Advanced level

Day 2

Each problem is worth 7 points.

These problems are to be kept confidential till Monday, 18th May 2020, 1200 hours (GMT).

Problem 5

Let $\mathbb{Q}$ denote the set of rational numbers. Determine all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that, for all $x, y \in \mathbb{Q}$,

$$f(x)f(y+1) = f(xf(y)) + f(x)$$

Problem 6

Decide whether there exist infinitely many triples (a, b, c) of positive integers such that all prime factors of $a! + b! + c!$ are smaller than 2020.

Problem 7

Each integer in $\{1, 2, 3, \ldots, 2020\}$ is coloured in such a way that, for all positive integers a and b such that $a + b \leq 2020$, the numbers a, b and $a + b$ are not coloured with three different colours. Determine the maximum number of colours that can be used.

Problem 8

Let ABC be an acute scalene triangle, with the feet of A, B, C onto BC, CA, AB being D, E, F respectively. Let W be a point inside ABC whose reflections over BC, CA, AB are W_a, W_b, W_c respectively. Finally, let N and I be the circumcentre and incentre of $W_aW_bW_c$ respectively. Prove that, if N coincides with the nine-point centre of DEF, the line WI is parallel to the Euler line of ABC.

Note: If XYZ is a triangle with circumcentre O and orthocentre H, then the line OH is called the Euler line of XYZ and the midpoint of OH is called the nine-point centre of XYZ.

May 17th, 2020 *Time: 5 hours*