Roots and Powers

Contents

1 Exponents and Radicals

In this unit, students will explore concepts related to roots and powers, including estimating roots, classifying numbers, converting between mixed and entire radicals, operations with radicals, laws of exponents, and real-world applications.

1.1 Estimating Roots

Why Estimate Roots?

Estimating roots is an essential skill because not all numbers have perfect square or cube roots. This technique allows us to approximate values for irrational roots, which are critical in real-world applications such as engineering, physics, and computer science. By estimating roots, we develop a deeper understanding of how numbers behave and learn to work with values that cannot be precisely calculated.

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Key Points to Remember

- Perfect squares and cubes have exact roots, such as $\sqrt{16} = 4$ and $\sqrt[3]{27} = 3$.
- For non-perfect squares or cubes, estimate between the nearest perfect squares or cubes.
- Approximate values can be refined using calculators or iterative methods.

Estimate $\sqrt{50}$.

Estimate $\sqrt[3]{85}$.

Practice Problems

Estimate the following roots. Show your reasoning for each problem:

1. $\sqrt{45}$
2. $\sqrt{75}$
3. $\sqrt{20}$
4. $\sqrt{2}$
5. $\sqrt{90}$
6. $\sqrt{150}$
7. $\sqrt[3]{30}$
8. $\sqrt[3]{200}$
9. $\sqrt[3]{500}$
10. $\sqrt[3]{10}$
11. $\sqrt{12}$
12. $\sqrt[3]{64}$
13. $\sqrt[3]{8}$
14. $\sqrt{28}$
15. $\sqrt{120}$
16. $\sqrt[3]{40}$
17. $\sqrt[3]{15}$
18. $\sqrt{99}$
19. $\sqrt[3]{216}$
20. $\sqrt{1.5}$

1.2 Classifying Numbers

Understanding Number Systems

Numbers can be categorized into distinct groups based on their properties. These categories help us understand and work with numbers more effectively. Below is a brief explanation of the main number systems:

- **Natural Numbers ($\mathbb{N}$)**: Counting numbers starting from 1 $(1, 2, 3, \ldots)$.
- **Whole Numbers ($\mathbb{W}$)**: Natural numbers including 0 $(0, 1, 2, \ldots)$.
- **Integers ($\mathbb{Z}$)**: Whole numbers and their negatives $(\ldots, -2, -1, 0, 1, 2, \ldots)$.
- **Rational Numbers ($\mathbb{Q}$)**: Numbers that can be written as fractions $(\frac{p}{q})$ where $q \neq 0$, including terminating and repeating decimals.
- **Irrational Numbers**: Non-repeating, non-terminating decimals like π or $\sqrt{2}$.
- **Real Numbers ($\mathbb{R}$)**: All numbers on the number line, combining rational and irrational numbers.

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Classifying Numbers Classifying numbers involves determining which categories a given number belongs to. A single number can belong to multiple categories. For example, 3 is a natural number, whole number, integer, rational, and real number.

Classify the number $\sqrt{16}$.

Classify the number $-\frac{5}{2}$.

Practice Problems

Classify each of the following numbers into as many categories as possible ($\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or Irrational):

1. 7
2. 0
3. -3
4. $\frac{2}{3}$
5. 5.1
6. $-\sqrt{25}$
7. π
8. $\sqrt{64}$
9. $-0.\overline{333}\ldots$
10. -4
11. $\sqrt{2}$
12. $1.414213...$
13. $-\frac{7}{4}$
14. 8.0
15. $0.\overline{121212}...$
16. -2
17. $-\sqrt{9}$
18. $\frac{1}{8}$
19. 15
20. $\sqrt{50}$

1.3 Converting Radicals

Understanding Conversions

Radicals can be written in different forms for easier computation and comparison. Conversions involve switching between entire radicals and mixed radicals:

- **Entire Radical**: The expression under the root sign contains all factors (e.g., $\sqrt{50}$).
- **Mixed Radical**: A simplified form where part of the expression is moved outside the root (e.g., $5\sqrt{2}$).
- Conversion simplifies expressions or prepares them for addition, subtraction, or comparison.

