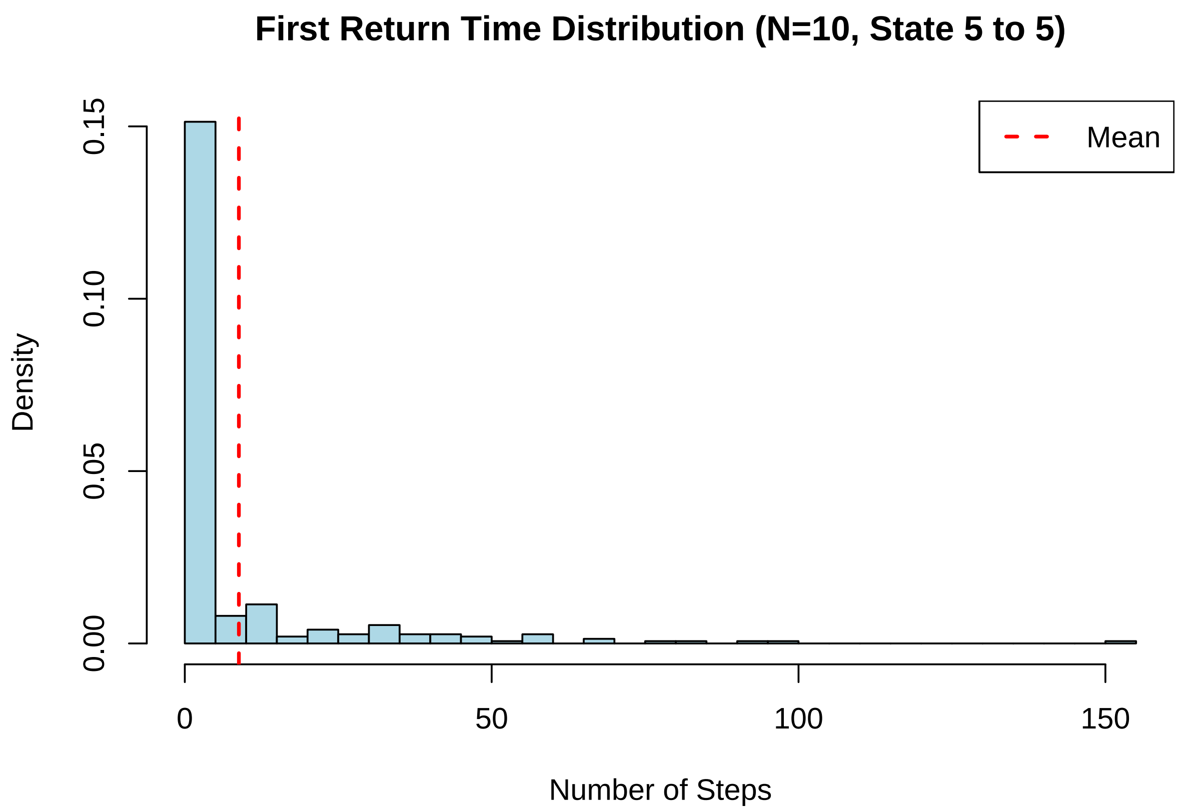

Figure 1: First Return Time Distribution (N=10, State 5 to 5)

