By using n-dimensional Levy's theorem i.e., Theorem 3.12 we arrive at the following multi-dimensional Girsanov's theorem.

Theorem 3.20 (Girsanov's theorem, n-dimensional). *Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \dots, B_n(t)\big) : 0 \le t \le T$ be a n-dimensional Brownian motion associated with a given filtration $\{\mathcal{F}_t\}$. Let $\mathbf{\Theta}(t) = \big(\theta_1(t), \dots, \theta_n(t)\big)$ be a n-dimensional adapted process such that*

$$Z(t) = \exp\Big\{ -\int_0^t \sum_{i=1}^n \theta_i(s)\, dB_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^d \theta_i^2(s)\, ds \Big\}$$

becomes a martingale. Then the stochastic process, defined by

$$\bar{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \mathbf{\Theta}(s)\, ds \quad 0 \le t \le T$$

is a n-dimensional Brownian motion under the probability measure $\tilde{\mathbb{P}}$, where

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

We have n-dimensional version of Corollary 3.19.

Corollary 3.21. *Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \dots, B_m(t)\big) : 0 \le t \le T$ be a m-dimensional Brownian motion relative to a filtration $\{\mathcal{F}_t\}$, and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{L}_n^1[0,T]$, $\mathbf{\Theta}(t) = ((\theta_{ij})) \in \mathcal{Y}_{n\times m}(0,T)$. Let $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t), \dots, Y_n(t)\big)$ be a n-dimensional Ito process of the form*

$$d\mathbf{Y}(t) = \mathbf{b}(t)\, dt + \mathbf{\Theta}(t)\, d\mathbf{B}(t), \quad 0 \le t \le T.$$

Suppose there exist $\boldsymbol{u} = \big(u_1, u_2, \dots, u_m\big) \in \mathcal{Y}_{1\times m}(0,T)$ and $\mathbf{a} \in \mathbb{L}_n^1(0,T)$ such that

$$\mathbf{\Theta}(t)\boldsymbol{u}(t) = \mathbf{b}(t) - \mathbf{a}(t)$$

and

$$Z(t) = \exp\Big\{ -\int_0^t \sum_{i=1}^m u_i(s)\, dB_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^m u_i^2(s)\, ds \Big\}$$

becomes a martingale. Then the stochastic process, defined by

$$\bar{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \mathbf{u}(s)\, ds \quad 0 \le t \le T$$

is a m-dimensional Brownian motion under the probability measure $\tilde{\mathbb{P}}$, where

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

Moreover, in terms of $\bar{\mathbf{B}}(t)$, the n-dimensional process $\mathbf{Y}(t)$ has the following representation:

$$d\mathbf{Y}(t) = \mathbf{a}(t)\, dt + \mathbf{\Theta}(t)\, d\bar{\mathbf{B}}(t).$$

Example 3.15. *Consider a 2-dimensional Ito process $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t)\big)$ given by*

$$dY_1(t) = 2dt + dB_1(t) + dB_2(t), \quad dY_2(t) = 4dt + +dB_1(t) - dB_2(t)$$

where $\mathbf{B}(t) = \big(B_1(t), B_2(t)\big)$ is a 2-dimensional Brownian motion. Find a probability measure $\bar{\mathbb{P}}$ such that $\mathbf{Y}(t)$ is a martingale with respect to $\bar{\mathbb{P}}$.

Solution: *Given Ito process can be written in the form*

$$d\mathbf{Y}(t) = \mathbf{b}(t)\,dt + \mathbf{\Theta}(t)\,d\mathbf{B}(t), \quad 0 \le t \le T.$$

where $\mathbf{b} = (2,4) \in \mathbb{L}^1_2[0,T]$ *and* $\mathbf{\Theta}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathcal{Y}_{2\times 2}(0,T)$. *Let* $\boldsymbol{u} = \big(u_1, u_2\big) \in \mathcal{Y}_{1\times 2}(0,T)$ *such that* $\mathbf{\Theta}(t)\boldsymbol{u}(t) = \mathbf{b}(t)$. *Then* $u_1(t) = 3$ *and* $u_2(t) = -1$. *Note that the process*

$$Z(t) = \exp\Big\{ - \int_0^t \sum_{i=1}^2 u_i(s)\,dB_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^2 u_i^2(s)\,ds \Big\}$$

becomes a martingale (Novikov condition holds trivially). Define a probability measure $\bar{\mathbb{P}}$ *as*

$$\bar{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

Then by Corollary 3.21, the stochastic process

$$\bar{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \mathbf{u}(s)\,ds \quad 0 \le t \le T$$

is a 2-dimensional Brownian motion under the probability measure $\bar{\mathbb{P}}$. *Moreover,* $\mathbf{Y}(t)$ *can be written as*

$$d\mathbf{Y}(t) = \mathbf{\Theta}(t)\,d\bar{\mathbf{B}}(t).$$

This shows that $\mathbf{Y}(t)$ *is a martingale with respect to the probability measure* $\bar{\mathbb{P}}$.

