Q1:

a): $F = \int_0^T B^2(s)\, ds$.

define $f(t) = \int_0^t B^2(s)\, ds$

$df(t) = B^2(t)\, dt$

$E[F] = E\left[\int_0^T B^2(s)\, ds\right] = \int_0^T E[B^2(s)]\, ds$

$= \int_0^T s\, ds = \frac{T^2}{2}$.

(continued later)

b) $Y(t) = \log(X(t))$.

$\Rightarrow X(t) = e^{Y(t)}$

$\Rightarrow dX(t) = e^{Y(t)} dY(t) + 0 \cdot dt + \frac{1}{2} \cdot e^{Y(t)} dY \cdot dY$

$\Rightarrow \cancel{e^{Y(t)}} dY(t) + \frac{1}{2} \cancel{e^{Y(t)}} dY \cdot dY = 2(3 - Y(t)) \cancel{e^{Y(t)}} dt + 4 \cancel{e^{Y(t)}} dB(t)$

$\Rightarrow dY + \frac{1}{2} dY \cdot dY = 2(3 - Y)\, dt + 4\, dB(t)$

$e^{2t} Y = Z$

$Y = Z e^{-2t}$

$dY = e^{-2t} dZ - 2Z e^{-2t} dt + 0 \cancel{\frac{1}{2} \cdot dt}$

$\Rightarrow \boxed{e^{-2t} dZ \cancel{- 2Z e^{-2t} dt} + \frac{1}{2} dZ \cdot dZ = 6\, dt \cancel{- 2Z e^{-2t} dt} + 4\, dB(t)} \quad (1)$

$\Rightarrow$

$\Rightarrow Z(t) = Z(0) - (e^{2t} - 1) + \int_0^t 4 e^{2s}\, dB(s)$

$\Rightarrow Y(t) = e^{-2t} \underbrace{Y(0)}_{\log(X(0))} - (1 - e^{-2t}) + e^{-2t} \int_0^t 4 e^{2s}\, dB(s)$

$\Rightarrow \boxed{X(t) = \exp\left(-(1 - e^{-2t}) + e^{-2t} \int_0^t 4 e^{2s}\, dB(s)\right)}$

The factor of -1 comes from noting that $dZ \cdot dZ$ is $dt \times$ something & only $4\, dB(t)$ contributes to it

$Y(t) = \log(X(t))$

$= [illegible] - (1 - e^{-2t}) + e^{-2t} \underbrace{\int_0^t 4e^{2s}\, dB(s)}$

Ito integral $\Rightarrow$ expectation $= \int_0^t 4e^{2s}\, dB(s) = 0$

So $E[Y(t)] = [illegible] - (1 - e^{-2t})$

For distribution, note that the Ito integral is integral of ~~constant~~ deterministic function, so it is normally distributed

mean $= 0$

variance $= \int_0^t (4e^{2s})^2\, ds = \int_0^t 16 e^{4s}\, ds$

$= \frac{e^{4t} - 1}{4} \times 16 = 4(e^{4t} - 1)$

So ~~mean of~~ $Y(t)$ is normally distributed with mean $[illegible] - (1 - e^{-2t})$ & variance $e^{-4t} \cdot (4(e^{4t} - 1))$

$= \boxed{4(1 - e^{-4t})}$

$\Rightarrow Y(t) \sim N\left([illegible](1 - e^{-2t}),\ 4(1 - e^{-4t})\right)$

Note: We got Z by guessing the form of Z as $dZ(t) = \alpha(t)dt + \beta(t)\, dB(t)$ & solving for α & β since $dZ \cdot dZ = \beta^2(t) \cdot dt$ makes it easier

It is possible to multiply eqn ① by dZ to get to the same conclusion.

Q2 (i) We can write it as (omitting the (t))

$$\begin{pmatrix} dY_1 \\ dY_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \beta \end{pmatrix} dt + \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \\ dB_3 \end{pmatrix}$$

We find u_1, u_2, u_3 such that

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 2 \\ \beta \end{pmatrix}$$

$$\Rightarrow \quad u_1 + 2u_2 + 3u_3 = 2$$
$$u_1 + 2u_2 + 2u_3 = \beta$$

$$\Rightarrow \quad u_3 = 2 - \beta$$

Now let $u_2 = \alpha$

So $u_1 + 2u_2 = 2 - 3(2-\beta) = 3\beta - 4$

$$\Rightarrow u_1 = 3\beta - 2\alpha - 4$$

Now consider

$$Z(t) = \exp\left(-\int_0^t \left((3\beta-2\alpha-4)\,dB_1 + \alpha\, dB_2 + (2-\beta)\,dB_3\right) - \int_0^t \left((3\beta-2\alpha-4)^2 + \alpha^2 + (2-\beta)^2\right) ds\right).$$

$$- \int_0 \left((3\beta - 2\alpha - 4\right.$$

Varying α over $\mathbb{R}$ generates an infinite family of stochastic processes Z.

Now by corollary to Girsanov's theorem, the measure Q

$$dQ = Z\, dP$$

satisfies

$$dY = \underbrace{\left[\begin{pmatrix} 2 \\ \beta \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right]}_{0} dt + \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \\ dB_3 \end{pmatrix}$$

$\Rightarrow$ Y is a martingale (since second term integrates to martingale as it is Ito integral)

$\Rightarrow$ Q is a martingale measure for $Y(t)$

& there are ∞-ly many such Q, each generated by a different γ.

(ii)

$$dX = \frac{1}{3}X^{1/3}\,dt + X^{2/3}\,dB$$

$$dX \cdot dX = X^{4/3}\,dt$$

$$Y = X^{1/3}$$

$$dY = \frac{1}{3}X^{-2/3}\,dX + \frac{1}{3}\cdot\left(-\frac{2}{3}\right)\times\frac{1}{2}\times X^{4/3}\times X^{-5/3}\,dt$$

$$= \frac{1}{3}X^{-2/3}\left(\cancel{\frac{1}{3}X^{1/3}dt} + X^{2/3}dB\right) \cancel{-\frac{1}{9}X^{-1/3}dt}$$

$$= \frac{1}{3}\,dB$$

$$Y(0) = 3^{1/3}$$

$$Y(t) - Y(0) = \frac{B(t) - B(0)}{3} = \frac{B(t)}{3}$$

$$\Rightarrow Y(t) = 3^{1/3} + \frac{B(t)}{3}$$

$$\Rightarrow \boxed{X(t) = \left(3^{1/3} + \frac{B(t)}{3}\right)^3}$$

Conditions in the existence & uniqueness theorem for SDE:

given: $a(t,x) = \frac{1}{3}x^{1/3}$ $\quad \sigma(t,x) = x^{2/3}$.

Conditions: $\exists\, C, D$ such that

$$\left|\frac{1}{3}x^{1/3}\right| + |x^{2/3}| \leq C(1+|x|).$$

$$\left|\frac{1}{3}(x^{1/3} - y^{1/3})\right| + |x^{2/3} - y^{2/3}| \leq D|x-y|$$

Note that for arbitrarily small x, y, this inequality is false. For example, $x = 8\varepsilon^3$, $y = \varepsilon^3$

$$\left|\frac{1}{3}\varepsilon\right| + |3\varepsilon^2| \leq 7D\varepsilon^3 \rightarrow \text{false for small } \varepsilon$$

So, we $\boxed{\text{can't}}$ use the existence & uniqueness theorem