Systems of Linear Equations

Contents

1 Systems of Linear Equations

In this unit, students will develop a comprehensive understanding of systems of linear equations, learning various methods to solve them. Each section builds foundational skills and offers progressively complex applications, from two-variable systems to Gaussian elimination for three-variable systems.

1.1 Solving Linear Equations

Steps to solve

Follow these steps to solve multistep linear equations that involve brackets (distributive property) and variables on both sides of the equation.

1. **Simplify Each Side Separately (if necessary).**
2. **Move All Variable Terms to One Side of the Equation.**
3. **Move Constant Terms to the Opposite Side.**
4. **Isolate the Variable.**
5. **Check the Solution (Optional).**

Example: Solve $3(x + 2) = 2x + 5$

Example: Solve $4(3x + 5) - 2(x - 7) = 5(2x + 4) - 3$

Practice Problems

Solve each of the following equations using the steps provided.

1. $4(x-3)=2x+10$
2. $5(2x+1)=3x+15$
3. $6+3(x+2)=2x+8$
4. $2(x-5)+4=3x-2$
5. $3x+4(2x-1)=5(x+3)$
6. $3(x+4)=x+18$
7. $2(3x-5)=4x+6$
8. $7x-4(2-x)=5x+10$
9. $6(x+3)=3(2x+4)+6$
10. $8+2(x-3)=5x-7$
11. $4x+3(x-2)=2(3x+1)$
12. $9(x-1)+3=3(3x-4)+10$
13. $5(2x-3)+6=3(3x+1)-8$
14. $2(x+7)+4x=3(2x+5)-1$
15. $6(3x-2)-5x=2(4x+6)-8$
16. $4(2x+3)-3(x+5)=3x+2(2x+4)$
17. $7x+4(3x-5)-2=3(5x+1)$
18. $3(4x-1)-2(2x+5)=x+10$
19. $5(3x+2)-4(2x-3)=6x+4$
20. $10x-3(2x+5)+4=2(5x-3)+6$

1.2 What is a System of Linear Equations?

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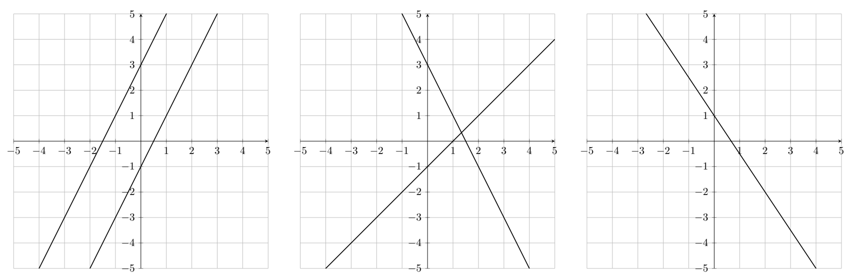
Objective

- Define a system of linear equations.
- Understand different types of solutions to a system.

Definition of system of equations

A **system of linear equations** is a set of two or more linear equations with the same variables. The solution to a system of equations is the point where the graphs of the equations intersect.

Types of Solutions



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Types of Solutions A system of two linear equations can have:

- **One solution**: The lines intersect at one point.
- **No solution**: The lines are parallel and never intersect.
- **Infinitely many solutions**: The lines coincide, meaning they lie on top of each other.

1.3 Solving by Graphing

In a system of linear equations, each equation represents a line on a coordinate plane. The solution to the system is a set of values for x and y that satisfy both equations simultaneously.

The intersection point of the two lines represents the only set of values for x and y that makes both equations true at the same time. In other words, the coordinates of the intersection point satisfy both equations in the system.

Steps for Solving Systems of Equations by Graphing

Solving a system of linear equations by graphing involves finding the point where the graphs of the equations intersect. Here's a step-by-step guide:

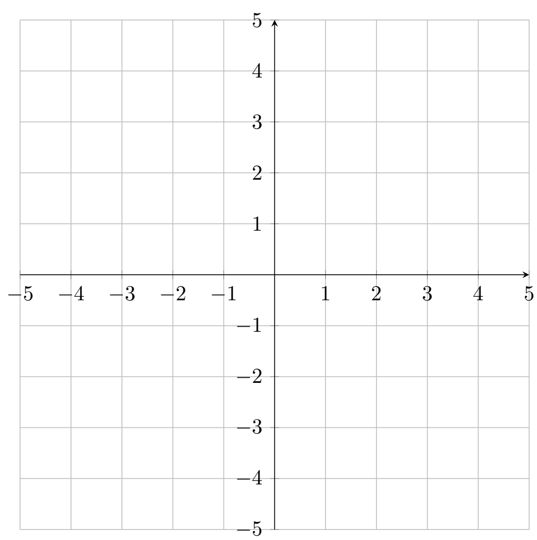
1. **Graph both Equations**
2. **Find the point where they meet**
3. **Verify the solution (optional).** Substitute the coordinates of the intersection point into the original equations to ensure they satisfy both equations.

