1. **Solution:**

<u>Claim:</u> Let $P(k)$ denote the statement 'for $2^k \leq n < 2^{k+1}$, we have $T(n) = 2^k$'. Then, $\forall\ k \in \mathbb{N} \cup \{0\}$, $P(k)$ is true.

<u>Proof:</u> Proof: We perform induction on k.
Base case: $k = 0$. In this case, $1 \leq n < 2$, so $n = 1$, and since $T(1) = 1 = 2^0 = 2^k$, we are done.
Inductive hypothesis: $P(i)$ is true for some $i \in \mathbb{N} \cup \{0\}$.
Inductive step: Suppose $2^{i+1} \leq n < 2^{i+2}$.

1. n is even. In this case, $T(n) = 2T(n/2)$. Since $2^i \leq n/2 < 2^{i+1}$, invoking the inductive hypothesis, we have $T(n/2) = 2^i$, so $T(n) = 2^{i+1}$.
2. n is odd. Here $n > 2^{i+1}$, so $T(n) = T(n-1) = 2T(\frac{n-1}{2})$. Note that $2^i \leq \frac{n-1}{2} < 2^{i+1}$, so $T(\frac{n-1}{2}) = 2^i$, so $T(n) = 2^{i+1}$.

Since in either case, $T(n) = i + 1$, $P(i) \implies P(i+1)$.
Thus by the principle of mathematical induction we are done. ■

2. **Solution:** I claim that the answer is false.
I'll show that for any constant $c > 0$, there exists a positive integer n_0 such that $g(n) > cf(n)$ for all $n > n_0$. Note that if $n > 2c$, we have $\frac{n}{2} \cdot 3^n > c \cdot 3^n$, and for $n > \log_{\frac{3}{2}}(20c)$, we have $\frac{n}{2} 3^n > 10c \cdot n \cdot 2^n$. Thus for $n > \max(2c, \log_{\frac{3}{2}}(20c))$, we have, by adding the above inequalities, $n \cdot 3^n > c \cdot (10n2^n + 3^n)$, as needed.