COL352 Lecture 17

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1 Recap

Grammar $\implies$ PDA.

2 Definitions

Definition 1

A (non-deterministic) pushdown automaton ((N)PDA) is a 6-tuple $(Q, \Sigma, \Gamma, \Delta, q_0, A)$ where

1. Q – finite nonempty set of states
2. Σ – finite nonempty input alphabet
3. Γ – finite stack alphabet
4. $q_0 \in Q$ – initial state
5. A – set of accepting states
6. $\Delta \subseteq Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times Q \times \Gamma_\epsilon$, where X_ϵ is defined as $X \cup \{\epsilon\}$

Note that in an NFA, $\Delta \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times Q$.

Definition 2

Let $P = (Q, \Sigma, \Gamma, \Delta, q_0, A)$ be a PDA. An instantaneous description (i.d.) of P is a tuple (q, x, α) where $q \in Q$, $x \in \Sigma^*$, $\alpha \in \Gamma^*$. (The set of instantaneous descriptions is $Q \times \Sigma^* \times \Gamma^*$).

Informally, it consists of the current state, the string left to be read, and the description of the current stack.

Definition 3

Let $P = (Q, \Sigma, \Gamma, \Delta, q_0, A)$ be a PDA. The relation $\vdash_P$ (read as "changes to") is defined on the set of i.d.s as follows:

If $(q, a, B, q', B') \in \Delta$, then $(q, ax, B\alpha) \vdash_P (q', x, B'\alpha)$, and no other pairs of i.d.s are related.

In other words:

$$(q, y, \beta) \vdash_P (q', y', \beta') \iff \exists a \in \Sigma_\epsilon, B \in \Gamma_\epsilon, \alpha \in \Gamma^*, B' \in \Gamma_\epsilon \text{ such that } y = ay', \beta = B\alpha, \beta' = B'\alpha, (q, a, B, q', B') \in \Delta$$

Definition 4

$\vdash_P^*$ is defined as the reflexive transitive closure of $\vdash$ (read as "changes to in finitely many steps").

Definition 5

$x \in \Sigma^*$ is said to be accepted by PDA $P = (Q, \Sigma^*, \Gamma, \Delta, q_0, A)$ iff

$$(q_0, x, \epsilon) \vdash_P^* (q, \epsilon, \alpha)$$

for some $q \in A$ and some $\alpha \in \Gamma^*$.

New in lecture 17

Definition 6

The language recognized by PDA P denoted by $\mathcal{L}(P)$ is $\{x \in \Sigma^* \mid P \text{ accepts } x\}$.

3 Content

Plan for the next 4 lectures is to show that L is generated by the grammar iff L is recognized by a PDA.

The forward implication will be done in the next two lectures.
Recall that $\Delta \subseteq Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times Q \times \Gamma_\epsilon$, $(q, a, B, q', B') \in \Delta$. q stands for the current state, a for current input, B for the current stack top to be popped, q' for the next state, and B' for the element to be pushed onto stack.

Suppose that instead, we do $\Delta \in Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times Q \times \Gamma^*$ (i.e., we can push strings onto the stack). Call such a PDA a fast PDA.

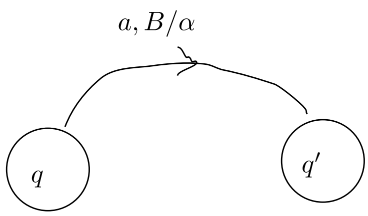
Theorem 1

L is recognized by a PDA $\iff$ L is recognized by a fast PDA.

Proof. The forward direction is obvious.

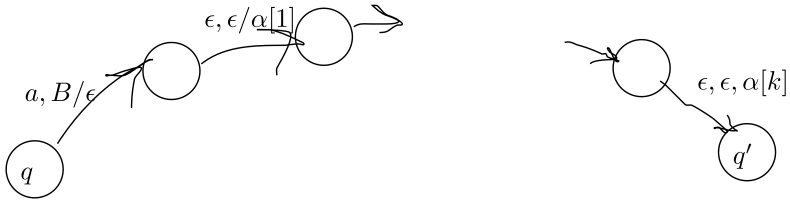
Now we look at a sketch of the proof of the non-obvious reverse direction.

L is recognized by a fast PDA, say P. A state transition looks like the following:



$a \in \Sigma_\epsilon, \alpha \in \Gamma^*$
$B \in \Gamma_\epsilon$

We do the following:



Check that P accepts x iff P' accepts x. Now since Δ was finite, Δ' is finite too. □

Corollary 1

In order to show that L generated by grammar $\implies$ L recognized by a PDA, it is sufficient to show that L generated by grammar $\implies$ L recognized by a fast PDA.