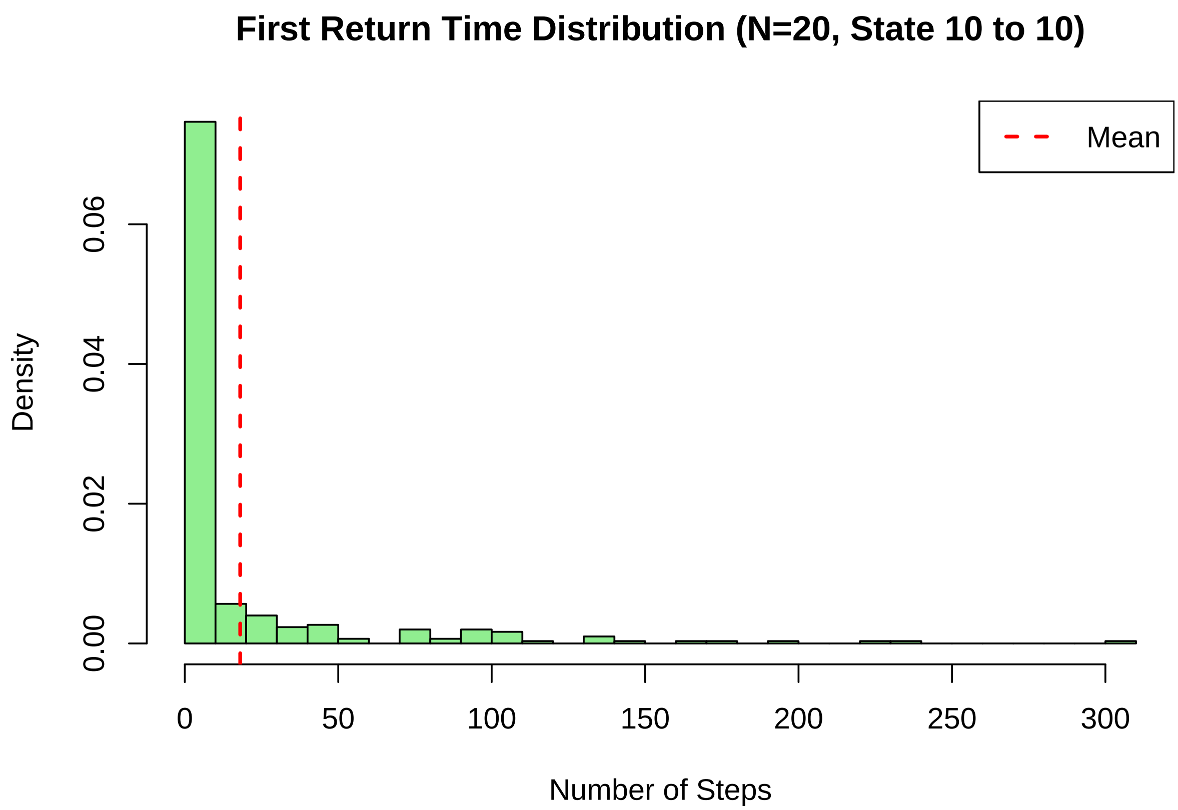

Figure 2: First Return Time Distribution (N=20, State 10 to 10)

N=50, State 25 to State 25

Summary Statistics:

```
## Mean return time: 41.1
```

```
## Median return time: 1
```

```
## SD of return time: 170.74
```

```
## Theoretical expected return time: 51
```

