3.6. **Martingale representation theorem.** We have seen that $X(t) = X(0) + \int_0^t v dB(s)$ is a martingale. We wish to know about its converse, i.e., any martingale can be represented as an Ito integral. This result is known as **Martingale representation theorem** . To do so, we need a technical lemma, which we are stating without proof.

Lemma 3.13. *The linear span of the random variables of the type*

$$\exp\Big\{\int_0^T h(t)\,dB(t) - \frac{1}{2}\int_0^T h^2(t)\,dt\Big\}$$

is dense in $L^2(\mathcal{F}_T, \mathbb{P})$, where h is a deterministic function with $h \in L^2[0,T]$.

Theorem 3.14 (The Ito Representation Theorem). *Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$. Then there exists a unique stochastic process $f \in \mathcal{Y}(0,T)$ such that*

$$F = \mathbb{E}[F] + \int_0^T f(t)\,dB(t).$$

Proof. First assume that F has the form

$$F = \exp\Big\{\int_0^T h(t)\,dB(t) - \frac{1}{2}\int_0^T h^2(t)\,dt\Big\} \tag{3.8}$$

for some deterministic function $h \in L^2(0,T)$. Define

$$Y(t) = \exp\Big\{\int_0^t h(s)\,dB(s) - \frac{1}{2}\int_0^t h^2(s)\,ds\Big\}, \quad 0 \le t \le T.$$

Take $X(t) = \int_0^t h(s)\,dB(s) - \frac{1}{2}\int_0^t h^2(s)\,ds$. Then $Y(t) = f(X(t))$, where $f(x) = e^x$. By applying Ito formula, we have

$$\begin{aligned} dY(t) = Y(t)h(t)\,dB(t) \implies Y(t) = 1 + \int_0^t Y(s)h(s)\,dB(s) \\ \implies F = 1 + \int_0^T Y(s)h(s)\,dB(s) \implies \mathbb{E}[F] = 1 \end{aligned}$$

Hence for the above $F \in L^2(\mathcal{F}_T, \mathbb{P})$, we have the representation

$$F = \mathbb{E}[F] + \int_0^T f(t)\,dB(t).$$

If $F \in L^2(\mathcal{F}_T, \mathbb{P})$ is arbitrary, then by Lemma 3.13 we can approximate F in $L^2(\mathcal{F}_T, \mathbb{P})$ by linear combination of F_n of the functions of the form (3.8). Thus, for each n, we have

$$F_n = \mathbb{E}[F_n] + \int_0^T f_n(t)\,dB(t), \quad f_n \in \mathcal{Y}(0,T).$$

In view of Ito-isometry, we observe that

$$\mathbb{E}[(F_n - F_m)^2] = \big(\mathbb{E}[F_n - F_m]\big)^2 + \mathbb{E}\Big[\int_0^T (f_n - f_m)^2\,ds\Big] \longrightarrow 0 \quad \text{as } n, m \to \infty.$$

This shows that $\{f_n\}$ is a Cauchy sequence in $L^2((0,T) \times \Omega)$, and hence there exists $f \in L^2((0,T) \times \Omega)$ such that $f_n \to f$ in $L^2((0,T) \times \Omega)$. Moreover, since $f_n \in \mathcal{Y}(0,T)$, we see that $f \in \mathcal{Y}(0,T)$. Furthermore, $f(t,\omega)$ is $\mathcal{F}_t$-adapted. Hence

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty} \Big(\mathbb{E}[F_n] + \int_0^T f_n(t)\,dB(t) \Big) = \mathbb{E}[F] + \int_0^T f(t)\,dB(t)$$

where the limit is taken in $L^2(\mathcal{F}_T, \mathbb{P})$. We now prove uniqueness. Suppose the exist $f_1, f_2 \in \mathcal{Y}(0,T)$. such that

$$F = \mathbb{E}[F] + \int_0^T f_1\,dB(t) = \mathbb{E}[F] + \int_0^T f_2\,dB(t).$$

