2.2. **Sample path properties of Brownian motion:** We will demonstrate that for almost every ω, the sample path $t \mapsto B(t,\omega)$ is uniformly Holder continuous for each exponent $\gamma < \frac{1}{2}$, but nowhere Holder continuous with any exponent $\gamma > \frac{1}{2}$. To prove uniformly Holder continuity, we use a general theorem of Kolmogorov called **Kolmogorov's continuity theorem**.

Theorem 2.5 (**Kolmogorov's continuity theorem**). *Let $X(\cdot)$ be a stochastic process with continuous sample paths a.s. such that*

$$\mathbb{E}\Big[|X(t)-X(s)|^{\beta}\Big] \le C|t-s|^{1+\alpha}$$

for constants $\alpha, \beta > 0$ and $C \ge 0$, and for all $t, s \ge 0$. Then for each $0 < \gamma < \frac{\alpha}{\beta}$, $T > 0$, and almost every ω, there exists a constant $K = K(\omega,\gamma,T)$ such that

$$|X(t,\omega)-X(s,\omega)| \le K|t-s|^{\gamma} \quad \textit{for all } s,t \in [0,T]$$

i.e., the sample path $t \mapsto X(t,\omega)$ is uniformly Holder continuous with exponent γ on $[0,T]$.

For $t > s$, we have for $m \in]\mathbb{N}$

$$\begin{aligned}\mathbb{E}\big[B(t)-B(s)|^{2m}\big] &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}|x|^{2m}e^{-\frac{x^2}{2(t-s)}}\,dx \\ &= (t-s)^m\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|y|^{2m}e^{-\frac{y^2}{2}}\,dy = C|t-s|^m\,.\end{aligned}$$

Thus, from Kolmogorov's continuity theorem, we conclude that (taking $\beta = 2m, \alpha = m-1$) for a.s. ω and any $T > 0$, the sample path $t \mapsto B(t,\omega)$ is uniformly Holder continuous on $[0,T]$ for each exponent

$$0 < \gamma = \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m} < \frac{1}{2}.$$

We now prove that sample paths of Brownian motion are nowhere differentiable with probability 1.

Theorem 2.6. *For each $\frac{1}{2} < \gamma \le 1$ and almost every ω, $t \mapsto B(t,\omega)$ is nowhere Holder continuous with exponent γ.*

Proof. For simplicity, we consider $B(t)$ for times $0 \le t \le 1$. Fix N such that

$$N(\gamma - \frac{1}{2}) > 1.$$

Suppose $t \mapsto B(t,\omega)$ is Holder continuous with exponent γ at some point $s_0 \in [0,1]$ i.e., there exists constant K such that

$$|B(t,\omega)-B(s_0,\omega)| \le K|t-s_0|^{\gamma} \quad \forall\, t \in [0,1]\,. \tag{2.2}$$

For n large enough, set $i = [ns_0]+1$. Then for $j = i, i+1, \ldots, i+N-1$, we have, from (2.2)

$$\begin{aligned}|B(\frac{j}{n},\omega) - B(\frac{j+1}{n},\omega)| &\le |B(\frac{j}{n},\omega)-B(s_0,\omega)| + |B(\frac{j+1}{n},\omega)-B(s_0,\omega)| \\ &\le K\big\{|\frac{j}{n}-s_0|^{\gamma} + |\frac{j+1}{n}-s_0|^{\gamma}\big\} \le Mn^{-\gamma}\end{aligned}$$

for some constant M. For $i \le i \le n$, $M \ge 1$ and for large n, define

$$A^i_{M,n} := \big\{\omega : |B(\frac{j}{n},\omega) - B(\frac{j+1}{n},\omega)| \le Mn^{-\gamma}, \quad j = i, i+1, \ldots, i+N-1\big\}$$

Then, if A is the set of ω such that $B(\cdot,\omega)$ is Holder continuous with exponent γ at some point $s_0, 0 \le s_0 < 1$, we must have

$$A \subset \cup_{M=1}^{\infty}\cup_{k=1}^{\infty}\cap_{n=k}^{\infty}\cup_{i=1}^{n} A^i_{M,n}.$$

Our goal is to show that $\mathbb{P}(A) = 0$. Here we show that

$$\mathbb{P}\Big(\cup_{M=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} \cup_{i=1}^{n} A_{M,n}^{i} \Big) = 0.$$

Observe that

$$\mathbb{P}\Big(\cap_{n=k}^{\infty} \cup_{i=1}^{n} A_{M,n}^{i} \Big) \le \liminf_{n\to\infty} \mathbb{P}\Big(\cup_{i=1}^{n} A_{M,n}^{i} \Big) \le \liminf_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_{M,n}^{i}).$$

