STOCHASTIC OF FINANCE (MTL733)

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1. Introduction

The theory of stochastic processes turns to be a useful tool in solving problems in various fields such as engineering, genetics, statistics, economics, finance, etc. The word *stochastic* means *random* or *chance*. Stochastic processes can be thought of collection of random variables indexes by some parameter.

Definition 1.1 (**Stochastic process**)**.** A real stochastic process is a collection of random variables $\{X(t) : t \in \mathbf{T}\}$ defined on a common probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with values in $\mathbb{R}$. For each $\omega \in \Omega$, the mapping $t \mapsto X(t, \omega)$ is the corresponding sample path. In general we observe a different sample path.

- $\mathbf{T}$ is called the index set of the process which is usually a subset of $\mathbb{R}$.
- $\{X(t) : t \in \mathbf{T}\}$ is said to be continuous stochastic process if its sample function $X(t, \omega)$ is a continuous function of $t \in \mathbf{T}$ for almost every $\omega \in \Omega$.

Definition 1.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A **filtration** is a family of sub-σ algebras $\{\mathcal{F}_t : t \geq 0\}$ of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.
- A stochastic process $X(t)$ is said to be **adapted to the filtration** $\{\mathcal{F}_t\}$ if for each t, $X(t)$ is $\mathcal{F}_t$-measurable i.e., $\{\omega \in \Omega : X(t, \omega) \in B\} \in \mathcal{F}_t$ for any Borel subset B of $\mathbb{R}$.

We think of $X(t)$ as the price of some asset at time t and $\mathcal{F}_t$ as the information obtained by watching all the prices in the market up to time t.

Example 1.1. *Let $X(\cdot)$ be a real-valued stochastic process. Define*

$$\mathcal{F}_t := \sigma(X(s) : 0 \leq s \leq t)$$

Then $\mathcal{F}_t$ is a filtration. Moreover, $X(t)$ is adapted to the filtration $\mathcal{F}_t$. This filtration is called the history of the process until (and including) time $t \geq 0$. Sometimes it is called **natural filtration** *generated by the process $X(\cdot)$.*

1.1. Conditional expectation.

We now recall definition of conditional expectation and its important properties.

Definition 1.3 (Conditional expectation)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{Y}$ is a sub-σ-algebra of $\mathcal{F}$. For any integrable random variable X, we define conditional expectation of X given the σ-algebra $\mathcal{Y}$, denoted by $\mathbb{E}(X|\mathcal{Y})$ to be a $\mathcal{Y}$-measurable random variable such that

$$\int_A X \, d\mathbb{P} = \int_A \mathbb{E}(X|\mathcal{Y}) \, d\mathbb{P} \quad \forall \, A \in \mathcal{Y}.$$

One can easily check that

i) $\mathbb{E}[\mathbb{E}(X|\mathcal{Y})] = \mathbb{E}[X]$.
ii) $\mathbb{E}(X|\mathcal{W}) = \mathbb{E}[X]$, where $\mathcal{W} = \{\emptyset, \Omega\}$ is the trivial σ-algebra.

Theorem 1.1. *Let X be an integrable random variable. Then for each σ-algebra $\mathcal{Y} \subset \mathcal{F}$, the conditional expectation $\mathbb{E}(X|\mathcal{Y})$ exists and is unique up to $\mathcal{Y}$-measurable sets of probability zero.*

Properties of conditional expectation: We now state some important properties of conditional expectation:

a) For any $\mathcal{Y}$-measurable integrable random variable, we have $\mathbb{E}(X|\mathcal{Y}) = X$ a.s.
b) **Linearity:** for $a, b \in \mathbb{R}$, $\mathbb{E}(aX + bY|\mathcal{Y}) = a\mathbb{E}(X|\mathcal{Y}) + b\mathbb{E}(Y|\mathcal{Y})$ a.s..
c) If X is $\mathcal{Y}$-measurable and XY is integrable, then $\mathbb{E}(XY|\mathcal{Y}) = X\mathbb{E}(Y|\mathcal{Y})$ a.s..
d) **Tower property:** For $\mathcal{W} \subset \mathcal{Y}$, we have
$$\mathbb{E}(X|\mathcal{W}) = \mathbb{E}(\mathbb{E}(X|\mathcal{Y})|\mathcal{W}) = \mathbb{E}(\mathbb{E}(X|\mathcal{W})|\mathcal{Y}) \text{ a.s..}$$
e) **Monotonicity:** $X \leq Y$ a.s. implies that $\mathbb{E}(X|\mathcal{Y}) \leq \mathbb{E}(Y|\mathcal{Y})$ a.s.

Lemma 1.2 (Conditional Jensen's inequality:)**.** *Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is convex with $\mathbb{E}[|\phi(X)|] < +\infty$. Then*
$$\phi(\mathbb{E}(X|\mathcal{Y})) \leq \mathbb{E}(\phi(X)|\mathcal{Y}) \text{ a.s..}$$

Definition 1.4 (Martingale)**.** A stochastic process $\{X(t) : t \in \mathbf{T}\}$ is called a **martingale** with respect to a filtration $\{\mathcal{F}_t\}$ if

a) $\{X(t) : t \in \mathbf{T}\}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbf{T}}$
b) $\mathbb{E}[|X(t)|] < +\infty$ for all $t \in \mathbf{T}$.
c) for all $s \leq t$, $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s.

- If condition *c*) is replaced by: $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. for all $s \leq t$, then it is called **submartingale**.
- If condition *c*) is replaced by: $\mathbb{E}[X(t)|\mathcal{F}_s] \leq X(s)$ a.s. for all $s \leq t$, then it is called **supermartingale**.

Observe that if $X(t)$ is a martingale, then $\mathbb{E}[X(t)]$ is constant.

Example 1.2. *Let $\{X(t) : t \geq 0\}$ be a stochastic process with stationary and independent increments such that $\mathbb{E}[|X(t)|] < +\infty$ for all $t \geq 0$. Then $\{X(t) : t \geq 0\}$ is a martingale/submartingale/supermartingale with respect to its natural filtration if $\mathbb{E}[X(t)] = 0/\, \mathbb{E}[X(t)] \geq 0/\, \mathbb{E}[X(t)] \leq 0$ for all $t \geq 0$ respectively.*

Solution: *For $s \leq t$, we have*
$$\begin{aligned}\mathbb{E}[X(t)|\mathcal{F}_s] &= \mathbb{E}[X(t) - X(s) + X(s)|\mathcal{F}_s] = \mathbb{E}[X(t) - X(s)|\mathcal{F}_s] + \mathbb{E}[X(s)|\mathcal{F}_s] \\ &= \mathbb{E}[X(t) - X(s)] + X(s) = \mathbb{E}[X(t-s)] + X(s) \ \textit{a.s.}\end{aligned}$$
Thus, we have
$$\mathbb{E}[X(t)|\mathcal{F}_s] \begin{cases} = X(s) \ \textit{a.s.} & \textit{if } \mathbb{E}[X(t)] = 0 \\ \geq X(s) \ \textit{a.s.} & \textit{if } \mathbb{E}[X(t)] \geq 0 \\ \leq X(s) \ \textit{a.s.} & \textit{if } \mathbb{E}[X(t)] \leq 0. \end{cases}$$

Remark 1.1. Poisson process $N(t)$ with respect to its natural filtration is a submartingale .

Definition 1.5 (Stopping time)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space equipped with the filtration $\{\mathcal{F}_t\}$. A random time $\tau : \Omega \mapsto [0, \infty]$ is called a **stopping time** with respect to the filtration $\mathcal{F}_t$ if
$$\{\tau \leq t\} \in \mathcal{F}_t \quad \forall\, t \geq 0.$$

Every random time equal to a nonnegative constant is a stopping time.

2. Brownian motion

We will discuss a more useful stochastic process so called Brownian motion and its essential properties. R. Brown in 1826-27 observed the irregular motion of pollen particles suspended in water. He and others noted that

a) the path of a given particle is very irregular, having a tangent at no point, and
b) the motions of two distinct particles appear to be independent.

The mathematical description of the above observation can be given as follows:

Definition 2.1 (Brownian motion)**.** A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a Brownian motion if

a) $B(0) = 0$ a.s.
b) $B(t) - B(s)$ is $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$
c) for all times $0 < t_1 < t_2 < \ldots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent.
d) sample paths are continuous with probability 1.

