Stochastic of Finance Lecture 1

Contents

1 Recap

2 Content

Definition 1

Axiomatic definition of probability spaces

Let Ω be the set of all possible outcomes of a random experiment. Let $\mathcal{F} \subseteq 2^{\Omega}$ be a σ-algebra on Ω, i.e., the following hold:

1. $\Omega \in \mathcal{F}$.
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under countable union, that is, if $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Let P be a function (measure) $P : \mathcal{F} \to [0, 1]$ such that the following hold:

1. $P(A) \geq 0$ (trivially holds due to range).
2. P is σ-additive, i.e., if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a countable collection of disjoint subsets, then we have $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.
3. $P(\Omega) = 1$.

Then (Ω, F, P) is a probability space.

The classical probability comes from the formal definition when Ω is finite, $|\Omega| = n$, $P(\{\omega\}) = \frac{1}{n}$ where $\omega \in \Omega$, $\mathcal{F}$ is the power set of Ω, which also turns out to be a σ-field on Ω.

Definition 2

Random variables:

Given a probability space $(\Omega, \mathcal{F}, P)$, if $X : \Omega \to \mathbb{R}$ is a function such that $X^{-1}((-\infty, x]) \in \mathcal{F}\ \forall x \in \mathbb{R}$, then X is a random variable, or a measurable function w.r.t. $\mathcal{F}$.

Example 1

If F is the largest σ-field (power set but maybe for infinite sets), then any real-function is a random variable.

Example 2

A constant function is always a random variable.

Definition 3

Stochastic processes:

A stochastic process $\{X(t), t \in T\}$ is a collection of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$.

Example 3

Examples of real-life stochastic processes:

1. Price of some stock at the end of the day.
2. Number of trades made every second.
3. Market index at time t.
4. Number of companies registered in stock market at the end of the week.
5. Variance in a stock price in a day - measure on the random variables (since computed from the data). Usually we call observed information (and not computed information) a random variable. Note also time series. Nothing wrong with calling this a random variable, but this won't be the focus of the course.

Some stochastic processes have some important properties, as follows:

1. Independence (mutual, not pairwise) - can verify such assumptions.
2. Stationary - many times we can assume that data is stationary. Two types:
   (a) Wide sense
   (b) Strict sense (by default)
3. Memoryless property.
4. Martingale property - also useful with conditional expectations.

For more, revisit MTL106. Time homogeneous is similar to stationary.

Example 4

Poisson process

$\{N(t), t \geq 0\}$ - number of events occuring upto and including time t. Suppose $N(t) \sim \mathcal{P}(\lambda t)$ where $\mathcal{P}$ is the Poisson distribution, and λ is a fixed parameter. This stochastic process is called a Poisson process. Some properties:

1. Increments are independent.
2. Increments are stationary.
3. Satisfies the memoryless property.
4. Doesn't satisfy the martingale property.

We can derive a random variable that satisfies the martingale property from any random variable.

Example 5

Brownian motion/Wiener process

Let $\{W(t), t \geq 0\}$ be a stochastic process which satisfies the following conditions:

1. $W(0) = 0$
2. For fixed t, $W(t) \sim \mathcal{N}(0, t)$
3. Increments are independent.
4. Increments are stationary.