Stochastic of Finance Lecture 1

Contents

1 Recap

Definition 1

Axiomatic definition of probability spaces

Let Ω be the set of all possible outcomes of a random experiment. Let $\mathcal{F} \subseteq 2^{\Omega}$ be a σ-algebra on Ω, i.e., the following hold:

1. $\Omega \in \mathcal{F}$.
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under countable union, that is, if $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Let P be a function (measure) $P : \mathcal{F} \to [0, 1]$ such that the following hold:

1. $P(A) \geq 0$ (trivially holds due to range).
2. P is σ-additive, i.e., if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a countable collection of disjoint subsets, then we have $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.
3. $P(\Omega) = 1$.

Then (Ω, F, P) is a probability space.

The classical probability comes from the formal definition when Ω is finite, $|\Omega| = n$, $P(\{\omega\}) = \frac{1}{n}$ where $\omega \in \Omega$, $\mathcal{F}$ is the power set of Ω, which also turns out to be a σ-field on Ω.

Definition 2

Random variables:

Given a probability space $(\Omega, \mathcal{F}, P)$, if $X : \Omega \to \mathbb{R}$ is a function such that $X^{-1}((-\infty, x]) \in \mathcal{F}\ \forall x \in \mathbb{R}$, then X is a random variable, or a measurable function w.r.t. $\mathcal{F}$.

Example 1

If F is the largest σ-field (power set but maybe for infinite sets), then any real-function is a random variable.

Example 2

A constant function is always a random variable.

Definition 3

Stochastic processes:

A stochastic process $\{X(t), t \in T\}$ is a collection of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$.

Example 3

Examples of real-life stochastic processes:

1. Price of some stock at the end of the day.
2. Number of trades made every second.
3. Market index at time t.
4. Number of companies registered in stock market at the end of the week.
5. Variance in a stock price in a day - measure on the random variables (since computed from the data). Usually we call observed information (and not computed information) a random variable. Note also time series. Nothing wrong with calling this a random variable, but this won't be the focus of the course.

Some stochastic processes have some important properties, as follows:

1. Independence (mutual, not pairwise) - can verify such assumptions.
2. Stationary - many times we can assume that data is stationary. Two types:
   (a) Wide sense
   (b) Strict sense (by default)
3. Memoryless property.
4. Martingale property - also useful with conditional expectations.

For more, revisit MTL106. Time homogeneous is similar to stationary.

Example 4

Poisson process

$\{N(t), t \geq 0\}$ - number of events occuring upto and including time t. Suppose $N(t) \sim \mathcal{P}(\lambda t)$ where $\mathcal{P}$ is the Poisson distribution, and λ is a fixed parameter. This stochastic process is called a Poisson process. Some properties:

1. Increments are independent.
2. Increments are stationary.
3. Satisfies the memoryless property.
4. Doesn't satisfy the martingale property.

We can derive a random variable that satisfies the martingale property from any random variable.

Example 5

Brownian motion/Wiener process

Let $\{W(t), t \geq 0\}$ be a stochastic process which satisfies the following conditions:

1. $W(0) = 0$
2. For fixed t, $W(t) \sim \mathcal{N}(0, t)$
3. Increments are independent.
4. Increments are stationary.

2 Content

Definition 4

Filtration: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $\{\mathcal{F}_t \mid t \geq 0\}$ of sub σ-fields of $\mathcal{F}$ is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$.

HW: create some examples of filtrations.

Definition 5

If Ω is a space of functions on $T \subset \mathbb{R}^+$, then it comes with a natural filtration $\mathcal{F}_t = \sigma\{x(s), s \leq t\}$ where x is a stochastic process. That is, consider the set of all possible values of $x(s)$ where $s \leq t$, and generate a σ-field out of it.

Definition 6

Given a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_t \subset F$, a family $\{M(t), t \geq 0\}$ (this is a stochastic process) is called a martingale wrt $(\Omega, \mathcal{F}_t, P)$ if

1. For almost all $w \in \Omega$, we have $M(t, w)$ has left and right limits at every t and is continuous from the right.
2. For each $t \geq 0$, $M(t)$ (random variable) is a measurable function wrt $\mathcal{F}_t$ and integrable.
3. For $0 \leq s \leq t$, $\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$ almost everywhere/surely.

Example 6

Example for filtration:

The random experiment is tossing an unbiased coin infinitely many times.

We have $\Omega = \{HHH \ldots, HTH \ldots, \ldots\}$.

