Boundary conditions: In order to determine the solution, one needs boundary conditions at $x = 0$ and $x = \infty$. Substituting $x = 0$ in (5.10), we have

$$C_t(t, 0) = rC(t, 0) - - - \text{an ordinary differential equation.}$$

Solution to this ODE is given by $C(t, 0) = e^{rt}C(0, 0)$. Again, by substituting $t = T$ into this equation, and the fact that $C(T, 0) = \max\{0 - K, 0\}$, we have

$$0 = e^{rT}C(0, 0) \implies C(0, 0) = 0.$$

Hence one boundary condition is given by

$$C(t, 0) = 0 \quad \forall\, t \in [0, T]. \tag{5.12}$$

As $x \to \infty$, the function $C(t, x)$ grows without bound. In such case, we give boundary condition at $x = \infty$ by specifying the rate of growth. One such way to specify a boundary condition at $x = \infty$ for the European call is

$$\lim_{x\to\infty} \Big[C(t, x) - e^{-r(T-t)}K\Big] = 0 \quad \forall\, t \in [0, T]\,. \tag{5.13}$$

Solution of Black-Scholes-Merton equation: Let $h(x) = \max\{0, x - K\}$. Then by discounted Feynman-Kac formula, we see that the function

$$C(t, x) = e^{-r(T-t)}\mathbb{E}[h(Z^{t,x}(T))]$$

solves the PDE (5.10) with terminal condition (5.11), where the stochastic process $Z^{t,x}(s)$ is given by: for $s \ge t$

$$Z^{t,x}(s) = x + r\int_t^s Z^{t,x}(u)\,du + \sigma\int_t^s Z^{t,x}(u)\,dB(u).$$

Observe that $Z^{t,x}(s)$ is a geometric Brownian motion starting at x and time t and we have

$$Z^{t,x}(s) = x\exp\Big\{\big(r - \frac{\sigma^2}{2}\big)(s - t) + \sigma\big(B(s) - B(t)\big)\Big\}.$$

Observe that $\sigma\big(B(T) - B(t)\big) \sim \mathcal{N}(0, \sigma^2(T - t))$ and therefore

$$\sigma\big(B(T) - B(t)\big) = \sigma\sqrt{T - t}Y, \quad Y := \frac{B(T) - B(t)}{\sqrt{T - t}} \sim \mathcal{N}(0, 1).$$

Thus,

$$\mathbb{E}[h(Z^{t,x}(T))] = \mathbb{E}\Big[\max\{0, \exp\{\log(x) + \big(r - \frac{\sigma^2}{2}\big)(T - t) + \sigma\sqrt{T - t}Y\} - K\Big]$$

Note that the random variable inside the expectation is zero when Y is such that

$$\log(x) + \big(r - \frac{\sigma^2}{2}\big)(T - t) + \sigma\sqrt{T - t}Y < \log(K) \Leftrightarrow Y < -d_2,$$

where

$$d_2 := \frac{\log(\frac{x}{K}) + \big(r - \frac{\sigma^2}{2}\big)(T - t)}{\sigma\sqrt{T - t}}\,.$$

Thus, we get

$$\mathbb{E}[h(Z^{t,x}(T))] = \int_{-d_2}^{\infty}\Big[\exp\{\log(x) + \big(r - \frac{\sigma^2}{2}\big)(T - t) + \sigma\sqrt{T - t}y\} - K\Big]\phi(y)\,dy$$

$$= xe^{r(T-t)} \int_{-d_2}^{\infty} \exp\{-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}y\}\phi(y)\,dy - K\int_{-d_2}^{\infty}\phi(y)\,dy\,,$$

where ϕ is the probability density function of Y. Since ϕ is symmetry around *zero*, one has

$$K\int_{-d_2}^{\infty}\phi(y)\,dy = KN(d_2),$$

where N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz.$$

By using change of variable $z = y - \sigma\sqrt{T-t}$, we see that

$$\int_{-d_2}^{\infty} \exp\{-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}y\}\phi(y)\,dy$$
$$= \int_{-d^2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(y-\sigma\sqrt{T-t}\right)^2}{2}}\,dy = N(d_2 + \sigma\sqrt{T-t}) = N(d_1)\,,$$

