Existence proof: It is similar to the existence proof for ODE. Consider the Picard type iteration as follows:

$$\begin{aligned} Y^{(0)}(t) &= Z, \\ Y^{(k+1)}(t) &= Z + \int_0^t a(s, Y^{(k)}(s))\, ds + \int_0^t \sigma(s, Y^{(k)}(s))\, dB(s). \end{aligned}$$

Then for $k \geq 1$ and $t \in [0,T]$, we have similar to uniqueness proof

$$\mathbb{E}\big[|Y^{(k+1)}(t) - Y^{(k)}(t)|^2\big] \leq 2D^2(1+T) \int_0^t \mathbb{E}\big[|Y^{(k)}(s) - Y^{(k-1)}(s)|^2\big]\, ds.$$

Observe that

$$\mathbb{E}\big[|Y^{(1)}(t) - Y^{(0)}(t)|^2\big] \leq 2\mathbb{E}\Big[\Big(\int_0^t a(s,Z)\, ds\Big)^2\Big] + 2\mathbb{E}\Big[\int_0^t \sigma^2(s,Z)\, ds\Big] \leq tA_1$$

where the constant A_1 only depends on C, T and $\mathbb{E}[Z^2]$. Hence by induction on k, we get

$$\mathbb{E}\big[|Y^{(k+1)}(t) - Y^{(k)}(t)|^2\big] \leq \frac{A_2^{k+1} t^{k+1}}{(k+1)!}, \quad k \geq 0,\ t \in [0,T]$$

where $A_2 \equiv A_2(C, D, T, \mathbb{E}[Z^2])$. Thus, for $m > n \geq 0$, we get

$$\begin{aligned} &\big\|Y^{(m)}(t) - Y^{(n)}(t)\big\|_{L^2([0,T]\times\Omega)} = \Big\| \sum_{k=n}^{m-1} Y^{(k+1)}(t) - Y^{(k)}(t)\Big\|_{L^2([0,T]\times\Omega)} \\ &\leq \sum_{k=n}^{m-1} \big\|Y^{(k+1)}(t) - Y^{(k)}(t)\big\|_{L^2([0,T]\times\Omega)} \leq \sum_{k=n}^{m-1} \Big(\int_0^T \mathbb{E}\big[|Y^{(1)}(t) - Y^{(0)}(t)|^2\big]\, dt\Big)^{\frac{1}{2}} \\ &\leq \sum_{k=n}^{m-1} \Big(\int_0^T \frac{A_2^{k+1} t^{k+1}}{(k+1)!}\, dt\Big)^{\frac{1}{2}} = \sum_{k=n}^{m-1} \Big(\frac{A_2^{k+1} T^{k+2}}{(k+2)!}\Big)^{\frac{1}{2}} \to 0 \quad \text{as } m, n \to \infty. \end{aligned}$$

Thus, $\{Y^{(n)}(t)\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$ and hence it is convergent. Thus, there exists a $\mathcal{F}_t$-adapted stochastic process $X(t)$ such that

$$X(t) := \lim_{n\to\infty} Y^{(n)}(t) \quad \text{in } L^2([0,T]\times\Omega).$$

By using (4.3), Cauchy-Schwartz and Ito-isometry along with the fact that $X(t) := \lim_{n\to\infty} Y^{(n)}(t)$, we get, for all $t \in [0,T]$

$$\begin{aligned} \int_0^t a(s, Y^{(n)}(s))\, ds &\to \int_0^t a(s, X(s))\, ds, \\ \int_0^t \sigma(s, Y^{(n)}(s))\, dB(s) &\to \int_0^t \sigma(s, X(s))\, dB(s) \end{aligned}$$

in $L^2(\Omega)$. Passing to the limit as $k \to \infty$ in Picard iteration, we get

$$X(t) = Z + \int_0^t a(s, X(s)) + \int_0^t \sigma(s, X(s))\, dB(s). \tag{4.4}$$

It remains to show that $X(t)$ can be chosen to be continuous. Note that the right hand side of (4.4) has a continuous version. Denote it by $\tilde{X}(t)$. Then $\tilde{X}(t)$ is continuous and

$$\begin{aligned}\tilde{X}(t) &= Z + \int_0^t a(s, X(s)) + \int_0^t \sigma(s, X(s))\, dB(s) \text{ a.s.} \\ &= Z + \int_0^t a(s, \tilde{X}(s)) + \int_0^t \sigma(s, \tilde{X}(s))\, dB(s) \text{ a.s.}\end{aligned}$$