Key Rules for Conversion

- To convert from **entire to mixed**, factor the radicand into a perfect square (or cube, etc.) and a remaining factor.
- To convert from **mixed to entire**, multiply the coefficient back into the radicand.

Simplify $\sqrt{72}$.

Simplify $\sqrt[3]{96}$.

Rewrite $3\sqrt{5}$ as an entire radical.

Rewrite $2\sqrt[4]{10}$ as an entire radical.

Practice Problems

Convert the following radicals. Simplify where possible:

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Convert Entire Radicals to Mixed Radicals

1. $\sqrt{50}$
2. $\sqrt{98}$
3. $\sqrt{200}$
4. $\sqrt{32}$
5. $\sqrt{18}$
6. $\sqrt{45}$
7. $\sqrt{63}$
8. $\sqrt{24}$
9. $\sqrt{125}$
10. $\sqrt{48}$

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Convert Mixed Radicals to Entire Radicals

1. $4\sqrt{3}$
2. $2\sqrt{6}$
3. $5\sqrt{2}$
4. $3\sqrt{7}$
5. $6\sqrt{5}$
6. $2\sqrt{10}$

7. $7\sqrt{2}$
8. $4\sqrt{2}$
9. $5\sqrt{6}$
10. $3\sqrt{8}$

1.4 Simplifying Algebraic Radicals

Understanding Simplification of Algebraic Radicals

Simplifying algebraic radicals involves reducing expressions under the radical sign to their simplest form, following the properties of exponents and radicals. Special attention is required when variables are involved, particularly for even-index roots, as they introduce the need for absolute values to ensure the result remains mathematically accurate.

Simplifying Radicals with Variables

Example: Simplify $\sqrt{x^6}$ and $\sqrt[3]{x^6y^{10}}$

Example: Simplify $\sqrt{x^6}$

Simplify $\sqrt[4]{x^8}$.

Simplify $\sqrt[3]{x^9}$.

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Practice Problems: Simplifying Algebraic Radicals Simplify the following:

1. $\sqrt{x^4}$
2. $\sqrt[3]{x^6}$
3. $\sqrt{x^{10}}$
4. $\sqrt[4]{x^{12}}$
5. $\sqrt[5]{x^{15}}$
6. $\sqrt{x^8}$
7. $\sqrt[6]{x^{18}}$
8. $\sqrt[3]{x^9}$
9. $\sqrt{x^7}$
10. $\sqrt[4]{x^{16}}$
11. $\sqrt[4]{16x^8y^{12}z^4}$
12. $\sqrt[5]{243x^{15}y^{10}}$
13. $\sqrt{72x^6y^3}$
14. $\sqrt[6]{64x^{12}y^{18}}$
15. $\sqrt[4]{81x^4y^8z^{16}}$
16. $\sqrt[3]{125x^9y^6z^3}$
17. $\sqrt{45x^{10}y^7}$
18. $\sqrt[5]{1024x^{25}y^{15}z^{10}}$

1.5 Adding and Subtracting Radicals

Understanding Addition and Subtraction of Radicals

Adding or subtracting radicals is similar to combining like terms in algebra. You can only combine radicals with the same radicand (the value under the square root or other root).

- **Like Radicals:** Radicals with the same root index and radicand. For example, $3\sqrt{5}$ and $2\sqrt{5}$ are like radicals.
- **Unlike Radicals:** Radicals with different radicands cannot be directly added or subtracted. For example, $\sqrt{2}$ and $\sqrt{3}$ are unlike radicals.
- **Simplification:** Simplify each radical before combining. For instance, $2\sqrt{50} = 2 \cdot 5\sqrt{2} = 10\sqrt{2}$.