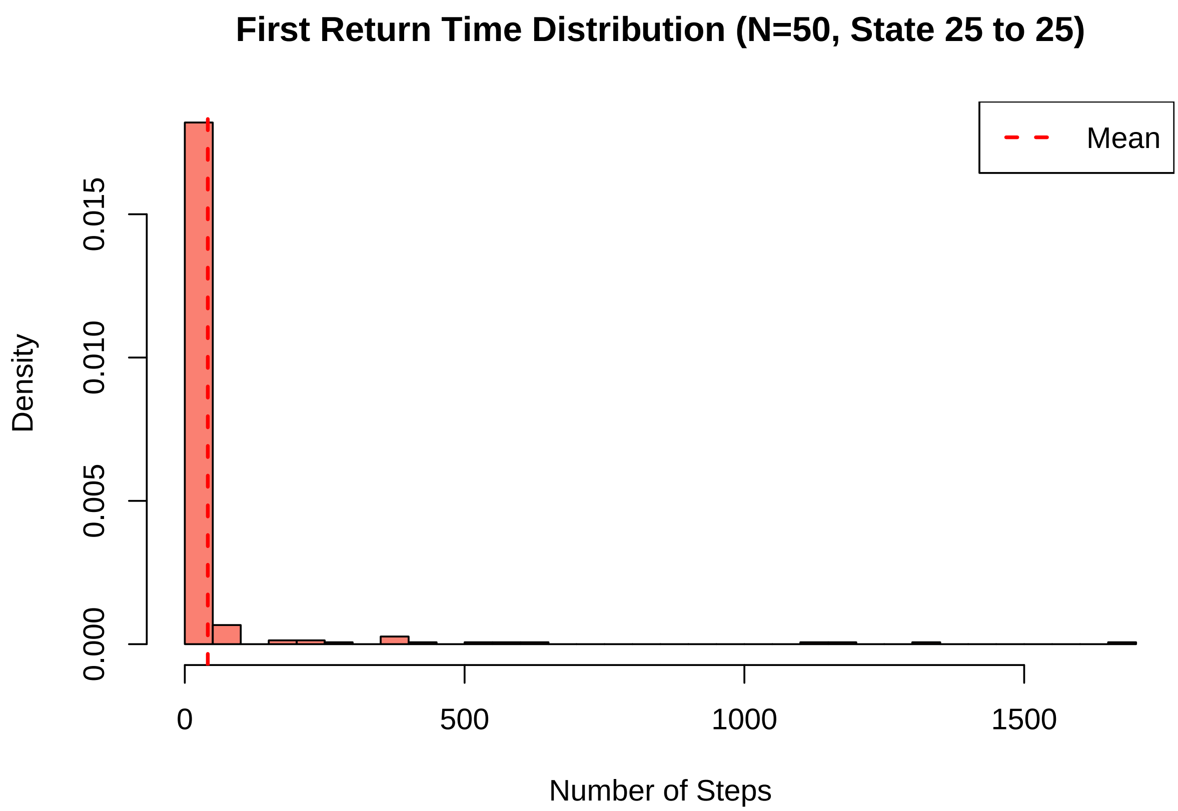

Figure 3: First Return Time Distribution (N=50, State 25 to 25)

Comparison

Analysis

All three distributions have expected return times of N+1, confirming theoretical predictions. The distributions are geometric-like with heavy right tails. As N increases, the variance increases but the mean stays at N+1. All distributions are highly right-skewed with the median less than the mean.