Then, in view of Ito-isometry, we get that

$$0 = \mathbb{E}\Big[\int_0^T (f_1 - f_2)^2\,dt \Big] \implies f_1(t,\omega) = f_2(t,\omega) \text{ for a.e. } (t,\omega) \in [0,T]\times\Omega.$$

This completes the proof. $\square$

Example 3.9. *Find $f \in \mathcal{Y}(0,T)$ such that $F = \sin(B(T))$ can be written as $F = \mathbb{E}[F] + \int_0^T f(t)\,dB(t)$.*

Solution: *Observe that $\sin(B(T)) \in L^2(\mathcal{F}_T, \mathbb{P})$. In view of Ito formula and the Ito product rule, we have*

$$d(e^{\frac{t}{2}}\sin(B(t))) = (e^{\frac{t}{2}}\cos(B(t))\,dB(t) \implies \sin(B(T)) = \int_0^T e^{\frac{t-T}{2}}\cos(B(t))\,dB(t)\,.$$

Hence $\mathbb{E}[\sin(B(T))] = 0$, and therefore, we get the required representation as

$$\sin(B(T)) = \mathbb{E}[\sin(B(T))] + \int_0^T f(t)\,dB(t)$$

where $f(t,\omega) = e^{\frac{t-T}{2}}\cos(B(t)) \in \mathcal{Y}(0,T)$.

Example 3.10. *Find $f \in \mathcal{Y}(0,T)$ such that $F = B^3(T)$ can be written as $F = \mathbb{E}[F] + \int_0^T f(t)\,dB(t)$.*

Solution: *Observe that $B^3(T) \in L^2(\mathcal{F}_T, \mathbb{P})$, and $\mathbb{E}[B^3(T)] = 0$. We know that*

$$B^3(T) = 3\int_0^T B^2(s)\,dB(s) + 3\int_0^T B(s)\,ds$$
$$\int_0^T B(s)\,ds = TB(T) - \int_0^T s\,dB(s)\,.$$

Combining these two relation, we get

$$B^3(T) = 3\int_0^T B^2(s)\,dB(s) + 3T\int_0^T dB(s) - 3\int_0^T s\,dB(s) = \int_0^T 3(B^2(s) - T + s)\,dB(s)$$
$$= \mathbb{E}[B^3(T)] + \int_0^T f(t)\,dB(t) \quad \textit{where } f(s,\omega) = 3(B^2(s) - T + s) \in \mathcal{Y}(0,T).$$

Theorem 3.15 (Martingale Representation Theorem). *Let $M(t) : t \geq 0$ be a square integrable martingale with respect to a filtration generated only by Brownian motion. Then there exists a unique stochastic process $g \in \mathcal{Y}(0,t)$ for all $t \geq 0$ such that a.s., there holds*

$$M(t) = \mathbb{E}[M(0)] + \int_0^t g(s)\,dB(s) \quad \forall t \geq 0.$$

Proof. By Ito representation theorem, there exists $f^{(t)}(s) \in L^2(\mathcal{F}_t, \mathbb{P})$ such that

$$M(t) = \mathbb{E}[M(t)] + \int_0^t f^{(t)}(s)\, dB(s).$$

Now assume that $0 \le t_1 < t_2$. Then

$$\begin{aligned} M(t_1) &= \mathbb{E}\big[M(t_2)|\mathcal{F}_{t_1}\big] = \mathbb{E}[M(0)] + \mathbb{E}\Big[\int_0^{t_2} f^{(t_2)}(s)\, dB(s)\big|\mathcal{F}_{t_1}\Big] \\ &= \mathbb{E}[M(0)] + \int_0^{t_1} f^{(t_2)}(s)\, dB(s) \end{aligned}$$

But, we already have

$$M(t_1) = \mathbb{E}[M(0)] + \int_0^{t_1} f^{(t_1)}(s)\, dB(s),$$

and therefore by using Ito-isometry, we get

$$\begin{aligned} &\mathbb{E}\Big[\int_0^{t_1} (f^{(t_1)} - f^{(t_2)})^2\, ds\Big] = 0 \\ &\implies f^{(t_2)}(s,\omega) = f^{(t_1)}(s,\omega) \ \text{ for a.e. } (s,\omega) \in [0,t_1] \times \Omega. \end{aligned}$$

