Department of Mathematics, IIT Delhi

MTL733: Assignment-1

Q.1) Let $B(t) : t \geq 0$ be a Brownian motion. Define a stochastic process

$$\bar{B}(t) := \frac{1}{c} B(c^2 t), \quad c > 0.$$

Show that $\bar{B}(t) : t \geq 0$ is also a Brownian motion.

Q.2) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$, and Y be a random variable such that $\mathbb{E}[|Y|] < +\infty$. Define a stochastic process

$$X(t) := \mathbb{E}\big[Y|\mathcal{F}_t\big].$$

Show that $X(t)$ is a martingale.

Q.3) For a Brownian motion $B(\cdot)$, define the processes, for $t \geq 0$

$$X(t) := B^3(t) - 3tB(t); \quad Y(t) := B^4(t) - 6tB^2(t) + 3t^2.$$

Show that both $X(t)$ and $Y(t)$ are martingale with respect to the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$.

Q.4) Let $X(t) : t \geq 0$ be a martingale with respect to a given filtration. Show that

$$\mathbb{E}[X(t)] = \mathbb{E}[X(0)] \quad \forall\, t \geq 0.$$

Is converse true? Justify your answer.

Q.5) A stochastic process $X(t, \cdot) : \Omega \to \mathbb{R}$ is **continuous in mean square** if $\mathbb{E}[|X(t)|^2] < +\infty$ for all $t \geq 0$ and

$$\lim_{s \to t} \mathbb{E}\big[|X(s) - X(t)|^2\big] = 0 \quad \forall\, t \geq 0.$$

Show that $f(B(t))$ is continuous in mean square where f is Lipschitz continuous function and $B(t)$ is Brownian motion.

Q.6) Let $B(t)$ be Brownian motion and $\mathcal{F}_t$ be its natural filtration. Show that the process $X(t)$ defined by

$$X(t) := 2\exp\{3B(t) + 4t\}$$

is a Markov process.

Q.7) For any function $f : [0, T] \to \mathbb{R}$, we define the p-th variation of f up to time T as

$$\langle f, f\rangle_p(T) := \lim_{\|\Pi\|\to .0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^p$$

where $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ is a partition of $[0, T]$. Show that p-th variation of Brownian motion equals to zero for all $3 \le p < \infty$ and almost every ω.

Q.8) For $r \in \mathbb{R}^+$, consider the stochastic process

$$X(t) = 3\exp\{(r-2)t + 2B(t)\}, \quad t \ge 0$$

where $B(t)$ is Brownian motion. Show that for any $t > 0$,

$$\mathbb{E}\Big[e^{-rt}\big(X(t) - 3\big)^+\Big] = 3N(m_+) - 3e^{-rt}N(m_-)$$

where $N(\cdot)$ is the cumulative standard normal distribution function and $m_\pm = \frac{\sqrt{t}}{2}\big(r \pm 2\big)$.

Q.9) Consider a stochastic process

$$X(t) := 2t + B(t), \quad t \ge 0$$

where $B(t)$ is a Brownian motion.

a) Show that the process

$$Z(t) := \exp\{\sigma X(t) - (2\sigma + \frac{1}{2}\sigma^2)t\}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \le s \le t)$.

b) For $m > 0$, define the first passage time of $X(t)$ in level m

$$\tau_m := \min\{t \ge 0 : X(t) = m\}.$$

Show that

$$\mathbb{E}\Big[\exp\{\sigma m - (2\sigma + \frac{1}{2}\sigma^2)\tau_m\}\mathbf{1}_{\{\tau_m < +\infty\}}\Big] = 1.$$

c) Show that

$$\mathbb{E}\big[e^{-\alpha\tau_m}\big] = \exp\{2m - m\sqrt{2\alpha + 4}\}, \quad \forall\ \alpha > 0.$$

Moreover, prove that $\mathbb{E}[\tau_m]. < +\infty$.

Q.10) Define $X(t) := e^{-t}B(e^{2t})$, where $B(\cdot)$ is a Brownian motion. Show that

$$\mathbb{E}[X(t)X(s)] = e^{-|t-s|}, \quad \forall\ -\infty < s, t < \infty.$$

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MTL733: Assignment-2

Q.1) Let $X(t) = \int_0^t B(s)\,ds$ be a stochastic process. Show that

$$\mathbb{E}[X^2(t)] = \frac{t^3}{3}, \quad \forall\, t \geq 0.$$

Q.2) Explain whether the stochastic process

$$X(t) = t^2 B(t) - 2\int_0^t sB(s)\,ds$$

is a martingale or not (with respect to the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$).