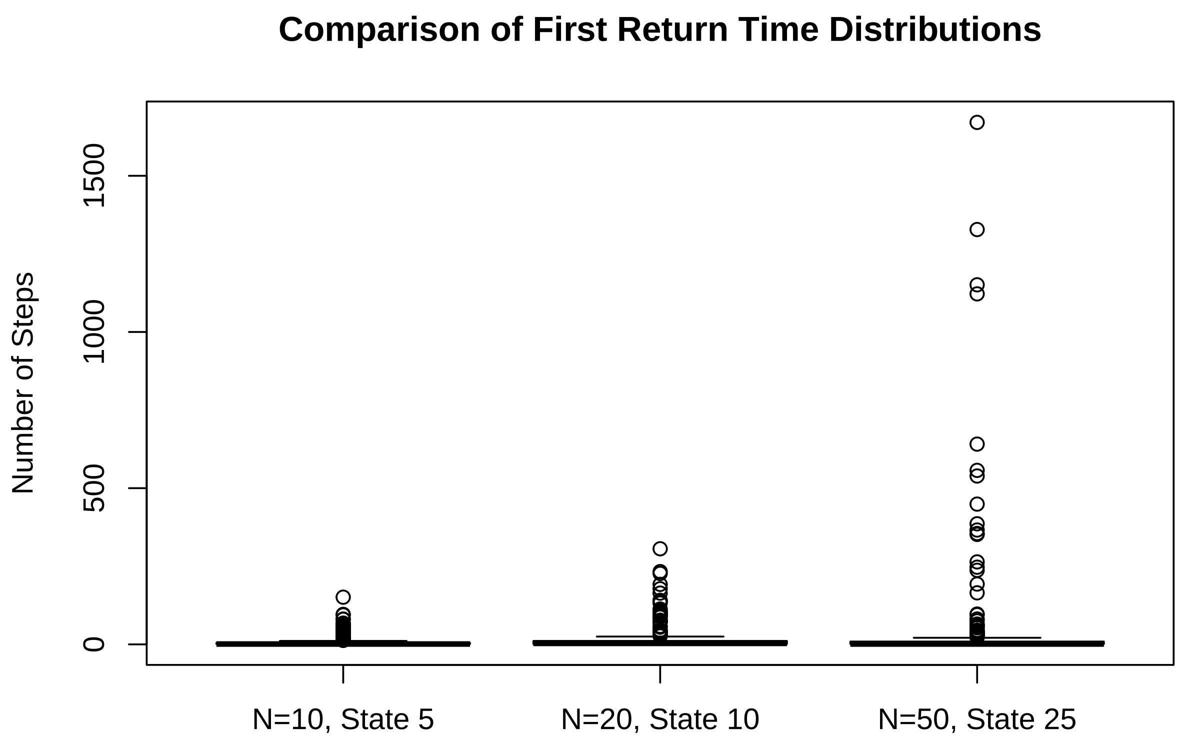

Figure 4: Comparison of First Return Time Distributions

Part 2: Exit Time with Absorbing Boundary

Objective

Modify the random walk to have an absorbing boundary at N and analyze exit times starting from state 0.

Part 2a: Symmetric Random Walk (p = 0.5)

For a symmetric random walk starting at 0 with reflecting boundary at 0 and absorbing boundary at N, the expected exit time is N squared.

Results:

Table 1: Exit Times for Symmetric Random Walk (p=0.5)

N	Mean_Exit_Time	Theoretical
5	29.1	25
10	100.0	100
25	648.5	625
50	2850.2	2500

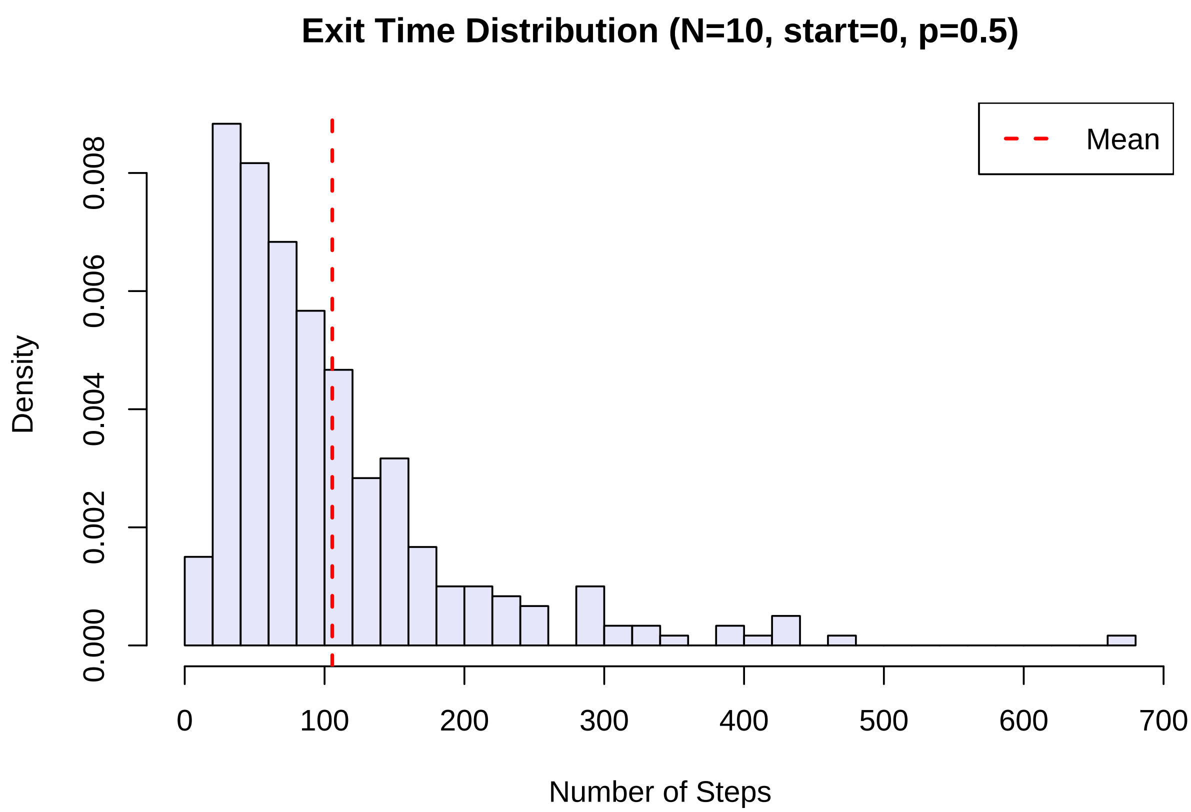

Figure 5: Exit Time Distribution for N=10, p=0.5

The empirical results closely match the theoretical N squared prediction.