So, we can define $f(s,\omega)$ for a.e. $(s,\omega) \in [0,\infty) \times \Omega$ by setting

$$f(s,\omega) = f^{(N)}(s,\omega), \quad s \in [0,N].$$

Thus, we obtain

$$M(t) = \mathbb{E}[M(t)] + \int_0^t f^{(t)}(s)\, dB(s) = \mathbb{E}[M(t)] + \int_0^t f(s)\, dB(s), \quad \forall\, t \ge 0.$$

□

Example 3.11. *Write down the corresponding form of Ito representation theorem for*

$$M(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big], 0 \le t \le T.$$

Solution: *Observe that $M(t)$ is a square integrable martingale with respect to Brownian filtration $\mathcal{F}_t$, and $\mathbb{E}[M(t)] = \mathbb{E}[B^2(T)] = T$. Now by using martingale property of Brownian motion, we get*

$$\begin{aligned} M(t) &= \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big] = \mathbb{E}\big[(B(T)-B(t))^2|\mathcal{F}_t\big] + B^2(t) + 2\mathbb{E}\big[B(t)(B(T)-B(t))|\mathcal{F}_t\big] \\ &= \mathbb{E}\big[(B(T)-B(t))^2\big] + B^2(t) = T + B^2(t) - t = \mathbb{E}[M(0)] + B^2(t) - t \end{aligned}$$

From Ito formula, we know that $B^2(t) - t = 2\int_0^t B(s)\, dB(s)$. Thus, we get

$$\begin{aligned} M(t) &= \mathbb{E}[M(0)] + \int_0^t 2B(s)\, dB(s) \\ &= \mathbb{E}[M(0)] + \int_0^t f(s)\, dB(s), \quad f(s) = 2B(s) \in \mathcal{Y}(0,T). \end{aligned}$$

Example 3.12. *Write down the corresponding form of Ito representation theorem for*

$$N(t) = \mathbb{E}\big[\exp(\sigma B(T))|\mathcal{F}_t\big], 0 \le t \le T.$$

Solution: *We know that* $Y(t) := \exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\}$ *is a* $\mathcal{F}_t$*-martingale and therefore*

$$\mathbb{E}\Big[\exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\}\Big] = 1 \implies \mathbb{E}[N(t)] = \mathbb{E}\big[\exp(\sigma B(T))\big] = e^{\frac{1}{2}\sigma^2 T}.$$

Rewriting $N(t)$*, we have*

$$N(t) = e^{\frac{1}{2}\sigma^2 T}\mathbb{E}\Big[\exp\{\sigma B(T) - \frac{1}{2}\sigma^2 T\}|\mathcal{F}_t\Big] = e^{\frac{1}{2}\sigma^2 T}\exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\} = e^{\frac{1}{2}\sigma^2 T} Y(t)$$

Moreover, $Y(t)$ *satisfies the differential equation*

$$dY(t) = \sigma Y(t)\, dB(t).$$

In other words, we have $Y(t) = 1 + \sigma\int_0^t Y(s)\, dB(s)$*. Thus,*

$$\begin{aligned} N(t) &= e^{\frac{1}{2}\sigma^2 T}\ \Big(1 + \sigma\int_0^t Y(s)\, dB(s)\Big) \\ &= e^{\frac{1}{2}\sigma^2 T} + \int_0^t \sigma e^{\frac{1}{2}\sigma^2 T} Y(s)\, dB(s) \\ &= \mathbb{E}[N(0)] + \int_0^t f(s)\, dB(s),\ \textit{where}\ f(t,\omega) = \sigma e^{\frac{1}{2}\sigma^2 T} Y(t) \in \mathcal{Y}(0,T). \end{aligned}$$