Notice in particular that

$$\mathbb{E}[B(t)] = 0, \quad \mathbb{E}[B^2(t)] = t \text{ for each time } t \geq 0.$$

Again, from the definition, we see that for all $t > 0$ and $a \geq b$

$$\mathbb{P}(a \leq B(t) \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}}\, dx\,.$$

Moreover, we can find the joint probability as follows: for $0 < t_1 < t_2 < \ldots < t_n$ and $a_i \leq b_i$ $i = 1, 2, \ldots, n$, the joint probability is given by

$$\begin{aligned}
&\mathbb{P}\Big(a_1 \leq B(t_1) \leq b_1, a_2 \leq B(t_2) \leq b_2, \ldots, a_n \leq B(t_n) \leq b_n\Big) \\
&= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_n}^{b_n} g(x_1, t_1|0) g(x_2, t_2 - t_1|x_1) \ldots g(x_n, t_n - t_{n-1}|x_{n-1}) dx_n\, dx_{n-1} \ldots dx_2\, dx_1
\end{aligned}$$

where $g(x, t|y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$.

Remark 2.1. In view of Example 1.2, it is easy to see that Brownian motion is a martingale with respect to its natural filtration.

Theorem 2.1. *Let $B(t)$ be a Brownian motion. Then the processes $X(t) := B^2(t) - t$ and $M(t) := \exp(\sigma B(t) - \frac{1}{2}\sigma^2 t)$, $\sigma \in \mathbb{R}$ are martingale with respect to its natural filtration $\mathcal{F}_t$. $M(t)$ is called* **exponential martingale**.

Proof. First we prove that $X(t)$ is a martingale. For $t \geq s \geq 0$, we have

$$\begin{aligned}
\mathbb{E}[X(t)|\mathcal{F}_s] &= \mathbb{E}[B^2(t) - B^2(s) + B^2(s) - t|\mathcal{F}_s] = \mathbb{E}[(B^2(t) - B^2(s))|\mathcal{F}_s] + B^2(s) - t \\
&= \mathbb{E}[\{(B(t) - B(s))^2 + 2B(s)(B(t) - B(s))\}|\mathcal{F}_s] + B^2(s) - t \\
&= \mathbb{E}[\{(B(t) - B(s))^2|\mathcal{F}_s] + 2\mathbb{E}[B(s)(B(t) - B(s))|\mathcal{F}_s] + B^2(s) - t
\end{aligned}$$

Since $B(t) - B(s)$ is $\mathcal{F}_s$ independent and $B(s)$ is $\mathcal{F}_s$-measurable, by using properties of conditional expectation, we have

$$\begin{aligned}
&\mathbb{E}[\{(B(t) - B(s))^2|\mathcal{F}_s] = \mathbb{E}[(B(t) - B(s))^2] = t - s \\
&\mathbb{E}[B(s)(B(t) - B(s))|\mathcal{F}_s] = B(s)\mathbb{E}[(B(t) - B(s))|\mathcal{F}_s] = B(s)\mathbb{E}[B(t) - B(s)] = 0.
\end{aligned}$$

Thus, we obtain

$$\mathbb{E}[X(t)|\mathcal{F}_s] = (t - s) + B^2(s) - t = X(s) \text{ a.s.}$$

This shows that $X(t)$ is a martingale.
We now show that $M(t)$ is a martingale. First of all, $M(t)$ is integrable. Indeed,

$$\mathbb{E}[e^{\sigma B(t)}] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x} e^{-\frac{x^2}{2t}}\, dx = e^{\frac{1}{2}\sigma^2 t} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma t)^2}{2t}}\, dx = e^{\frac{1}{2}\sigma^2 t}$$

$$\implies \mathbb{E}[M(t)] = 1.$$

Since for any $t \ge s \ge 0$, $B(t) - B(s) \sim \mathcal{N}(0, t-s)$, by using similar calculation, we see that

$$\mathbb{E}\Big[\exp\big\{\sigma(B(t) - B(s))\big\}\Big] = \exp\big\{\frac{1}{2}\sigma^2(t-s)\big\}.$$

We have

$$\begin{aligned}
\mathbb{E}\Big[M(t)|\mathcal{F}_s\Big] &= \mathbb{E}\Big[\exp\{\sigma(B(t)-B(s))\}\exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}|\mathcal{F}_s\Big] \\
&= \exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}\mathbb{E}\Big[\exp\{\sigma(B(t)-B(s))\}|\mathcal{F}_s\Big] \\
&= \exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}\mathbb{E}\big[\exp\{\sigma(B(t)-B(s))\}\big] \\
&= \exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}\exp\big\{\frac{1}{2}\sigma^2(t-s)\big\} = M(s) \text{ a.s.}
\end{aligned}$$

Thus, $M(t)$ is a martingale. □

Lemma 2.2. *Let $B(\cdot)$ be a one-dimensional Brownian motion. Then*

$$\mathbb{E}[B(t)B(s)] = t \wedge s = \min\{t, s\} \quad t, s \ge 0.$$

Proof. Assume that $t \ge s \ge 0$. Then

$$\begin{aligned}
\mathbb{E}[B(t)B(s)] &= \mathbb{E}[(B(s) + B(t) - B(s))B(s)] = \mathbb{E}[B^2(s)] + \mathbb{E}[(B(t)-B(s))B(s)] \\
&= s + \mathbb{E}[B(t) - B(s)]\mathbb{E}[B(s)] = s = t \wedge s.
\end{aligned}$$

In the above, we have used the fact that $B(s)$ is normally distributed with mean zero and variance s and $B(t) - B(s)$ is independent of $B(s)$. □

Example 2.1. *Let $B(t)$ be a Brownian motion . Define a stochastic process*

$$X(t) = \begin{cases} 0, & t = 0 \\ tB(\frac{1}{t}), & t > 0. \end{cases}$$

Then $X(t)$ is a Brownian motion. Indeed, for $t > s$, we have

$$X(t) - X(s) = (t-s)B(\frac{1}{t}) + s\big(B(\frac{1}{t}) - B(\frac{1}{s})\big)$$

Observe that

$$s\big(B(\frac{1}{t}) - B(\frac{1}{s})\big) \sim \mathcal{N}\big(0, s^2(\frac{1}{s} - \frac{1}{t})\big), \quad (t-s)B(\frac{1}{t}) \sim \mathcal{N}\big(0, \frac{(t-s)^2}{t}\big).$$

Moreover, $s\big(B(\frac{1}{t}) - B(\frac{1}{s})\big)$ and $(t-s)B(\frac{1}{t})$ are independent. Hence $X(t) - X(s)$ is a normally distributed random variable with mean zero and variance $s^2\big(\frac{1}{s} - \frac{1}{t}\big) + \frac{(t-s)^2}{t} = t - s$. In other words, the increments $X(t) - X(s)$ is $\mathcal{N}(0, t-s)$ for all $t \ge s > 0$. Next we show that it has independent increments. Note that

$$\begin{aligned}
\mathrm{Cov}(X(t), X(s)) &= \mathbb{E}[X(t)X(s)] = st\mathbb{E}[B(\frac{1}{t})B(\frac{1}{s})] = st\mathrm{Cov}(B(\frac{1}{t}), B(\frac{1}{s})) \\
&= st\min(\frac{1}{t}, \frac{1}{s}) = \min(s, t).
\end{aligned}$$

Let $s < t$. Then

$$\mathrm{Cov}(X(s), X(t) - X(s)) = \mathrm{Cov}(X(s), X(t)) - \mathrm{Cov}(X(s), X(s)) = \min(s,t) - s = 0\,.$$

Since $X(s)$ and $X(t) - X(s)$ are normal random variables, and $\mathrm{Cov}(X(s), X(t) - X(s)) = 0$, we conclude that they are independent. It remains to show that sample paths are continuous with *probability* 1. *Observe that $X(t)$ is continuous for $t>0$. We show that $X(t)$ is continuous at $t=0$.*

$$\lim_{t\to 0} X(t) = \lim_{t\to 0} tB(\frac{1}{t}) = \lim_{n\to\infty} \frac{1}{n} B(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \{B(i)-B(i-1)\}$$

Note that $B(i)-B(i-1)$ are **i.i.d** *sequence of random variables with mean zero. Hence by SLLN, we get*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \{B(i)-B(i-1)\} = 0 \quad a.s.$$

This completes the proof.

