COL352 Lecture 28

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1 Recap

Started closure properties last time.

2 Definitions

Definition 1

A language L is said to be co-Turing recognizable if its complement is Turing recognizable.

Definition 2

The membership language of Turing machines is defined as

$$A_{TM} = \{(\langle M \rangle, x) \mid \text{Turing machine } M \text{ accepts string } x\}$$

Definition 3

A DTM recognizing A_{TM} is called a universal TM.

3 Content

4 Closure properties

Decidable languages The class of decidable languages is closed under $\cap, \cup, \cdot, *$ and complementation.

1. $\cap$ – copy x to tape 2, simulate M_1 on tape 1, M_2 on tape 2, and if both accept x, accept x else reject x.
2. $\cup$ – similar.
3. $\cdot$ – construct 2-tape NTM for $L_1 \cdot L_2$: Non-deterministically cut a suffix of x from tape 1, and paste on tape 2, and then simulate both.
4. $*$ – while $x \neq \epsilon$: non-deterministically cut a nonempty prefix of x from tape 1, and paste on tape 2, simulate M on tape 2, and if M rejects its input, then reject x. After the while loop is over, accept x.
5. complementation – swap q_{acc}, q_{rej} in the DTM (not the NTM).

Turing-recognizable languages The class of Turing recognizable languages is closed under $\cap, \cup, \cdot, *$ – same constructions in case of decidable languages, just take care of the possibility of non-terminating runs in the proof (just stay positive and think the iff version instead of arguing about the strings not in the language).

Question 1

Is the class of Turing recognizable languages closed under complementation?

Question 2

Is there a language which is recognizable but not decidable?

Question 3

Give an explicit unrecognizable language.

Note 1

co-Turing recognizable is not the same as unrecognizable.

For instance, if L is regular, then L^c is also regular, so both L, L^c are recognizable as well as co-recognizable.

Note 2

L is co-recognizable is equivalent to saying that there exists a Turing machine M such that $x \notin L$ iff M on x halts and rejects – for a construction, swap accepting and rejecting states in the TM of L^c.

Observation 1

If L is decidable then L is recognizable.

If L is decidable then L is co-recognizable (since L decidable $\implies$ L^c decidable $\implies$ L^c recognizable).

Theorem 1

A language L is decidable iff L is recognizable as well as co-recognizable.

Proof. The forward direction is as in the previous observation.

For the backward diretion, suppose L is recognizable as well as co-recognizable. Let M be a DTM which recognizes L and M' be a DTM which recognizes L^c. Consider the 2-tape DTM D which on input x does the following:

1. Copy x from tape 1 to tape 2.
2. Simulate M on tape 1 and M' on tape 2 in parallel.
3. If M halts earlier, abort M' and follow M's decision. Else if M' halts earlier, abort M and invert M''s decision. The other case never arises, since either x is in L or in L^c, so either M accepts x or M' accepts x.

Note that if D accepts x, then $x \in L$ and if D rejects x, then $x \in L$.

By construction, D is a decider for L, so L is decidable. $\square$

Goal: Construct an unrecognizable language.

Recall the proof of the fact that $2^{\mathbb{N}}$ is uncountable from COL202.

Flipping the incidence matrix diagonal is equivalent to considering $\{i \mid i \notin S_i\}$ and this set is different from each S_j.

Recall that the set of TMs over any fixed finite alphabet Σ is countable (using binary encoding).

Let M_w be the DTM whose description is w where $w \in \Sigma$, and if w is not the description of any DTM, then we define M_w to be the DTM where $q_0 = q_{rej}$, $\Gamma = \Sigma \cup \{\textvisiblespace\}$, $Q = \{q_{acc}, q_{rej}\}$.

Now consider the set of strings w_i in Σ^*, and for each of those, consider M_{w_i} in Σ^*. Let A be a matrix where $a_{ij} = 1$ iff M_{w_i} accepts w_j, and 0 otherwise.

