Example 3.6. *Solve the (2-dimensional) stochastic differential equation*

$$dX_1 = X_2(t)dt + \alpha dB_1(t), \quad dX_2(t) = -X_1(t)\,dt + \beta\,dB_2(t)$$

where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion, and α, β are constants.
Solution: *We can re-write the given equation as 2-dimensional Ito process:*

$$d\mathbf{X}(t) = \boldsymbol{A}\mathbf{X}(t)\,dt + \boldsymbol{G}\,d\mathbf{B}(t)$$

where $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\boldsymbol{G} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$ *and* $d\mathbf{B}(t) = \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$. *We apply Ito formula with* $\mathbf{u}(t, x_1, x_2) = \exp(-t\boldsymbol{A}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ *to have*

$$d[\exp(-t\boldsymbol{A})\mathbf{X}(t)] = -\boldsymbol{A}\exp(-t\boldsymbol{A})\mathbf{X}(t)\,dt + \exp(-t\boldsymbol{A})d\mathbf{X}(t)$$

where $\exp(t\boldsymbol{A}) = \sum_{n=0}^{\infty} \frac{t^n \boldsymbol{A}^n}{n!}$. *Again, from the given equation, we have*

$$\exp(-t\boldsymbol{A})d\mathbf{X}(t) = \exp(-t\boldsymbol{A})\boldsymbol{A}\mathbf{X}(t)\,dt + \exp(-t\boldsymbol{A})\boldsymbol{G}\,d\mathbf{B}(t)$$

Combining last two equations, we have

$$\begin{aligned} & d[\exp(-t\boldsymbol{A})\mathbf{X}(t)] = \exp(-t\boldsymbol{A})\boldsymbol{G}\,d\mathbf{B}(t) \\ & \implies \mathbf{X}(t) = \exp(t\boldsymbol{A})\Big[\mathbf{X}(0) + \int_0^t \exp(-s\boldsymbol{A})\boldsymbol{G}\,d\mathbf{B}(s)\Big]. \end{aligned}$$

Since, $\boldsymbol{A}^2 = -\mathbf{I}$, *one has*

$$X_1(t) = X_1(0)\cos(t) + X_2(0)\sin(t) + \alpha\int_0^t \cos(t-s)\,dB_1(s) + \beta\int_0^t \sin(t-s)\,dB_2(s)\,,$$

$$X_2(t) = -X_1(0)\sin(t) + X_2(0)\cos(t) - \alpha\int_0^t \sin(t-s)\,dB_1(s) + \beta\int_0^t \cos(t-s)\,dB_2(s)\,.$$

3.5. **Recognizing a Brownian motion.** We have seen that Brownian motion is a continuous paths martingale and its quadratic variation up to time t is t i.e., $[B, B](t) = t$. These conditions may characterize any stochastic process to be a Brownian motion.

Theorem 3.11 (**Levy, one dimensional**). *Let $\{M(t : t \geq 0\}$ be a continuous paths martingale relative to a given filtration such that $M(0) = 0$ and $[M, M](t) = t$ for all $t \geq 0$. Then, $M(t)$ is a Brownian motion.*

Proof. We need to show that $M(t)$ is normally distributed with mean 0 and variance t. Since moment generating function uniquely determine the distribution of a stochastic process, we basically show that

$$\mathbb{E}[\exp(uM(t))] = e^{\frac{1}{2}u^2 t}, \quad \forall u \in \mathbb{R}.$$

To do so, we will use Ito formula. Observe that, in the proof of Ito formula for Brownian motion, we have used two important facts of Brownian motion namely it has continuous paths and quadratic variation up to time t is t. Since given stochastic process M has these two properties, the Ito-formula may be applied to M. Hence for any function $f(t, x)$ whose derivatives exist and are continuous, one has

$$df(t, M(t)) = f_t(t, M(t))\,dt + f_x(t, M(t))\,dM(t) + \frac{1}{2}f_{xx}(t, M(t))\,dt$$

$$\implies f(t, M(t)) = f(0,0) + \int_0^t \{f_t(s, M(s)) + \frac{1}{2} f_{xx}(s, M(s))\}\, ds + \int_0^t f_x(s, M(s))\, dM(s)$$

Since M is a martingale, the stochastic integral $\int_0^t f_x(s, M(s))\, dM(s)$ is also a martingale and hence

$$\mathbb{E}\Big[\int_0^t f_x(s, M(s))\, dM(s)\Big] = 0.$$

Thus, we have, after taking expectation in the above integral equation

$$\mathbb{E}\big[f(t, M(t))\big] = f(0,0) + \mathbb{E}\Big[\int_0^t \{f_t(s, M(s)) + \frac{1}{2} f_{xx}(s, M(s))\}\, ds\Big]. \tag{3.7}$$