2.1. **First and Quadratic Variation.** Let f be a function defined on $[0,T]$ and $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = T\}$ be a partition of $[0,T]$. We define **first-variation** of f up to time T as

$$\mathrm{FV}_T(f) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \big|f(t_{j+1}) - f(t_j)\big|.$$

The **quadratic variation** of f up to time T is defined by

$$[f,f](T) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \big(f(t_{j+1}) - f(t_j)\big)^2.$$

Observe that if $f:[0,T]\to\mathbb{R}$ such that $|f'|$ is Riemann integrable on $[0,T]$, then

$$\mathrm{FV}_T(f) = \int_0^T |f'(t)|\,dt.$$

Indeed, by MVT, there exists $t_j^* \in (t_j, t_{j+1})$ such that

$$f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j).$$

This implies that

$$\begin{aligned}
&\sum_{j=0}^{n-1} \big|f(t_{j+1}) - f(t_j)\big| = \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1}-t_j)\\
&\implies \mathrm{FV}_T(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \big|f(t_{j+1}) - f(t_j)\big| = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1}-t_j) = \int_0^T |f'(t)|\,dt\,.
\end{aligned}$$

Notice that if f has continuous derivative then $[f,f](T)=0$. Indeed, by using MVT we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} \big(f(t_{j+1}) - f(t_j)\big)^2 = \sum_{j=0}^{n-1} |f'(t_j^*)|^2 (t_{j+1}-t_j)^2 \le \|\Pi\| \sum_{j=0}^{n-1} \|f'(t_j^*)|^2 (t_{j+1}-t_j)\\
&\implies [f,f](T) \le \lim_{\|\Pi\|\to 0} \Big\{\|\Pi\| \sum_{j=0}^{n-1} \|f'(t_j^*)|^2 (t_{j+1}-t_j)\Big\} = \lim_{\|\Pi\|\to 0} \|\Pi\| \cdot \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \|f'(t_j^*)|^2 (t_{j+1}-t_j)\\
&\qquad = \lim_{\|\Pi\|\to 0} \|\Pi\| \int_0^T \|f'(t)|^2\,dt\,.
\end{aligned}$$

Since f' is continuous, $\int_0^T |f'(t)|^2\,dt$ is finite and hence we obtain that $[f,f](T)=0$.

Theorem 2.3. *Let $B(\cdot)$ be a one-dimensional Brownian motion. Then $[B,B](T)=T$ for all $T\ge 0$ a.s.*

Proof. Let $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = T\}$ be a partition of $[0, T]$. Consider the random variable $Q_T^n := \sum_{j=0}^{n-1}(B(t_{j+1}) - B(t_j))^2$. Then

$$\begin{aligned}\mathbb{E}\big[(Q_T^n - T)^2\big] &= \mathbb{E}\Big[\Big(\sum_{j=0}^{n-1}(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\Big)^2\Big] \\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\mathbb{E}\Big[\Big((B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\Big)\Big((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)\Big)\Big].\end{aligned}$$

Thanks to independent increments and the fact that $B(t) - B(s) \sim \mathcal{N}(0, t - s)$ for all $t \geq s \geq 0$, we have for $k \neq j$

$$\mathbb{E}\Big[\Big((B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\Big)\Big((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)\Big)\Big] = 0.$$

Thus, we get that

$$\mathbb{E}\big[(Q_T^n - T)^2\big] = \sum_{k=0}^{n-1}\mathbb{E}\Big[(Y_k^2 - 1)^2(t_{k+1} - t_k)^2\Big]$$

where

$$Y_k := \frac{B(t_{k+1}) - B(t_k)}{\sqrt{t_{k+1} - t_k}} \sim \mathcal{N}(0, 1).$$

Hence, for some constant $C > 0$, we have

$$\mathbb{E}\big[(Q_T^n - T)^2\big] \leq C\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \implies \lim_{\|\Pi\|\to 0}\mathbb{E}(Q_T^n - T)^2 = 0.$$

Therefore, there exists a sub-sequence along which the convergence is almost surely. Thus, $[B, B](T) = T$ for all $T \geq 0$ a.s. $\square$

Remark 2.2. Brownian motion accumulates $b - a$ units of quadratic variation over the interval $[a, b]$.

Lemma 2.4. *For any $T > 0$, the first variation of Brownian motion $B(\cdot)$ up to time T is infinite almost surely.*

Proof. Let $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = T\}$ be a partition of $[0, T]$. Observe that

$$\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)^2 \leq \sup_{0\leq k\leq n-1}|B(t_{k+1}) - B(t_k)|\sum_{j=0}^{n-1}|B(t_{j+1}) - B(t_j)|$$

We know that $\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)^2 \to T$ a.s. Pick an ω for which this holds and the sample paths of Brownian motion is continuous. Then

$$0 < T \leq \lim_{\|\Pi\|\to 0}\sup_{0\leq k\leq n-1}|B(t_{k+1}) - B(t_k)|\sum_{j=0}^{n-1}|B(t_{j+1}) - B(t_j)|$$

Since $\lim_{\|\Pi\|\to 0}\sup_{0\leq k\leq n-1}|B(t_{k+1}) - B(t_k)| = 0$, we see that the sample paths have infinite variation with probability one:

$$\mathrm{FV}_T(B) := \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\big|B(t_{j+1}) - B(t_j)\big| = \infty.$$

$\square$

We now show how to use the quadratic variation of Brownian motion to identify the volatility σ in asset-price model used in the Black-Scholes-Merton option-pricing formula. Consider the **geometric Brownian motion**

$$S(t) = S(0)\exp\{\sigma B(t) + (\alpha - \frac{1}{2}\sigma^2)t\}$$

where α and $\sigma > 0$ are constant. Let $0 \leq T_1 < T_2$ be given and suppose we observe geometric Brownian motion $S(t)$ for $T_1 \leq t \leq T_2$. Let $\Pi = \{T_1 = t_0 < t_1 < \ldots < t_n = T_2\}$ be a partition of $[T_1, T_2]$. Then we have

$$\begin{aligned} &\log \frac{S(t_{j+1})}{S(t_j)} = \sigma\big(B(t_{j+1}) - B(t_j)\big) + (\alpha - \frac{1}{2}\sigma^2)(t_{j+1} - t_j) \\ \Longrightarrow &\sum_{j=0}^{n-1}\Big(\log\frac{S(t_{j+1})}{S(t_j)}\Big)^2 = \sigma^2\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)^2 + (\alpha - \frac{1}{2}\sigma^2)^2\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2 \\ &\qquad + 2\sigma(\alpha - \frac{1}{2}\sigma^2)\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)(t_{j+1} - t_j) \\ &\qquad \equiv \sigma^2\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \end{aligned} \tag{2.1}$$

Observe that

$$\lim_{\|\Pi\|\to 0}\mathcal{A}_1 = T_2 - T_1, \qquad \lim_{\|\Pi\|\to 0}\mathcal{A}_2 = 0.$$

We claim that $\lim_{\|\Pi\|\to 0}\mathcal{A}_3 = 0$. Indeed,

$$\begin{aligned} &\big|\big(B(t_{j+1}) - B(t_j)\big)(t_{j+1} - t_j)\big| \leq \max_{0\leq k\leq n-1}\big|B(t_{k+1}) - B(t_k)\big|(t_{j+1} - t_j) \\ \Longrightarrow &\Big|\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)(t_{j+1} - t_j)\Big| \leq \max_{0\leq k\leq n-1}\big|B(t_{k+1}) - B(t_k)\big|\sum_{j=0}^{n-1}(t_{j+1} - t_j) \\ &\qquad = (T_2 - T_1)\max_{0\leq k\leq n-1}\big|B(t_{k+1}) - B(t_k)\big| \end{aligned}$$

Since $B(\cdot)$ is continuous on $[T_1, T_2]$, we see that $\max_{0\leq k\leq n-1}\big|B(t_{k+1}) - B(t_k)\big|$ converges to zero as $\|\Pi\| \to 0$. Thus $\lim_{\|\Pi\|\to 0}\mathcal{A}_3 = 0$. Hence, from (2.1) we conclude that when the maximum step size $\|\Pi\|$ is small, we approximate the volatility σ as

$$\sigma^2 \approx \frac{1}{T_2 - T_1}\sum_{j=0}^{n-1}\Big(\log\frac{S(t_{j+1})}{S(t_j)}\Big)^2.$$

2.2. **Sample path properties of Brownian motion:** We will demonstrate that for almost every ω, the sample path $t \mapsto B(t,\omega)$ is uniformly Holder continuous for each exponent $\gamma < \frac{1}{2}$, but nowhere Holder continuous with any exponent $\gamma > \frac{1}{2}$. To prove uniformly Holder continuity, we use a general theorem of Kolmogorov called **Kolmogorov's continuity theorem**.