Example: Solve by Graphing

Solve the system by graphing:

$$y = x + 2$$

$$y = -x + 4$$

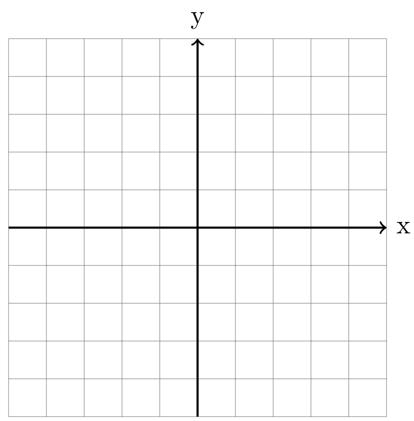


Practice Problems

1. Graph and solve:

$$y = -2x + 3$$

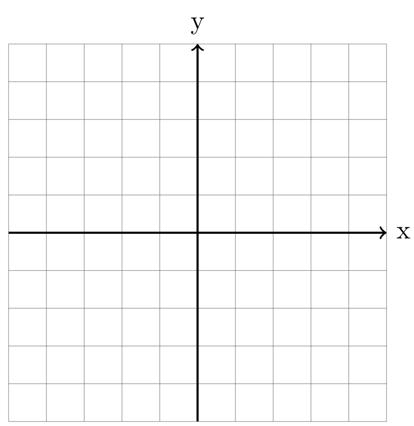
$$y = x - 1$$



2. Graph and solve:

$$y = x$$

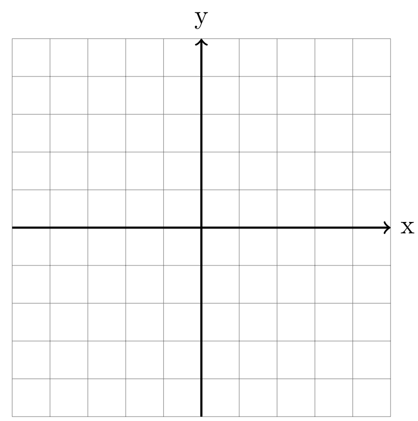
$$y = -x + 2$$



3. Graph and solve:

$$4y = 3x - 2$$

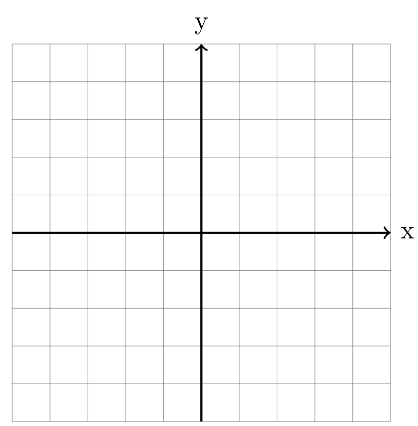
$$y = -x + 5$$



4. Graph and solve:

$$y = 3x - 2$$

$$y = -x + 5$$



1.4 Solving by Substitution

The substitution method works by isolating one variable in one of the equations and then substituting that expression into the other equation. This substitution allows us to reduce the system to a single equation with one variable, making it simpler to solve.

By substituting, we are essentially expressing one variable in terms of the other, allowing us to solve for one variable at a time. Once we find the value of the first variable, we substitute it back into one of the original equations to find the second variable. This ensures that the solution satisfies both equations simultaneously, giving us the correct point of intersection.

