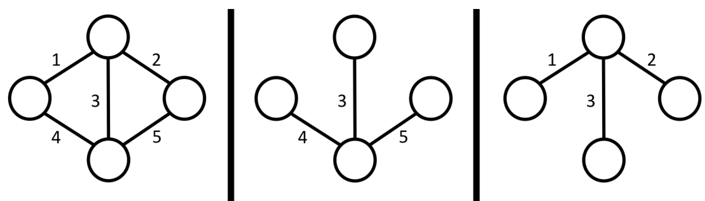
There are 2 questions for a total of 10 points.

---

1. (3 points) Let us define the "max-weight" of a spanning tree T of a strongly connected, weighted, undirected graph G to be the weight of the maximum weight edge in the spanning tree T. Also, let us define a "minmax spanning tree" of a strongly connected, weighted, undirected graph G to be a spanning tree with minimum value of max-weight.

   (For example, consider a graph below on the left. The max-weight of the spanning tree in the middle is 5. The minmax spanning tree of G is shown on the right.)



Graph G | A spanning tree with max-weight 5 | A minmax spanning tree of G

   Prove or disprove the following statement:

   Let G be any strongly connected, weighted, undirected graph. Any minimum spanning tree of G is also a minmax spanning tree of G.

Solution:

I will show that this property is true.

Suppose that this is not true, and thus there exists a MST (say T) of G which has an edge e with higher weight than the edge e' with the maximum weight in a minmax spanning tree (say T') of G. We try adding e to the tree T'. Since T' is a tree, and e is not in T' (by weight of e), the addition induces a cycle. Hence, there is a cycle C in the graph with the edge e, and all edges in the cycle have a weight $<$ weight of e.

Now we claim that this leads to a contradiction.

When we remove the edge e from T, it breaks T into two trees T_1 and T_2.

Consider the set of vertices in the cycle C in T_1 (say S_1) and the set of vertices in the cycle C in T_2 (say S_2). Note that both of these are non-empty, since the endpoints of e are not in the same tree.

Then adding any edge of the cycle between one vertex in S_1 and another vertex in S_2 joins the trees T_1 and T_2 and is a spanning tree, with a weight strictly less than that of the spanning tree T, which is a contradiction since T is, by definition, an MST of G.

Hence, we have shown that our assumption was false, and thus any minimum spanning tree is also a minmax spanning tree of G.

2. (7 points) There are n jobs that are supposed to be scheduled on a single machine. With each job i, there is an associated duration $t(i)$ that denotes the time that job i will take to execute on the machine. Each job i also has an associated weight $w(i)$ that denotes the importance of this job. Given a schedule, which is an ordering of jobs to be executed on the machine, the completion time of job i is the time at which this job is finished by the machine. Let the completion time of job i be denoted by $C(i)$. The cost V of a schedule is defined as $V = \sum_{i=1}^{n} w(i) \cdot C(i)$. You are supposed to design an algorithm for giving a schedule with minimum cost.

   (For example, consider three jobs with their duration and weight given in the table below. If the schedule is $(1, 2, 3)$, then $C(1) = 2, C(2) = 6$, and $C(3) = 9$. So, the cost of this schedule is $2 \cdot 3 + 1 \cdot 6 + 2 \cdot 9 = 30$.)

Job	**Duration**	**Weight**
1	2	3
2	4	1
3	3	2

Answer the following:

(a) Give the schedule for the above example that has the minimum cost.

> **Answer:** Order of job numbers: $1, 3, 2$, cost $= 25$.
> These jobs should be completed one after the other without any time gap.

(b) Consider the following greedy strategy for this problem:
Trial greedy strategy: Do the jobs in increasing order of duration (breaking ties arbitrarily).
Show that this strategy does not always outputs a schedule with minimum cost.

> **Answer:** Consider the following set of jobs:

Job	Duration	Weight
1	2	1
2	3	2
3	4	3

> If we do them in the increasing order of duration, we get a cost of 39. However, if we do them in the reverse order, we get a cost of 35, which is strictly less than 39, and thus the greedy algorithm doesn't always output a schedule with minimum cost since there is a schedule with a smaller cost.

(c) Design a greedy algorithm for this problem. Give proof of correctness using "modify-the-solution" method. Give a running time analysis of your algorithm.

> **Answer:**
>
> **Greedy strategy:** Among all remaining jobs, run the job with the least value of $d(i)/w(i)$, and recursively run the remaining jobs with the same strategy.
>
> **Proof of correctness:**
>
> Exchange lemma: Given any solution OS, we can construct a solution OS' whose first decision is given by the greedy strategy and it is at least as good as OS.
>
> Consider any solution OS. We will construct a solution OS' with the first decision matching the greedy decision.
>
> Suppose the job with least d/w is at index i in the solution OS, and let the corresponding duration and weight be d, w respectively. Let the job at the first index in OS have duration and weight d', w' respectively.
>
> Let $OS = o_1, o_2, ..., o_n$. Then $OS' = o_i, o_1, ..., o_{i-1}, o_{i+1}, ..., o_n$. This is valid since we consider all jobs in non-overlapping fashion and because of validity of OS.
>
> Let d_i = duration of job i and let w_i be the weight of job i and let $a_i = d_i/w_i$.
>
> Then $V(OS') - V(OS)$ is (upon simplification) $w_1 w_i(a_i - a_1) + w_2 w_i(a_i - a_2) + \ldots + w_{i-1} w_i(a_i - a_{i-1})$. Since $a_i \leq a_r$ for all r, we have that $V(OS) \geq V(OS')$, and our exchange lemma is proved.
>
> Now we prove by induction the fact that for any problem the greedy strategy gives the optimal solution
>
> Base case: For a problem with no jobs, we are trivially done.

Induction step: Let P be an instance of a problem with n elements. Let $GS(P)$ be the greedy solution for P, and $OS(P)$ be any solution. Define OS' as in the exchange lemma (valid since we schedule all jobs) and let OS' = first job + OS'' and $GS(P)$ = first job + GS'. Then we have

$V(GS(P))$ = cost of first element $\times$ sum of weights + $V(GS') \leq$ cost of first element $\times$ sum of weights + $V(OS'') = V(OS') \leq V(OS)$, where the first inequality comes from the induction hypothesis.

Algorithm:

```
function SOLVE(d[1..n], w[1..n])
    let a be the identity permutation on [1..n].
    sort a in non-decreasing order of d/w.
    return a
end function
```

Runtime analysis:

Sorting takes $O(n \log n)$ time where n is the number of jobs, and returning takes $O(n)$ time, so the whole algorithm takes $O(n \log n)$ time.