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MTL733: Assignment-4

Q.1) Consider the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$. A trading strategy $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing if and only if $\tilde{V}_\Psi(t)$ can be expressed for all $t \in [0, T]$ as

$$\tilde{V}_\Psi(t) = V_\Psi(0) + \int_0^t \psi(u)\, d\tilde{S}(u)$$

where $\tilde{S}(t)$ is the discounted stock price process.

Q.2) Consider a market $X(t) = \big(S_0(t), S(t)\big)$ given by

$$dS_0(t) = 3S_0(t)\,dt,\ S_0(0) = 1; \quad dS(t) = 2S(t)\,dt + 5S(t)\,dB(t),\ S(0) = 1.$$

Show that the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$, given by

$$\psi_0(t) = \frac{1}{3}\big(e^{-3t} - 1\big), \quad \psi(t) = \int_0^t \exp\{-5B(u) + \frac{21}{2}u\}\, du$$

is self-financing.

Q.3) Consider a market $X(t) = \big(S_0(t), S(t)\big)$ where

$$dS_0(t) = 0,\ S_0(0) = 1; \quad dS(t) = tS(t)\,dt + 5S(t)\,dB(t),\ S(0) = 4.$$

Examine whether the market has arbitrage opportunity or not.

Q.4) Show that there exists an arbitrage if and only if there is a way to start with $V_\Psi(0) > 0$ and at a later time T have a portfolio value satisfying

$$\mathbb{P}\Big(V_\Psi(T) \geq \frac{V_\Psi(0)}{D(T)}\Big) = 1, \quad \mathbb{P}\Big(V_\Psi(T) > \frac{V_\Psi(0)}{D(T)}\Big) > 0. \tag{0.1}$$

Q.5) Use the Black-Scholes formula to price a European call option for a stock whose price today is \$75 with expiry date 3 months from now, strike price \$70 and volatility 20%. The risk-free interest is 7% per year.

a) Find the value of the European call option, and compute *delta* and *gamma* for this option.

b) Find the value of the European put option and compute *delta* and *gamma* for this option.

Q.6) Let Q be a risk-neutral measure of a given market $X(t) = \big(S_0(t), S(t)\big)$. The value at time zero of a European call on a stock whose initial price is $S(0) = x$ is given by

$$C(0,x) = \mathbb{E}_Q\Big[e^{-rT}\max\{0, S(T) - K\}\Big].$$

Show that there exists a probability measure $\bar{Q}$ such that

a) Q and $\bar{Q}$ are equivalent,

b) $\bar{B}(t) = \tilde{B}(t) - \sigma t$ is a Brownian motion under $\bar{Q}$, where $\tilde{B}(t)$ is a Brownian motion under the risk-neutral measure Q,

c) The delta of the call option $C_x(0,x)$ can be written as follows:

$$C_x(0,x) = \bar{Q}\big(S(T) > K\big) = \bar{Q}\Big\{-\frac{\bar{B}(T)}{\sqrt{T}} < d\Big\} = N(d)$$

where $N(\cdot)$ is the cumulative standard normal distribution and

$$d := \frac{\log(\frac{x}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Q.7) Let Q be a probability measure defined by

$$Q(A) = \int_A Z(T)\, d\mathbb{P}, \text{ where } Z(t) = \exp\Big\{-\int_0^t \theta(s)\, dB(s) - \frac{1}{2}\int_0^t \theta^2(s)\, ds\Big\}.$$

Suppose that the filtration $\{\mathcal{F}_t\}$ is generated by the Brownian motion $B(t) : 0 \le t \le T$. Let $\widetilde{M}$ be a martingale under Q. We will show that Martingale representation theorem holds for $\widetilde{M}$.

a) Show that $\widetilde{M}(t)Z(t)$ is a martingale under $\mathbb{P}$.

b) Find the differential of $(Z(t))^{-1}$ and $\widetilde{M}(t)$.

c) Show that Martingale representation theorem holds for $\widetilde{M}$.

Q.8) Suppose that a stock sells today for \$100, the value of the call option is \$6, the value of the put option is \$5 and both options have the same strike price, \$100, with one year expiry time. What is the risk-free interest rate?

Q.9) A call option on a non-divident-paying stock has a market price of \$2.5. The stock price is \$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is implied volatility?