COL352 Lecture 5

Contents

1 Recap

Recall the informal definition of a run on NFA. We formalize this today.

2 Content

Definition 1. *Let $N = (Q, \Sigma, \Delta, Q_0, A)$ be an NFA, and $x \in \Sigma^*$ be a string. A run of N on x is a sequence $q_0, x_1, q_1, \dots, x_m, q_m$ such that*

1. *Each $q_i \in Q$*
2. *Each $x_i \in \Sigma \cup \{\epsilon\}$, such that $x = x_1 x_2 \dots x_m$.*
3. *$q_0 \in Q_0$*
4. *$\forall i : (q_{i-1}, x_i, q_i) \in \Delta$*

Definition 2. *$q_0, x_1, \dots, q_{m-1}, x_m, q_m$ is an accepting run if $q_m \in A$, otherwise it is a rejecting run.*

Definition 3. *N accepts x if there exists an accepting run of N on x.*

Definition 4. $\mathcal{L}(N) = \{x \in \Sigma^* \mid N \text{ accepts } x\}$

Question 1. *Is it possible to create an NFA for any regular language?*

Answer 1. *Yes.*

Claim 1. *If L is recognized by a DFA, then L is recognized by some NFA.*

Proof. Suppose $D = (Q, \Sigma, \delta, q_0, A)$ recognizes the language L.
We construct $N = (Q, \Sigma, \Delta, \{q_0\}, A)$, with $\Delta = \{(q, a, q') \in Q \times \Sigma \times Q \mid q' = \delta(q, a)\}$.
Then N also recognizes D. ∎

Question 2. *Is it possible to create a DFA for any language that is accepted by an NFA?*

Answer 2. *Yes.*

Theorem 2.1. *Let N be any NFA. There exists a DFA D such that $\mathcal{L}(N) = \mathcal{L}(D)$.*

Proof. We shall break this into two steps.

1. Step 1:

   Theorem 2.2. *Let N be any NFA. There exists an NFA N' without ϵ transitions such that $\mathcal{L}(N) = \mathcal{L}(D)$.*

2. Step 2:

   Theorem 2.3. *Let N' be any NFA without ϵ transitions. There exists a DFA D such that $\mathcal{L}(N) = \mathcal{L}(D)$.*

First we prove step 1.

$x \in \mathcal{L}(N)$. Let $q_0 x_1 q_1 x_2 q_2 \ldots x_m q_m$ be an accepting run.

$x = x[1]x[2]\ldots x[n]$, with $x[i] \in \Sigma$.

$\langle x[i] \rangle$ is a subsequence of $\langle x_i \rangle$.

For any $j \notin \{i_1, i_2, \ldots, i_n\}$, $x_j = \epsilon$, and $x_{i_k} = x[k]$. (basically compression).
High-level idea - compress $x_{i_{k-1}+1} \ldots x_{i_k}$ into a single step.

Now coming to the official proof.

Let $N = (Q, \Sigma, \Delta, Q_0, A)$. Define NFA N' as follows:

$$N' = (Q, \Sigma, \Delta', Q_0, A')$$

Here,

$$\Delta' = \{(q, a, q') \in Q \times \Sigma \times Q \mid \exists q'' \in Q \text{ such that } q'' \text{ is reachable from } q \text{ by } \epsilon \text{ transitions and } (q'', a, q') \in \Delta\}$$

(note: this corresponds to the compression in the high-level idea)

$$A' = \{q \in Q \mid \exists q' \in A \text{ such that } q' \text{ is reachable from } q \text{ by } \epsilon \text{ transitions of } \Delta\}$$

(note: this corresponds to compressing the last part of an accepting run)

Claim 2. $\mathcal{L}(N) \subseteq \mathcal{L}(N')$

Proof. Suppose $x \in \mathcal{L}(N)$. Then there exists an accepting run $q_0, x_1, q_1, \ldots, x_m, q_m$ of N on x.

Let $i_1 < i_2 < \cdots < i_n$ be all the indices such that $x_i \in \Sigma$ (i.e., if $j \notin \{i_1, \ldots, i_n\}$, then $x_j = \epsilon$).

Let's construct an accepting run of N' on x. Consider the sequence $q_0 x[1] q_{i_1} x[2] \ldots x[n] q_{i_n}$.

Now note that:

1. $(q_{i_{j-1}}, x[j], q_{i_j}) \in \Delta'$

   Proof. This follows directly from the definition of Δ'; indeed, in the original sequence, we have $q_{i_{j-1}} \epsilon q_{i_{j-1}+1} \epsilon \ldots \epsilon q)i_j - 1$
   so using $q = q_{i_{j-1}}, q'' = q_{i_j - 1}, q' = q_{i_j}$ in the definition, we are done. ■

2. $q_{i_n} \in A'$

   Proof. Note that all transitions after this are ϵ transitions, and hence the accepting state q_m is reachable from q_{i_n} using ϵ transitions, and hence $q_{i_n} \in A'$. ■

Hence we get an accepting run of N' on x, whence we are done. ■

Claim 3. $\mathcal{L}(N') \subseteq \mathcal{L}(N)$

Proof. Exercise.

Suppose $x \in \mathcal{L}(N')$. Then there exists an acepting run $q_0, x[1]', q_2, \ldots, x[n']', q_{n'}$ of N' on x. Note that since there are no ϵ transitions, we have $x[i]' = x[i]$, and $n = n'$.

Now consider any $(q_{i-1}, x[i], q_i)$. Then since this is in D', we must have some q'' such that q'' is reachable from q_{i-1} by ϵ transitions and $(q'', x[i], q_i) \in D$.

Consider the corresponding path from q_{i-1} to q'', say $q_{i-1}\epsilon q'_{i-1,1}\epsilon \dots \epsilon q'' = q'_{i-1,l_{i-1}}$.

Also note that the last state q_n is an accepting state in N', hence there exists a path starting from q_n to some accepting state q'' in A consisting solely of ϵ transitions, say $q_n\epsilon q'_{n,1}\epsilon \dots \epsilon q'_{n,l_n} = q''$.

Consider the path

$$q_0\epsilon q'_{0,1}\epsilon \dots \epsilon q_{0,l_0}x[1]q_1\epsilon q_{1,1}\epsilon \dots \epsilon q_{1,l_1}x[2]q_2 \dots q_n\epsilon q'_{n,1}\epsilon \dots \epsilon q'_{n,l_n}$$

Then the last state q'_{n,l_n} is accepting as discussed above, and all transitions are in Δ (by definition of Δ'), and the string corresponding to this run is $\epsilon^{l_0}x[1]\epsilon^{l_1}x[2]\dots\epsilon^{l_{n-1}}x[n]\epsilon^{l_n} = x$, as needed.

Hence we get an accepting run of N on x, whence we are done.

$\blacksquare$

Now these two claims together show that $\mathcal{L}(N') = \mathcal{L}(N)$, whence we have completed the proof of theorem 2.2. $\blacksquare$