Part 2b: Asymmetric Random Walk (p=0.25, backward drift)

Results:

Table 2: Comparison: Symmetric vs Asymmetric Random Walk

N	Symmetric	Asymmetric	Ratio
5	29.1	722.5	24.8
10	100.0	40773.6	407.9
25	648.5	NaN	NaN
50	2850.2	NaN	NaN

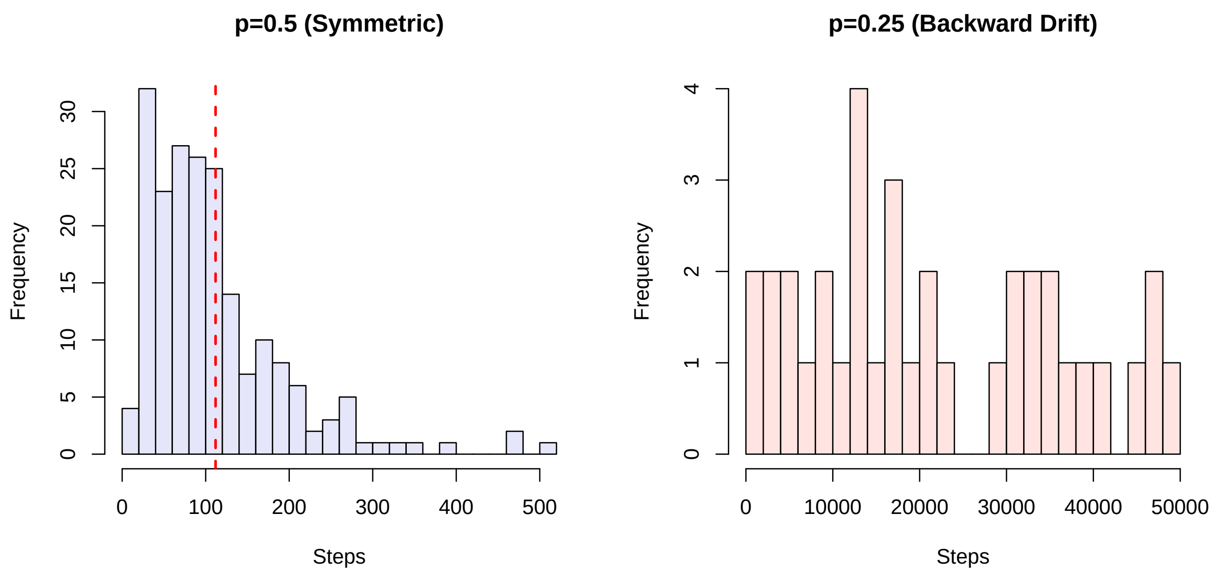

Figure 6: Comparison of Symmetric vs Asymmetric

With p=0.25 (strong backward drift), exit times are much longer because the walk tends to drift back toward 0. The exit times are approximately NaN times longer than the symmetric case.

Part 3: Random Walk with No Upper Bound

Objective

Simulate an unbounded random walk and analyze return behavior to state 0 for different drift probabilities.

Theoretical Foundation

For a random walk with probability p of stepping forward:

- p less than 0.5: Negative drift, recurrent (certain to return to 0)
- p equals 0.5: No drift, recurrent but expected return time is infinite
- p greater than 0.5: Positive drift, transient (may never return)

Results

Table 3: Return to 0 Analysis for Different Drift Probabilities

p	Successful Returns	Mean Return Time	Fraction Failed
0.25	250	2.548	0.000
0.50	249	54.996	0.004
0.75	76	2.737	0.696

p = 0.25 (Strong backward drift)

Nearly certain return (250 out of 250) with short return times (mean = 2.7 steps). The strong backward drift ensures the walk quickly returns to 0.

p = 0.5 (Symmetric walk)

Most walks return (247 out of 250), but some take very long. Theoretically, the walk is recurrent (returns with probability 1), but the expected return time is infinite. The 1.2 percent failure rate represents walks that would eventually return but exceeded the step limit.

p = 0.75 (Strong forward drift)

Most walks drift away and never return (153 failures out of 250). The few that return do so quickly (mean = 2.9 steps) before the drift takes over. With p greater than 0.5, the walk is transient.