Theorem 2.5 (**Kolmogorov's continuity theorem**). *Let $X(\cdot)$ be a stochastic process with continuous sample paths a.s. such that*

$$\mathbb{E}\Big[|X(t)-X(s)|^{\beta}\Big] \le C|t-s|^{1+\alpha}$$

for constants $\alpha,\beta>0$ and $C\ge 0$, and for all $t,s\ge 0$. Then for each $0<\gamma<\frac{\alpha}{\beta}$, $T>0$, and almost every ω, there exists a constant $K=K(\omega,\gamma,T)$ such that

$$|X(t,\omega)-X(s,\omega)| \le K|t-s|^{\gamma} \quad \textit{for all } s,t\in[0,T]$$

i.e., the sample path $t\mapsto X(t,\omega)$ is uniformly Holder continuous with exponent γ on $[0,T]$.

For $t>s$, we have for $m \in]\mathbb{N}$

$$\begin{aligned}\mathbb{E}\big[B(t)-B(s)|^{2m}\big] &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}|x|^{2m}e^{-\frac{x^2}{2(t-s)}}\,dx \\ &= (t-s)^m\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|y|^{2m}e^{-\frac{y^2}{2}}\,dy = C|t-s|^m\,.\end{aligned}$$

Thus, from Kolmogorov's continuity theorem, we conclude that (taking $\beta=2m,\alpha=m-1$) for a.s. ω and any $T>0$, the sample path $t\mapsto B(t,\omega)$ is uniformly Holder continuous on $[0,T]$ for each exponent

$$0<\gamma=\frac{m-1}{2m}=\frac{1}{2}-\frac{1}{2m}<\frac{1}{2}.$$

We now prove that sample paths of Brownian motion are nowhere differentiable with probability 1.

Theorem 2.6. *For each $\frac{1}{2}<\gamma\le 1$ and almost every ω, $t\mapsto B(t,\omega)$ is nowhere Holder continuous with exponent γ.*

Proof. For simplicity, we consider $B(t)$ for times $0\le t\le 1$. Fix N such that

$$N(\gamma-\frac{1}{2})>1.$$

Suppose $t\mapsto B(t,\omega)$ is Holder continuous with exponent γ at some point $s_0\in[0,1]$ i.e., there exists constant K such that

$$|B(t,\omega)-B(s_0,\omega)| \le K|t-s_0|^{\gamma} \quad \forall\, t\in[0,1]\,. \tag{2.2}$$

For n large enough, set $i=[ns_0]+1$. Then for $j=i,i+1,\dots,i+N-1$, we have, from (2.2)

$$\begin{aligned}|B(\frac{j}{n},\omega)-B(\frac{j+1}{n},\omega)| &\le |B(\frac{j}{n},\omega)-B(s_0,\omega)|+|B(\frac{j+1}{n},\omega)-B(s_0,\omega)| \\ &\le K\big\{|\frac{j}{n}-s_0|^{\gamma}+|\frac{j+1}{n}-s_0|^{\gamma}\big\} \le Mn^{-\gamma}\end{aligned}$$

for some constant M. For $i\le i\le n$, $M\ge 1$ and for large n, define

$$A^i_{M,n} := \big\{\omega : |B(\frac{j}{n},\omega)-B(\frac{j+1}{n},\omega)| \le Mn^{-\gamma}, \quad j=i,i+1,\dots,i+N-1\big\}$$

Then, if A is the set of ω such that $B(\cdot,\omega)$ is Holder continuous with exponent γ at some point $s_0, 0\le s_0<1$, we must have

$$A\subset \cup_{M=1}^{\infty}\cup_{k=1}^{\infty}\cap_{n=k}^{\infty}\cup_{i=1}^{n}A^i_{M,n}.$$

Our goal is to show that $\mathbb{P}(A)=0$. Here we show that

$$\mathbb{P}\Big(\cup_{M=1}^{\infty}\cup_{k=1}^{\infty}\cap_{n=k}^{\infty}\cup_{i=1}^{n}A_{M,n}^{i}\Big)=0.$$

Observe that

$$\mathbb{P}\Big(\cap_{n=k}^{\infty}\cup_{i=1}^{n}A_{M,n}^{i}\Big)\le\liminf_{n\to\infty}\mathbb{P}\Big(\cup_{i=1}^{n}A_{M,n}^{i}\Big)\le\liminf_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}(A_{M,n}^{i}).$$

Since $B(\frac{j}{n})-B(\frac{j+1}{n})\sim\mathcal{N}(0,\frac{1}{n})$ and $B(\frac{j}{n})-B(\frac{j+1}{n}), j=i,i+1,\ldots,i+N-1$ are independent, we see that

$$\begin{aligned}\mathbb{P}(A_{M,n}^{i})&=\Big\{\mathbb{P}\Big(|B(\frac{1}{n})|\le Mn^{-\gamma}\Big)\Big\}^{N}=\Big\{\frac{\sqrt{n}}{\sqrt{2\pi}}\int_{-Mn^{-\gamma}}^{Mn^{-\gamma}}e^{-\frac{nx^2}{2}}\,dx\Big\}^{N}\\&=\Big\{\frac{1}{\sqrt{2\pi}}\int_{-Mn^{\frac{1}{2}-\gamma}}^{Mn^{\frac{1}{2}-\gamma}}e^{-\frac{y^2}{2}}\,dy\Big\}^{N}\le Cn^{(\frac{1}{2}-\gamma)N}\quad\text{for some constant } C.\end{aligned}$$

Thus, we get

$$\mathbb{P}\Big(\cap_{n=k}^{\infty}\cup_{i=1}^{n}A_{M,n}^{i}\Big)\le\liminf_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}(A_{M,n}^{i})\le\liminf_{n\to\infty}\sum_{i=1}^{n}Cn^{(\frac{1}{2}-\gamma)N}=0\quad(\because\ N(\gamma-\frac{1}{2})>1)$$

Since the above relation holds for all k and M, we conclude that

$$\mathbb{P}\Big(\cup_{M=1}^{\infty}\cup_{k=1}^{\infty}\cap_{n=k}^{\infty}\cup_{i=1}^{n}A_{M,n}^{i}\Big)=0\implies\mathbb{P}(A)=0.$$

□

Remark 2.3. From Theorem 2.6, we conclude the followings:

a) Sample paths of Brownian motion is nowhere differentiable. Indeed, if $B(\cdot,\omega)$ is differentiable at some point s, then $B(\cdot,\omega)$ would be Holder continuous at point s with exponent $\gamma=1$ which contradicts Theorem 2.6.
b) Sample path of Brownian motion is infinite variation. Because, if $B(\cdot,\omega)$ were finite variation on some sub-interval, then it would be differentiable almost everywhere there– which is NOT possible.

2.3. **Markov property of Brownian motion.**

Definition 2.2 (Markov process)**.** Let $\{X(t):t\ge 0\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$. We say that $X(t)$ is a Markov process if for any Borel measurable function f, there is a Borel measurable function g such that

$$\mathbb{E}\big[f(X(t))|\mathcal{F}_s\big]=g(X(s))\quad\forall\,t>s\ge 0.$$

We show that Brownian motion is a Markov process. For its proof, we need the following independence lemma.

