# COL352 Lecture 10

## Contents



## 1 Recap

Proof of the fact that $\{0^n 1^n \mid n \in \mathbb{N}\}$ is not regular.
We constructed a long enough string depending on $|Q|$, and used pigeonhole principle to claim some state was visited twice in the first $n$ transitions. Then we pumped using the loop, and found a $k$ such that pumping $k$ times results in a string not in $L$, which gives a contradiction by the definition of the DFA $D$.

## 2 Definitions

## 3 Content

**Note 1**

Plan: Abstract out ideas from the last proof to get a necessary condition for $L$ to be regular, or equivalently, a sufficient condition for $L$ to be not regular.

Suppose $L \subseteq \Sigma^*$ is a regular language, and $D = (Q, \Sigma, \delta, q_0, A)$ be a DFA recognizing $L$.

Let $p = |Q|$. Consider $s \in L$ with $|s| = n \geq p$. Let $q_i = \hat{\delta}(q_0, s[1]s[2]\ldots s[i])$.

By the pigeonhole principle, $\exists i, j$ such that $i < j \leq p$ and $q_i = q_j$.

$\hat{\delta}(q_i, s[i+1]\ldots s[j]) = q_j = q_i$.

Let $x = s[1]\ldots s[i], y = s[i+1]\ldots s[j], x = s[j+1]\ldots s[n]$. So we have $s = xyz$, $|y| > 0, |xy| \leq p$.

$\hat{\delta}(q_i, y) = q_i, \hat{\delta}(q_0, x) = q_i, \hat{\delta}(q_i, z) = q_n \in A$.

$\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z) = \hat{\delta}(q_i, z) = q_n \in A$. So $xz \in L$.

$\hat{\delta}(q_0, xy^k z) = q_n \in A$.

Thus $\forall k \in \mathbb{N} \cup \{0\}$, we have $xy^k z \in L$.

**Lemma 0.1**

**Pumping Lemma for Regular Languages**

Let $L \in \Sigma^*$ be a regular language. Then $\exists p \in \mathbb{N}$ such that $\forall s \in L$ with $|s| \geq p$, $\exists x, y, z \in \Sigma^*$ such that

1. $s = xyz$
2. $|y| > 0$
3. $|xy| < p$
4. $\forall i \in \mathbb{N} \cup \{0\}$, $xy^i z \in L$

*Proof.* Let $D$ be a DFA for $L$. Set $p = |Q|$ and proceed as in the previous paragraph. $\square$

**Note 2**

This holds even when $|L|$ is finite, just take $p >$ the length of the longest string in $L$. Then this statement is vacuously true.

**Lemma 0.2**

**Contrapositive of the Pumping Lemma for Regular languages**

Let $L \in \Sigma^*$ be any regular language. If $\forall p \in \mathbb{N}, \exists s \in L$ with $|s| \geq p$, such that $\forall x, y, z \in \Sigma^*$,

$$(s = xyz) \wedge (|y| > 0) \wedge (|xy| \leq p) \implies \exists i \in \mathbb{N} \cup \{0\} \text{ such that } xy^i z \in L$$

Then $L$ is not regular.

**Note 3**

Guideline: Look at it like this sort of a game.

$\forall$ corresponds to moves by opponent. $\exists$ corresponds to moves by you.

| Opponent | You |
|---|---|
| Gives $p \in \mathbb{N}$ | |
| | $s : s \in L, \|s\| \geq p$ |
| $x, y, z$ such that $s = xyz, y \neq \epsilon, \|xy\| \leq p$ | |
| | $i \in \mathbb{N} \cup \{0\}$ |

You win if $xy^i \notin L$, opponent otherwise.

The pumping lemma says that if $L$ is regular, then the opponent has a winning strategy, i.e., if you have a winning strategy, then $L$ is not regular.

**Example 1**

Consider $\Sigma = \{0, 1\}, L = \{s \in \Sigma^* \mid \#0 = \#1 \text{ in } s\}$.

*Answer.*

| Opponent | You |
|---|---|
| $p$ | |
| | $0^p 1^p$ |
| $0^p 1^p = xyz, y \neq \epsilon, \|xy\| \leq p$ | |
| | $i \neq 1$ |

$|xy| = q \leq p$, so $xy = 0^q$. Then done.

**Example 2**

Consider $\Sigma = \{0, 1\}, L = \{ww \in \Sigma^* \mid w \in \Sigma^*\}$.

*Answer.*

| Opponent | You |
|---|---|
| $p$ | |
| | $0^p 1^p 0^p 1^p$ |
| $x, y, z...$ | |
| | any $i \neq 1$ |

**Example 3**

Consider $\Sigma = \{0, 1\}, L = \{0^i 1^j \in \Sigma^* \mid i > j\}$.

*Answer.*

| Opponent | You |
| --- | --- |
| $p$ | |
| | $0^{p+1}1^p$ |
| $x, y, z...$ | |
| | $i = 0$ |

Note that when we play this, by a similar argument as above, we have $|y| > 0$, so the length of first prefix of 0 is $\leq p$, which gives a contradiction.

**Example 4**

Consider $\Sigma = \{a, b, c\}, L = \{a^{n_1} b^{n_2} c^{n_3} \in \Sigma^* \mid n_1 = 1 \implies n_2 = n_3\}$.

*Answer.* The opponent always has a winning strategy. If we give $ab^n c^n$, then the opponent gives us $x = \epsilon, y = a, z = b^n c^n$. If we give $a^m b^n c^p$, the opponent gives us $x = \epsilon, y = a^m, z = b^n c^p$.