Stochastic of Finance Lecture 1

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1 Recap

Definition 1

Axiomatic definition of probability spaces

Let Ω be the set of all possible outcomes of a random experiment. Let $\mathcal{F} \subseteq 2^{\Omega}$ be a σ-algebra on Ω, i.e., the following hold:

1. $\Omega \in \mathcal{F}$.
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under countable union, that is, if $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Let P be a function (measure) $P : \mathcal{F} \to [0, 1]$ such that the following hold:

1. $P(A) \geq 0$ (trivially holds due to range).
2. P is σ-additive, i.e., if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a countable collection of disjoint subsets, then we have $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.
3. $P(\Omega) = 1$.

Then (Ω, F, P) is a probability space.

The classical probability comes from the formal definition when Ω is finite, $|\Omega| = n$, $P(\{\omega\}) = \frac{1}{n}$ where $\omega \in \Omega$, $\mathcal{F}$ is the power set of Ω, which also turns out to be a σ-field on Ω.

Definition 2

Random variables:

Given a probability space $(\Omega, \mathcal{F}, P)$, if $X : \Omega \to \mathbb{R}$ is a function such that $X^{-1}((-\infty, x]) \in \mathcal{F}\ \forall x \in \mathbb{R}$, then X is a random variable, or a measurable function w.r.t. $\mathcal{F}$.

Example 1

If F is the largest σ-field (power set but maybe for infinite sets), then any real-function is a random variable.

Example 2

A constant function is always a random variable.

Definition 3

Stochastic processes:

A stochastic process $\{X(t), t \in T\}$ is a collection of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$.

Example 3

Examples of real-life stochastic processes:

1. Price of some stock at the end of the day.
2. Number of trades made every second.
3. Market index at time t.
4. Number of companies registered in stock market at the end of the week.
5. Variance in a stock price in a day - measure on the random variables (since computed from the data). Usually we call observed information (and not computed information) a random variable. Note also time series. Nothing wrong with calling this a random variable, but this won't be the focus of the course.

Some stochastic processes have some important properties, as follows:

1. Independence (mutual, not pairwise) - can verify such assumptions.
2. Stationary - many times we can assume that data is stationary. Two types:
   (a) Wide sense
   (b) Strict sense (by default)
3. Memoryless property.
4. Martingale property - also useful with conditional expectations.

For more, revisit MTL106. Time homogeneous is similar to stationary.

Example 4

Poisson process

$\{N(t), t \geq 0\}$ - number of events occuring upto and including time t. Suppose $N(t) \sim \mathcal{P}(\lambda t)$ where $\mathcal{P}$ is the Poisson distribution, and λ is a fixed parameter. This stochastic process is called a Poisson process. Some properties:

1. Increments are independent.
2. Increments are stationary.
3. Satisfies the memoryless property.
4. Doesn't satisfy the martingale property.

We can derive a random variable that satisfies the martingale property from any random variable.

Example 5

Brownian motion/Wiener process

Let $\{W(t), t \geq 0\}$ be a stochastic process which satisfies the following conditions:

1. $W(0) = 0$
2. For fixed t, $W(t) \sim \mathcal{N}(0, t)$
3. Increments are independent.
4. Increments are stationary.

Definition 4

Filtration: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $\{\mathcal{F}_t \mid t \geq 0\}$ of sub σ-fields of $\mathcal{F}$ is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$.

HW: create some examples of filtrations.

Definition 5

If Ω is a space of functions on $T \subset \mathbb{R}^+$, then it comes with a natural filtration $\mathcal{F}_t = \sigma\{x(s), s \leq t\}$ where x is a stochastic process. That is, consider the set of all possible values of $x(s)$ where $s \leq t$, and generate a σ-field out of it.

Definition 6

Given a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_t \subset F$, a family $\{M(t), t \geq 0\}$ (this is a stochastic process) is called a martingale wrt $(\Omega, \mathcal{F}_t, P)$ if

1. For almost all $w \in \Omega$, we have $M(t, w)$ has left and right limits at every t and is continuous from the right.
2. For each $t \geq 0$, $M(t)$ (random variable) is a measurable function wrt $\mathcal{F}_t$ and integrable.
3. For $0 \leq s \leq t$, $\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$ almost everywhere/surely.