Q.3) Let $m_k(t) = \mathbb{E}\big[B^k(t)\big],\ k = 0, 1, 2, 3, \ldots$. Use Ito-formula to prove that

$$m_k(t) = \frac{1}{2}k(k-1)\int_0^t m_{k-2}(s)\,ds, \quad k \geq 2.$$

Deduce that

$$\mathbb{E}[B^4(t)] = 3t^2; \quad \mathbb{E}[B^6(t)] = 15t^3.$$

Q.4) For c, α constants, define a stochastic process $X(t) := \exp\{ct + \alpha B(t)\}$. Show that

$$dX(t) = \big(c + \frac{1}{2}\alpha^2\big)X(t)\,dt + \alpha X(t)\,dB(t).$$

Let $m(t) := \mathbb{E}[X(t)]$. Then show that $m(t)$ satisfies the ODE

$$\begin{cases} m'(t) = \big(c + \frac{1}{2}\alpha^2\big)m(t) \\ m(0) = 1. \end{cases}$$

Show that if $c < -\frac{1}{2}\alpha^2$, then $\lim_{t\to\infty} m(t) = 0$.

Q.5) Let $X(t)$ be an Ito process given by $X(t) = X(0) + \int_0^t v(s)\,dB(s)$. Then show that $M(t) := X^2(t) - \int_0^t v^2(s)\,ds$ is a martingale.

Q.6) Let $X(t)$ be an Ito process given by

$$dX(t) = u(t)dt + dB(t).$$

Define $Y(t) = X(t)M(t)$ where $M(t)$ is a stochastic process given as

$$M(t) = \exp\Big\{ - \int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u^2(s)\, ds\Big\}.$$

Use Ito formula to show that $Y(t)$ is a martingale. In particular, show that

$$Z(t) = (t + B(t))e^{-B(t)-\frac{t}{2}}$$

is a martingale.

Q.7) Using Ito formula, show that the process $Y(t) = e^{B(t)-\frac{t}{2}}$ is an Ito process with the differential form:

$$dY(t) = Y(t)\, dB(t)$$

Q.8) Write down the differential form of $\sin(B(t))$. Using Ito formula, show that the processes $X(t) = e^{\frac{t}{2}} \sin(B(t))$ and $Y(t) = e^{\frac{t}{2}} \cos(B(t))$ are martingale.

Q.9) Let $S(t)$ be a positive stochastic process that satisfies the generalized geometric Brownian motion differential equation

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)\, dB(t)$$

for some adapted processes $\alpha(\cdot)$ and $\sigma(\cdot)$. Show that $S(t)$ is given by the formula

$$S(t) = S(0)\exp\Big\{ \int_0^t \sigma(s)\, dB(s) + \int_0^t \big(\alpha(s) - \frac{1}{2}\sigma^2(s)\big)\, ds\Big\}.$$

Q.10) Let $R(t)$ a stochastic process satisfying the Vasicek interest rate equation

$$dR(t) = (\alpha - \beta R(t))\, dt + \sigma dB(t)$$

where α, β and σ are positive constants and $R(0)$ is nonrandom. Show that $R(t)$ is given by

$$R(t) = e^{-\beta t}\Big\{R(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta s}\, dB(s)\Big\}.$$

Let $m(t) = \mathbb{E}[R(t)]$. Show that $m(t)$ is decreasing if $R(0) > \frac{\alpha}{\beta}$, and increasing if $R(0) < \frac{\alpha}{\beta}$. Moreover, prove that $\lim_{t\to\infty} m(t) = \frac{\alpha}{\beta}$.

Q.11) Show that

$$(1-t)\int_0^t \frac{1}{1-r}, dB(r) = B(t) - \int_0^t \Big(\int_0^s \frac{1}{1-r}, dB(r)\Big)\, ds, \quad 0 \le t < 1.$$

Q.12) Let $u = u(t, x)$ be a smooth solution of the diffusion equation

$$u_t(t, x) + \frac{1}{2}u_{xx}(t, x) = 0.$$

Show that for each time $t > 0$, $\mathbb{E}[u(t, B(t))] = u(0, 0)$.