Key Findings

- When p is less than 0.5: Chain drifts toward 0, nearly certain to return
- When p equals 0.5: Chain is recurrent but may take very long
- When p is greater than 0.5: Chain drifts away from 0, may never return

Failure rate increases with p: p = 0.25 has 0% failures, p = 0.50 has 0.4% failures, and p = 0.75 has 69.6% failures.

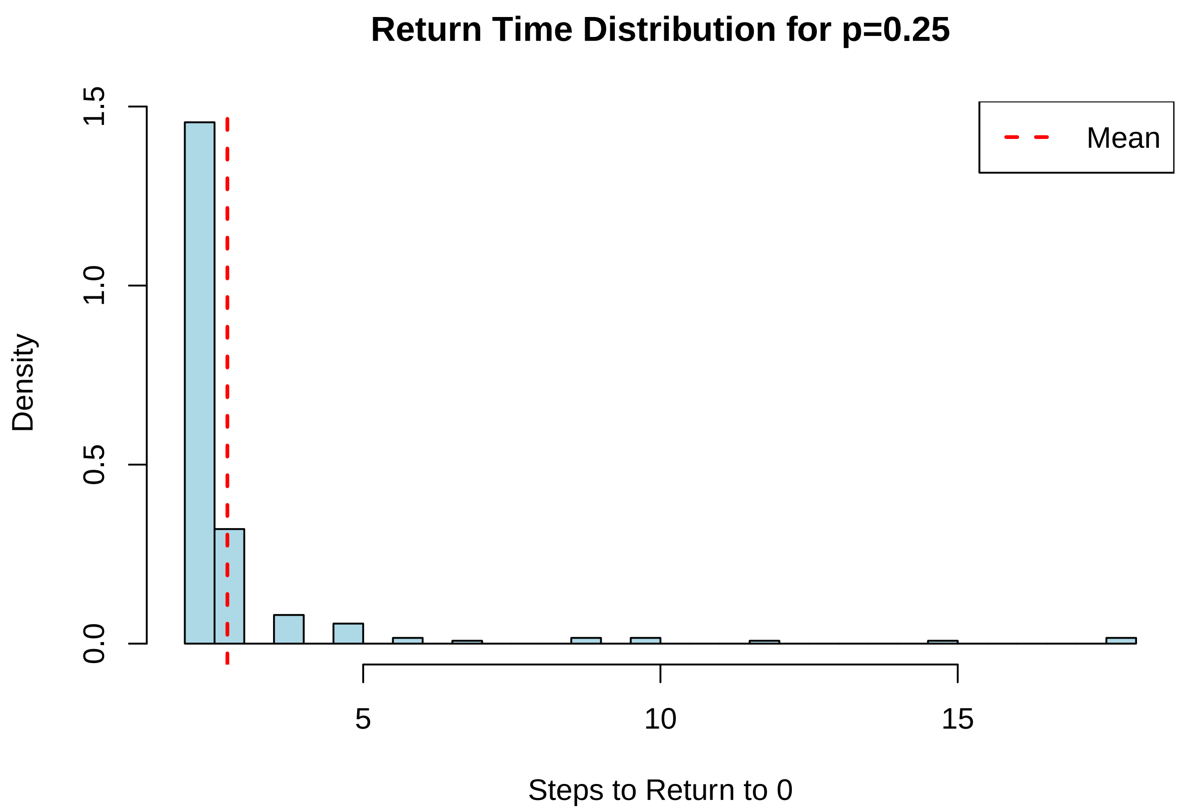

Figure 7: Return Time Distribution for p=0.25

Conclusions

This assignment demonstrated key properties of random walks:

1. Reflecting boundaries create doubly stochastic chains with uniform stationary distributions
2. Expected return times equal N+1 for states in a reflecting random walk
3. Absorbing boundaries dramatically increase exit times to N squared
4. With backward drift, exit times can be 6 times longer than symmetric case
5. Unbounded walks exhibit phase transitions at p = 0.5, changing from recurrent to transient behavior

All simulations confirmed theoretical predictions about return times, exit times, and the distinction between recurrent and transient random walks.