Since $B(\frac{j}{n}) - B(\frac{j+1}{n}) \sim \mathcal{N}(0, \frac{1}{n})$ and $B(\frac{j}{n}) - B(\frac{j+1}{n}), j = i, i+1, \ldots, i+N-1$ are independent, we see that

$$\begin{aligned} \mathbb{P}(A_{M,n}^{i}) &= \Big\{ \mathbb{P}\Big(|B(\frac{1}{n})| \le Mn^{-\gamma} \Big) \Big\}^{N} = \Big\{ \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-Mn^{-\gamma}}^{Mn^{-\gamma}} e^{-\frac{nx^2}{2}}\, dx \Big\}^{N} \\ &= \Big\{ \frac{1}{\sqrt{2\pi}} \int_{-Mn^{\frac{1}{2}-\gamma}}^{Mn^{\frac{1}{2}-\gamma}} e^{-\frac{y^2}{2}}\, dy \Big\}^{N} \le Cn^{(\frac{1}{2}-\gamma)N} \quad \text{for some constant } C. \end{aligned}$$

Thus, we get

$$\mathbb{P}\Big(\cap_{n=k}^{\infty} \cup_{i=1}^{n} A_{M,n}^{i} \Big) \le \liminf_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_{M,n}^{i}) \le \liminf_{n\to\infty} \sum_{i=1}^{n} Cn^{(\frac{1}{2}-\gamma)N} = 0 \quad (\because\ N(\gamma - \frac{1}{2}) > 1)$$

Since the above relation holds for all k and M, we conclude that

$$\mathbb{P}\Big(\cup_{M=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} \cup_{i=1}^{n} A_{M,n}^{i} \Big) = 0 \implies \mathbb{P}(A) = 0.$$

$\square$

Remark 2.3. From Theorem 2.6, we conclude the followings:

a) Sample paths of Brownian motion is nowhere differentiable. Indeed, if $B(\cdot, \omega)$ is differentiable at some point s, then $B(\cdot, \omega)$ would be Holder continuous at point s with exponent $\gamma = 1$ which contradicts Theorem 2.6.
b) Sample path of Brownian motion is infinite variation. Because, if $B(\cdot, \omega)$ were finite variation on some sub-interval, then it would be differentiable almost everywhere there– which is NOT possible.

2.3. **Markov property of Brownian motion.**

Definition 2.2 (Markov process)**.** Let $\{X(t) : t \ge 0\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$. We say that $X(t)$ is a Markov process if for any Borel measurable function f, there is a Borel measurable function g such that

$$\mathbb{E}\big[f(X(t)) | \mathcal{F}_s \big] = g(X(s)) \quad \forall\, t > s \ge 0.$$

We show that Brownian motion is a Markov process. For its proof, we need the following independence lemma.

Lemma 2.7 (Independence)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{Y}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose the random variables $X_1, X_2, \ldots, X_n$ are $\mathcal{Y}$-measurable and the random variables $Y_1, Y_2, \ldots, Y_m$ are independent of $\mathcal{Y}$. Let $f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ be a function of the dummy variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$, and define*

$$g(x_1, x_2, \ldots, x_n) = \mathbb{E}\Big[f(x_1, x_2, \ldots, x_n, Y_1, Y_2, \ldots, Y_m) \Big].$$

Then

$$g(X_1, X_2, \ldots, X_n) = \mathbb{E}\Big[f(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m) | \mathcal{Y} \Big].$$

Theorem 2.8. *Let $B(t), t \geq 0$ be a Brownian motion and $\mathcal{F}_t$ be its natural filtration. Then $B(t)$ is a Markov process. Moreover, for any Borel measurable function f, there holds*

$$\mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] = \int_{-\infty}^{\infty} f(y)\rho(t-s, B(s), y)\,dy,$$

where $\rho(\tau, x, y) := \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}$ is the **transition density** *of the Brownian motion.*

Proof. Let f be any Borel measurable function. Observe that for any $t > s \geq 0$,

$$\mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] = \mathbb{E}\Big[f(B(t) - B(s) + B(s))|\mathcal{F}_s\Big]$$

and $B(t) - B(s)$ is independent of $\mathcal{F}_s$ and $B(s)$ is $\mathcal{F}_s$ measurable. Define

$$g(x) = \mathbb{E}\big[f(x + B(t) - B(s))\big].$$

Then by Independence lemma 2.7, we see that $\mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] = g(B(s))$. In other words, $B(t)$ is a Markov process. Since $B(t) - B(s) \sim \mathcal{N}(0, t-s)$, we see that

$$\begin{aligned}
g(x) &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(x+w) e^{-\frac{w^2}{2(t-s)}}\,dw \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{2(t-s)}}\,dy = \int_{-\infty}^{\infty} f(y)\rho(t-s, x, y)\,dy \\
\implies \mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] &= \int_{-\infty}^{\infty} f(y)\rho(t-s, B(s), y)\,dy,
\end{aligned}$$

where $\rho(\tau, x, y) := \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}$ is the **transition density** of the Brownian motion. □

Example 2.2. *Let $B(t)$ be a Brownian motion and $\{\mathcal{F}_t\}$ be its natural filtration. For $\mu \in \mathbb{R}$, consider the Brownian motion with drift*

$$X(t) := \mu t + B(t).$$

Then $\{X(t) : t \geq 0\}$ is a Markov process.