Lemma 2.7 (Independence)**.** *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\mathcal{Y}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose the random variables $X_1,X_2,\ldots,X_n$ are $\mathcal{Y}$-measurable and the random variables $Y_1,Y_2,\ldots,Y_m$ are independent of $\mathcal{Y}$. Let $f(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m)$ be a function of the dummy variables $x_1,x_2,\ldots,x_n$ and $y_1,y_2,\ldots,y_m$, and define*

$$g(x_1,x_2,\ldots,x_n)=\mathbb{E}\Big[f(x_1,x_2,\ldots,x_n,Y_1,Y_2,\ldots,Y_m)\Big].$$

Then

$$g(X_1,X_2,\ldots,X_n)=\mathbb{E}\Big[f(X_1,X_2,\ldots,X_n,Y_1,Y_2,\ldots,Y_m)|\mathcal{Y}\Big].$$

Theorem 2.8. *Let $B(t), t \geq 0$ be a Brownian motion and $\mathcal{F}_t$ be its natural filtration. Then $B(t)$ is a Markov process. Moreover, for any Borel measurable function f, there holds*

$$\mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] = \int_{-\infty}^{\infty} f(y)\rho(t-s, B(s), y)\, dy,$$

where $\rho(\tau, x, y) := \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}$ is the **transition density** *of the Brownian motion.*

Proof. Let f be any Borel measurable function. Observe that for any $t > s \geq 0$,

$$\mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] = \mathbb{E}\Big[f(B(t) - B(s) + B(s))|\mathcal{F}_s\Big]$$

and $B(t) - B(s)$ is independent of $\mathcal{F}_s$ and $B(s)$ is $\mathcal{F}_s$ measurable. Define

$$g(x) = \mathbb{E}\big[f(x + B(t) - B(s))\big].$$

Then by Independence lemma 2.7, we see that $\mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] = g(B(s))$. In other words, $B(t)$ is a Markov process. Since $B(t) - B(s) \sim \mathcal{N}(0, t-s)$, we see that

$$\begin{aligned}
g(x) &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(x+w) e^{-\frac{w^2}{2(t-s)}}\, dw \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{2(t-s)}}\, dy = \int_{-\infty}^{\infty} f(y)\rho(t-s, x, y)\, dy \\
\implies \mathbb{E}\Big[f(B(t))|\mathcal{F}_s\Big] &= \int_{-\infty}^{\infty} f(y)\rho(t-s, B(s), y)\, dy,
\end{aligned}$$

where $\rho(\tau, x, y) := \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}$ is the **transition density** of the Brownian motion. □

Example 2.2. *Let $B(t)$ be a Brownian motion and $\{\mathcal{F}_t\}$ be its natural filtration. For $\mu \in \mathbb{R}$, consider the Brownian motion with drift*

$$X(t) := \mu t + B(t).$$

Then $\{X(t) : t \geq 0\}$ is a Markov process.

To see this, let $0 \leq s < t$ and f be any Borel measurable function. Then

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = \mathbb{E}\big[f(\mu t + B(s) + B(t) - B(s))|\mathcal{F}_s\big].$$

Since $\mu t + B(s)$ is $\mathcal{F}_s$-measurable and $B(t) - B(s)$ is $\mathcal{F}_s$-independent, we get

$$\begin{aligned}
\mathbb{E}[f(X(t))|\mathcal{F}_s] &= \mathbb{E}[f(a + B(t-s))]\big|_{a=\mu t + B(s)} \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(\mu t + B(s) + x) e^{-\frac{x^2}{2(t-s)}}\, dx \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-B(s)-\mu t)^2}{2(t-s)}}\, dy \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-B(s)-\mu s-\mu(t-s))^2}{2(t-s)}}\, dy \\
&= \int_{-\infty}^{\infty} f(y)\rho(t-s, X(s), y)\, dy = g(X(s))
\end{aligned}$$

where $\rho(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x-\mu\tau)^2}{2\tau}}$ and $g(x) = \int_{-\infty}^{\infty} f(y)\rho(t-s, x, y)\, dy$.

2.4. **First passage time of Brownian motion and reflexion principle:** We now discuss the first passage time in which Brownian motion reaches to level m first time. For any real number m, define the first passage time as

$$\tau_m := \min\{t \geq 0 : B(t) = m\}. \tag{2.3}$$

Notice that

$$\{\tau_m \leq t\} = \{\exists\, s \in [0,t] : B(s) = a\} \in \mathcal{F}_t$$

where $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$. τ_m is a stopping time with respect to the filtration $\mathcal{F}_t$. Here we state some important property of Brownian motion associated with τ_m without proof.

Lemma 2.9. *$B(\tau_m + s) - B(\tau_m) = B(\tau_m + s) - m$ is also a Brownian motion, independent from $B(t)$, $t \leq \tau_m$.*

Theorem 2.10. *For $m \in \mathbb{R}^+$, let τ_m be the first passage time of Brownian motion to level m. Then τ_m is finite almost surely.*

Proof. For any $\sigma > 0$, consider the exponential martingale $M(t)$ given in Theorem 2.1. Since τ_m is a stopping time, we have

$$1 = M(0) = \mathbb{E}[M(t \wedge \tau_m)] = \mathbb{E}\Big[\exp\big\{\sigma B(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)\big\}\Big] \tag{2.4}$$

It is easy to see that

$$\lim_{t \to \infty} \exp\{-\frac{1}{2}\sigma^2 (t \wedge \tau_m)\} = \mathbf{1}_{\{\tau_m < \infty\}} \exp\{-\frac{1}{2}\sigma^2 \tau_m\}$$

where $\mathbf{1}_A$ is the indicator function on A defined as

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Notice that, if $\tau_m < \infty$, then for large t, $\exp\{\sigma B(t \wedge \tau_m)\} = e^{\sigma m}$, but for $\tau_m = \infty$, we do not know what happens to $\exp\{\sigma B(t \wedge \tau_m)\}$ as $t \to \infty$. Observe that since $\sigma > 0$ and $m > 0$, for $t \leq \tau_m$, the get the following upper bound:

$$0 \leq \exp\{\sigma B(t \wedge \tau_m)\} \leq e^{\sigma m}.$$

Due to this bound, we conclude that $\exp\big\{\sigma B(t \wedge \tau_m) - \frac{1}{2}\sigma^2(t \wedge \tau_m)\big\}$ converges to zero as $t \to \infty$ for $\tau_m = \infty$. Combining these analysis, we finally have

$$\lim_{t \to \infty} \exp\big\{\sigma B(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)\big\} = \mathbf{1}_{\{\tau_m < \infty\}} \exp\{\sigma m - \frac{1}{2}\sigma^2 \tau_m\} \tag{2.5}$$

Passing to the limit in (2.4) as $t \to \infty$ along with (2.5), we get

$$\begin{aligned} 1 &= \lim_{t \to \infty} \mathbb{E}\Big[\exp\big\{\sigma B(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)\big\}\Big] \\ &= \mathbb{E}\Big[\lim_{t \to \infty} \exp\big\{\sigma B(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)\big\}\Big] \\ &= \mathbb{E}\Big[\mathbf{1}_{\{\tau_m < \infty\}} \exp\{\sigma m - \frac{1}{2}\sigma^2 \tau_m\}\Big]. \end{aligned}$$

Since this holds for all $\sigma > 0$, passing to the limit as $\sigma \to 0$, we have

$$\mathbb{E}\big[\mathbf{1}_{\{\tau_m < \infty\}}\big] = 1 \implies \mathbb{P}(\tau_m < \infty) = 1.$$

In other words, τ_m is finite almost surely. ∎

We now discuss the distribution of

$$\bar{M}(t) := \sup_{0 \le s \le t} B(s)$$

for any given t. This stochastic process is used in pricing barrier option.

Theorem 2.11 (Reflection principle)**.** *Let $B(t)$ be Brownian motion. Then for every $m \ge 0$,*

$$\mathbb{P}(\bar{M}(t) \ge m) = 2\mathbb{P}(B(t) \ge m) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx\,.$$

Proof. We have

$$\mathbb{P}(B(t) \ge m) = \mathbb{P}(B(t) \ge m, \bar{M}(t) \ge m) + \mathbb{P}(B(t) \ge m, \bar{M}(t) < m)$$

Since $\bar{M}(t) \ge B(t)$, we have $\mathbb{P}(B(t) \ge m, \bar{M}(t) < m) = 0$. Now

$$\mathbb{P}(B(t) \ge m, \bar{M}(t) \ge m) = \mathbb{P}(B(t) \ge m | \bar{M}(t) \ge m)\mathbb{P}(\bar{M}(t) \ge m).$$

Observe that $\bar{M}(t) \ge m$ if and only if $\tau_m \le t$. Thus,

$$\mathbb{P}(B(t) \ge m, \bar{M}(t) \ge m) = \mathbb{P}(B(t) \ge m | \tau_m \le t)\mathbb{P}(\bar{M}(t) \ge m).$$

Observe that

$$\mathbb{P}(B(t) \ge m | \tau_m \le t) = \mathbb{P}(B(\tau_m + (t - \tau_m)) - m \ge 0 | \tau_m \le t)$$

In view of Lemma 2.9, $B(\tau_m + (t - \tau_m)) - m, \tau_m \le t$ is a Brownian motion and therefore, we get that

$$\mathbb{P}(B(\tau_m + (t - \tau_m)) - m \ge 0 | \tau_m \le t) = \frac{1}{2}$$

since the Brownian motion satisfies $\mathbb{P}(B(t) \ge 0) = \frac{1}{2}$ for every t. Combining all these, we get

$$\mathbb{P}(\bar{M}(t) \ge m) = 2\mathbb{P}(B(t) \ge m) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx.$$

□

Remark 2.4. Since $\bar{M}(t) \ge m$ if and only if $\tau_m \le t$, from Theorem 2.11, we see that the distribution function $F_{\tau_m}(\cdot)$ and probability density function $f_{\tau_m}(\cdot)$ are given by

$$F_{\tau_m}(t) = 2\int_m^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx; \quad f_{\tau_m}(t) = \frac{1}{\sqrt{2\pi t^3}} m e^{-\frac{m^2}{2t}}.$$

respectively.

