We now discuss the distribution of

$$\bar{M}(t) := \sup_{0\le s\le t} B(s)$$

for any given t. This stochastic process is used in pricing barrier option.

Theorem 2.11 (Reflection principle). *Let $B(t)$ be Brownian motion. Then for every $m \ge 0$,*

$$\mathbb{P}(\bar{M}(t) \ge m) = 2\mathbb{P}(B(t) \ge m) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx\,.$$

Proof. We have

$$\mathbb{P}(B(t) \ge m) = \mathbb{P}(B(t) \ge m, \bar{M}(t) \ge m) + \mathbb{P}(B(t) \ge m, \bar{M}(t) < m)$$

Since $\bar{M}(t) \ge B(t)$, we have $\mathbb{P}(B(t) \ge m, \bar{M}(t) < m) = 0$. Now

$$\mathbb{P}(B(t) \ge m, \bar{M}(t) \ge m) = \mathbb{P}(B(t) \ge m | \bar{M}(t) \ge m)\mathbb{P}(\bar{M}(t) \ge m).$$

Observe that $\bar{M}(t) \ge m$ if and only if $\tau_m \le t$. Thus,

$$\mathbb{P}(B(t) \ge m, \bar{M}(t) \ge m) = \mathbb{P}(B(t) \ge m | \tau_m \le t)\mathbb{P}(\bar{M}(t) \ge m).$$

Observe that

$$\mathbb{P}(B(t) \ge m | \tau_m \le t) = \mathbb{P}(B(\tau_m + (t-\tau_m)) - m \ge 0 | \tau_m \le t)$$

In view of Lemma 2.9, $B(\tau_m + (t-\tau_m)) - m, \tau_m \le t$ is a Brownian motion and therefore, we get that

$$\mathbb{P}(B(\tau_m + (t-\tau_m)) - m \ge 0 | \tau_m \le t) = \frac{1}{2}$$

since the Brownian motion satisfies $\mathbb{P}(B(t) \ge 0) = \frac{1}{2}$ for every t. Combining all these, we get

$$\mathbb{P}(\bar{M}(t) \ge m) = 2\mathbb{P}(B(t) \ge m) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx.$$

□

Remark 2.4. Since $\bar{M}(t) \ge m$ if and only if $\tau_m \le t$, from Theorem 2.11, we see that the distribution function $F_{\tau_m}(\cdot)$ and probability density function $f_{\tau_m}(\cdot)$ are given by

$$F_{\tau_m}(t) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx; \quad f_{\tau_m}(t) = \frac{1}{\sqrt{2\pi t^3}} m e^{-\frac{m^2}{2t}}.$$

respectively.

We now establish joint probability distribution and density function of $\bar{M}(t)$ and $B(t)$.

Proposition 2.12. *For every $m > 0$, $y \ge 0$,*

$$\mathbb{P}(\bar{M}(t) \ge m, B(t) \le m - y) = \mathbb{P}(B(t) > m + y).$$

Moreover, the joint density function of $\bar{M}(t)$ and $B(t)$ is given by

$$f_{(\bar{M}(t),B(t))}(m, w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \le m,\ m > 0\,.$$

Proof. For every $m > 0$, $y \geq 0$, we have

$$\begin{aligned}
\mathbb{P}(B(t) > m+y) &= \mathbb{P}(B(t) > m+y, \bar{M}(t) \geq m) + \mathbb{P}(B(t) > m+y, \bar{M}(t) < m) \\
&= \mathbb{P}(B(t) > m+y, \bar{M}(t) \geq m) \\
&= \mathbb{P}(B(t) > m+y | \bar{M}(t) \geq m)\mathbb{P}(\bar{M}(t) \geq m) \\
&= \mathbb{P}(B(\tau_m + (t-\tau_m)) - m > y | \tau_m \leq t)\mathbb{P}(\bar{M}(t) \geq m)
\end{aligned}$$