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Steps for Adding and Subtracting Radicals

1. Simplify each radical expression, if possible.
2. Identify like radicals (same radicand and root index).
3. Combine the coefficients of like radicals.

Simplify $3\sqrt{7} + 2\sqrt{7}$.

Simplify $5\sqrt{2} - 3\sqrt{2}$.

Simplify $2\sqrt{18}+\sqrt{8}$.

Simplify $4\sqrt[3]{2}+3\sqrt[3]{2}$.

Simplify $5\sqrt[4]{3}-2\sqrt[4]{3}$.

Simplify $2\sqrt{5}+3\sqrt[3]{4}-\sqrt{5}$.

Simplify $2\sqrt[3]{16} + 4\sqrt[3]{2}$.

Practice Problems

Simplify each of the following by adding or subtracting radicals:

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Square Roots

1. $3\sqrt{12} + 5\sqrt{3}$
2. $2\sqrt{50} - \sqrt{8}$
3. $4\sqrt{18} + 3\sqrt{8} - \sqrt{72}$
4. $6\sqrt{45} - 3\sqrt{20}$
5. $\sqrt{98} + 2\sqrt{24} - \sqrt{50}$

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Cube Roots

6. $4\sqrt[3]{16} - 2\sqrt[3]{2}$
7. $3\sqrt[3]{27} + 5\sqrt[3]{9}$
8. $2\sqrt[3]{250} - 3\sqrt[3]{125} + \sqrt[3]{500}$
9. $5\sqrt[3]{64} - 2\sqrt[3]{32} + \sqrt[3]{8}$
10. $\sqrt[3]{216} + 4\sqrt[3]{54} - 2\sqrt[3]{27}$

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Fourth Roots

11. $3\sqrt[4]{16} + 2\sqrt[4]{81}$
12. $4\sqrt[4]{32} - \sqrt[4]{128}$
13. $2\sqrt[4]{256} + 5\sqrt[4]{16} - \sqrt[4]{64}$
14. $6\sqrt[4]{81} + 3\sqrt[4]{27} - 2\sqrt[4]{243}$
15. $\sqrt[4]{16} + 2\sqrt[4]{256} - \sqrt[4]{81}$

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Mixed Radicals

16. $3\sqrt{5} + 2\sqrt[3]{4} - \sqrt{5}$
17. $2\sqrt[3]{9} + 4\sqrt[4]{16} - \sqrt[3]{27}$
18. $5\sqrt[4]{81} - 3\sqrt{3} + 2\sqrt[3]{8}$
19. $6\sqrt[3]{64} + 2\sqrt{25} - 3\sqrt[3]{27}$
20. $\sqrt[4]{16} + 3\sqrt[3]{8} - 2\sqrt{4} + \sqrt[3]{64}$

1.6 Exponent Laws

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Understanding Exponent Laws Exponent laws help simplify and solve expressions involving powers. Each law provides a systematic way to handle operations on exponents.

Product Rule

Rule: $x^a \cdot x^b = x^{a+b}$ When multiplying terms with the same base, add their exponents.

Simplify $x^3 \cdot x^5$.

Simplify $2a^4 \cdot 3a^2$.

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Practice Problems Simplify the following:

1. $x^2 \cdot x^4$
2. $a^5 \cdot a^3$
3. $2x^3 \cdot 3x^2$
4. $y \cdot y^6$
5. $m^4 \cdot m^7$
6. $(3x^2) \cdot (5x^3)$
7. $p^6 \cdot p^2$
8. $z^3 \cdot z$
9. $(2a) \cdot (4a^5)$
10. $(7k^3) \cdot (k^4)$

Quotient Rule

Rule: $\frac{x^a}{x^b} = x^{a-b}$ When dividing terms with the same base, subtract their exponents.

Simplify $\frac{x^6}{x^3}$.

Simplify $\frac{12a^7}{4a^5}$.

