Release date: March 6, 2021 Deadline: March 13, 2021: 23:00

Read the instructions carefully.

This homework is primarily about proving that certain languages L are not regular. For this, we have the Pumping Lemma and the Myhill-Nerode Theorem at our disposal. Recall that the Pumping Lemma merely gives a sufficient condition for non-regularity. In some cases, using closure properties might give a much cleaner proof: assume that L is regular, then argue that some other language L' must also be regular, then apply Pumping Lemma to show that L' is, in fact, not regular. Remember that you can use any claim proven in class and in the previous quizzes and homeworks without reproducing its proof.

1. Prove that the language $\{x \mid x \text{ is the binary representation of } 3^{n^2} \text{ for some } n \in \mathbb{N}\}$ is not regular.

2. Prove that the language $\{0^m 1^n \mid m \neq n\} \subseteq \{0,1\}^*$ is not regular.

   As a challenge, construct a clean proof using the pumping lemma only. However, no extra credit will be given for this.

3. Construct the minimal DFA $D = (Q, \{0,1\}, \delta, q_0, A)$ that recognizes the language

   $$\{x \in \{0,1\}^* \mid x \text{ is the binary representation of a number coprime with } 6\}.$$

   Prove its minimality by giving a string $z_{q,q'}$ for each pair of distinct states $q, q' \in Q$ such that exactly one of $\widehat{\delta}(q, z_{q,q'})$ and $\widehat{\delta}(q', z_{q,q'})$ is in A. (Proof of correctness of your automaton is not required.)

4. Let $L_k \subseteq \{0,1\}^*$ be the language defined as $L_k = \{x \mid |x| \geq k \text{ and the EXOR of the last } k \text{ bits of } x \text{ is } 1\}$. Prove that any DFA that recognizes L_k has at least 2^k states. (By the way, observe that L_k is recognized by an NFA with $O(k)$ states.)

5. We all know that the set of strings over the alphabet $\{a, b\}$ containing an equal number of occurrences of ab and ba is regular. However, what if the alphabet is $\{a, b, c\}$? Prove that the language

   $$\{x \in \{a,b,c\}^* \mid x \text{ contains an equal number of occurrences of } ab \text{ and } ba\}$$

   is not regular. Here are some hints.

   1. Take help of the regular expression $(abc \cup bac)^*$.
   2. Use closure under inverse homomorphisms from Homework 1.

6. Prove that for any infinite regular language L, there exist two infinite regular languages L_1, L_2 such that $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$. Here are some hints.

   1. Let D be any DFA and q be any one of its states. Prove informally that the language

      $$L_q = \{x \mid x \in \mathcal{L}(D) \text{ and the run of } D \text{ on } x \text{ visits } q \text{ an odd number of times}\}$$

      is regular.
   2. Recall the proof of the pumping lemma.