Approximation Algorithms Lecture 13

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APTAS for bin packing.

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2.1 Linear program for vertex cover

Our template would be the following: make an integer program, do a suitable relaxation to get a linear program, then solve it to get an optimal solution of the LP, then round this fractional solution to get an integer solution, which could be a solution to the original problem too.

So for vertex cover, we have a graph $G = (V, E)$, and we want to pick a set of vertices, which covers all edges.

Let $x_v = 1$ if $v \in V$ is in the vertex cover, and 0 otherwise.

What should the constraints on these variables be to ensure that the vertices v with $x_v = 1$ form a vertex cover?

$\forall e = (u, v) \in E : x_u + x_v \geq 1$.

The objective function is $\sum_{v \in V} x_v$.

So the integer program becomes something like the following:

$$\begin{aligned} &\min \sum_{v \in V} x_v \\ &\forall e = (u, v) \in E : x_u + x_v \geq 1 \\ &\forall v \in V : x_v \in \{0, 1\} \end{aligned}$$

Note that the integer program always corresponds to a vertex cover.

Relaxing this to a linear program gives us:

$$\begin{aligned} &\min \sum_{v \in V} x_v \\ &\forall e = (u, v) \in E : x_u + x_v \geq 1 \\ &\forall v \in V : 0 \leq x_v \leq 1 \end{aligned}$$

Claim 0.1

The constraint $x_v \leq 1$ is redundant.

Proof. Replace all $x_v > 1$ by $x_v = 1$, and note that it doesn't violate any constraint, but only decreases the value of the objective function. $\square$

Hence, the linear program is equivalent to

$$\begin{aligned} &\min \sum_{v \in V} x_v \\ &\forall e = (u, v) \in E : x_u + x_v \geq 1 \\ &\forall v \in V : x_v \geq 0 \end{aligned}$$

Let's try to write the dual program of this.

Note 1

Note that the canonical forms are the following:

$$\begin{aligned} &\max c^\top x \\ &Ax \leq b \\ &x \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\min c^\top x \\ &Ax \geq b \\ &x \geq 0 \end{aligned}$$

To get the multipliers, we do the following: the objective function has coefficient of u being 1.

So the dual is:

$$\begin{aligned} &\min \sum_{e \in E} y_e \\ &\forall u \in V : \sum_{e \in \delta(u)} y_e \leq 1 \\ &\forall e \in E : y_e \geq 0 \end{aligned}$$

where $\delta(u)$ is the set of edges incident to a vertex u.

Suppose primal is a minimization problem, then dual is a maximization problem.

So the value of the dual will always be on the left of the value of the primal.

Now let's come back to the primal.

$$\begin{aligned} &\min \sum_{v \in V} x_v \\ &\forall e = (u, v) \in E : x_u + x_v \geq 1 \\ &\forall v \in V : x_v \geq 0 \end{aligned}$$

Let x^* be the optimal solution to this linear program.

Then we have $\sum_{v \in V} x_v^* \leq \mathbf{OPT}_{VC}$, since we have only relaxed the constraints. Note that x^* is not integral, i.e., it could be fractional.

(For bipartite, x^* is integral, and in general it is half-integral, and the dual corresponds to the maximum independent set).

We round x^* to an integral solution as follows: if $x{*}_v \geq 1/2$, round it to 1, else to 0.

Is $\overline{x}$ a feasible solution to the integer program? Yes, since if $\overline{x}_v = 0$ and $\overline{x}_u = 0$, then $x_v^* < \frac{1}{2}$ and $x_u^* < \frac{1}{2}$, which would violate the inequality.

Now let's look at the quality of the solution. We have the size of the vertex cover as:

$$\begin{aligned}\sum_{v \in V} \overline{x}_v &\leq \sum_{v \in V} 2x_v^* \\ &\leq 2 \cdot \mathbf{OPT}_{VC}\end{aligned}$$

This gives us a 2-approximation.

Now suppose we have weights on the vertices, i.e., a function $w : V \to \mathbb{R}^+$, and we want a minimum weight vertex cover.

We need to just change the objective function by replacing x_v by $w(v)x_v$, and the rest of the proof holds.

Let's look at what the dual program becomes. We just need to change the RHS from 1 to $w(u)$.

2.2 Linear program for set cover

Let $U = \{e_1, \ldots, e_n\}$, and $S_1, \ldots, S_m \subseteq U$.

x_i is the variable corresponding to S_i such that $x_i = 1$ if S_i is in the set cover, and 0 otherwise.

Now we need: $\forall e_j \in U : \sum_{i:e_j \in S_i} x_i \geq 1$, and $x_i \in \{0,1\} \forall i$.

Our objective function is $\sum_i x_i$.

The corresponding integer program is:

$$\begin{aligned}&\min \sum_i x_i \\ &\forall e_j \in U : \sum_{i:e_j \in S_i} x_i \geq 1 \\ &x_i \in \{0,1\} \forall i\end{aligned}$$

By the same argument as before, we can relax it to:

$$\begin{aligned}&\min \sum_i x_i \\ &\forall e_j \in U : \sum_{i:e_j \in S_i} x_i \geq 1 \\ &x_i \geq 0 \forall i\end{aligned}$$

Consider the dual of this. Coefficient of x_i in the linear combination should not exceed the coefficient of x_i in the objective function.

$$\begin{aligned}&\max \sum_j y_j \\ &\forall i \quad \sum_{j | e_j \in S_i} y_j \leq 1 \\ &\forall j \quad y_j \geq 0\end{aligned}$$

Note that this aligns precisely with the argument we were trying to make for the set cover problem.

We are doing something like this: relax integer program to linear program, then make a feasible solution to the dual program, and use that as an approximation.

2.3 Set cover through LP rounding

1. Solve the primal LP to get x^*, where $0 \le x^* \le 1$.
2. Randomized rounding - Pick set i with probability $X_i^* \cdot 2H_n$.

Claim 0.2

The sets picked form a set cover with high probability.

Proof. Consider an element e_j. It lies in some sets, say $i_1, \ldots, i_k$. If any of the sets has $x_i^* \ge \frac{1}{2H_n}$, we are done. Otherwise:

$$\begin{aligned}
P(e_i \text{ is not covered}) &= \prod_i (1 - 2x_i^* H_n) \\
&\le \prod_i e^{-2x_i^* H_n} \\
&= e^{\sum_{i : e_j \in S_i} 2x_i^* H_n} \\
&\le e^{-2H_n} \\
&\le \frac{1}{n^2}
\end{aligned}$$

So the probability that some element is not covered is $\le \frac{1}{n}$ by the union bound. Hence the probability that this is a set cover is $\ge 1 - \frac{1}{n}$. $\square$

Number of sets picked is

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{S_i}\right] &= \sum_i 2x_i^* H_n \\
&\le 2H_n \cdot \mathbf{OPT}
\end{aligned}$$