Steps for Solving Systems of Linear Equations by Substitution

To solve a system of linear equations using substitution, follow these steps:

1. **Isolate one variable in one of the equations.**
2. **Substitute the expression into the other equation.**
3. **Solve for the remaining variable.**
4. **Substitute back to find the value of the other variable.**
5. **Check the solution (optional).**

Example: Solve by Substitution

Solve the system:

$$y = 2x + 1$$

$$3x + y = 9$$

Practice Problems

1. Solve by substitution:

$$y = 2x + 4$$
$$3x - y = 5$$

2. Solve by substitution:

$$y = -2x + 3$$
$$x + y = 7$$

3. Solve by substitution:

$$y = 4x - 1$$
$$2x + y = 9$$

4. Solve by substitution:

$$y = -\frac{1}{2}x + 6$$
$$x + 2y = 10$$

5. Solve by substitution:

$$y = 7x + 2$$
$$3x - y = -4$$

6. Solve by substitution:

$$y = -3x + 5$$
$$4x + y = 7$$

7. Solve by substitution:

$$y = 6x - 3$$
$$2x + 3y = 12$$

8. Solve by substitution:

$$y = \frac{3}{4}x + 1$$
$$x - y = 2$$

9. Solve by substitution:

$$y = -4x + 8$$
$$5x + y = 11$$

10. Solve by substitution:

$$y = 2x - 7$$
$$x + y = 3$$

11. Solve by substitution:

$$2y + 3x = 12$$
$$x - y = 4$$

12. Solve by substitution:

$$4x - y = 7$$
$$3x + 2y = 13$$

13. Solve by substitution:

$$5x + 2y = 9$$
$$x + 3y = 7$$

14. Solve by substitution:

$$3y - x = 5$$
$$4x + y = 14$$

1.5 Solving by Elimination

The elimination method works by adding or subtracting the equations in the system to eliminate one of the variables. By aligning the coefficients of one variable (either by multiplication or direct setup), we can add or subtract the equations to cancel out that variable, reducing the system to a single equation with one variable.

Once we have a single equation, we can solve for the remaining variable. Then, we substitute this value back into one of the original equations to solve for the other variable. This process gives us a solution that satisfies both equations, representing the intersection point of the two lines. The elimination method is particularly useful when the equations are already aligned for elimination or when manipulating coefficients is straightforward.

Steps for Solving Systems of Linear Equations by Elimination

To solve a system of linear equations using Elimination, follow these steps:

1. **Arrange both equations in the form** $Ax + By = C$
2. **Multiply one or both equations (if necessary)**
3. **Add the equations**
4. **Solve for the remaining variable.**
5. **Substitute back the value obtained**
6. **Check the solution (optional)**

Example: Solve by Elimination

Solve the system:

$$2x + 3y = 6$$
$$-2x + y = -4$$

Example: Solve by Elimination

Solve the system:

$$5x + 7y = 12$$

$$3x + 5y = 8$$

Example: Solve by Elimination

Solve the system:

$$2x + 11y = 61$$

$$-7x + 5y = 4$$

Practice Problems

1. Solve by elimination:
$$3x + 2y = 12$$
$$5x - 2y = 10$$

2. Solve by elimination:
$$4x + 3y = 18$$
$$2x - 3y = 6$$

3. Solve by elimination:
$$6x + y = 25$$
$$3x - y = 7$$

4. Solve by elimination:
$$7x - 2y = 15$$
$$5x + 2y = 3$$

5. Solve by elimination:
$$8x + 5y = 20$$
$$6x - 5y = 10$$

6. Solve by elimination:
$$2x - 3y = 9$$
$$4x + 3y = 3$$

7. Solve by elimination:
$$3x + 4y = 16$$
$$5x - 4y = 2$$

8. Solve by elimination:
$$7x + 3y = 19$$
$$2x - 3y = 4$$

9. Solve by elimination:
$$5x + y = 9$$
$$3x - y = 1$$

10. Solve by elimination:
$$4x + 6y = 24$$
$$2x - 6y = 12$$

11. Solve by elimination:
$$6x + 7y = 29$$
$$4x - 7y = 7$$

12. Solve by elimination:
$$9x + 2y = 17$$
$$3x - 2y = 5$$

13. Solve by elimination:
$$5x + 3y = 18$$
$$7x - 3y = 10$$

14. Solve by elimination:
$$4x - 5y = 6$$
$$6x + 5y = 30$$

15. Solve by elimination:
$$3x + 6y = 21$$
$$5x - 2y = 9$$

1.6 Solving by Cramer's Rule EXTENDING - NON REQUIRED

Cramer's Rule is a method for solving systems of linear equations using determinants. It is particularly useful for small systems, such as a system of two equations in two variables, as it provides a straightforward formula to find the solution.

This method works only when the coefficient matrix of the system is square (same number of equations as variables) and has a nonzero determinant. If the determinant of the coefficient matrix is zero, the system either has no solution or infinitely many solutions, and Cramer's Rule cannot be applied.