Since $B(\tau_m + (t-\tau_m)) - m$ is a Brownian motion, by symmetry, we have

$$\begin{aligned}
&\mathbb{P}(B(\tau_m + (t-\tau_m)) - m > y | \tau_m \leq t) \\
&= \mathbb{P}(B(\tau_m + (t-\tau_m)) - m < -y | \tau_m \leq t) \\
&= \mathbb{P}(B(t) < m - y | \tau_m \leq t) = \mathbb{P}(B(t) < m-y | \bar{M}(t) \geq m)
\end{aligned}$$

Thus, we have

$$\mathbb{P}(B(t) > m+y) = \mathbb{P}(B(t) < m-y | \bar{M}(t) \geq m)\mathbb{P}(\bar{M}(t) \geq m) = \mathbb{P}(\bar{M}(t) \geq m, B(t) \leq m-y).$$

To establish the second part, we write the above equality in terms of density functions. We have

$$\begin{aligned}
&\int_m^\infty \int_{-\infty}^{m-y} f_{(\bar{M}(t),B(t))}(u,v)\,du\,dv = \frac{1}{\sqrt{2\pi t}} \int_{m+y}^\infty e^{-\frac{z^2}{2t}}\,dz \\
\implies &\int_m^\infty \int_{-\infty}^{w} f_{(\bar{M}(t),B(t))}(u,v)\,du\,dv = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}}\,dz, \quad w \leq m,\ m > 0.
\end{aligned}$$

Differentiating with respect to m, we have

$$\int_{-\infty}^{w} f_{(\bar{M}(t),B(t))}(m,v)\,dv = \frac{t}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}.$$

Again differentiating with respect to w, we get the joint density function as

$$f_{(\bar{M}(t),B(t))}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \leq m,\, m > 0\,.$$

This completes the proof. □

Theorem 2.13. *For any $\alpha > 0$, we have*

$$\mathbb{E}[\exp(-\alpha\tau_m)] = e^{-m\sqrt{2\alpha}}\,.$$

Furthermore, $\mathbb{E}[\tau_m] = \infty$.

Proof. In view of the proof of Theorem 2.10, we have shown that

$$1 = \mathbb{E}\Big[\mathbf{1}_{\{\tau_m < \infty\}} \exp\{\sigma m - \frac{1}{2}\sigma^2 \tau_m\}\Big]$$

holds for any $\sigma > 0$. Since $\mathbb{P}(\tau_m < \infty) = 1$, we actually get

$$\mathbb{E}\Big[\exp\{\sigma m - \frac{1}{2}\sigma^2\tau_m\}\Big] = 1 \implies \mathbb{E}\big[\exp\{-\frac{1}{2}\sigma^2\tau_m\}\big] = \exp(-\sigma m)\,.$$

Taking $\sigma = \sqrt{2\alpha}$ in the above equality, we get

$$\mathbb{E}[\exp(-\alpha\tau_m)] = e^{-m\sqrt{2\alpha}}\,. \tag{2.6}$$

Differentiating (2.6) with respect to α and then sending $\alpha \to 0$ in the resulting expression, we finally get that $\mathbb{E}[\tau_m] = \infty$. This completes the proof. □

3. Ito integrals:

We would like to define $\int_0^T G(s)dB(s)$ for some wide class of stochastic processes G. Observe that the integral $\int_0^T G(s)dB(s)$ simply cannot be understood as an ordinary integral (Riemann-Stieltjes sense) as the paths $t \mapsto B(t,\omega)$ is of infinite variation and nowhere differentiable for almost every ω. Let us first think about what might be an appropriate definition for $\int_0^T G(s)dB(s)$. Suppose $\Pi^n := \{0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = T\}$ are partitions of $[0,T]$ such that $\|\Pi^n\| \to 0$ as $n \to \infty$. We define corresponding Riemann sum approximation: for $\lambda \in [0,1]$

