Department of Mathematics, IIT Delhi

MTL733: Assignment-2

Q.1) Let $X(t) = \int_0^t B(s)\,ds$ be a stochastic process. Show that

$$\mathbb{E}[X^2(t)] = \frac{t^3}{3}, \quad \forall\, t \geq 0.$$

Q.2) Explain whether the stochastic process

$$X(t) = t^2 B(t) - 2\int_0^t sB(s)\,ds$$

is a martingale or not (with respect to the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$).

Q.3) Let $m_k(t) = \mathbb{E}\big[B^k(t)\big],\ k = 0, 1, 2, 3, \ldots$. Use Ito-formula to prove that

$$m_k(t) = \frac{1}{2}k(k-1)\int_0^t m_{k-2}(s)\,ds, \quad k \geq 2.$$

Deduce that

$$\mathbb{E}[B^4(t)] = 3t^2; \quad \mathbb{E}[B^6(t)] = 15t^3.$$

Q.4) For c, α constants, define a stochastic process $X(t) := \exp\{ct + \alpha B(t)\}$. Show that

$$dX(t) = \big(c + \frac{1}{2}\alpha^2\big)X(t)\,dt + \alpha X(t)\,dB(t).$$

Let $m(t) := \mathbb{E}[X(t)]$. Then show that $m(t)$ satisfies the ODE

$$\begin{cases} m'(t) = \big(c + \frac{1}{2}\alpha^2\big)m(t) \\ m(0) = 1. \end{cases}$$

Show that if $c < -\frac{1}{2}\alpha^2$, then $\lim_{t\to\infty} m(t) = 0$.

Q.5) Let $X(t)$ be an Ito process given by $X(t) = X(0) + \int_0^t v(s)\,dB(s)$. Then show that $M(t) := X^2(t) - \int_0^t v^2(s)\,ds$ is a martingale.

Q.6) Let $X(t)$ be an Ito process given by

$$dX(t) = u(t)dt + dB(t).$$

Define $Y(t) = X(t)M(t)$ where $M(t)$ is a stochastic process given as

$$M(t) = \exp\Big\{ - \int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u^2(s)\, ds\Big\}.$$

Use Ito formula to show that $Y(t)$ is a martingale. In particular, show that

$$Z(t) = (t + B(t))e^{-B(t)-\frac{t}{2}}$$

is a martingale.

Q.7) Using Ito formula, show that the process $Y(t) = e^{B(t)-\frac{t}{2}}$ is an Ito process with the differential form:

$$dY(t) = Y(t)\, dB(t)$$

Q.8) Write down the differential form of $\sin(B(t))$. Using Ito formula, show that the processes $X(t) = e^{\frac{t}{2}} \sin(B(t))$ and $Y(t) = e^{\frac{t}{2}} \cos(B(t))$ are martingale.

Q.9) Let $S(t)$ be a positive stochastic process that satisfies the generalized geometric Brownian motion differential equation

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)\, dB(t)$$

for some adapted processes $\alpha(\cdot)$ and $\sigma(\cdot)$. Show that $S(t)$ is given by the formula

$$S(t) = S(0)\exp\Big\{ \int_0^t \sigma(s)\, dB(s) + \int_0^t \big(\alpha(s) - \frac{1}{2}\sigma^2(s)\big)\, ds\Big\}.$$

Q.10) Let $R(t)$ a stochastic process satisfying the Vasicek interest rate equation

$$dR(t) = (\alpha - \beta R(t))\, dt + \sigma dB(t)$$

where α, β and σ are positive constants and $R(0)$ is nonrandom. Show that $R(t)$ is given by

$$R(t) = e^{-\beta t}\Big\{R(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta s}\, dB(s)\Big\}.$$

Let $m(t) = \mathbb{E}[R(t)]$. Show that $m(t)$ is decreasing if $R(0) > \frac{\alpha}{\beta}$, and increasing if $R(0) < \frac{\alpha}{\beta}$. Moreover, prove that $\lim_{t\to\infty} m(t) = \frac{\alpha}{\beta}$.