Cramer's Rule calculates each variable in the system as a fraction where the numerator is a determinant that replaces one column of the coefficient matrix with the constants, and the denominator is the determinant of the coefficient matrix.

Steps for Solving Systems of Linear Equations by Cramer's Rule

To solve a system of two linear equations using Cramer's Rule, follow these steps:

1. **Write the system in standard form:**

$$ax + by = e \quad \text{and} \quad cx + dy = f$$

   where a, b, c, d, e, and f are constants.

2. **Form the coefficient matrix:**

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

3. **Calculate the determinant of the coefficient matrix:**

$$\det(\mathbf{A}) = ad - bc$$

   If $\det(\mathbf{A}) = 0$, Cramer's Rule cannot be applied.

4. **Find the determinants for the numerators:** Replace the first column of $\mathbf{A}$ with the constants to find the determinant for x:

$$\mathbf{A}_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \det(\mathbf{A}_x) = ed - bf$$

   Replace the second column of $\mathbf{A}$ with the constants to find the determinant for y:

$$\mathbf{A}_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}, \quad \det(\mathbf{A}_y) = af - ec$$

5. **Solve for each variable:**

$$x = \frac{\det(\mathbf{A}_x)}{\det(\mathbf{A})}, \quad y = \frac{\det(\mathbf{A}_y)}{\det(\mathbf{A})}$$

6. **Check the solution (optional).**

Example: Solve by Cramer's Rule

Solve the system:

$$2x + 3y = 8$$
$$x - 4y = -6$$

Example: Solve by Cramer's Rule

Solve the system:

$$8x + 4y = -4$$
$$6x + 2y = -2$$

Practice Problems

1. Solve using Cramer's Rule:
$$3x + 2y = 7$$
$$5x - y = 8$$

2. Solve using Cramer's Rule:
$$4x - 3y = 10$$
$$2x + y = 1$$

3. Solve using Cramer's Rule:
$$x + 2y = 5$$
$$3x - y = 4$$

4. Solve using Cramer's Rule:
$$6x - 4y = 12$$
$$3x + 2y = -1$$

5. Solve using Cramer's Rule:
$$2x + 3y = 6$$
$$5x - y = 11$$

6. Solve using Cramer's Rule:
$$7x + 5y = 2$$
$$4x - 3y = -3$$

7. Solve using Cramer's Rule:
$$x - y = 1$$
$$2x + 3y = 5$$

8. Solve using Cramer's Rule:
$$2x + y = 4$$
$$3x - 2y = -6$$

9. Solve using Cramer's Rule:
$$x + 4y = 7$$
$$5x - y = 9$$

10. Solve using Cramer's Rule:
$$3x + 2y = 6$$
$$4x - y = -1$$

11. Solve using Cramer's Rule:
$$2x - y = 3$$
$$x + y = 4$$

12. Solve using Cramer's Rule:
$$5x + y = 8$$
$$x - 2y = 3$$

13. Solve using Cramer's Rule:
$$4x + 3y = 10$$
$$3x - y = 1$$

14. Solve using Cramer's Rule:
$$2x + y = -5$$
$$x - 3y = 7$$

1.7 Application Problems

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Type of Application Problems **3 main types**

1. cost and quantity
2. Motion
3. Mixing Solutions

Cost and Quantity

Example : Solving a Cost and Quantity Problem

A student buys 3 notebooks and 2 pens for \$8. A second student buys 1 notebook and 4 pens for \$6. How much does each item cost?

Example : Solving a Cost and Quantity Problem

A café customer buys 2 croissants and 3 muffins for \$14. Another customer buys 4 croissants and 2 muffins for \$16. How much does each pastry cost?