Q.13) Let $B(t)$ be a Brownian motion. Define a stochastic process

$$M(t) := \int_0^t \operatorname{sign}(B(s))\, dB(s), \text{ where } \operatorname{sign}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0. \end{cases}$$

a) Show that $M(t)$ is a Brownian motion and $\mathbb{E}[M(t)B(t)] = 0$.
b) Show that $\mathbb{E}[M(t)B^2(t)] \neq \mathbb{E}[M(t)]\mathbb{E}[B^2(t)]$. Explain, whether $M(t)$ and $B(t)$ are independent or not.

Q.14) Let $\mathbf{X}(t) = \big(X_1(t), X_2(t)\big)$ be a 2-dimensional stochastic process given by the SDE

$$\begin{aligned} dX_1(t) &= X_2(t)\, dt \\ dX_2(t) &= \Big(-\frac{R}{L} X_2(t) - \frac{1}{CL} X_1(t) + \frac{g(t)}{L} \Big)\, dt + \frac{\alpha}{L}\, dB(t) \end{aligned}$$

where $B(t)$ is a one-dimensional Brownian motion, R, L, C, α are positive constants and $g(t)$ is a given adapted process. Show that $\mathbf{X}(t)$ is given by the following formula:

$$\mathbf{X}(t) = \exp(t\boldsymbol{A})\Big\{ \mathbf{X}(0) + \exp(-t\boldsymbol{A})\mathbf{K}B(t) + \int_0^t \exp(-s\boldsymbol{A})\big[\mathbf{H}(s) + \boldsymbol{A}\mathbf{K}B(s)\big]\, ds \Big\},$$

where

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}, \quad \mathbf{H}(t) = \begin{pmatrix} 0 \\ \frac{g(t)}{L} \end{pmatrix}.$$

Q.15) Show that the process $\mathbf{X}(t) = \big(X_1(t), X_2(t)\big)$, defined by

$$X_1(t) = a\cos(B(t)), \quad X_2(t) = b\sin(B(t)), \quad a, b > 0$$

is a solution of the SDE

$$d\mathbf{X}(t) = -\frac{1}{2}\mathbf{X}(t)\, dt + \boldsymbol{M}\mathbf{X}(t)\, dB(t)$$

where $\boldsymbol{M} = \begin{pmatrix} 0 & -\frac{a}{b} \\ \frac{b}{a} & 0 \end{pmatrix}$ and $B(t)$ is a one-dimensional Brownian motion.

Q.16) Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \ldots, B_m(t)\big)$ be a m-dimensional Brownian motion. For any $\boldsymbol{g} = ((g_{ij})) \in \mathcal{Y}_{n\times m}$ with $g_{ij} > 0$, define a stochastic process.

$$g_i(t) := \Big(\sum_{j=1}^m g_{ij}^2(t) \Big)^{\frac{1}{2}} \quad 1 \leq i \leq n.$$

a) For each i, show that the process $M_i(t) := \sum_{j=1}^m \int_0^t \frac{g_{ij}(s)}{g_i(s)}\, dB_j(s)$ is a one-dimensional Brownian motion.
b) Explain whether the process $\mathbf{M}(t) = \big(M_1(t), M_2(t), \ldots, M_n(t)\big)$ is a n-dimensional Brownian motion or not.

Q.17) Find the process $f(t,\omega) \in \mathcal{Y}(0,T)$ such that $F = \mathbb{E}[F] + \int_0^T f(t,\omega)\, dB(t)$ for $F = B^2(T)$ and $F = e^{B(T)}$.

Q.18) Find the Ito representation form for the martingales:

i) $X(t) := B^3(t) - 3tB(t),\ t \geq 0$

ii) $Y(t) := B^4(t) - 6tB^2(t) + 3t^2,\ t \geq 0$

iii) $Z(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big],\ 0 \leq t \leq T.$

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MTL733: Assignment-3

Q.1) Find the process $f(t,\omega) \in \mathcal{Y}(0,T)$ such that $F = \mathbb{E}[F] + \int_0^T f(t,\omega)\,dB(t)$ for $F = B^2(T)$ and $F = e^{B(T)}$.