Let A_H be the collection of samples starting with H in the first toss.

Let A_T be the collection of samples starting with T in the second toss.

Let A_{HH} be the collection of samples starting with H in the first toss and H in the second toss.

Let A_{HT} be the collection of samples starting with H in the first toss and T in the second toss.

Consider the trivial σ-field $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Using the first toss, we construct the σ-field $\mathcal{F}_1 = \{\emptyset, A_H, A_T, \Omega\}$.

Using the second toss, we construct the σ-field $\mathcal{F}_2 = \{\emptyset, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH} \cup A_{HT}, \ldots, A_{HH}^c, \ldots, \Omega\}$.

Note that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. Also note that $\lim_{n\to\infty} \mathcal{F}_n = \mathcal{F}_\infty = \mathcal{F}$.

Example 7

Non-example for martingale:

Let $\{N(t), t \geq 0\}$ be a Poisson process on $(\Omega, \mathcal{F}, P)$.

Now that the parameter space is contained in $\mathbb{R}^+$, we have a natural filtration $\mathcal{F}_t = \sigma\{N(s), s \leq t\}$.

Note the following properties:

1. $N(t, w)$ is right continuous at t for $w \in \Omega$.
2. $N(t)$ is a measurable function wrt $\mathcal{F}_t$ and integrable.
3. $\mathbb{E}[N(t) \mid \mathcal{F}_s] = \mathbb{E}[N(t) - N(s) + N(s) \mid \mathcal{F}_s] = N(s) + \lambda(t - s)$.

Therefore this is not a martingale wrt the given filtration.

Example 8

$\{W(t), t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ with the natural filtration. The first two conditions hold as usual. The third condition:

$\mathbb{E}[W(t) \mid \mathcal{F}_s] = \mathbb{E}[W(t) - W(s) + W(s) \mid \mathcal{F}_s] = \mathbb{E}[W(t) - W(s)] + \mathbb{E}[W(s) \mid \mathcal{F}_s] = 0 + \mathbb{E}[W(s)]$ since $W(t) \sim \mathcal{N}(0, t)$ and $\mathbb{E}[W(s) \mid \mathcal{F}_s] = \mathbb{E}[W(s)]$.

Hence brownian motion is a martingale wrt the natural filtration $\{\mathcal{F}(t), t \geq 0\}$.

Definition 7

Sub-martingale: If $\mathbb{E}[X(t) \mid \mathcal{F}(s)] \geq X(s)$ a.e., then it is called a sub-martingale.

Definition 8

Super-martingale: If $\mathbb{E}[X(t) \mid \mathcal{F}(s)] \leq X(s)$ a.e., then it is called a super-martingale.

Poisson process is a sub-martingale.

Example 9

Let $\{X_n \mid n = 0, 1, 2, \dots\}$, where $X_n =$ the amount at the end of the n^{th} game. Y_i is the payoff of the i^{th} game, where $P[Y_i = 1] = P[Y_i = -1] = \frac{1}{2}$. Suppose $X_0 = A$.

We have $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n \mid \mathcal{F}_n] + \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n] + \mathbb{E}[Y_{n+1}] = \mathbb{E}[X_n]$. So this stochastic process is a martingale. Also note that $\mathbb{E}[X_n] = \mathbb{E}[X_0 + Y_1 + \cdots + Y_n] = A + 0 + \cdots + 0 = A$.

Definition 9

Markov Property

Let $\{X(t) \mid t \geq 0\}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$. If for $0 \leq s \leq t$, $P(X(t) \mid X(u), 0 \leq u \leq s) = P(X(t) \mid X(s))$, then this stochastic process is a Markov process, and is said to have the Markov property.

The same can be done for discrete processes.

For instance, verify that $P(X_{n+1} = x_{n+1} \mid X_0 = A, X_1 = x_1, \dots, X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$, which gives us that the random walk is also a Markov process.

Example 10

Consider the Poisson process $\{N(t) \mid t \geq 0\}$. Then we have $P(N(t) = k \mid N(u), 0 \leq u \leq s) = P(N(t) = k \mid N(s))$ due to independent increments, so this is a Markov process.

Any process with independent increments is a Markov process.

A Markov process is a 1st order dependent process.

More generally, an auto-regressive process $AR(r)$ is a process where X_n depends on $X_{n-1}, \dots, X_{n-r}$.

Example 11

Consider Brownian motion $\{W(t) \mid t \geq 0\}$. This has independent increments, so this is a Markov process.

More properties: Nowhere differentiable property and so on.