Example 6

Example for filtration:

The random experiment is tossing an unbiased coin infinitely many times.

We have $\Omega = \{HHH\ldots, HTH\ldots, \ldots\}$.

Let A_H be the collection of samples starting with H in the first toss.

Let A_T be the collection of samples starting with T in the second toss.

Let A_{HH} be the collection of samples starting with H in the first toss and H in the second toss.

Let A_{HT} be the collection of samples starting with H in the first toss and T in the second toss.

Consider the trivial σ-field $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Using the first toss, we construct the σ-field $\mathcal{F}_1 = \{\emptyset, A_H, A_T, \Omega\}$.

Using the second toss, we construct the σ-field $\mathcal{F}_2 = \{\emptyset, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH} \cup A_{HT}, \ldots, A_{HH}^c, \ldots, \Omega\}$.

Note that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. Also note that $\lim_{n\to\infty} \mathcal{F}_n = \mathcal{F}_\infty = \mathcal{F}$.

Example 7

Non-example for martingale:

Let $\{N(t), t \geq 0\}$ be a Poisson process on $(\Omega, \mathcal{F}, P)$.

Now that the parameter space is contained in $\mathbb{R}^+$, we have a natural filtration $\mathcal{F}_t = \sigma\{N(s), s \leq t\}$.

Note the following properties:

1. $N(t, w)$ is right continuous at t for $w \in \Omega$.
2. $N(t)$ is a measurable function wrt $\mathcal{F}_t$ and integrable.
3. $\mathbb{E}[N(t) \mid \mathcal{F}_s] = \mathbb{E}[N(t) - N(s) + N(s) \mid \mathcal{F}_s] = N(s) + \lambda(t - s)$.

Therefore this is not a martingale wrt the given filtration.

Example 8

$\{W(t), t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ with the natural filtration. The first two conditions hold as usual. The third condition:

$\mathbb{E}[W(t) \mid \mathcal{F}_s] = \mathbb{E}[W(t) - W(s) + W(s) \mid \mathcal{F}_s] = \mathbb{E}[W(t) - W(s)] + \mathbb{E}[W(s) \mid \mathcal{F}_s] = 0 + \mathbb{E}[W(s)]$ since

$W(t) \sim \mathcal{N}(0,t)$ and $\mathbb{E}[W(s) \mid \mathcal{F}_s] = \mathbb{E}[W(s)]$.

Hence brownian motion is a martingale wrt the natural filtration $\{\mathcal{F}(t), t \geq 0\}$.

Definition 7

Sub-martingale: If $\mathbb{E}[X(t) \mid \mathcal{F}(s)] \geq X(s)$ a.e., then it is called a sub-martingale.

Definition 8

Super-martingale: If $\mathbb{E}[X(t) \mid \mathcal{F}(s)] \leq X(s)$ a.e., then it is called a super-martingale.

Poisson process is a sub-martingale.

Example 9

Let $\{X_n \mid n = 0, 1, 2, \ldots\}$, where $X_n =$ the amount at the end of the n^{th} game. Y_i is the payoff of the i^{th} game, where $P[Y_i = 1] = P[Y_i = -1] = \frac{1}{2}$. Suppose $X_0 = A$.

We have $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n \mid \mathcal{F}_n] + \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n] + \mathbb{E}[Y_{n+1}] = \mathbb{E}[X_n]$. So this stochastic process is a martingale. Also note that $\mathbb{E}[X_n] = \mathbb{E}[X_0 + Y_1 + \cdots + Y_n] = A + 0 + \cdots + 0 = A$.

Definition 9

Markov Property

Let $\{X(t) \mid t \geq 0\}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$. If for $0 \leq s \leq t$, $P(X(t) \mid X(u), 0 \leq u \leq s) = P(X(t) \mid X(s))$, then this stochastic process is a Markov process, and is said to have the Markov property.

The same can be done for discrete processes.