Q.2) Find the Ito representation form for the martingales:

i) $X(t) := B^3(t) - 3tB(t),\ t \geq 0$

ii) $Y(t) := B^4(t) - 6tB^2(t) + 3t^2,\ t \geq 0$

iii) $Z(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big],\ 0 \leq t \leq T.$

Q.3) Let X be a standard normal random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Find a probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that the random variable

$$Y = X + \theta, \quad 0 \neq \theta \in \mathbb{R}$$

becomes a standard normal under the measure $\bar{\mathbb{P}}$.

Q.4) Consider a 2-dimensional Ito process $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t)\big)$ given by

$$dY_1(t) = dB_1(t) + 3dB_2(t), \quad dY_2(t) = dt - dB_1(t) - 2dB_2(t)$$

where $\mathbf{B}(t) = \big(B_1(t), B_2(t)\big)$ is a 2-dimensional Brownian motion. Find a probability measure $\bar{\mathbb{P}}$ such that $\mathbb{P}$ and $\bar{\mathbb{P}}$ are equivalent, and $\mathbf{Y}(t)$ is a martingale with respect to $\bar{\mathbb{P}}$.

Q.5) Suppose $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t)\big) \in \mathbb{R}^2$ is given by

$$dY_1(t) = \beta_1(t)\,dt + dB_1(t) + 2dB_2(t) + 3dB_3(t)$$
$$dY_2(t) = \beta_2(t)\,dt + dB_1(t) + 2dB_2(t) + 2dB_3(t)$$

where β_1, β_2 are bounded adapted processes and $\mathbf{B}(t) = \big(B_1(t), B_2(t), B_3(t)\big)$ is 3-dimensional Brownian motion. Show that there are infinitely many equivalent martingale measures Q for $\mathbf{Y}(t)$.

Q.6) Let $B(t)$ be a 1-dimensional Brownian motion. Use Girsanov's theorem to evaluate

$$\mathbb{E}\Big[(B^2(T) - T)\exp\{-\int_0^T s^2\,dB(s)\}\Big], \quad \text{for any } T > 0.$$

Q.7) Let $\mathbf{B}(t) := \big(B_1(t), B_2(t)\big) : 0 \le t \le T$ be a 2-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that there exists a probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that the stochastic process $\bar{\mathbf{B}}(t) = \big(\bar{B}_1(t), \bar{B}_2(t)\big) : 0 \le t \le T$ given by

$$\bar{B}_1(t) = B_1(t), \quad \bar{B}_2(t) = B_2(t) + \int_0^t B_1(s)\, ds$$

is a 2-dimensional Brownian motion under $\bar{\mathbb{P}}$. Show that

$$\bar{\mathrm{Cov}}(B_1(T), B_2(T)) \neq \mathrm{Cov}(B_1(T), B_2(T))$$

Q.8) Show that solution of the SDE

$$dX(t) = \kappa(\alpha - \log(X(t)))X(t)\, dt + \sigma X(t)\, dB(t); \quad X(0) = x > 0$$

is given by the formula

$$X(t) = \exp\Big\{e^{-\kappa t}\ln(x) + \big(\alpha - \frac{\sigma^2}{2\kappa}\big)\big(1 - e^{-\kappa t}\big) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\, dB(s)\Big\},$$

where $\sigma, \kappa, \alpha, x$ are positive constant. Find the mean of $X(t)$.

Q.9) Consider a nonlinear SDE of the form

$$dX(t) = f(t, X(t))\, dt + \alpha X(t)\, dB(t), \quad X(0) = x \tag{0.1}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given continuous deterministic function, and $\alpha \in \mathbb{R}$ is a constant.

a) Show that
$$d(F(t)X(t)) = F(t)f(t, X(t))\, dt,$$
where the process $F(t)$ is given by $F(t) = \exp\{-\alpha B(t) + \frac{\alpha^2 t}{2}\}$.

b) Define the process $Y(t) = F(t)X(t)$ so that $X(t) = (F(t))^{-1}Y(t)$. Deduce that $Y(t)$ satisfies a deterministic differential equation in the function $t \mapsto Y(t,\omega)$ for each $\omega \in \Omega$.

Q.10) Use **Q. 9)** to solve the following SDEs:

i) $dX(t) = \frac{1}{X(t)}\, dt + \alpha X(t)\, dB(t); \quad X(0) = x > 0$, where α is a constant.

ii) $dX(t) = X^\gamma(t)\, dt + 4X(t)\, dB(t); \quad X(0) = x > 0$, where γ is a constant.

Q.11) For any positive, smooth function f, show that the process

$$M(t) := f(B(t))\exp\{-\frac{1}{2}\int_0^t f''(B(s))\, ds\}$$

is a martingale.