4. Stochastic Differential Equation and connection with PDEs

Let $B(t)$ be a m-dimensional Brownian motion and Z be a random variable independent of $B(\cdot)$. Consider a stochastic differential equation (**SDE**) of the type

$$\begin{cases} dX(t) = a(t, X(t))\,dt + \sigma(t, X(t))\,dB(t) \\ X(0) = Z \end{cases} \tag{4.1}$$

where $a : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given functions. We now discuss existence and uniqueness of solution of the SDE. Let $\mathcal{F}_t$ be the filtration generated by Z and $B(t)$ i.e., $\mathcal{F}_t = \sigma(Z, B(s) : 0 \le s \le t)$. Let us first define the solution concept.

Definition 4.1. Let $B(t)$ be a m-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Z be a random variable on it, independent of $B(\cdot)$. An $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ on $(\Omega, \mathcal{F})$ is a (strong) solution of the SDE (4.1) if

i) $X(t)$ is $\mathcal{F}_t$-adapted stochastic process where $\mathcal{F}_t = \sigma(Z, B(s) : 0 \le s \le t)$.
ii) For all $0 \le t \le T$, there holds

$$X(t) = Z + \int_0^t a(s, X(s))\,ds + \int_0^t \sigma(s, X(s))\,dB(s)$$

where the integrals on the right hand side are well-defined.

We now prove a technical lemma so called **Gronwall's lemma** which is useful in our uniqueness proof.

Lemma 4.1. *Let $\phi(\cdot)$ and $f(\cdot)$ are nonnegative continuous function on $[0,T]$ and $C \geq 0$ be a constant. If $\phi(t) \leq C + \int_0^t f(s)\phi(s)\,ds$ for all $t \in [0,T]$, then*

$$\phi(t) \leq Ce^{\int_0^t f(s)\,ds}, \quad t \in [0,T].$$

Proof. Let $h(t) = C + \int_0^t f(s)\phi(s)\,ds$. Then by given condition, $\phi(t) \leq h(t)$ and hence, since f is nonnegative, $f(t)\phi(t) \leq f(t)h(t)$. We have

$$\begin{aligned}
&h'(t) = f(t)\phi(t) \leq f(t)h(t)\\
&\implies \big(h'(t) - h(t)f(t)\big)e^{-\int_0^t f(s)\,ds} \leq 0\\
&\implies \Big(h(t)e^{-\int_0^t f(s)\,ds}\Big)' \leq 0\\
&\implies h(t)e^{-\int_0^t f(s)\,ds} \leq h(0)e^{-\int_0^0 f(s)\,ds} = C\\
&\implies h(t) \leq Ce^{\int_0^t f(s)\,ds}\\
&\implies \phi(t) \leq Ce^{\int_0^t f(s)\,ds}.
\end{aligned}$$

This completes the proof. □

We now prove existence and uniqueness theorem for SDE.

Theorem 4.2. *Let $a : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|a(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^d,\ t \in [0,T] \tag{4.2}$$

$$|a(t,x) - a(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|, \quad x,y \in \mathbb{R}^n,\ t \in [0,T] \tag{4.3}$$

for some constants C and D and $Z \in L^2(\Omega)$. Then the SDE (4.1) has a unique continuous solution $X(t)$ in the sense of Definition 4.1. Moreover, there exists $K = K(C,T)$ such that

$$\mathbb{E}[|X(t)|^2] \leq K(1 + \mathbb{E}[Z^2])e^{Kt}.$$

Proof. We first prove its uniqueness. Suppose there exist two solutions X and Y of (4.1) with continuous paths a.s. Then for all $t \in [0,T]$, we have

$$X(t) - Y(t) = \int_0^t \big(a(s,X(s)) - b(s,Y(s))\big)\,ds + \int_0^t \big(\sigma(s,X(s)) - \sigma(s,Y(s))\big)\,dB(s)\,.$$

We now use Ito-isometry, Cauchy-Schwartz inequality, together with the assumption (4.3) to have

$$\begin{aligned}
&\mathbb{E}\Big[|X(t) - Y(t)|^2\Big]\\
&\leq 2\mathbb{E}\Big[\Big(\int_0^t \big(a(s,X(s)) - a(s,Y(s))\big)\,ds\Big)^2 + \Big(\int_0^t \big(\sigma(s,X(s)) - \sigma(s,Y(s))\big)\,dB(s)\Big)^2\Big]\\
&\leq 2T\mathbb{E}\Big[\int_0^t \big(a(s,X(s)) - a(s,Y(s))\big)^2\,ds\Big] + 2\mathbb{E}\Big[\int_0^t \big(\sigma(s,X(s)) - \sigma(s,Y(s))\big)^2\,ds\Big]\\
&\leq 2D^2(T+1)\int_0^t \mathbb{E}\Big[|X(s) - Y(s)|^2\Big]\,ds
\end{aligned}$$

Hence by Grownwall's lemma, $\mathbb{E}\Big[|(X(t)-Y(t)|^2\Big]=0$ for all $t\in[0,T]$. Hence

$$\mathbb{P}\Big(|X(t)-Y(t)|=0 \text{ for all } t\in\mathbb{Q}\cap[0,T]\Big)=1$$

where $\mathbb{Q}$ denotes the rational numbers. By continuity $t\mapsto |X(t)-Y(t)|$, it follows that

$$\mathbb{P}\Big(X(t)=Y(t) \text{ for all } t\in[0,T]\Big)=1.$$

This completes the uniqueness proof.