For instance, verify that $P(X_{n+1} = x_{n+1} \mid X_0 = A, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$, which gives us that the random walk is also a Markov process.

Example 10

Consider the Poisson process $\{N(t) \mid t \geq 0\}$. Then we have $P(N(t) = k \mid N(u), 0 \leq u \leq s) = P(N(t) = k \mid N(s))$ due to independent increments, so this is a Markov process.

Any process with independent increments is a Markov process.

A Markov process is a 1st order dependent process.

More generally, an auto-regressive process $AR(r)$ is a process where X_n depends on $X_{n-1}, \ldots, X_{n-r}$.

Example 11

Consider Brownian motion $\{W(t) \mid t \geq 0\}$. This has independent increments, so this is a Markov process.

More properties: Nowhere differentiable property and so on.

2 Content

2.1 Conditional expectation

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and X be an integrable random variable. Suppose γ is a sub-σ-algebrao of $\mathcal{F}$. Then $\mathbb{E}[X \mid \gamma]$ is a γ-measurable random variable such that $\int_A X dP = \int_A E[X \mid \gamma] dP$ for all A.

Theorem 1

Let X be an integrable random variable. Then for each σ-algebra $\gamma \subset \mathcal{F}$, the conditional expectation $\mathbb{E}[X \mid \gamma]$ exists and is unique upto γ-measurable sets of probability 0.

Properties of conditional expectation

1. $\mathbb{E}[X \mid \gamma] = X$ almost surely when X is γ-measurable.
2. Linear combinations work
3. If X is γ measurable and XY is integrable, then $\mathbb{E}[XY \mid \gamma] = X\mathbb{E}[Y \mid \gamma]$ a.s.
4. Tower property: $W \subset \gamma \subset \mathcal{F}$. Then $\mathbb{E}[X \mid W] = \mathbb{E}[\mathbb{E}[X \mid W] \mid \gamma] = \mathbb{E}[\mathbb{E}[X \mid \gamma] \mid W]$
5. $X \leq Y$ a.s. implies $\mathbb{E}[X \mid \gamma] \leq \mathbb{E}[Y \mid \gamma]$.

Lemma 1.1

(Conditional Jensen's inequality)

$\Phi : \mathbb{R} \to \mathbb{R}$, convex such that $\mathbb{E}[|\Phi(X)|] < +\infty$ satisfies $\Phi(\mathbb{E}[X \mid \gamma]) \leq \mathbb{E}[\Phi(X) \mid \gamma]$.

2.2 Brownian Motion

$\{B(t) : t \geq 0\}$ is called Brownian motion if

1. $B(0) = 0$ a.s.
2. Independent increments: $0 < t_1 < \cdots < t_n$. Then $B(t_1), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1})$ are independent.
3. $B(t) - B(s) \sim N(0, t-s)$ for all $t \geq s \geq 0$.
4. Sample paths are continuous with probability 1.

That is, we have:

$$P(a \leq B(t) \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b \exp(-\frac{x^2}{2t})dx$$

Now we have $P(a_1 \leq B(t_1) \leq b_1, ..., a_n \leq B(t_n) \leq b_n)$ equals

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} g(x_1, t_1 \mid 0) \cdot g(x_2, t_2 - t_1 \mid x_1) \cdots g(x_n, t_n - t_{n-1} \mid x_n) dx_n \cdots dx_1$$

where $g(x, t \mid y) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(x-y)^2}{2t})$.

$B(t)$ is a martingale with respect to its natural filtration $\mathcal{F}_t = \sigma(B(s) \mid 0 \leq s \leq t)$.

Theorem 2

Let $B(t)$ be a Brownian motion. Then the processes

1. $X(t) = B^2(t) - t$.
2. $M(t) = \exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\}$, $\sigma \in \mathbb{R}^+$

are both martingales wrt to its natural filtration.

Proof. We prove for the second martingale, which is also called the exponential martingale.

$\mathbb{E}[\exp(\sigma B(t))] = \exp(\frac{1}{2}\sigma^2 t)$ using standard computation. So we have $\mathbb{E}[\exp(\sigma B(t) - \frac{1}{2}\sigma^2 t)] = 1$, hence it is integrable.