We now establish joint probability distribution and density function of $\bar{M}(t)$ and $B(t)$.

Proposition 2.12. *For every $m > 0$, $y \ge 0$,*

$$\mathbb{P}(\bar{M}(t) \ge m, B(t) \le m - y) = \mathbb{P}(B(t) > m + y).$$

Moreover, the joint density function of $\bar{M}(t)$ and $B(t)$ is given by

$$f_{(\bar{M}(t), B(t))}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \le m,\ m > 0\,.$$

Proof. For every $m > 0$, $y \geq 0$, we have

$$\begin{aligned}\mathbb{P}(B(t) > m + y) &= \mathbb{P}(B(t) > m + y, \bar{M}(t) \geq m) + \mathbb{P}(B(t) > m + y, \bar{M}(t) < m) \\ &= \mathbb{P}(B(t) > m + y, \bar{M}(t) \geq m) \\ &= \mathbb{P}(B(t) > m + y | \bar{M}(t) \geq m)\mathbb{P}(\bar{M}(t) \geq m) \\ &= \mathbb{P}(B(\tau_m + (t - \tau_m)) - m > y | \tau_m \leq t)\mathbb{P}(\bar{M}(t) \geq m)\end{aligned}$$

Since $B(\tau_m + (t - \tau_m)) - m$ is a Brownian motion, by symmetry, we have

$$\begin{aligned}&\mathbb{P}(B(\tau_m + (t - \tau_m)) - m > y | \tau_m \leq t) \\ &= \mathbb{P}(B(\tau_m + (t - \tau_m)) - m < -y | \tau_m \leq t) \\ &= \mathbb{P}(B(t) < m - y | \tau_m \leq t) = \mathbb{P}(B(t) < m - y | \bar{M}(t) \geq m)\end{aligned}$$

Thus, we have

$$\mathbb{P}(B(t) > m+y) = \mathbb{P}(B(t) < m-y | \bar{M}(t) \geq m)\mathbb{P}(\bar{M}(t) \geq m) = \mathbb{P}(\bar{M}(t) \geq m, B(t) \leq m-y).$$

To establish the second part, we write the above equality in terms of density functions. We have

$$\begin{aligned}&\int_m^\infty \int_{-\infty}^{m-y} f_{(\bar{M}(t),B(t))}(u,v)\,du\,dv = \frac{1}{\sqrt{2\pi t}} \int_{m+y}^\infty e^{-\frac{z^2}{2t}}\,dz \\ \Longrightarrow &\int_m^\infty \int_{-\infty}^{w} f_{(\bar{M}(t),B(t))}(u,v)\,du\,dv = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}}\,dz, \quad w \leq m, \ m > 0.\end{aligned}$$

Differentiating with respect to m, we have

$$\int_{-\infty}^{w} f_{(\bar{M}(t),B(t))}(m,v)\,dv = \frac{t}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}.$$

Again differentiating with respect to w, we get the joint density function as

$$f_{(\bar{M}(t),B(t))}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \leq m, \, m > 0\,.$$

This completes the proof. □

Theorem 2.13. *For any $\alpha > 0$, we have*

$$\mathbb{E}[\exp(-\alpha\tau_m)] = e^{-m\sqrt{2\alpha}}\,.$$

Furthermore, $\mathbb{E}[\tau_m] = \infty$.

Proof. In view of the proof of Theorem 2.10, we have shown that

$$1 = \mathbb{E}\Big[\mathbf{1}_{\{\tau_m < \infty\}} \exp\{\sigma m - \frac{1}{2}\sigma^2\tau_m\}\Big]$$

holds for any $\sigma > 0$. Since $\mathbb{P}(\tau_m < \infty) = 1$, we actually get

$$\mathbb{E}\Big[\exp\{\sigma m - \frac{1}{2}\sigma^2\tau_m\}\Big] = 1 \implies \mathbb{E}\big[\exp\{-\frac{1}{2}\sigma^2\tau_m\}\big] = \exp(-\sigma m)\,.$$

Taking $\sigma = \sqrt{2\alpha}$ in the above equality, we get

$$\mathbb{E}[\exp(-\alpha\tau_m)] = e^{-m\sqrt{2\alpha}}\,. \tag{2.6}$$

Differentiating (2.6) with respect to α and then sending $\alpha \to 0$ in the resulting expression, we finally get that $\mathbb{E}[\tau_m] = \infty$. This completes the proof. □

3. Ito integrals:

We would like to define $\int_0^T G(s)dB(s)$ for some wide class of stochastic processes G. Observe that the integral $\int_0^T G(s)dB(s)$ simply cannot be understood as an ordinary integral (Riemann-Stieltjes sense) as the paths $t \mapsto B(t,\omega)$ is of infinite variation and nowhere differentiable for almost every ω. Let us first think about what might be an appropriate definition for $\int_0^T G(s)dB(s)$. Suppose $\Pi^n := \{0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = T\}$ are partitions of $[0,T]$ such that $\|\Pi^n\| \to 0$ as $n \to \infty$. We define corresponding Riemann sum approximation: for $\lambda \in [0,1]$

$$R_n = R_n(\Pi^n, \lambda) := \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n))$$

where $\tau_k^n := (1-\lambda)t_k^n + \lambda t_{k+1}^n,\ k = 0,1,2\ldots,m_n-1$. We claim that

$$\lim_{n\to\infty} R_n = \frac{B^2(T)}{2} + (\lambda - \frac{1}{2})T$$

where the limit is taken in $L^2(\Omega)$. Indeed, by using the identity $a^2-b^2 = (a-b)^2+2b(a-b)$, we have

$$\begin{aligned}
R_n :&= \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&= \frac{B^2(T)}{2} - \frac{1}{2}\sum_{k=0}^{m_n-1} (B^2(t_{k+1}^n) - B^2(t_k^n)) + \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&= \frac{B^2(T)}{2} - \frac{1}{2}\sum_{k=0}^{m_n-1} (B(t_{k+1}^n) - B(t_k^n))^2 - \sum_{k=0}^{m_n-1} B(t_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&\qquad + \sum_{k=0}^{m_n-1} B(\tau_k^n)(B(t_{k+1}^n) - B(t_k^n)) \\
&= \frac{B^2(T)}{2} \underbrace{- \frac{1}{2}\sum_{k=0}^{m_n-1} (B(t_{k+1}^n) - B(t_k^n))^2}_{:=\mathcal{A}} + \underbrace{\sum_{k=0}^{m_n-1} (B(\tau_k^n) - B(t_k^n))^2}_{:=\mathcal{B}} \\
&\qquad + \underbrace{\sum_{k=0}^{m_n-1} (B(t_{k+1}^n) - B(\tau_k^n))(B(\tau_k^n) - B(t_k^n))}_{:=\mathcal{C}}
\end{aligned}$$

