5.2. **Pricing under the risk-neutral measure:** So far, we have derived the Black-Scholes-Merton equation for the value of a European call by asking what initial capital $V_\Psi(0)$ and portfolio process $\psi(t)$ an agent would need in order to hedge a short position in the call i.e., in order to have

$$V_\Psi(T) = \max\{0, S(T) - K\}.$$

We now ask the following general question: *what initial capital $V_\Psi(0)$ and portfolio process $\psi(t), 0 \le t \le T$, an agent would need in order to hedge a short position to have*

$$V_\Psi(T) = V(T) \ a.s. \tag{5.15}$$

where $V(T)$ is a $\mathcal{F}_T$-measurable given random variable.

Suppose our agent wishes to choose initial capital $V_\Psi(0)$ and portfolio strategy $\psi(t), 0 \le t \le T$ such that (5.15) holds. Let Q be the risk-neutral measure. Then the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$ is a martingale under Q i.e,

$$D(t)V_\Psi(t) = \mathbb{E}_Q\big[D(T)V_\Psi(T)\big|\mathcal{F}_t\big] = \mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_t\big], \tag{5.16}$$

where in the last equality, we have used (5.15). The value $V_\Psi(t)$ of the hedging portfolio in (5.16) is the capital required at time t in order to successfully complete the hedge of the short position in the derivative security with payoff $V(T)$. We call this price of the derivative security at time by $V(t)$. Thus we have

$$D(t)V(t) = \mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_t\big], \ 0 \le t \le T.$$

Since $D(t)$ is $\mathcal{F}_t$-measurable and bounded, by using the properties of conditional expectation and the definition of discounted process $D(t)$, we get

$$V(t) = \mathbb{E}_Q\Big[\exp\{-\int_t^T r(s)\,ds\}V(T)\Big|\mathcal{F}_t\Big]. \tag{5.17}$$

Equation (5.17) is known as **risk-neutral pricing formula**. Therefore, we determine the *correct initial capital* to be

$$V(0) = \mathbb{E}_Q\big[D(T)V(T)\big]$$

and the value of the hedging portfolio at every time t, $0 \le t \le T$, to be $V(t)$ given in the risk-neutral pricing formula (5.17). It remains to find the portfolio value process $\psi(t)$. **To do so, we consider the following assumption: the filtration $\{\mathcal{F}_t\}$ is generated by the Brownian motion $B(t) : 0 \le t \le T$.** We first claim that the process $D(t)V(t)$ is a Q-martingale. Indeed, for $0 \le s \le t \le T$, one has

$$\mathbb{E}_Q\big[D(t)V(t)\big|\mathcal{F}_s\big] = \mathbb{E}_Q\Big[\mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_t\big]\Big|\mathcal{F}_s\Big] = \mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_s\big] = D(s)V(s)\,.$$

Observe that the filtration $\{\mathcal{F}_t\}$ is generated by $B(t) : 0 \le t \le T$, not the Q-Brownian motion $\tilde{B}(t) : 0 \le t \le T$. Still the martingale representation theorem holds for $D(t)V(t)$ (one needs to work to show this, see e.g. Assignment-4). Hence there exists a adapted process $\tilde{f}(t) : 0 \le t \le T$ such that

$$D(t)V(t) = D(0)V(0) + \int_0^t \tilde{f}(d)\,d\tilde{B}(s), \quad 0 \le t \le T\,.$$

Since $D(0) = 1$, we have

$$D(t)V(t) = V(0) + \int_0^t \tilde{f}(s)\, d\tilde{B}(s), \quad 0 \le t \le T\,. \tag{5.18}$$

Again, we know that

$$\tilde{V}_\Psi(t) := D(t)V(t) = V_\Psi(0) + \int_0^t \psi(s)\sigma(s)D(s)S(s)\, d\tilde{B}(s)\,. \tag{5.19}$$

Let us assume that the volatility $\sigma(t)$ does not vanishes. In order to have $V_\Psi(t) = V(t)$ for all t, we should have, comparing to (5.18) and (5.19)

$$V_\Psi(0) = V(0)$$

and choose the $\psi(t)$ to satisfy

$$\begin{aligned}\psi(s)\sigma(s)D(s)S(s) &= \tilde{f}(s)\\ \implies \psi(s) &= \frac{\tilde{f}(s)}{\sigma(s)D(s)S(s)}, \quad 0 \le s \le T.\end{aligned}$$

With these choices, we have a hedge for a short position in the derivative security with payoff $V(T)$ at time T.

Remark 5.6. i): Our key assumption is that the volatility $\sigma(t)$ is not zero. If the volatility vanishes, then the randomness of the Brownian motion does not enter the stock, although it may still enter in the payoff $V(T)$ of the derivative security. In this case, the stock is no longer an effective hedging instrument.
ii): There is no randomness in the derivative security apart from the Brownian motion randomness, which can be hedged by trading the stock.

Under these two assumptions, every $\mathcal{F}_T$-measurable derivative security can be hedged. Such a model is said to be **complete**.