We know that $B(t) - B(s) \sim N(0, t-s)$ for $t \geq s \geq 0$.

We can show that $\mathbb{E}[\exp(\sigma(B(t) - B(s)))] = \exp(\frac{1}{2}\sigma^2(t-s))$.

So we can show that $\mathbb{E}[M(t) \mid \mathcal{F}_s] = \mathbb{E}[\exp(\sigma(B(t) - B(t))) \exp(\sigma B(s) - \frac{1}{2}\sigma^2 t) \mid \mathcal{F}_s] = \exp(\sigma B(s) - \frac{1}{2}\sigma^2 t) \cdot \mathbb{E}[\exp(\sigma(B(t) - B(s))) \mid \mathcal{F}_s] = \exp(\sigma B(s) - \frac{1}{2}\sigma^2 t) \cdot \mathbb{E}[\exp(\sigma(B(t) - B(s)))] = \exp(\sigma B(s) - \frac{1}{2}\sigma^2 t) \cdot \exp(\frac{1}{2}\sigma^2(t - s)^2) = M(s)$, a.s.

Hence $M(t)$ is a martingale. $\square$

Lemma 2.1

Let $B(\cdot)$ be a one-dimensional Brownian motion. Then $\mathbb{E}[B(t)B(s)] = \min t, s$, with $t, s \geq 0$. (Covariance).

How to define other brownian motions in terms of a brownian motion?

Example 12

Let $B(t)$ be a Brownian motion. Define a stochastic process: $X(t) = 0$ if $t = 0$, $t \cdot B(1/t)$ if $t > 0$.

Then we claim that $X(t)$ is a standard Brownian motion.

Proof. Suppose $t > s$. Then we have $X(t) - X(s) = (t-s)B(1/t) + s(B(1/t) - B(1/s))$. Note that the second part is $\sim N(0, s^2 \cdot (1/s - 1/t))$, and the first term is normally distributed with distribution $N(0, \frac{(t-s)^2}{t})$. Moreover, these are independent, so their sum has variance added, i.e., it is normally distributed with distribution $N(0, s^2(1/s - 1/t) + (t-s)^2/t) = N(0, t-s)$.

Now we only need to show that increments are independent.

$Cov(X(t), X(s)) = \mathbb{E}[X(t)X(s)] = stE[B(1/t)B(1/s)] = st \min(1/s, 1/t) = \min(s, t)$.

Let $s < t$, then $Cov(X(s), X(t) - X(s)) = Cov(X(s), X(t)) - Cov(X(s), X(s)) = \min(s, t) - s = 0$. Then these are independent (since normally distributed - check).

Now we need to check that they are continuous with probability 1. At $t = 0$, we have $\lim_{t\to 0} X(t) = \lim_{t\to 0} tB(\frac{1}{t}) = \lim_{n\to\infty} \frac{B(n)}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} B(i) - B(i-1)$.

Let $Y(i) = B(i) - B(i-1)$. All these are i.i.d. By SLLN (strong law of large numbers), we have that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i = 0$ a.s., so we are done. $\square$

2.2.1 First and quadratic variation

Let $f : [0, T] \to \mathbb{R}$. $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$.

Definition 10

First variation of f upto time T is defined as $FV_T(f) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$.

Definition 11

The quadratic variation of f upto time T is defined by $[f, f](T) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (f(t_{j+1}) - f(t_j))^2$.

Note 1

If $f : [0, T] \to \mathbb{R}$ such that $|f'|$ is Riemann integrable on $[0, T]$, then $FV_T(f) = \int_0^T |f'(t)|dt$. For a proof, using mean value theorem, we have $f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j)$ for some $t_j^* in[t_j, t_{j+1}]$.

So we have $\sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)| = \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j)$, and hence

$FV_T(f) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j)$, which is the Riemann integral of $|f'|$ on $[0, T]$.

Note 2

If f has continuous derivative, then $[f, f](T) = 0$.