Theorem 2

Every language which is generated by a grammar is also recognized by a fast PDA.

We begin by an example.

Example 1

Consider $N = \{A, B\}$, $\Sigma = \{0, 1\}$, $R = \{A \to 0BB, A \to 1, B \to A1A, B \to 0\}$, with initial nonterminal as A.

We construct the PDA as follows.

$Q = \{q_i, q, q_a\}, A = \{q_a\}$, initial state $= q_i$, $\Sigma = \{0, 1\}, \Gamma = N \cup \Gamma \cup \{\perp\}$.

$\Delta = \Delta_{special} \cup \Delta_{match} \cup \Delta_{produce}$.

$\Delta_{match} = \{(q, 0, 0, q, \epsilon), (q, 1, 1, q, \epsilon)\}$. This corresponds to removing a from the stack, where $a \in \Sigma$.

$\Delta_{produce} = \{(q, \epsilon, A, q, BB0), (q, \epsilon, A, q, 1), (q, \epsilon, B, q, A1A), (q, \epsilon, B, q, 0)\}$. This corresponds to the production rules.

$\Delta_{special} = \{(q_i, \epsilon, \epsilon, q, \perp A), (q, \epsilon, \perp, q_a, \epsilon)\}$. This corresponds to initialization and cleanup.

Consider the following leftmost derivation of 01110: $A \implies 0BB \implies 0A1AB \implies 011AB \implies 0111B \implies 01110$.

Then we have

$$\begin{aligned}(q_i, 01110, \epsilon) &\vdash (q, 01110, A\bot)\\ &\vdash (q, 01110, A\bot)\\ &\vdash (q, 01110, 0BB\bot)\\ &\vdash (q, 1110, BB\bot)\\ &\vdash (q, 1110, A1AB\bot)\\ &\vdash (q, 1110, 11AB\bot)\\ &\vdash (q, 110, 1AB\bot)\\ &\vdash (q, 10, AB\bot)\\ &\vdash (q, 10, 1B\bot)\\ &\vdash (q, 0, B\bot)\\ &\vdash (q, 0, 0\bot)\\ &\vdash (q, \epsilon, \bot)\\ &\vdash (q_a, \epsilon, \epsilon)\end{aligned}$$

Proof.

Continuation from lecture 17.

We proceed to formalize the proof.

Fast PDA construction We construct the fast PDA P from grammar $G = (N, \Sigma, R, S)$ as follows: $P = (\{q_{init}, q, q_{accept}\}, \Sigma, \Gamma, \Delta, q_{init}, \{q_{accept}\})$, where

1. $\Gamma = N \cup \Sigma \cup \{\bot\}$
2. $\Delta = \Delta_{match} \cup \Delta_{produce} \cup \Delta_{special}$ where
   (a) $\Delta_{match} = \{(q, a, a, q, \epsilon) \mid a \in \Sigma\}$
   (b) $\Delta_{produce} = \{(q, \epsilon, A, q, rev(\alpha)) \mid (A \to \alpha) \in R\}$ (this is finite since R is finite)
   (c) $\Delta_{special} = \{(q_{init}, \epsilon, \epsilon, q, \bot S), (q, \epsilon, \bot, q_{accept}, \epsilon)\}$

We need to prove that $\forall x, S \overset{*}{\Longrightarrow} x \iff P$ accepts x (actually $(q_{init}, x, \epsilon) \vdash^* (q_{accept}, \epsilon, \epsilon)$).

Observe that $(q_{init}, x, \epsilon) \vdash^* (q_{accept}, \epsilon, \epsilon)$ iff $(q, x, S\bot) \vdash^* (q, \epsilon, \bot)$.

Need to show that $\forall x : S \overset{*}{\Longrightarrow} x \iff (q, x, S\bot) \vdash^* (q, \epsilon, \bot)$.

Claim 2.1

$\forall \alpha \in (N \cup \Sigma)^*, \forall x \in \Sigma^* : \alpha \overset{*}{\Longrightarrow} x \iff (q, x, \alpha\bot) \vdash^* (q, \epsilon, \bot)$

To show this we break this into two claims.

Claim 2.2

$\forall \alpha \in (N \cup \Sigma)^*, \forall x \in \Sigma^* : \alpha \overset{*}{\Longrightarrow} x \Longleftarrow (q, x, \alpha\bot) \vdash^* (q, \epsilon, \bot)$

Proof. By induction on the number of transitions in the (shortest) run of P from $(q, x, \alpha\bot)$ to $(q, \epsilon, \bot)$.