Practice Problems

1. A concert offers two types of tickets: VIP tickets and general admission tickets. VIP tickets cost \$50 each, and general admission tickets cost \$30 each. If 200 tickets are sold for a total of \$7,000, how many of each type of ticket were sold?
2. A teacher buys notebooks and pens for her class. Each notebook costs \$3, and each pen costs \$1. She spends a total of \$75 on 40 items. How many notebooks and how many pens did she buy?
3. At a fruit market, apples cost \$2 each, and bananas cost \$1 each. A customer buys a total of 35 pieces of fruit for \$60. How many apples and how many bananas did they buy?
4. A group of friends buys popcorn and drinks at a movie theater. Each popcorn costs \$5, and each drink costs \$3. They buy a total of 12 items and spend \$46. How many popcorns and how many drinks did they buy?
5. A shopper buys oranges and grapefruits. Each orange costs \$0.50, and each grapefruit costs \$1. She buys 30 pieces of fruit and spends \$22.50. How many oranges and how many grapefruits did she buy?
6. A school sells t-shirts and hats as a fundraiser. Each t-shirt costs \$10, and each hat costs \$5. They sell a total of 100 items and earn \$750. How many t-shirts and how many hats did they sell?
7. A restaurant receives an order for sandwiches and salads. Each sandwich costs \$8, and each salad costs \$6. The total order is for 20 items, costing \$140. How many sandwiches and how many salads were ordered?
8. A customer buys hardback and paperback books. Each hardback book costs \$15, and each paperback book costs \$10. The customer buys a total of 8 books and spends \$100. How many hardback and how many paperback books did they buy?
9. A toy store sells action figures and puzzles. Each action figure costs \$12, and each puzzle costs \$8. A customer buys 15 toys for a total of \$160. How many action figures and how many puzzles did they buy?
10. At a cafeteria, sandwiches cost \$4 each, and drinks cost \$2 each. A group buys a total of 18 items and spends \$52. How many sandwiches and how many drinks did they buy?

Motion Application Problems

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Understanding the Relationship between Speed, Distance, and Time

In motion problems, there is a fundamental relationship between speed, distance, and time, represented by the equation:

$$\text{Distance} = \text{Speed} \times \text{Time}$$

This equation is often written in the form:

$$d = rt$$

where:

- d represents the **distance** traveled,
- r represents the **rate** (or speed) of travel, and
- t represents the **time** taken.

Example : Motion Problem with Two Moving Objects

Two trains start from towns 200 miles apart and travel towards each other. Train A travels at 60 mph, and Train B travels at 40 mph. How long will it take for the two trains to meet?

Example : Motion Problem with Headwind and Tailwind

A plane travels 600 miles with a tailwind in 2 hours. The return trip against the headwind takes 3 hours. What is the speed of the plane in still air, and what is the speed of the wind?

Example : Motion Problem with River Current

A boat travels 30 miles downstream in 2 hours with the help of a current. The return trip upstream against the current takes 3 hours. What is the speed of the boat in still water, and what is the speed of the current?

Example : Motion Problem

Alex drove a total of 150 km in 5 hours. For part of the trip, he drove at a speed of 40 km/h, and for the rest, he drove at 20 km/h. **How far did he drive at 40 km/h?**

Practice Problems

1. Two cars start from points 300 miles apart and travel towards each other. Car A travels at 50 mph, and Car B travels at 70 mph. How long will it take for the two cars to meet?
2. A plane travels 800 miles with a tailwind in 2 hours. The return trip against the headwind takes 4 hours. What is the speed of the plane in still air, and what is the speed of the wind?
3. A boat travels 45 miles downstream in 1.5 hours with the help of a current. The return trip upstream against the current takes 2 hours. What is the speed of the boat in still water, and what is the speed of the current?
4. Samantha took 6 hours to travel a total distance of 84 km. She spent part of the time rowing at a speed of 12 km/h, and for the rest, she paddled at 4 km/h. **How far did she row at 12 km/h?**
5. Two cyclists start from points 90 miles apart and travel towards each other. Cyclist A rides at 12 mph, and Cyclist B rides at 18 mph. How long will it take for the two cyclists to meet?
6. A plane travels 500 miles with a tailwind in 5 hours. The return trip against the headwind takes 7 hours. What is the speed of the plane in still air, and what is the speed of the wind?
7. A swimmer swims 2 miles downstream in 0.5 hours with the help of a current. The return trip upstream against the current takes 1 hour. What is the swimmer's speed in still water, and what is the speed of the current?
8. Marcus hiked a total distance of 120 km in 10 hours. Part of the time, he hiked uphill at 8 km/h, and the remainder he hiked downhill at 6 km/h. **How far did he hike downhill?**
9. Two runners start from points 6 miles apart and run towards each other. Runner A runs at 5 mph, and Runner B runs at 7 mph. How long will it take for the two runners to meet?
10. A boat travels 40 miles downstream in 2 hours with the help of a current. The return trip upstream against the current takes 4 hours. What is the speed of the boat in still water, and what is the speed of the current?
11. A plane travels 900 miles with a tailwind in 3 hours. The return trip against the headwind takes 6 hours. What is the speed of the plane in still air, and what is the speed of the wind?
12. Julia took 9 hours to travel a distance of 180 km. She spent part of the time driving at 30 km/h, and for the rest, she biked at 20 km/h. **How far did she drive?**

Solutions and Mixtures

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Understanding the Relationship in Solution and Mixture Problems

Example : Mixing Solutions Problem

A chemist needs to prepare 10 liters of a 20% saline solution by mixing a 30% saline solution with a 10% saline solution. How much of each solution should be used?