Define $\textbf{DIAG} = \{w \in \Sigma^* \mid M_w \text{ does not accept } w\}$.

Theorem 2

DIAG is unrecognizable.

Proof. Suppose not. Let R be a DTM that recognizes **DIAG**. Then $R = M_w$ for some $w \in \Sigma^*$. Does R accept w?

If M_w accepts w, then by the definition of **DIAG**, $w \notin$ **DIAG**. But M_w recognizes **DIAG**, so by assumption since M_w accepts w, we have $w \in$ **DIAG**, which is a contradiction.

If M_w doesn't accept w, it follows that $w \notin$ **DIAG** from the fact that M_w recognizes **DIAG**. But from the definition of **DIAG**, since M_w doesn't accept w, $w \in$ **DIAG**, which is a contradiction. ☐

We shall look at the membership language.

Theorem 3

A_{TM} is recognizable.

Proof.(?) Simulate M on x. If M accepts x, then accept $(\langle M \rangle, x)$, if it rejects, then reject, else run forever.

Design a TM U for A_{TM} as follows:

Take $Q_U \supseteq Q_M$, $\Gamma_U \supseteq \Gamma_M$ and so on. This is not a valid proof, since it potentially has infinitely many states (due to considering all possible M). ☐

Goal: Design TM U to recognize A_{TM}.

Σ_U : alphabet of U, with $|\Sigma_U| \geq 10$.

$(\langle M \rangle, x)$ where $M = (Q_M, \Sigma_M, \Gamma_M, \cdots)$ and $x \in \Sigma_M^*$.

Has to be converted to Σ_U^*. Σ_M, Γ_M, Q_M could be of much larger size that Σ_U.

Each $a \in \Sigma_M$ is represented by a string over Σ_U – this gives an encoding of x.

Encoding M – ($|Q_M|, |\Gamma_M|,$ Indicator vector of $\Sigma_M \subseteq \Gamma_M,$ encoded as bit vector of size $|\Gamma_M|$, encodings of $q_0^M, q_{acc}^M, q_{rej}^M$).

$\forall q \in Q_M \setminus \{q_{acc}^M, q_{rej}^M\}, a \in \Gamma_M$, $\delta(q, a) = (r, b, dir)$ is encoded as $(enc(q), enc(a), enc(r), enc(b), enc(dir))$.

Theorem 4

A_{TM} is recognizable.

Proof. Construct a 3-tape Turing machine U which on input $(\langle M \rangle, x)$ does the following:

1. Preprocess –
   (a) Cut $\langle M \rangle$ from tape 1 and paste it on tape 2.
   (b) Copy the encoding of q_0^M from tape 2 to tape 3.
   (c) Check that $x \in \Sigma_M^*$. If not, reject.
2. Simulation – Now the stack looks like x on tape 1, encoding of M on tape 2, and encoding of current state of M on tape 3. A character of x is a string in Γ_U.
   (a) Search tape 2 for the next transition.
   (b) Implement the transition (overwrite tape 3 with the new state, tape 1 with the new symbol in Γ_M.

(c) If tape 3 contains the encoding of q_{acc}^M (resp q_{rej}^M), then accept (resp. reject).

Then U recognizes A_{TM}. Observe that U is not a decider. ☐

Question 4

Is $\overline{\mathbf{DIAG}}$ recognizable?

$\overline{\mathbf{DIAG}} = \{w \mid M_w \text{ accepts } w\}$.

Theorem 5

$\overline{\mathbf{DIAG}}$ is recognizable, and hence $\mathbf{DIAG}$ is co-recognizable.

Proof. Replace w by (w, w) and simulate a universal TM on the result. ☐

Theorem 6

A_{TM} is undecidable.

Proof. Suppose A_{TM} was decidable. Then $\overline{\mathbf{DIAG}}$ would have been decidable from the previous proof, which would imply $\mathbf{DIAG}$ is a decidable language. Can also flip decision to get decider for $\mathbf{DIAG}$. ☐