In view of the assumption (4.2) and Ito-isometry, we see that

$$\begin{aligned}\mathbb{E}[|X(t)|^2] &\le 3\Big\{\mathbb{E}[Z^2] + TC^2 \int_0^t (1+\mathbb{E}[|X(s)|^2])\, ds + 2C^2 \int_0^t (1+\mathbb{E}[|X(s)|^2])\, ds\Big\} \\ &\le K(1+\mathbb{E}[Z^2]) + K\int_0^t \mathbb{E}[|X(s)|^2]\, ds\end{aligned}$$

for some constant $K = K(C,T)$. Hence from Gronwall's lemma, we get

$$\mathbb{E}[|X(t)|^2] \le K(1+\mathbb{E}[Z^2])e^{Kt}.$$

□

Example 4.1. *Solve the stochastic differential equation:*

$$dX(t) = (m - X(t))\, dt + \sigma dB(t), \quad X(0) = Z$$

where Z is non-random and $m, \sigma \in \mathbb{R}$. Show that variance of the solution $X(t)$ tends to $\frac{\sigma^2}{2}$ as $t \to \infty$.

Solution: *The SDE can be written as in the form of (4.1) with $a(t,x) = m - x$ and $\sigma(t,x) = \sigma$. It is easy to see that the functions $a(t,x)$ and $\sigma(t,x)$ satisfy the conditions (4.2) and (4.3). Hence by existence and uniqueness theorem, given SDE has a unique strong solution $X(t)$. By applying Ito product rule, we have*

$$\begin{aligned}& d(X(t)e^t) = e^t dX(t) + X(t)e^t\, dt = me^t\, dt + \sigma e^t\, dB(t) \\ \implies\ & X(t)e^t = Z + \int_0^t me^s\, ds + \int_0^t \sigma e^s\, dB(s) \\ \implies\ & X(t) = m(1-e^{-t}) + Ze^{-t} + \sigma \int_0^t e^{-(t-s)}\, dB(s).\end{aligned}$$

Thus the unique solution is given by

$$X(t) = m(1-e^{-t}) + Ze^{-t} + \sigma \int_0^t e^{-(t-s)}\, dB(s).$$

Since Ito-integral is martingale, we have

$$\mathbb{E}[X(t)] = m(1-e^{-t}) + Ze^{-t}.$$

By using Ito-isometry, we have

$$\begin{aligned}\mathrm{Var}(X(t)) &= \mathbb{E}[(X(t))^2] - (\mathbb{E}[X(t)])^2 = \mathbb{E}\Big[\Big(\sigma \int_0^t e^{-(t-s)}\, dB(s)\Big)^2\Big] \\ &= \sigma^2 \int_0^t e^{2(s-t)}\, ds = \frac{\sigma^2}{2}(1-e^{-2t}) \to \frac{\sigma^2}{2} \text{ as } t \to \infty.\end{aligned}$$

Example 4.2. *Solve the SDE: for $r, \alpha \in \mathbb{R}$*

$$dX(t) = r\,dt + \alpha X(t)\,dB(t), \quad X(0) = Z,\ \mathbb{E}[Z^2] < \infty$$

Solution: *The SDE can be written as in the form of (4.1) with $a(t,x) = r$ and $\sigma(t,x) = \alpha x$. It is easy to see that the functions $a(t,x)$ and $\sigma(t,x)$ satisfy the conditions (4.2) and (4.3). Hence by existence and uniqueness theorem, given SDE has a unique strong solution $X(t)$. We now find explicit solution of the given SDE. Consider a stochastic process given by*

$$Y(t) = \exp\Big\{ -\alpha B(t) + \frac{1}{2}\alpha^2 t \Big\}.$$

Then $Y(t)$ satisfies the differential form

$$dY(t) = \alpha^2 Y(t)\,dt - \alpha Y(t)\,dB(t).$$

By using Ito product rule, we get

$$\begin{aligned} d(X(t)Y(t)) &= X(t)dY(t) + Y(t)dX(t) - \alpha^2 Y(t)X(t)\,dt \\ &= X(t)\Big(\alpha^2 Y(t)\,dt - \alpha Y(t)\,dB(t)\Big) + Y(t)\Big(r\,dt + \alpha X(t)\,dB(t)\Big) - \alpha^2 Y(t)X(t)\,dt \\ &= rY(t)\,dt \\ &\implies X(t) = \frac{1}{Y(t)}\left(Z + r\int_0^t Y(s)\,ds\right). \end{aligned}$$

Thus the unique solution of the given SDE is given by

$$X(t) = \exp\Big\{\alpha B(t) - \frac{1}{2}\alpha^2 t\Big\}Z + \int_0^t \exp\Big\{\alpha(B(t) - B(s)) - \frac{1}{2}\alpha^2(t-s)\Big\}\,ds.$$