Q.11) Show that

$$(1-t)\int_0^t \frac{1}{1-r}, dB(r) = B(t) - \int_0^t \Big(\int_0^s \frac{1}{1-r}, dB(r)\Big)\, ds, \quad 0 \le t < 1.$$

Q.12) Let $u = u(t, x)$ be a smooth solution of the diffusion equation

$$u_t(t, x) + \frac{1}{2}u_{xx}(t, x) = 0.$$

Show that for each time $t > 0$, $\mathbb{E}[u(t, B(t))] = u(0, 0)$.

Q.13) Let $B(t)$ be a Brownian motion. Define a stochastic process

$$M(t) := \int_0^t \operatorname{sign}(B(s))\,dB(s), \text{ where } \operatorname{sign}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0. \end{cases}$$

a) Show that $M(t)$ is a Brownian motion and $\mathbb{E}[M(t)B(t)] = 0$.
b) Show that $\mathbb{E}[M(t)B^2(t)] \neq \mathbb{E}[M(t)]\mathbb{E}[B^2(t)]$. Explain, whether $M(t)$ and $B(t)$ are independent or not.

Q.14) Let $\mathbf{X}(t) = \big(X_1(t), X_2(t)\big)$ be a 2-dimensional stochastic process given by the SDE

$$dX_1(t) = X_2(t)\,dt$$
$$dX_2(t) = \Big(-\frac{R}{L}X_2(t) - \frac{1}{CL}X_1(t) + \frac{g(t)}{L}\Big)\,dt + \frac{\alpha}{L}\,dB(t)$$

where $B(t)$ is a one-dimensional Brownian motion, R, L, C, α are positive constants and $g(t)$ is a given adapted process. Show that $\mathbf{X}(t)$ is given by the following formula:

$$\mathbf{X}(t) = \exp(t\boldsymbol{A})\Big\{\mathbf{X}(0) + \exp(-t\boldsymbol{A})\mathbf{K}B(t) + \int_0^t \exp(-s\boldsymbol{A})\big[\mathbf{H}(s) + \boldsymbol{A}\mathbf{K}B(s)\big]\,ds\Big\},$$

where

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}, \quad \mathbf{H}(t) = \begin{pmatrix} 0 \\ \frac{g(t)}{L} \end{pmatrix}.$$

Q.15) Show that the process $\mathbf{X}(t) = \big(X_1(t), X_2(t)\big)$, defined by

$$X_1(t) = a\cos(B(t)), \quad X_2(t) = b\sin(B(t)), \quad a, b > 0$$

is a solution of the SDE

$$d\mathbf{X}(t) = -\frac{1}{2}\mathbf{X}(t)\,dt + \boldsymbol{M}\mathbf{X}(t)\,dB(t)$$

where $\boldsymbol{M} = \begin{pmatrix} 0 & -\frac{a}{b} \\ \frac{b}{a} & 0 \end{pmatrix}$ and $B(t)$ is a one-dimensional Brownian motion.

Q.16) Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \ldots, B_m(t)\big)$ be a m-dimensional Brownian motion. For any $\boldsymbol{g} = ((g_{ij})) \in \mathcal{Y}_{n\times m}$ with $g_{ij} > 0$, define a stochastic process.

$$g_i(t) := \Big(\sum_{j=1}^m g_{ij}^2(t)\Big)^{\frac{1}{2}} \quad 1 \leq i \leq n.$$

a) For each i, show that the process $M_i(t) := \sum_{j=1}^m \int_0^t \frac{g_{ij}(s)}{g_i(s)}\,dB_j(s)$ is a one-dimensional Brownian motion.
b) Explain whether the process $\mathbf{M}(t) = \big(M_1(t), M_2(t), \ldots, M_n(t)\big)$ is a n-dimensional Brownian motion or not.

Q.17) Find the process $f(t,\omega) \in \mathcal{Y}(0,T)$ such that $F = \mathbb{E}[F] + \int_0^T f(t,\omega)\,dB(t)$ for $F = B^2(T)$ and $F = e^{B(T)}$.

Q.18) Find the Ito representation form for the martingales:

i) $X(t) := B^3(t) - 3tB(t),\ t \geq 0$

ii) $Y(t) := B^4(t) - 6tB^2(t) + 3t^2,\ t \geq 0$

iii) $Z(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big],\ 0 \leq t \leq T.$