Base case: 0 transitions, then $\alpha = \epsilon, x = \epsilon$, then $\alpha \overset{*}{\Longrightarrow} x$.

Inductive case: $n > 0$ transitions. Then $\exists x', \alpha'$ such that $(q, x, \alpha\bot) \vdash (q, x', \alpha'\bot) \vdash^* (q, \epsilon, \bot)$, where the second part takes $n - 1$ transitions.

1. Case 1: If T is a match transition, $x = ax'$ and $\alpha = a\alpha'$ for some $a \in \Sigma$. By inductive hypothesis, $\alpha' \overset{*}{\Longrightarrow} x'$. But $\alpha = a\alpha' \overset{*}{\Longrightarrow} ax' = x$, so $\alpha \overset{*}{\Longrightarrow} x$.
2. Case 2: If T is a produce transition, $x = x'$, $\exists A \in N, \beta, \gamma \in (N \cup \Sigma)^*$ such that $\alpha = A\beta, \alpha' = \gamma\beta$,

and $(A \to \gamma) \in R$. (basically pop off a non-terminal from the stack, and push the reverse of the production rule RHS on the stack). By the inductive hypothesis, $\alpha' \overset{*}{\Longrightarrow} x' = x$, i.e., $\gamma\beta \overset{*}{\Longrightarrow} x$. But $\alpha = A\beta \implies \gamma\beta$ (because $(A \to \gamma) \in R$), so $\alpha \overset{*}{\Longrightarrow} x$.

□

Claim 2.3

$\forall \alpha \in (N \cup \Sigma)^*, \forall x \in \Sigma^* : \alpha \overset{*}{\Longrightarrow} x \implies (q, x, \alpha\bot) \vdash^* (q, \epsilon, \bot)$

Proof. By induction on the number of productions, say n, in the (shortest) (leftmost) derivation of x from α, and then on $|x|$. We have the following two inductive hypotheses:

1. $\forall \alpha \in (N \cup \Sigma)^*, \forall x \in \Sigma^*$: if $\alpha' \overset{*}{\Longrightarrow} x'$ in $< n$ productions, then $(q, x', \alpha\bot) \vdash^* (q, \epsilon, \bot)$.
2. $\forall \alpha \in (N \cup \Sigma)^*, \forall x \in \Sigma^*$: if $\alpha' \overset{*}{\Longrightarrow} x'$ in n productions, and $|x'| < |x|$, then $(q, x', \alpha'\bot) \vdash^* (q, \epsilon, \bot)$.

This is a standard double induction, similar to nested loops.

Note that $|\alpha| > 0, |x| > 0$ because we are in the inductive case.

1. Case 1: $\alpha = a\alpha'$ for some $a \in \Sigma$.

   In this case, $x = ax'$ for some $x' \in \Sigma^*$, and $\alpha' \overset{*}{\Longrightarrow} x'$ in n productions.

   Using a match transition, we have $(q, x, \alpha\bot) = (q, ax', a\alpha'\bot) \vdash (q, x', \alpha'\bot)$. By inductive hypothesis 2, $(q, x', \alpha'\bot) \vdash^* (q, \epsilon, \bot)$, and hence, $(q, x, \alpha\bot) \vdash^* (q, \epsilon, \bot)$.

2. Case 2: $\alpha = A\alpha'$ for some $A \in N$.

   So $\exists \gamma \in (N \cup \Sigma)^*$ such that $(A \to \gamma) \in R$ and $\gamma\alpha' \overset{*}{\Longrightarrow} x$ in $n - 1$ productions.

   Using a produce transition, we get $(q, x, \alpha\bot) = (q, x, A\alpha'\bot) \vdash (q, x, \gamma\alpha'\bot)$. By inductive hypothesis 1, $(q, x, \gamma\alpha'\bot) \vdash^* (q, \epsilon, \bot)$. Hence, $(q, x, \alpha\bot) \vdash^* (q, \epsilon, \bot)$.

Now for the base case for inductive hypothesis 1, $\alpha \overset{*}{\Longrightarrow} x$ in 0 steps, so $\alpha = x$. If $\alpha = x = \epsilon$ then it's trivial. Else, go to case 1.

For the base case for inductive hypothesis 2, if $\alpha \overset{*}{\Longrightarrow} x$ in n steps and $x = \epsilon$, either $\alpha = \epsilon$ or it isn't. If it is, then it's trivial, else α is a string of non-terminals only, so go to case 2. □

□

Corollary 2

Every language which is generated by a grammar is also recognized by a PDA.