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Practice Problems Simplify the following:

1. $\frac{x^5}{x^2}$
2. $\frac{a^7}{a^4}$
3. $\frac{10y^6}{2y^3}$
4. $\frac{p^8}{p}$
5. $\frac{m^9}{m^6}$
6. $\frac{4x^4}{2x^2}$
7. $\frac{z^5}{z^3}$
8. $\frac{k^3}{k^3}$
9. $\frac{6a^6}{3a^4}$
10. $\frac{5b^4}{b^2}$

Power Rule

Rule: $(x^a)^b = x^{a \cdot b}$ When raising a power to another power, multiply the exponents.

Simplify $(x^3)^2$.

Simplify $(2a^4)^3$.

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Practice Problems Simplify the following:

1. $(x^2)^3$
2. $(a^4)^2$
3. $(y^5)^3$
4. $(p^6)^2$
5. $(3x^3)^2$
6. $(2y^4)^3$
7. $(z^2)^5$
8. $(k^7)^1$
9. $(5a^3)^2$
10. $(4b^2)^3$

Zero Exponent Rule

Rule: $x^0 = 1$ Any base raised to the power of zero equals 1, provided the base is nonzero.

Simplify x^0.

Simplify $(3a^4)^0$.

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Practice Problems Simplify the following:

1. 5^0
2. x^0
3. $(2a^3)^0$
4. z^0
5. $(3x^4)^0$
6. $(5p^2)^0$
7. k^0
8. $(7y^5)^0$
9. m^0
10. $(4b^3)^0$

Negative Exponent Rule

Rule: $x^{-a} = \frac{1}{x^a}$ A negative exponent indicates the reciprocal of the base raised to the positive exponent.

Simplify x^{-3}.

Simplify $(2a^{-4})$.

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Practice Problems Simplify the following:

1. x^{-2}
2. a^{-3}
3. $(2y^{-4})$
4. p^{-1}
5. m^{-5}
6. $(3x^{-3})$
7. z^{-4}
8. $(5k^{-2})$
9. b^{-6}
10. $(4a^{-3})$

Fractional Powers

Fractional powers are another way of expressing roots. They follow specific rules that simplify operations involving powers and roots.

Fractional Powers:

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

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Examples and Applications

Simplify $27^{\frac{1}{3}}$.

Simplify $16^{\frac{3}{4}}$.

Simplify $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$.

Simplify $\frac{x^{\frac{5}{4}}}{x^{\frac{3}{4}}}$.

Simplify $(x^{\frac{2}{3}})^3$.

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Practice Problems Simplify the following using fractional power rules:

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Basic Operations with Fractional Exponents

1. $64^{\frac{1}{2}}$
2. $8^{\frac{1}{3}}$
3. $81^{\frac{3}{4}}$
4. $125^{\frac{2}{3}}$
5. $49^{\frac{1}{2}}$

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Product and Quotient Rules

6. $x^{\frac{1}{4}} \cdot x^{\frac{3}{4}}$
7. $\frac{y^{\frac{5}{6}}}{y^{\frac{2}{6}}}$
8. $x^{\frac{2}{3}} \cdot x^{\frac{1}{3}}$
9. $\frac{a^{\frac{7}{5}}}{a^{\frac{4}{5}}}$
10. $m^{\frac{3}{4}} \cdot m^{\frac{1}{2}}$

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Power Rule

11. $(x^{\frac{1}{2}})^2$
12. $(y^{\frac{3}{4}})^4$
13. $(a^{\frac{2}{5}})^5$
14. $(z^{\frac{4}{3}})^3$
15. $(x^{\frac{2}{3}})^{\frac{3}{2}}$