$$R_n = R_n(\Pi^n, \lambda) := \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n))$$

where $\tau_k^n := (1-\lambda)t_k^n + \lambda t_{k+1}^n,\ k = 0,1,2\ldots,m_n - 1$. We claim that

$$\lim_{n\to\infty} R_n = \frac{B^2(T)}{2} + (\lambda - \frac{1}{2})T$$

where the limit is taken in $L^2(\Omega)$. Indeed, by using the identity $a^2-b^2 = (a-b)^2+2b(a-b)$, we have

$$\begin{aligned}
R_n :&= \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&= \frac{B^2(T)}{2} - \frac{1}{2}\sum_{k=0}^{m_n-1} (B^2(t_{k+1}^n) - B^2(t_k^n)) + \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&= \frac{B^2(T)}{2} - \frac{1}{2}\sum_{k=0}^{m_n-1} (B(t_{k+1}^n) - B(t_k^n))^2 - \sum_{k=0}^{m_n-1} B(t_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&\quad + \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&= \frac{B^2(T)}{2} \underbrace{- \frac{1}{2}\sum_{k=0}^{m_n-1} (B(t_{k+1}^n) - B(t_k^n))^2}_{:=\mathcal{A}} + \underbrace{\sum_{k=0}^{m_n-1} (B(\tau_k^n) - B(t_k^n))^2}_{:=\mathcal{B}} \\
&\quad + \underbrace{\sum_{k=0}^{m_n-1} (B(t_{k+1}^n) - B(\tau_k^n))(B(\tau_k^n) - B(t_k^n))}_{:=\mathcal{C}}
\end{aligned}$$

Since the quadratic variation of Brownian motion up to time T is T, we see that $\mathcal{A} \to \frac{T}{2}$ in $L^2(\Omega)$ as $n \to \infty$. Again, by a similar argument , we can show easily that $\mathcal{B} \to \lambda T$ as $n \to \infty$. By independent increments of Brownian motion, we see that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{C}^2\big] &= \sum_{k=0}^{m_n-1} \mathbb{E}\big[(B(t_{k+1}^n) - B(\tau_k^n))^2\big]\mathbb{E}\big[(B(\tau_k^n) - B(t_k^n))^2\big] \\
&= \sum_{k=0}^{m_n-1} \lambda(1-\lambda)(t_{k+1}^n - t_k^n)^2 \le \|\Pi^n\|\lambda(1-\lambda)T
\end{aligned}$$

$$\Longrightarrow \lim_{n\to\infty} \mathbb{E}\big[(\mathcal{C}-0)^2\big] = 0\,.$$

Combining all these we get

$$\lim_{n\to\infty} R_n = \frac{B^2(T)}{2} + (\lambda - \frac{1}{2})T.$$

So, unlike Riemann-Stieltjes integral-it does make a difference here what point τ_k^n we choose. The following two choices are to be most useful:

i) $\tau_k^n = t_k^n$ (left end point corresponding to $\lambda = 0$), which leads to **Ito integral**, denoted by $\int_0^T BdB(s)$ and the value is given by $\int_0^T G(s)dB(s) = \frac{B^2(T)}{2} - \frac{T}{2}$.

ii) $\tau_k^n = \frac{t_k^n + t_{k+1}^n}{2}$ (the mid point corresponding to $\lambda = \frac{1}{2}$), which leads to **Stratonovich integral**, denoted by $\int_0^T B \circ dB(t)$, and the value is given by

$$\int_0^T B \circ dB(t) = \frac{B^2(T)}{2}.$$

3.1. **Ito integral:** Let us first describe the class of functions for which Ito integral will be defined. Let $\mathcal{Y} = \mathcal{Y}[0,T]$ be the class of functions $f(t,\omega) : [0,T]\times\Omega \to \mathbb{R}$ such that it is jointly measurable, $f(t)$ is $\mathcal{F}_t$-adapted and $\mathbb{E}\Big[\int_0^T f^2(t)\,dt\Big] < +\infty$.