where

$$d_1 = d_2 + \sigma\sqrt{T-t}.$$

Therefore, the solution of the Black-Scholes-Merton equation (5.10) with the terminal conditions (5.11) is given by the formula

$$\begin{aligned} C(t,x) &= e^{-r(T-t)}\mathbb{E}[h(Z^{t,x}(T))] \\ &= e^{-r(T-t)}\Big\{xe^{r(T-t)}N(d_2+\sigma\sqrt{T-t}) - KN(d_2)\Big\} \\ &= xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big)\,, \end{aligned} \tag{5.14}$$

where

$$d_\pm(\tau,x) := \frac{1}{\sigma\sqrt{\tau}}\Big[\log(\frac{x}{K}) + \big(r \pm \frac{\sigma^2}{2}\big)\tau\Big].$$

Example 5.3. *Consider a European call option with strike price* \$70 *and 2 years to expiry. The stock price. is* \$50 *and risk-free interest rate is* 8% *per year, and the volatility is* 20% *annually. What is the value of call option.*
Solution: *Here the parameter values are*

$$K = 70,\ r = 0.08,\ T = 2,\ S = 50,\ \sigma = 0.2\,.$$

Thus

$$d_1 := \frac{\log(\frac{50}{70}) + \big(0.08 - \frac{(0.02)^2}{2}\big)(2)}{0.2\sqrt{2}} = -0.4825$$
$$\implies N(d_1) = 1 - N(-d_1) = 1 - 0.685 = 0.315.$$
$$N(d_2) = N(d_1 - 0.2\sqrt{2}) = N(-0.765) = 0.22\,.$$

Therefore, the value of call option is given by

$$C = 50N(d_1) - 70e^{-0.16}N(d_2) = 2.63.$$

Greeks: The option pricing formula depends on five parameters namely S, K, T, r and σ. It is important to analyze the change of option price with respect to these parameters. These variations are known as **Greeks**.

Hedging/ delta of a call option: The *delta* of a European call option is the rate of change of its value with respect to the underlying asset price. The number of shares in the hedge is $\psi_H(t) = \partial_x C(t, S(t))$, and thus, we need to calculate $\partial_x C(t,x)$. Now

$$\partial_x C(t,x) = \frac{\partial}{\partial x}\Big[e^{-r(T-t)}\mathbb{E}[h(Z^{t,x}(T))]\Big] = e^{-r(T-t)}\mathbb{E}\Big[h'(Z^{t,x}(T))\frac{\partial}{\partial x}Z^{t,x}(T)\Big].$$

Observe that

$$\frac{\partial}{\partial x}Z^{t,x}(T) = \frac{\partial}{\partial x}\Big[x\exp\Big\{\big(r-\frac{\sigma^2}{2}\big)(s-t)+\sigma\big(B(s)-B(t)\big)\Big\}\Big] = Z^{t,1}(T).$$

Since we are replication a call option, a.e $x \in \mathbb{R}^*$

$$h'(x) = \begin{cases} 1, & \text{if } x > K \\ 0, & \text{if } x < K. \end{cases}$$

Thus,

$$\begin{aligned}
\partial_x C(t,x) &= e^{-r(T-t)}\mathbb{E}\Big[h'(Z^{t,x}(T))Z^{t,1}(T)\Big] \\
&= e^{-r(T-t)}\mathbb{E}\Big[Z^{t,1}(T)\mathbf{1}_{\{Z^{t,x}(T)>K\}}\Big] \\
&= e^{-r(T-t)}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{\big(r-\frac{\sigma^2}{2}\big)(T-t)+\sigma\sqrt{T-t}y-\frac{y^2}{2}\}\,dy \\
&= \int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy = N(d_1).
\end{aligned}$$

Thus, we have proved the following theorem.