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Mixed Operations

16. $(x^{\frac{3}{4}} \cdot x^{\frac{1}{4}}) \div x^{\frac{1}{2}}$
17. $\frac{(a^{\frac{5}{3}})^3}{a^5}$
18. $(y^{\frac{2}{5}} \cdot y^{\frac{3}{5}})^2$
19. $\frac{z^{\frac{7}{4}}}{(z^{\frac{3}{2}})^2}$
20. $(x^{\frac{2}{3}} \cdot x^{\frac{4}{3}})^{\frac{1}{2}}$

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More difficult problems

1. $\frac{2^{-3} \cdot 4^{\frac{1}{2}}}{4^{-1} \cdot 2^{\frac{3}{2}}}$
2. $\left(\frac{5^{-2} \cdot 3^{\frac{3}{2}}}{3^{-1} \cdot 2^{\frac{1}{2}}}\right)^2 \cdot \frac{2^{\frac{3}{2}} \cdot 5^3}{5^{-3} \cdot 2^{\frac{1}{2}}}$
3. $\frac{3^{-1} \cdot 6^{\frac{2}{3}}}{2^{\frac{1}{3}} \cdot 3^{\frac{1}{2}}} \div \left(\frac{4^{-1} \cdot 2^{\frac{1}{2}}}{2^{\frac{3}{2}} \cdot 4^{-\frac{3}{2}}}\right)$
4. $\left(\frac{7^{\frac{1}{2}} \cdot 3^{-3}}{3^{\frac{2}{3}} \cdot 7^{-1}}\right)^3 \cdot \frac{5^{-2} \cdot 4^{\frac{3}{2}}}{4^{-1} \cdot 5^{\frac{1}{2}}}$
5. $\frac{x^{-1} y^{\frac{1}{3}}}{x^{\frac{2}{3}} y^{-1}} \cdot \left(\frac{y^{\frac{3}{2}} x^{-\frac{1}{2}}}{x^{\frac{1}{3}} y^{-\frac{2}{3}}}\right)^2$
6. $\frac{4^{-\frac{1}{2}} \cdot 2^{-3}}{2^{-1} \cdot 4^{\frac{1}{3}}} \div \left(\frac{8^{-1} \cdot 4^{\frac{2}{3}}}{4^{-2} \cdot 8^{\frac{1}{3}}}\right)$
7. $\left(\frac{a^{\frac{1}{2}} b^{-3}}{b^2 a^{-1}} \div \frac{a^{-\frac{1}{3}} b^{\frac{1}{3}}}{b^{\frac{3}{2}} a^{\frac{2}{3}}}\right)^2 \cdot \frac{b^{-\frac{1}{2}} a^2}{a^{-\frac{3}{2}} b^{\frac{1}{3}}}$
8. $\frac{m^{-\frac{2}{3}} n^{\frac{1}{2}}}{n^{-1} m^{\frac{3}{2}}} \cdot \left(\frac{n^{\frac{3}{4}} m^{-\frac{1}{4}}}{m^{\frac{2}{3}} n^{-\frac{3}{2}}}\right)$
9. $\left(\frac{5^{-\frac{3}{2}} x^{\frac{1}{3}} y^{-2}}{x^{-\frac{1}{2}} y^{\frac{1}{3}}} \cdot \frac{y^{-\frac{1}{2}} x^2}{x^{\frac{1}{2}} y^{-\frac{3}{2}}}\right) \div \frac{x^{-\frac{3}{2}} y^{\frac{2}{3}}}{y^{-1} x^{\frac{1}{2}}}$
10. $\frac{3^{\frac{1}{2}} \cdot 6^{-1}}{2^{-\frac{3}{2}} \cdot 3^{-1}} \cdot \left(\frac{9^{\frac{1}{3}} \cdot 2^{-\frac{1}{2}}}{3^{-2} \cdot 4^{\frac{2}{3}}}\right)$

1.7 Multiplying and Dividing Radicals

Understanding Multiplication and Division of Radicals

Radicals can be multiplied and divided using their properties, provided the indices (root types) are the same. These operations are essential for simplifying expressions and solving equations.