Definition 3.1 (Step/Elementary process)**.** A process $G \in \mathcal{Y}$ is called step/elementary process if there exists a partition $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_m = T\}$ of $[0,T]$ such that

$$G(t,\omega) = \sum_{k=0}^{m-1} G_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t).$$

G_k is $\mathcal{F}_{t_k}$-measurable random variable.

Definition 3.2 (Stochastic integral of elementary process)**.** For elementary process G as above, we define the Ito stochastic integral of G as

$$\int_0^T GdB(t) := \sum_{k=0}^{m-1} G_k(B(t_{k+1}) - B(t_k)).$$

We can easily see that for all constants $a, b \in \mathbb{R}$ and all elementary processes $G, H \in \mathcal{Y}$,

$$\int_0^T (aG + bH)dB(t) = a\int_0^T GdB(t) + b\int_0^T HdB(t).$$

We claim that

$$\mathbb{E}\Big[\int_0^T GdB(t)\Big] = 0.$$

Indeed, we have

$$\mathbb{E}\Big[\int_0^T GdB(t)\Big] = \mathbb{E}\Big[\sum_{k=0}^{m-1} G_k(B(t_{k+1}) - B(t_k))\Big] = \sum_{k=0}^{m-1}\mathbb{E}\Big[G_k(B(t_{k+1}) - B(t_k))\Big]$$

Since G_k is $\mathcal{F}_{t_k}$-measurable and $(B(t_{k+1}) - B(t_k))$ is $\mathcal{F}_{t_k}$-independent, we get

$$\mathbb{E}\Big[G_k(B(t_{k+1}) - B(t_k))\Big] = \mathbb{E}[G_k]\mathbb{E}\big[(B(t_{k+1}) - B(t_k))\big] = 0.$$

Hence , we have $\mathbb{E}\Big[\int_0^T GdB(t)\Big] = 0$.

Lemma 3.1 (Ito-isometry)**.** *If $G \in \mathcal{Y}$ is elementary, then*

$$\mathbb{E}\Big[\Big(\int_0^T G\,dB(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T G^2(t)\,dt\Big].$$

This is called Ito-isometry.

Proof. Since $G \in \mathcal{Y}$ is elementary, there exists a partition $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_m = T\}$ of $[0, T]$ such that

$$G(t,\omega) = \sum_{k=0}^{m-1} G_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t).$$

Thus,

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^T G\,dB(t)\Big)^2\Big] &= \mathbb{E}\Big[\Big(\sum_{k=0}^{m-1} G_k(B(t_{k+1}) - B(t_k))\Big)^2\Big] \\
&= \sum_{k,j=0}^{m-1} \mathbb{E}\Big[G_jG_k(B(t_{k+1}) - B(t_k))(B(t_{j+1}) - B(t_j))\Big] \\
&= \sum_{k,j=0}^{m-1} \mathbb{E}\Big[G_jG_k\Delta B_k\Delta B_j\Big]
\end{aligned}$$

where $\Delta B_j = B(t_{j+1}) - B(t_j)$. Now if $j < k$, ΔB_k is independent of $G_jG_k\Delta B_j$ and therefore, we have

$$\mathbb{E}\Big[G_jG_k\Delta B_k\Delta B_j\Big] = \begin{cases} 0, & k \neq j \\ \mathbb{E}[G_k^2](t_{k+1} - t_k), & j = k. \end{cases}$$

Hence, we have

$$\mathbb{E}\Big[\Big(\int_0^T G\,dB(t)\Big)^2\Big] = \sum_{k=0}^{n-1} \mathbb{E}[G_k^2](t_{k+1} - t_k) = \mathbb{E}\Big[\sum_{k=0}^{n-1} G_k^2(t_{k+1} - t_k)\Big] = \mathbb{E}\Big[\int_0^T G^2(t)\,dt\Big].$$

□