5.3. **Put-Call Parity and pricing formula for European put option:** Consider a model with a unique risk-neutral measure Q and constant interest rate r. According to the risk-neutral pricing formula, the price at time t of a European call expiring at time T is

$$C(t) = \mathbb{E}_Q\Big[e^{-r(T-t)} \max\{0, S(T) - K\}\big|\mathcal{F}_t\Big]$$

where $S(T)$ is the underlying asset price at time T and K is the strike price of the call. Similarly, the price at time t of a European put expiring at time T is

$$P(t) = \mathbb{E}_Q\Big[e^{-r(T-t)} \max\{0, K - S(T)\}\big|\mathcal{F}_t\Big]\,.$$

Since

$$x - K = \max\{0, x - K\} - \max\{0, K - x\},$$

one has

$$\begin{aligned}C(t) &= e^{-r(T-t)}\mathbb{E}_Q\Big[(S(T) - K)\big|\mathcal{F}_t\Big] + e^{-r(T-t)}\mathbb{E}_Q\Big[\max\{0, K - S(T)\}\big|\mathcal{F}_t\Big]\\ &= e^{rt}\mathbb{E}_Q\Big[(D(T)S(T)\big|\mathcal{F}_t\Big] - Ke^{-r(T-t)} + P(t)\,.\end{aligned}$$

Since the discounted stock price $\tilde{S}(t) := D(t)S(t)$ is a martingale under Q, one has $\mathbb{E}_Q\Big[(D(T)S(T)\big|\mathcal{F}_t\Big] = e^{-rt}S(t)$ and hence

$$C(t) - P(t) = S(t) - Ke^{-r(T-t)}.$$

A forward contract with delivery price K obligates its holder to buy one share of the stock at expiration time T in exchange for payment K. At the expiration, the value of forward contract is $S(T) - K$. Let $f(t,x)$ denotes the value of forward contract at earlier time $t \in [0,T]$ if the stock price at time t is $S(t) = x$. Then

$$f(t,x) = x - Ke^{-r(T-t)}.$$

Since the value at time T of the forward contact agrees with the value of the portfolio that is long one at call and short one put, we must have

$$f(t,x) = C(t,x) - P(t,x), \quad x \ge 0, \quad 0 \le t \le T. \tag{5.20}$$

The relation (5.20) is called **Put-Call-Parity** relation.

5.3.1. **Black-Scholes-Merton put formula:** We have deduced Black-Scholes-Merton call formula for a European call option. By using Put-Call-Parity formula, we now ready to deduce the Black-Scholes-Merton put formula. Indeed, we have

$$\begin{aligned}
P(t,x) &= C(t,x) - f(t,x) \\
&= xN(d_1) - Ke^{-r(T-t)}N(d_2) - x + Ke^{-r(T-t)} \\
&= -x\big\{1 - N(d_1)\big\} + Ke^{-r(T-t)}\big\{1 - N(d_2)\big\} \\
&= -xN(-d_1) + Ke^{-r(T-t)}N(-d_2)\,.
\end{aligned} \tag{5.21}$$

Hedging/ delta of a put option: From the Put-Call-Parity formula, we see that

$$P_x(t,x) = C_x(t,x) - 1 = N(d_1) - 1 < 0.$$

Therefore, the value of a European put option is decreasing as the underlying asset price increases.

Gamma of a put option: The gamma of a European put option, denoted by (Γ_P), is given by

$$\Gamma_P := P_{xx}(t,x) = C_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_1).$$

Theta of a put option: The theta of a European put option, denoted by (θ_P), is given by

$$\begin{aligned}
\theta_P :&= P_t(t,x) = C_t(t,x) + rKe^{-r(T-t)} \\
&= -\frac{\sigma x}{2\sqrt{T-t}}N'(d_1) - rKe^{-r(T-t)}N(d_1) + rKe^{-r(T-t)} \\
&= rKe^{-r(T-t)}N(-d_2) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_1)\,.
\end{aligned}$$

Rho of a put option: The rho of a European put option ρ_P is given by

$$\rho_P = P_r(t,x) = C_r(t,x) = K(T-t)e^{-r(T-t)}N(d_2) > 0.$$

Vega of a put option: The vega of a European put option ν_P is given by

$$\nu_P = P_\sigma(t,x) = C_\sigma(t,x) = x\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}.$$

Vega is always positive. An increase in the volatility will lead to an increase in the put option value.

Remark 5.7. The value function $f(t,x)$ of a forward contract satisfies the same Black-Scholes-Merton PDE satisfied by $C(t,x)$. Indeed, since $f(t,x) = x - Ke^{-r(T-t)}$, one has

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\Big) f(t,x) \\ &= rKe^{-r(T-t)} + \frac{1}{2}\sigma^2 x^2 \cdot 0 + rx \cdot 1 - r\big(x - Ke^{-r(T-t)}\big) = 0\,.\end{aligned}$$

By the Put-Call-Parity formula, we also conclude that the value function of a European put option $P(t,x)$ satisfies the same Black-Scholes-Merton PDE satisfied by $C(t,x)$, European call option.