For any fixed $u \in \mathbb{R}$, consider the function

$$f(t, x) = \exp\big(ux - \frac{1}{2}u^2 t\big).$$

One can easily show that

$$f_t(t, x) + \frac{1}{2} f_{xx}(t, x) = 0 \quad \forall (t, x).$$

Therefore, with this choice of $f(t, x)$, we get

$$\mathbb{E}\Big[\exp\big(uM(t) - \frac{1}{2}u^2 t\big)\Big] = 1 \implies \mathbb{E}[\exp(uM(t))] = e^{\frac{1}{2}u^2 t}.$$

This completes the proof. □

Example 3.7. *Let* $\mathbf{B}(\cdot) = \big(B_1(\cdot), B_2(\cdot)\big)$ *be a 2-dimensional Brownian motion. Define a stochastic process*

$$M(t) := \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t), \quad |\rho| \le 1.$$

Show that $M(t)$ *is a Brownian motion.*

Solution: *Observe that* $M(t)$ *is a continuous paths martingale with* $M(0) = 0$*. If we show that quadratic variation of* M *is* t*, then according to one dimensional Levy's theorem* $M(t)$ *will be a Brownian motion. Now*

$$\begin{aligned} dM(t)dM(t) &= \rho^2\, dB_1(t)\, dB_1(t) + 2\rho\sqrt{1-\rho^2} dB_1(t)\, dB_2(t) + (1 - \rho^2) dB_2(t)\, dB_2(t) \\ &= \rho^2\, dt + 0 + (1 - \rho^2)\, dt = dt \\ &\implies [M, M](t) = t \quad \forall\, t \ge 0\,. \end{aligned}$$

Thus, $M(t)$ *is a brownian motion.*

Theorem 3.12 (Levy, n-dimensional). *Let* $\mathbf{M}(t) = \big(M_1(t), M_2(t), \ldots, M_n(t)\big)$ *be a* n*-dimensional martingale relative to a filtration. Assume that for* $i = 1, 2, \ldots, n$, $M_i(0) = 0$ *and* $M_i(\cdot)$ *has continuous paths and* $[M_i, M_j](t) = \delta_{ij} t$, $1 \le i, j \le n$*. Then* $\mathbf{M}(t) = \big(M_1(t), M_2(t), \ldots, M_n(t)\big)$ *is a* n*-dimensional Brownian motion.*

Example 3.8. *Suppose* $B_1(t)$ *and* $B_2(t)$ *are Brownian motions such that*

$$dB_1(t)\, dB_2(t) = \rho(t)\, dt$$

where $\rho(\cdot)$ *is a stochastic process with values in* $(-1, 1)$*. Define two stochastic processes* $M_1(t)$ *and* $M_2(t)$ *as follows:*

$$M_1(t) = B_1(t), \quad B_2(t) = \int_0^t \rho(s)\, dM_1(s) + \int_0^t \sqrt{1 - \rho^2(s)}\, dM_2(s)$$

Show that $\mathbf{M}(t) = \big(M_1(t), M_2(t)\big)$ *is a* 2*-dimensional Brownian motion.*

Solution: *Assume that* $\big(M_1(t), M_2(t)\big)$ *is a martingales given by the differential form*

$$dM_1(t) = dB_1(t), \quad dM_2(t) = \alpha(t)\, dB_1(t) + \beta(t)\, dB_2(t)$$

where $\alpha(\cdot)$ *and* $\beta(\cdot)$ *are adapted processes such that* $[M_i, M_j](t) = \delta_{ij} t$ *for* $1 \leq i, j \leq 2$. *That is we need to choose* $\alpha(t)$ *and* $\beta(t)$ *such that*

$$\begin{aligned} &\alpha^2(t) + \beta^2(t) + 2\alpha(t)\beta(t)\rho(t) = 1 \\ &\alpha(t) + \beta(t)\rho(t) = 0 \end{aligned}$$

Solving these two equation, we obtain

$$\alpha(t) = -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}, \quad \beta(t) = \frac{1}{\sqrt{1-\rho^2(t)}}.$$

Thus, by Levy's theorem 3.12, the process $\mathbf{M}(t) = \big(M_1(t), M_2(t)\big)$ *is a* 2*-dimensional Brownian motion. It remains to show that*

$$B_2(t) = \int_0^t \rho(s)\, dM_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dM_2(s).$$

Indeed, we have

$$\begin{aligned} dM_2(t) &= -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}\, dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}}\, dB_2(t) \\ \Longrightarrow\ dB_2(t) &= \rho(t) dB_1(t) + \sqrt{1-\rho^2(t)}\, dM_2(t) \\ \Longrightarrow\ B_2(t) &= \int_0^t \rho(s) dB_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dM_2(s) \\ &= \int_0^t \rho(s)\, dM_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dM_2(s). \end{aligned}$$