Since the quadratic variation of Brownian motion up to time T is T, we see that $\mathcal{A} \to \frac{T}{2}$ in $L^2(\Omega)$ as $n \to \infty$. Again, by a similar argument , we can show easily that $\mathcal{B} \to \lambda T$ as $n \to \infty$. By independent increments of Brownian motion, we see that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{C}^2\big] &= \sum_{k=0}^{m_n-1} \mathbb{E}\big[(B(t_{k+1}^n) - B(\tau_k^n))^2\big]\mathbb{E}\big[(B(\tau_k^n) - B(t_k^n))^2\big] \\
&= \sum_{k=0}^{m_n-1} \lambda(1-\lambda)(t_{k+1}^n - t_k^n)^2 \le \|\Pi^n\|\lambda(1-\lambda)T
\end{aligned}$$

$$\implies \lim_{n\to\infty} \mathbb{E}\big[(\mathcal{C}-0)^2\big] = 0\,.$$

Combining all these we get

$$\lim_{n\to\infty} R_n = \frac{B^2(T)}{2} + (\lambda - \frac{1}{2})T.$$

So, unlike Riemann-Stieltjes integral-it does make a difference here what point τ_k^n we choose. The following two choices are to be most useful:

i) $\tau_k^n = t_k^n$ (left end point corresponding to $\lambda = 0$), which leads to **Ito integral**, denoted by $\int_0^T BdB(s)$ and the value is given by $\int_0^T G(s)dB(s) = \frac{B^2(T)}{2} - \frac{T}{2}$.

ii) $\tau_k^n = \frac{t_k^n + t_{k+1}^n}{2}$ (the mid point corresponding to $\lambda = \frac{1}{2}$), which leads to **Stratonovich integral**, denoted by $\int_0^T B \circ dB(t)$, and the value is given by

$$\int_0^T B \circ dB(t) = \frac{B^2(T)}{2}.$$

3.1. **Ito integral:** Let us first describe the class of functions for which Ito integral will be defined. Let $\mathcal{Y} = \mathcal{Y}[0,T]$ be the class of functions $f(t,\omega) : [0,T] \times \Omega \to \mathbb{R}$ such that it is jointly measurable, $f(t)$ is $\mathcal{F}_t$-adapted and $\mathbb{E}\Big[\int_0^T f^2(t)\,dt\Big] < +\infty$.

Definition 3.1 (Step/Elementary process)**.** A process $G \in \mathcal{Y}$ is called step/elementary process if there exists a partition $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_m = T\}$ of $[0,T]$ such that

$$G(t,\omega) = \sum_{k=0}^{m-1} G_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t).$$

G_k is $\mathcal{F}_{t_k}$-measurable random variable.

Definition 3.2 (Stochastic integral of elementary process)**.** For elementary process G as above, we define the Ito stochastic integral of G as

$$\int_0^T GdB(t) := \sum_{k=0}^{m-1} G_k(B(t_{k+1}) - B(t_k)).$$

We can easily see that for all constants $a, b \in \mathbb{R}$ and all elementary processes $G, H \in \mathcal{Y}$,

$$\int_0^T (aG + bH)dB(t) = a\int_0^T GdB(t) + b\int_0^T HdB(t).$$

We claim that

$$\mathbb{E}\Big[\int_0^T GdB(t)\Big] = 0.$$

Indeed, we have

$$\mathbb{E}\Big[\int_0^T GdB(t)\Big] = \mathbb{E}\Big[\sum_{k=0}^{m-1} G_k(B(t_{k+1}) - B(t_k))\Big] = \sum_{k=0}^{m-1}\mathbb{E}\Big[G_k(B(t_{k+1}) - B(t_k))\Big]$$

Since G_k is $\mathcal{F}_{t_k}$-measurable and $(B(t_{k+1}) - B(t_k))$ is $\mathcal{F}_{t_k}$-independent, we get

$$\mathbb{E}\Big[G_k(B(t_{k+1}) - B(t_k))\Big] = \mathbb{E}[G_k]\mathbb{E}\big[(B(t_{k+1}) - B(t_k))\big] = 0.$$

Hence , we have $\mathbb{E}\Big[\int_0^T GdB(t)\Big] = 0$.

Lemma 3.1 (Ito-isometry). *If $G \in \mathcal{Y}$ is elementary, then*
$$\mathbb{E}\Big[\Big(\int_0^T G\,dB(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T G^2(t)\,dt\Big].$$
This is called Ito-isometry.

Proof. Since $G \in \mathcal{Y}$ is elementary, there exists a partition $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_m = T\}$ of $[0, T]$ such that
$$G(t,\omega) = \sum_{k=0}^{m-1} G_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t).$$
Thus,
$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T G\,dB(t)\Big)^2\Big] &= \mathbb{E}\Big[\Big(\sum_{k=0}^{m-1} G_k(B(t_{k+1}) - B(t_k))\Big)^2\Big] \\ &= \sum_{k,j=0}^{m-1} \mathbb{E}\Big[G_j G_k (B(t_{k+1}) - B(t_k))(B(t_{j+1}) - B(t_j))\Big] \\ &= \sum_{k,j=0}^{m-1} \mathbb{E}\Big[G_j G_k \Delta B_k \Delta B_j\Big]\end{aligned}$$
where $\Delta B_j = B(t_{j+1}) - B(t_j)$. Now if $j < k$, ΔB_k is independent of $G_j G_k \Delta B_j$ and therefore, we have
$$\mathbb{E}\Big[G_j G_k \Delta B_k \Delta B_j\Big] = \begin{cases} 0, & k \neq j \\ \mathbb{E}[G_k^2](t_{k+1} - t_k), & j = k. \end{cases}$$
Hence, we have
$$\mathbb{E}\Big[\Big(\int_0^T G\,dB(t)\Big)^2\Big] = \sum_{k=0}^{n-1} \mathbb{E}[G_k^2](t_{k+1} - t_k) = \mathbb{E}\Big[\sum_{k=0}^{n-1} G_k^2(t_{k+1} - t_k)\Big] = \mathbb{E}\Big[\int_0^T G^2(t)\,dt\Big].$$
$\square$

For any elementary process $G \in \mathcal{Y}$, we define indefinite Ito integral of G as

$$I_G(t) := \int_0^t G\, dB(s) = \sum_{j=0}^{m-1} G_j\big(B(t_{j+1} \wedge t) - B(t_j \wedge t)\big), \quad 0 \le t \le T.$$

Theorem 3.2. *$I_G(t)$ is a martingale.*

Proof. Let $0 \le s \le t \le T$ be given. Assume that s and t are in different sub-interval of the partition Π.. i.e., there exist partition points t_l and t_k with $t_l < t_k$ such that $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. Then, we have

$$\begin{aligned} I_G(t) = &\underbrace{\sum_{j=0}^{l-1} G_j(B(t_{j+1}) - B(t_j))}_{:=\mathcal{A}} + \underbrace{G_l(B(t_{l+1}) - B(t_l))}_{:=\mathcal{B}} + \underbrace{\sum_{j=l+1}^{k-1} G_j(B(t_{j+1}) - B(t_j))}_{:=\mathcal{C}} \\ &+ \underbrace{G_k\big(B(t) - B(t_k)\big)}_{:=\mathcal{D}} \end{aligned}$$

$$I_G(s) = \sum_{j=0}^{l-1} G_j(B(t_{j+1}) - B(t_j)) + G_l(B(s) - B(t_l)).$$

We must show that $\mathbb{E}\big[I_G(t)|\mathcal{F}_s\big] = I_G(s)$ a.s. Observe that $\mathcal{A}$ is $\mathcal{F}_s$-measurable and hence $\mathbb{E}\big[\mathcal{A}|\mathcal{F}_s\big] = \mathcal{A}$. Since $B(\cdot)$ is martingale and G_l is $\mathcal{F}_s$-measurable, we have

$$\mathbb{E}\big[\mathcal{B}|\mathcal{F}_s\big] = G_l\mathbb{E}\big[(B(t_{l+1}) - B(t_l))|\mathcal{F}_s\big] = G_l\big(\mathbb{E}\big[B(t_{l+1})|\mathcal{F}_s\big] - B(t_l)\big) = G_l(B(s) - B(t_l))\,.$$

By using tower property of conditional expectation and the fact that $t_j \ge t_{l+1} > s$, we get

$$\begin{aligned} \mathbb{E}\big[\mathcal{C}|\mathcal{F}_s\big] &= \sum_{j=l+1}^{k-1} \mathbb{E}\big[G_j(B(t_{j+1}) - B(t_j))|\mathcal{F}_s\big] \\ &= \sum_{j=l+1}^{k-1} \mathbb{E}\Big[\Big(\mathbb{E}\big[G_j(B(t_{j+1}) - B(t_j))|\mathcal{F}_{t_j}\big]\Big)|\mathcal{F}_s\Big] \\ &= \sum_{j=l+1}^{k-1} \mathbb{E}\Big[\Big(G_j\mathbb{E}\big[(B(t_{j+1}) - B(t_j))|\mathcal{F}_{t_j}\big]\Big)|\mathcal{F}_s\Big] \\ &= \sum_{j=l+1}^{k-1} \mathbb{E}\Big[G_j\,\underbrace{\mathbb{E}\big[B(t_{j+1}) - B(t_j))\big]}_{=0}\,|\mathcal{F}_s\Big] = 0. \end{aligned}$$

