Midterm

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2.1 Statement

Let x^* be an optimum solution to the linear relaxation of the integer program for minimum vertex cover in $G = (V, E)$. Suppose $x_i^* \in \{0, 1/2, 1\}, 1 \leq i \leq n$ where x_i is the variable associated with $v_i \in V$. Further assume that the vertices of G are colored with 4 colors such that every pair of adjacent vertices have different colors. How will you round the solution x^* to obtain a 1.5-approximation for the minimum vertex cover. Give an algorithm and analyse its guarantee.

2.2 Solution

Since the statement mentions that we are already given such a colouring, we will assume that the input additionally contains the colouring information for all vertices in the graph.

Without loss of generality, let the colours be red, blue, green, yellow. Our algorithm will be as follows:

1. Find a colour c with the most number of vertices coloured with that colour which have $x_v^* = \frac{1}{2}$ (i.e., c is a majority colour among vertices v which have $x_v^* = \frac{1}{2}$).
2. Initialize set S with the empty set.
3. For each vertex v in V, add v to S if and only either
   - $x_v^* = 1$, or
   - $x_v^* = \frac{1}{2}$ and the colour of v is not c

   (i.e., round an x_v^* with value $\frac{1}{2}$ to 1 if and only if the colour of v is not c, else round it down to 0).
4. Return S.

For the sake of convenience, we suppose that the colour c that was picked is blue.

Claim 2.1

S is a vertex-cover.

Proof. Suppose there is an edge which has not been covered. Then there exists an edge (u, v) such that after rounding the solution from x^* to $\overline{x}$, we have $\overline{x}_u + \overline{x}_v < 1$. Since both of these are integers, this can happen iff both $\overline{x}_u, \overline{x}_v$ are 0. In the case that either of x_v^* and x_u^* is 0, the other variable must have been 1, which couldn't have been rounded down. And if either of them is 1, it isn't rounded down anyway, so the inequality is still satisfied.

Hence, the only possibility is that both of x_u^*, x_v^* are $\frac{1}{2}$, and both of $\overline{x}_u, \overline{x}_v$ are both rounded down to 0. This is only possible if both u, v were blue, which is impossible since the condition on the colouring guarantees that u, v must have had opposite colours.

This is a contradiction, and hence, S is a valid vertex cover. □

Claim 2.2

$|S| \leq \frac{3}{2} \cdot \mathbf{OPT}$

Proof. Let the set of vertices with $x_v^* = 1$ be A, the set with $x_v^* = 0$ be B, and the set with $x_v^* = \frac{1}{2}$ be C. Let D be the subset of C which consists of vertices which have colour blue.

Note that $\mathbf{OPT} \geq \mathbf{OPT}_{LP} = |A| + \frac{1}{2}|C|$, since the LP is a relaxed version of the original problem.

Now note that since blue is a majority colour among vertices in C, it must have at least $\frac{1}{4}|C|$ vertices.

Now consider the following inequalities:

$$
\begin{aligned}
\mathbf{ALG} &= |A| + |C| - |D| \\
&\leq |A| + \frac{3}{4} \cdot |C| \\
&\leq \frac{3}{2} \cdot |A| + \frac{3}{4} \cdot |C| \\
&= \frac{3}{2} \cdot \left(|A| + \frac{1}{2} \cdot |C| \right) \\
&= \frac{3}{2} \cdot \mathbf{OPT}_{LP} \\
&\leq \frac{3}{2} \cdot \mathbf{OPT}
\end{aligned}
$$

This completes the proof. $\square$

3 Problem 3

3.1 Statement

3.2 Solution

4 Problem 4

4.1 Statement

4.2 Solution

5 Problem 5

5.1 Statement

5.2 Solution