Theorem 3

If $B(t)$ is a one dimensional Brownian motion, then $[B, B](T) = T$ for all $T \geq 0$ a.s.

Proof. Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0, T]$. Then we have $Q_T^n = \sum_{j=0}^{n-1} (B(t_{j+1}) - B(t_j))^2$.

We have $\mathbb{E}[(Q_T^n - T)^2] = \mathbb{E}[(\sum_{j=0}^{n-1} (B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j))^2] = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \mathbb{E}[((B(t_{j+1}) - B(t_j))^2 - t_{j+1} + t_j) \cdot (\text{same thing for k})]$. For $k \neq j$, we can show that the terms are 0 (using symmetry and the fact that $B(t)^2 - t$ is also a Brownian motion).

This implies that $\mathbb{E}[(Q_T^n - T)^2] = \sum_{k=0}^{n-1} \mathbb{E}[(Y_k - 1)^2 (t_{k+1} - t_k)^2]$, where $Y_k = \frac{B(t_{k+1}) - B(t_k)}{\sqrt{t_{k+1} - t_k}} \sim N(0, 1)$.

So $\exists c > 0$ such that $\mathbb{E}[(Q_T^n - T)^2] \leq c \cdot \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2$, whence the limit as $||\Pi|| \to 0$ of the expectation is 0. Thus $[B, B](T) = T$ for all $T \geq 0$ a.s. (due to convergence in L_2). $\square$

Lemma 3.1

For any $t > 0$, the first variation of Brownian motion $B(\cdot)$ upto time T is infinite almost surely.

Proof. Consider the partition yet again. Look at the quadratic variation formula. We have $\sum_{j=0}^{n-1} (B(t_{j+1}) - B(t_j))^2 \leq \sup_{0 \leq k \leq n-1} |B(t_{k+1} - t_k)| \cdot \sum_{j=0}^{n-1} |B(t_{j+1}) - B(t_j)|$. Now since Brownian motion has a continuous sample path, the supremum is 0. So the sample path has infinite variation with probability 1. Because suppose it is finite, then it gives the inequality $T \leq 0$, which is a contradiction. $\square$

Definition 12

Geometric Brownian motion: $S(t) = S(0) \exp(\sigma B(t) + (\alpha - \frac{1}{2}\sigma^2)t)$ where α and $\sigma > 0$ are constants.

This is used for stock pricing models.

Consider $0 \leq T_1 < T_2$, and $T_1 \leq t \leq T_2$. Let Π be a partition $\{T_1 = t_0 < t_1 < \cdots < t_n = T_2\}$.

Then we have $\log\left(\frac{S(t_{j+1})}{S(t_j)}\right) = \sigma(B(t_{j+1}) - B(t_j)) + (\alpha - \sigma^2/2)(t_{j+1} - t_j)$.

Summing this from $j = 0$ to $n - 1$ gives $\sum_{j=0}^{n-1} \log^2\left(\frac{S(t_{j+1})}{S(t_j)}\right) = \sum_{j=0}^{n-1} (\sigma(B(t_{j+1}) - B(t_j)) + (\alpha - \sigma^2/2)(t_{j+1} - t_j))^2 = A_1 + A_2 + A_3$, where A_1 is term corresponding to difference of B squared, A_2 is the term corresponding to the other square and A_3 is the cross term. Then $\lim_{||\Pi|| \to 0} A_1 = \sigma^2(T_2 - T_1)$, that for A_2 is 0. We need the following claim:

Claim 3.1

$\lim_{||\Pi|| \to 0} A_3 = 0$

Proof. $|\sum_{j=0}^{n-1} (B() - B())(() - ())| \leq \sum_{j=0}^{n-1} |B() - B()||() - ()| \leq max|B() - B()| \sum_{j=0}^{n-1} |() - ()| = 0 \cdot (T_2 - T_1)$. $\square$

So we have $\sigma^2 \approx \frac{1}{T_2 - T_1} \sum_{j=0}^{n-1} \left(\log \frac{S(t_{j+1})}{S(t_j)}\right)^2$ as an approximation to the volatility.