COL352 Lecture 10

Contents

1 Recap

Proof of the fact that $\{0^n 1^n \mid n \in \mathbb{N}\}$ is not regular.
We constructed a long enough string depending on $|Q|$, and used pigeonhole principle to claim some state was visited twice in the first n transitions. Then we pumped using the loop, and found a k such that pumping k times results in a string not in L, which gives a contradiction by the definition of the DFA D.

2 Definitions

3 Content

> **Note 1**
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> Plan: Abstract out ideas from the last proof to get a necessary condition for L to be regular, or equivalently, a sufficient condition for L to be not regular.

Suppose $L \subseteq \Sigma^*$ is a regular language, and $D = (Q, \Sigma, \delta, q_0, A)$ be a DFA recognizing L.

Let $p = |Q|$. Consider $s \in L$ with $|s| = n \geq p$. Let $q_i = \hat{\delta}(q_0, s[1]s[2] \ldots s[i])$.

By the pigeonhole principle, $\exists i, j$ such that $i < j \leq p$ and $q_i = q_j$.

$\hat{\delta}(q_i, s[i+1] \ldots s[j]) = q_j = q_i$.

Let $x = s[1] \ldots s[i], y = s[i+1] \ldots s[j], x = s[j+1] \ldots s[n]$. So we have $s = xyz$, $|y| > 0, |xy| \leq p$.

$\hat{\delta}(q_i, y) = q_i, \hat{\delta}(q_0, x) = q_i, \hat{\delta}(q_i, z) = q_n \in A$.

$\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z) = \hat{\delta}(q_i, z) = q_n \in A$. So $xz \in L$.

$\hat{\delta}(q_0, xy^k z) = q_n \in A$.

Thus $\forall k \in \mathbb{N} \cup \{0\}$, we have $xy^k z \in L$.

> **Lemma 0.1**
>
> **Pumping Lemma for Regular Languages**
>
> Let $L \in \Sigma^*$ be a regular language. Then $\exists p \in \mathbb{N}$ such that $\forall s \in L$ with $|s| \geq p$, $\exists x, y, z \in \Sigma^*$ such that
>
> 1. $s = xyz$
> 2. $|y| > 0$
> 3. $|xy| < p$
> 4. $\forall i \in \mathbb{N} \cup \{0\}$, $xy^i z \in L$

Proof. Let D be a DFA for L. Set $p = |Q|$ and proceed as in the previous paragraph. $\square$

Note 2

This holds even when $|L|$ is finite, just take $p >$ the length of the longest string in L. Then this statement is vacuously true.

Lemma 0.2

Contrapositive of the Pumping Lemma for Regular languages

Let $L \in \Sigma^*$ be any regular language. If $\forall p \in \mathbb{N}, \exists s \in L$ with $|s| \geq p$, such that $\forall x, y, z \in \Sigma^*$,

$$(s = xyz) \wedge (|y| > 0) \wedge (|xy| \leq p) \implies \exists i \in \mathbb{N} \cup \{0\} \text{ such that } xy^iz \notin L$$

Then L is not regular.

Note 3

Guideline: Look at it like this sort of a game.

$\forall$ corresponds to moves by opponent. $\exists$ corresponds to moves by you.

Opponent	You
Gives $p \in \mathbb{N}$	
	$s : s \in L, \|s\| \geq p$
x, y, z such that $s = xyz, y \neq \epsilon, \|xy\| \leq p$	
	$i \in \mathbb{N} \cup \{0\}$

You win if $xy^i \notin L$, opponent otherwise.

The pumping lemma says that if L is regular, then the opponent has a winning strategy, i.e., if you have a winning strategy, then L is not regular.

Example 1

Consider $\Sigma = \{0, 1\}, L = \{s \in \Sigma^* \mid \#0 = \#1 \text{ in } s\}$.

Answer.

Opponent	You
p	
	0^p1^p
$0^p1^p = xyz, y \neq \epsilon, \|xy\| \leq p$	
	$i \neq 1$

$|xy| = q \leq p$, so $xy = 0^q$. Then done.

Example 2

Consider $\Sigma = \{0, 1\}, L = \{ww \in \Sigma^* \mid w \in \Sigma^*\}$.

Answer.

Opponent	You
p	
	$0^p1^p0^p1^p$
$x, y, z...$	
	any $i \neq 1$

Example 3

Consider $\Sigma = \{0,1\}, L = \{0^i 1^j \in \Sigma^* \mid i > j\}$.

Answer.

Opponent	You
p	
	$0^{p+1}1^p$
$x, y, z...$	
	$i = 0$

Note that when we play this, by a similar argument as above, we have $|y| > 0$, so the length of first prefix of 0 is $\leq p$, which gives a contradiction.

Example 4

Consider $\Sigma = \{a,b,c\}, L = \{a^{n_1} b^{n_2} c^{n_3} \in \Sigma^* \mid n_1 = 1 \implies n_2 = n_3\}$.

Answer. The opponent always has a winning strategy. If we give $ab^n c^n$, then the opponent gives us $x = \epsilon, y = a, z = b^n c^n$. If we give $a^m b^n c^p$, the opponent gives us $x = \epsilon, y = a^m, z = b^n c^p$. However this language is not regular (apply pumping lemma on the reverse of this language, or show that the language of strings s such that as is in L is regular if L is regular, and use the pumping lemma on that).