To see this, let $0 \leq s < t$ and f be any Borel measurable function. Then

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = \mathbb{E}\big[f(\mu t + B(s) + B(t) - B(s))|\mathcal{F}_s\big].$$

Since $\mu t + B(s)$ is $\mathcal{F}_s$-measurable and $B(t) - B(s)$ is $\mathcal{F}_s$-independent, we get

$$\begin{aligned}
\mathbb{E}[f(X(t))|\mathcal{F}_s] &= \mathbb{E}[f(a + B(t-s))]\big|_{a=\mu t + B(s)} \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(\mu t + B(s) + x) e^{-\frac{x^2}{2(t-s)}}\,dx \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-B(s)-\mu t)^2}{2(t-s)}}\,dy \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-B(s)-\mu s-\mu(t-s))^2}{2(t-s)}}\,dy \\
&= \int_{-\infty}^{\infty} f(y)\rho(t-s, X(s), y)\,dy = g(X(s))
\end{aligned}$$

where $\rho(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x-\mu\tau)^2}{2\tau}}$ and $g(x) = \int_{-\infty}^{\infty} f(y)\rho(t-s, x, y)\,dy$.

2.4. **First passage time of Brownian motion and reflexion principle:** We now discuss the first passage time in which Brownian motion reaches to level m first time. For any real number m, define the first passage time as

$$\tau_m := \min\{t \ge 0 : B(t) = m\}\,. \tag{2.3}$$

Notice that

$$\{\tau_m \le t\} = \{\exists\, s \in [0,t] : B(s) = a\} \in \mathcal{F}_t$$

where $\mathcal{F}_t = \sigma(B(s) : 0 \le s \le t)$. τ_m is a stopping time with respect to the filtration $\mathcal{F}_t$. Here we state some important property of Brownian motion associated with τ_m without proof.

Lemma 2.9. *$B(\tau_m + s) - B(\tau_m) = B(\tau_m + s) - m$ is also a Brownian motion, independent from $B(t)$, $t \le \tau_m$.*

Theorem 2.10. *For $m \in \mathbb{R}^+$, let τ_m be the first passage time of Brownian motion to level m. Then τ_m is finite almost surely.*

Proof. For any $\sigma > 0$, consider the exponential martingale $M(t)$ given in Theorem 2.1. Since τ_m is a stopping time, we have

$$1 = M(0) = \mathbb{E}[M(t \wedge \tau_m)] = \mathbb{E}\Big[\exp\big\{\sigma B(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)\big\}\Big] \tag{2.4}$$

It is easy to see that

$$\lim_{t\to\infty} \exp\{-\frac{1}{2}\sigma^2(t\wedge\tau_m)\} = \mathbf{1}_{\{\tau_m<\infty\}}\exp\{-\frac{1}{2}\sigma^2\tau_m\}$$

where $\mathbf{1}_A$ is the indicator function on A defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Notice that, if $\tau_m < \infty$, then for large t, $\exp\{\sigma B(t\wedge\tau_m)\} = e^{\sigma m}$, but for $\tau_m = \infty$, we do not know what happens to $\exp\{\sigma B(t\wedge\tau_m)\}$ as $t \to \infty$. Observe that since $\sigma > 0$ and $m > 0$, for $t \le \tau_m$, the get the following upper bound:

$$0 \le \exp\{\sigma B(t\wedge\tau_m)\} \le e^{\sigma m}.$$

Due to this bound, we conclude that $\exp\big\{\sigma B(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)\big\}$ converges to zero as $t\to\infty$ for $\tau_m = \infty$. Combining these analysis, we finally have

$$\lim_{t\to\infty} \exp\big\{\sigma B(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)\big\} = \mathbf{1}_{\{\tau_m<\infty\}}\exp\{\sigma m - \frac{1}{2}\sigma^2\tau_m\} \tag{2.5}$$

Passing to the limit in (2.4) as $t \to \infty$ along with (2.5), we get

$$\begin{aligned} 1 &= \lim_{t\to\infty}\mathbb{E}\Big[\exp\big\{\sigma B(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)\big\}\Big] \\ &= \mathbb{E}\Big[\lim_{t\to\infty}\exp\big\{\sigma B(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)\big\}\Big] \\ &= \mathbb{E}\Big[\mathbf{1}_{\{\tau_m<\infty\}}\exp\{\sigma m - \frac{1}{2}\sigma^2\tau_m\}\Big]\,. \end{aligned}$$

Since this holds for all $\sigma > 0$, passing to the limit as $\sigma \to 0$, we have

$$\mathbb{E}\big[\mathbf{1}_{\{\tau_m<\infty\}}\big] = 1 \implies \mathbb{P}(\tau_m < \infty) = 1.$$

In other words, τ_m is finite almost surely. $\square$