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MTL733: Assignment-4

Q.1) Consider the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$. A trading strategy $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing if and only if $\tilde{V}_\Psi(t)$ can be expressed for all $t \in [0, T]$ as

$$\tilde{V}_\Psi(t) = V_\Psi(0) + \int_0^t \psi(u)\, d\tilde{S}(u)$$

where $\tilde{S}(t)$ is the discounted stock price process.

Q.2) Consider a market $X(t) = \big(S_0(t), S(t)\big)$ given by

$$dS_0(t) = 3S_0(t)\,dt,\ S_0(0) = 1; \quad dS(t) = 2S(t)\,dt + 5S(t)\,dB(t),\ S(0) = 1.$$

Show that the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$, given by

$$\psi_0(t) = \frac{1}{3}\big(e^{-3t} - 1\big), \quad \psi(t) = \int_0^t \exp\{-5B(u) + \frac{21}{2}u\}\,du$$

is self-financing.

Q.3) Consider a market $X(t) = \big(S_0(t), S(t)\big)$ where

$$dS_0(t) = 0,\ S_0(0) = 1; \quad dS(t) = tS(t)\,dt + 5S(t)\,dB(t),\ S(0) = 4.$$

Examine whether the market has arbitrage opportunity or not.

Q.4) Show that there exists an arbitrage if and only if there is a way to start with $V_\Psi(0) > 0$ and at a later time T have a portfolio value satisfying

$$\mathbb{P}\Big(V_\Psi(T) \geq \frac{V_\Psi(0)}{D(T)}\Big) = 1, \quad \mathbb{P}\Big(V_\Psi(T) > \frac{V_\Psi(0)}{D(T)}\Big) > 0. \tag{0.1}$$

Q.5) Use the Black-Scholes formula to price a European call option for a stock whose price today is \$75 with expiry date 3 months from now, strike price \$70 and volatility 20%. The risk-free interest is 7% per year.

a) Find the value of the European call option, and compute *delta* and *gamma* for this option.
b) Find the value of the European put option and compute *delta* and *gamma* for this option.

Q.6) Let Q be a risk-neutral measure of a given market $X(t) = \big(S_0(t), S(t)\big)$. The value at time zero of a European call on a stock whose initial price is $S(0) = x$ is given by

$$C(0,x) = \mathbb{E}_Q\Big[e^{-rT}\max\{0, S(T) - K\}\Big].$$

Show that there exists a probability measure $\bar{Q}$ such that

a) Q and $\bar{Q}$ are equivalent,

b) $\bar{B}(t) = \tilde{B}(t) - \sigma t$ is a Brownian motion under $\bar{Q}$, where $\tilde{B}(t)$ is a Brownian motion under the risk-neutral measure Q,

c) The delta of the call option $C_x(0,x)$ can be written as follows:

$$C_x(0,x) = \bar{Q}\big(S(T) > K\big) = \bar{Q}\Big\{-\frac{\bar{B}(T)}{\sqrt{T}} < d\Big\} = N(d)$$

where $N(\cdot)$ is the cumulative standard normal distribution and

$$d := \frac{\log(\frac{x}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Q.7) Let Q be a probability measure defined by

$$Q(A) = \int_A Z(T)\, d\mathbb{P}, \text{ where } Z(t) = \exp\Big\{-\int_0^t \theta(s)\, dB(s) - \frac{1}{2}\int_0^t \theta^2(s)\, ds\Big\}.$$

Suppose that the filtration $\{\mathcal{F}_t\}$ is generated by the Brownian motion $B(t) : 0 \le t \le T$. Let $\widetilde{M}$ be a martingale under Q. We will show that Martingale representation theorem holds for $\widetilde{M}$.

a) Show that $\widetilde{M}(t)Z(t)$ is a martingale under $\mathbb{P}$.

b) Find the differential of $(Z(t))^{-1}$ and $\widetilde{M}(t)$.

c) Show that Martingale representation theorem holds for $\widetilde{M}$.

Q.8) Suppose that a stock sells today for \$100, the value of the call option is \$6, the value of the put option is \$5 and both options have the same strike price, \$100, with one year expiry time. What is the risk-free interest rate?

Q.9) A call option on a non-divident-paying stock has a market price of \$2.5. The stock price is \$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is implied volatility?