Example : Mixing Trail Mix Problem

A shop owner wants to create 10 pounds of a trail mix that is 30% almonds by mixing a trail mix that is 50% almonds with another that is 20% almonds. How much of each type of trail mix should be used?

Example : Mixture Bank Problem

Montey purchased \$20,000 in two types of savings bonds. One bond earned him 8% annually and the other bond earned 5% annually. The total interest earned was \$1360. How much money was invested at each rate?

Practice Problems

1. A chemist needs to prepare 10 liters of a 20% saline solution by mixing a 30% saline solution with a 10% saline solution. How much of each solution should be used?
2. A shop owner wants to create 10 pounds of a trail mix that is 30% almonds by mixing a trail mix that is 50% almonds with another that is 20% almonds. How much of each type of trail mix should be used?
3. Montey purchased \$20,000 in two types of savings bonds. One bond earned him 8% annually, and the other bond earned 5% annually. The total interest earned was \$1360. How much money was invested at each rate?
4. A baker is preparing 15 kg of dough that is 50% sugar by mixing a dough that is 60% sugar with another that is 40% sugar. How much of each type of dough should the baker use?
5. A gardener wants to create 50 liters of a fertilizer solution that is 25% concentrate by mixing a 40% concentrate fertilizer with a 10% concentrate fertilizer. How much of each solution should be used?
6. A jeweler wants to create 100 grams of an alloy that is 70% gold by mixing a 90% gold alloy with a 50% gold alloy. How much of each alloy should be used?
7. A dairy farmer wants to create 20 liters of milk that has 3% fat by mixing whole milk (4% fat) with skim milk (0% fat). How much of each type of milk should be used?

8. A winemaker wants to produce 200 liters of a wine blend that contains 12% alcohol by mixing a wine that is 15% alcohol with another that is 9% alcohol. How much of each wine should be used?

9. A chocolatier needs to make 5 kg of a chocolate mix that is 30% cocoa by mixing a chocolate that is 50% cocoa with another that is 20% cocoa. How much of each type of chocolate should be used?

10. A chemist needs to prepare 25 liters of a 40% acid solution by mixing a 50% acid solution with a 30% acid solution. How much of each solution should be used?

11. A butcher wants to make 30 pounds of ground beef that is 80% lean by mixing ground beef that is 90% lean with another that is 70% lean. How much of each type of ground beef should be used?

12. A pharmacist needs to prepare 1 liter of a 5% saline solution by mixing a 10% saline solution with a 2% saline solution. How much of each solution should be used?

13. A coffee shop wants to blend 40 pounds of coffee that is 30% premium beans by mixing coffee that is 40% premium beans with coffee that is 20% premium beans. How much of each type should be used?

14. A fruit vendor wants to create 50 pounds of a mix that is 60% apples by mixing one batch that is 80% apples with another that is 40% apples. How much of each type should be used?

15. A perfumer wants to make 10 liters of a perfume that is 15% essential oil by mixing a perfume that is 20% essential oil with another that is 10% essential oil. How much of each perfume should be used?

16. A painter needs to create 5 gallons of a paint that is 25% blue by mixing a paint that is 40% blue with another that is 10% blue. How much of each paint should be used?

17. A veterinarian wants to prepare 2 liters of a medicine solution that is 30% active ingredient by mixing a solution that is 50% active ingredient with another that is 10% active ingredient. How much of each solution should be used?

18. A blacksmith needs to create 300 grams of a metal alloy that is 60% steel by mixing a 70% steel alloy with another that is 50% steel. How much of each alloy should be used?

19. A baker needs to make 10 kg of flour mix that is 30% whole wheat by mixing flour that is 50% whole wheat with flour that is 10% whole wheat. How much of each type of flour should be used?

20. A confectioner needs to make 8 kg of a candy mix that is 25% sugar by mixing a candy that is 35% sugar with another that is 15% sugar. How much of each candy should be used?