Theorem 5.2. *The number of shares of the underlying stock in the hedging portfolio of a call option with strike price K at expiry date T is given by*

$$\psi_H(t) = N(d_1).$$

Remark 5.5. Since $N(\cdot)$ is the cumulative standard normal distribution, and the delta of a European call option $\psi_H(t) = \partial_x C(t, S(t)) = N(d_1)$, we see that

$$0 < \psi_H(t) < 1.$$

Therefore, the value of a European call option is always increasing as the underlying asset price increases. The delta of the put option is also given by the option's first derivative with respect to the underlying asset price. The delta of the put option is given by $\psi_H(t) - 1 < 0$.

Example 5.4. *Consider a European call option as described in the Example 5.3. Compute the delta for this option.*

Solution: *In the Example 5.3, we have seen that $N(d_1) = 0.315$, and hence the delta for this option is* 0.315.

Theta: the time decay factor The theta (Θ) of a European claim with value function $C(t, S(t))$ is the rate of change of option price with respect to the real time i.e.,

$$\Theta = \frac{\partial C}{\partial t}.$$

Thus we have

$$\Theta = xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}.$$

Observe that

$$\begin{aligned}
Ke^{-r(T-t)}N'(d_2) &= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} = Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1-\sigma.\sqrt{T-t})^2}{2}} \\
&= Ke^{-r(T-t)}e^{\sigma d_1\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \\
&= Ke^{-r(T-t)}e^{\sigma d_1\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_1).
\end{aligned}$$

In view of the definition of d_1 i.e.,

$$d_1 = \frac{\log(\frac{x}{K}) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

we get that

$$e^{\sigma d_1\sqrt{T-t}} = \frac{x}{K}e^{(r+\frac{\sigma^2}{2})(T-t)}$$

and hence we have

$$\begin{aligned}
\Theta &= xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{x}{K}e^{(r+\frac{\sigma^2}{2})(T-t)}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_1)\frac{\partial d_2}{\partial t} \\
&= xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - xN'(d_1)\frac{\partial d_2}{\partial t} \\
&= xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - xN'(d_1)\frac{\partial}{\partial t}\left[d_1 - \sigma\sqrt{T-t}\right] \\
&= -\frac{\sigma x}{2\sqrt{T-t}}N'(d_1) - rKe^{-r(T-t)}N(d_2).
\end{aligned}$$

Since both $N(\cdot)$ and $N'(\cdot)$ are positive, theta is always negative and therefore the value of a European call option is a decreasing function of time. Theta is not the same type of hedge parameter as delta. This is because although there is some uncertainty about the future stock price there is no uncertainty about the passage of time. It does not make sense to hedge against the effect of the passage of time on an option portfolio.

Gamma: the convexity factor The gamma (Γ) of a European call option is the sensitivity of delta with respect to asset price. Thus

$$\Gamma = \frac{\partial^2}{\partial x^2}C(t,x) = \frac{\partial}{\partial x}N(d_1) = N'(d_1)\frac{\partial}{\partial x}d_1 = \frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}}.$$

Note that $\Gamma > 0$. If gamma is small, then delta changes only slowly and adjustments in the hedge ration need only be made infrequently. However, if gamma is large, then the hedge ration delta is highly sensitive to changes in the price of the underlying security.

Rho: the interest rate factor It is the rate of change of the value of the financial derivative with respect to the interest rate. For a European call option, rho is given by

$$\rho := \frac{\partial C}{\partial r} = xN'(d_1)\frac{\partial d_1}{\partial r} + K(T-t)e^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r}\,.$$

Since

$$Ke^{-r(T-t)}N'd_2 = xN'(d_1), \quad \frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r},$$

we have

$$\rho = K(T-t)e^{-r(T-t)}N(d_2)\,.$$

ρ is always positive.

Vega: the volatility factor. This is the rate of change of value of the derivative with respect to the volatility of the underlying asset. For a European call option, the vega is given by

$$\begin{aligned}\nu := \frac{\partial C}{\partial \sigma} &= xN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma} \\ &= xN'(d_1)\Big\{\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\Big\} = xN'(d_1)\sqrt{T-t} \\ &= x\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}.\end{aligned}$$

Vega is always positive. An increase in the volatility will lead to an increase in the call option value.