Approximation Algorithms Assignment 3

Show that the following is an integer programming formulation for the minimum spanning tree (MST) problem. Assume we are given a graph $G = (V, E), |V| = n$, with cost function $c : E \to \Re^+$. For $A \subseteq E$, we will denote by $\alpha(A)$ the number of connected components in the graph $G_A = (V, A)$.

$$\begin{array}{rccl}
\text{minimize} & \sum_e c_e x_e & & \\
\text{subject to} & & & \\
\sum_{e \in A} x_e & = & n - \alpha(A) & A \subset E \\
\sum_e x_e & = & n - 1 & \\
x_e & \in & \{0, 1\} & e \in E
\end{array}$$

The rest of this exercise develops a proof that the LP-relaxation of this integer program is exact for the MST problem.

1. First it will be convenient to change the objective function of the IP to $\max \sum_e -c_e x_e$. Obtain the LP-relaxation and dual of this modified formulation.

2. Consider the primal solution produced by Kruskal's algorithm. Let $e_1, \ldots, e_m$ be the edges sorted by increasing cost, $|E| = m$. This algorithm greedily picks a maximal acyclic subgraph from this sorted list. Obtain a suitable dual feasible solution so that all complementary slackness conditions are satisfied.

3. Show that x is a feasible solution to the above-stated primal program iff it is a feasible solution to the following LP. That is, prove that this is also an exact relaxation for the MST problem.

$$\begin{array}{rccl}
\text{minimize} & \sum_e c_e x_e & & \\
\text{subject to} & & & \\
\sum_{e \in S} x_e & \leq & |S| - 1 & S \subset V \\
\sum_{e \in E} x_e & = & n - 1 & \\
x_e & \geq & 0 & e \in E
\end{array}$$

Note that for $S \subset V, e \in S$ iff Both endpoints of e are in S.