Beginner level

Each problem is worth 7 points.

Problem 1

Find all quadruples of real numbers (a, b, c, d) such that the equalities

$$X^2 + aX + b = (X - a)(X - c) \text{ and } X^2 + cX + d = (X - b)(X - d)$$

hold for all real numbers X.

Problem 2

The Bank of Zürich issues coins with an H on one side and a T on the other. Alice has n of these coins arranged in a line from left to right. She repeatedly performs the following operation: if some coin is showing its H side, Alice chooses a group of consecutive coins (whose size can be any integer between 1 to n, both inclusive) and flips all of them; otherwise, all coins show T and Alice stops. For instance, if $n = 3$, Alice may perform the following operations: $THT \to HTH \to HHH \to TTH \to TTT$. She might also choose to perform the operation $THT \to TTT$.

For each initial configuration C, let $m(C)$ be the minimal number of operations that Alice must perform. For example, $m(THT) = 1$ and $m(TTT) = 0$. Determine the largest value of $m(C)$ over all 2^n possible initial configurations C.

Problem 3

Let ABC be a triangle. The tangent, at A, to the circumcircle of ABC meets BC at point D. Point $X \neq D$ is on the line BC such that $AD = AX$, and point Y is the foot of the perpendicular from C onto AD. Prove that if $BD = BC$, then XY is parallel to AC.

Problem 4

Find all functions f that are defined on the set of all real numbers and take real values, such that for any real x, y, it holds that

$$f(x+y) = (-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y)$$

where $\lfloor x \rfloor$ denotes the largest integer that does not exceed x.

Problem 5

Let n, k be positive integers such that $k \leq 2^n$. A and B are playing the following variant of the guessing game. First, A secretly picks an integer x, such that $1 \leq x \leq n$. B will attempt to determine x by asking some questions, which are described as follows. In each turn, B chooses k distinct subsets of $\{1, 2, \cdots, n\}$, and then, for each chosen set S, asks the question "Is x in the set S?". Then A has to pick one question and tell both the question and its answer to B.

Find, with proof, all pairs (n, k) such that B could determine x in finitely many turns with absolute certainty.

Problem 6

For any integer n not equal to 1 or -1, define $p(n)$ as the smallest prime number[a] that divides n. In particular, $p(0) = 2$. We also define $p(1) = p(-1) = 1$. Suppose that a nonconstant polynomial[b] f with integer coefficients satisfies $p(f(n)) \leq p(n)$ for every positive integer n. Prove that $f(0) = 0$.

[a] A prime number is an integer greater than 1 whose only positive divisors are 1 and itself.

[b] A polynomial p is a function of the form $p(n) = a_0 + a_1 n + \cdots + a_k n^k$ where the a_k's are called the coefficients of the polynomial p, and k is a non-negative integer.