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Key Rules

- **Multiplication Rule:** $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$
- **Division Rule:** $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ $(b \neq 0)$.
- Simplify radicals as much as possible after performing the operation.
- Rationalize denominators where applicable (make the denominator a rational number by eliminating the radical).

Multiplying Radicals

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Steps for Multiplying Radicals

1. Multiply the radicands together if they have the same root index.
2. Simplify the resulting radical.

Simplify $\sqrt{5} \cdot \sqrt{20}$.

Simplify $\sqrt[3]{4} \cdot \sqrt[3]{16}$.

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Practice Problems: Multiplying Radicals Simplify the following:

1. $\sqrt{2} \cdot \sqrt{8}$
2. $\sqrt{3} \cdot \sqrt{12}$
3. $\sqrt{5} \cdot \sqrt{25}$
4. $\sqrt[3]{7} \cdot \sqrt[3]{14}$
5. $\sqrt{6} \cdot \sqrt{24}$
6. $\sqrt[4]{9} \cdot \sqrt[4]{27}$
7. $\sqrt{15} \cdot \sqrt{5}$
8. $3\sqrt{2} \cdot 4\sqrt{3}$
9. $\sqrt{10} \cdot \sqrt{50}$
10. $\sqrt[3]{3} \cdot \sqrt[3]{81}$

Dividing Radicals

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Steps for Dividing Radicals

1. Divide the radicands if the radicals have the same root index.
2. Simplify the resulting radical.
3. Rationalize the denominator if it contains a radical.

Simplify $\frac{\sqrt{48}}{\sqrt{3}}$.

Simplify $\frac{5}{\sqrt{2}}$.

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Practice Problems Simplify the following:

1. $\frac{\sqrt{27}}{\sqrt{3}}$
2. $\frac{\sqrt{50}}{\sqrt{2}}$
3. $\frac{\sqrt[3]{32}}{\sqrt[3]{4}}$
4. $\frac{\sqrt{18}}{\sqrt{6}}$
5. $\frac{5}{\sqrt{3}}$
6. $\frac{\sqrt{72}}{\sqrt{8}}$
7. $\frac{\sqrt[4]{16}}{\sqrt[4]{2}}$
8. $\frac{\sqrt[3]{27}}{\sqrt[3]{9}}$
9. $\frac{7}{\sqrt{5}}$
10. $\frac{\sqrt{45}}{\sqrt{15}}$

Mixed Operations with Radicals

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Steps for Mixed Operations

1. Follow the order of operations: parentheses, exponents (including radicals), multiplication/division, addition/subtraction.
2. Simplify each operation involving radicals step by step.

Simplify $3\sqrt{2} \cdot \frac{\sqrt{18}}{\sqrt{3}} - \sqrt{8}$.

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Practice Problems Simplify the following:

1. $\sqrt{6} \cdot \sqrt{15} - \sqrt{10}$
2. $3\sqrt[3]{8} \cdot \frac{\sqrt[3]{64}}{\sqrt[3]{4}}$
3. $2\sqrt{12} - \frac{\sqrt{48}}{\sqrt{4}}$
4. $5\sqrt[4]{81} \cdot \sqrt[4]{3} - \sqrt[4]{27}$
5. $\sqrt{18} \cdot \sqrt{2} + \frac{\sqrt{50}}{\sqrt{2}}$
6. $\frac{4}{\sqrt{3}} + \sqrt{27} \cdot \sqrt{3}$
7. $\sqrt[3]{9} \cdot \frac{\sqrt[3]{81}}{\sqrt[3]{3}} - \sqrt[3]{27}$
8. $\sqrt{45} \cdot \sqrt{5} - \frac{\sqrt{20}}{\sqrt{5}}$
9. $6\sqrt[4]{16} \cdot \sqrt[4]{8} + \sqrt[4]{32}$
10. $\frac{2\sqrt{10}}{\sqrt{5}} - \sqrt{4}$