Similar argument reveals that $\mathbb{E}\big[\mathcal{D}|\mathcal{F}_s\big] = 0$. Combining all these, we have

$$\begin{aligned} \mathbb{E}\big[I_G(t)|\mathcal{F}_s\big] &= \mathbb{E}\big[\mathcal{A}|\mathcal{F}_s\big] + \mathbb{E}\big[\mathcal{B}|\mathcal{F}_s\big] + \mathbb{E}\big[\mathcal{C}|\mathcal{F}_s\big] + \mathbb{E}\big[\mathcal{D}|\mathcal{F}_s\big] \\ &= \sum_{j=0}^{l-1} G_j(B(t_{j+1}) - B(t_j)) + G_l(B(s) - B(t_l)) + 0 + 0 = I_G(s) \quad \text{a.s.} \end{aligned}$$

□

Since $I_G(t)$ is a martingale and $I_G(0) = 0$, we have $\mathbb{E}\big[I_G(t)\big] = 0$ for all $0 \le t \le T$. Moreover, by using Ito-isometry, one can easily see that

$$\mathrm{Var}(I_G(t)) = \mathbb{E}\big[I_G^2(t)\big] = \mathbb{E}\Big[\int_0^t G^2(r)\,dr\Big].$$

Next we turn to. the quadratic variation of the Ito-integral $I_G(t)$.

Theorem 3.3. *The quadratic variation of $I_G(\cdot)$ up to time t is given by*

$$[I_G, I_G](t) = \int_0^t G^2(u)\,du.$$

Proof. Suppose $t_k \le t < t_{k+1}$ where $\{t_k\}_{k=0}^m$ is a partition points of $[0,T]$ such that $G = G_k$ on $[t_k, t_{k+1})$. We first compute the quadratic variation on one of the sub-interval $[t_j, t_{j+1}]$. Let $\Pi_j = \{t_j = s_0 < s_1 < s_2 < \ldots < s_{\bar{m}} = t_{j+1}\}$ is a partition of $[t_j, t_{j+1}]$. Then

$$\begin{aligned}\sum_{i=0}^{\bar{m}-1} |I_G(s_{i+1}) - I_G(s_i)|^2 &= \sum_{i=0}^{\bar{m}-1} G_j^2\big(B(s_{i+1}) - B(s_i)\big)^2 = G_j^2 \sum_{i=0}^{\bar{m}-1} \big(B(s_{i+1}) - B(s_i)\big)^2 \\ &\to G_j^2(t_{j+1}) - t_j) = \int_{t_j}^{t_{j+1}} G^2(u)\,du\end{aligned}$$

as $\bar{m} \to \infty$. Analogously, the quadratic variation accumulated by the Ito integral between times t_k and t is $\int_{t_k}^t G^2(u)\,du$. Adding up all these pieces, we obtain

$$[I_G, I_G](t) = \int_0^t G^2(u)\,du.$$

□

3.2. **Ito integral for general integrands.** We now use Ito isometry to extend the definition of Ito integral from elementary functions to functions in $\mathcal{Y}$. We will do this several steps.

Step 1: Let $g \in \mathcal{Y}$ be and a.s. bounded and continuous process. Then there exist elementary functions $g_n \in \mathcal{Y}$ such that $\mathbb{E}\Big[\int_0^T (g - g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$. Indeed, by fixing a partition $\Pi_n = \{t_j^n\}$ of $[0,T]$ such that $\Delta_n = \max\{(t_{j+1}^n - t_j^n)\}$ tends to zero as $n \to \infty$, we define

$$g_n(t) = g(t_j^n), \quad t_j^n \le t < t_{j+1}^n.$$

Since a.s., $t \mapsto g(t,\omega)$ is continuous, $g_n(t) \to g(t)$ a.s. Since g is bounded a.s., by bounded convergence theorem, we get $\mathbb{E}\Big[\int_0^T (g - g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$.

Step 2: Let $g \in \mathcal{Y}$ be bounded a.s. Then there exists a sequence of a.s. bounded continuous process $g_n \in \mathcal{Y}$ such that $\mathbb{E}\Big[\int_0^T (g - g_n)^2\Big] \to 0 \quad \text{as } n \to \infty$. To show this, we define

$$g_n(t) := n \int_{t-\frac{1}{n}}^t g(r)\,dr.$$

Since g is a.s. bounded, g_n also a.s. bounded. Moreover, for any $t, s \in [0,T]$, we have $|g_n(t) - g_n(s)| \le 2n|t-s|M$, where $M > 0$ such that $|g| \le M$ a.s. Since $g \in \mathcal{Y}$, $g_n \in \mathcal{Y}$.

Furthermore, $g_n(t) \to g(t)$ a.s. Hence, by bounded convergence theorem, we conclude that $\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$.

Step 3: For any $g \in \mathcal{Y}$, there exist a sequence of a.s. bounded process $g_n \in \mathcal{Y}$ such that $\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$. Indeed, by defining

$$g_n(t) = \begin{cases} -n, & g(t) < -n \\ g(t), & -n \le g(t) \le n \\ n, & g(t) > n \end{cases}$$

we see that $g_n \in \mathcal{Y}$ and a.s. bounded. Moreover, $g_n \to g$ a.e. (t,ω). Observe that, since $|g_n(t)| \le |g(t)|$,

$$\int_0^T (g_n - g)^2\,dt \le 2\int_0^T g_n^2\,dt + 2\int_0^T g^2\,dt = 4\int_0^T g^2\,dt.$$

Since $\mathbb{E}\Big[\int_0^T g^2(s)\,ds\Big] < +\infty$, by dominated convergence theorem, we conclude that $\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$.

In view of **Steps** 1-3, we arrive at the following lemma.

Lemma 3.4 (Approximation by step processes)**.** *For any $g \in \mathcal{Y}$, there exists a sequence of a.s bounded elementary processes $g_n \in \mathcal{Y}$ such that*

$$\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \textit{as } n \to \infty.$$

Thanks to Ito-isometry, we see that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T (g_n - g_m)\,dB(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T (g_n-g_m)^2\,dt\Big] \\
&\le 2\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] + 2\mathbb{E}\Big[\int_0^T (g-g_m)^2\,dt\Big] \to 0 \quad \text{as } n,m \to \infty.
\end{aligned}$$

Hence $\{\int_0^T g_n\,dB(t)\}_n$ is a Cauchy sequence in $L^2(\Omega)$. Hence $\lim_{n\to\infty}\int_0^T g_n\,dB(t)$ exists.

Definition 3.3 (The Ito integral)**.** For any $g \in \mathcal{Y}$, the Ito integral of g is defined by

$$\int_0^t g\,dB(t) := \lim_{n\to\infty}\int_0^T g_n\,dB(t) \quad (\text{limit in } L^2(\Omega))$$

where $\{g_n\}$ is a sequence of elementary functions such that

$$\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty.$$

In view of Definition 3.3, Lemma 3.1 and Theorem 3.2 , we arrive at the following:

Theorem 3.5. *For any $g \in \mathcal{Y}$,*

$$\mathbb{E}\Big[\Big(\int_0^T g\,dB(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T g^2(t)\,dt\Big] \quad (\textit{ Ito-isometry})$$

Moreover, the process $I_g(t) := \int_0^t g(s)\,dB(s),\ 0 \le t \le T$ is a martingale.

In view of Ito-isometry and the identity $2ab = (a+b)^2 - a^2 - b^2$, we see that for $f, g \in \mathcal{Y}$

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T f\,dB(t)\int_0^T g\,dB(t)\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[\Big(\int_0^T (f+g)\,dB(t)\Big)^2 - \Big(\int_0^T f\,dB(t)\Big)^2 - \Big(\int_0^T g\,dB(t)\Big)^2\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[\int_0^T (f+g)^2\,dt - \int_0^T f^2\,dt - \int_0^T g^2\,dt\Big] = \mathbb{E}\Big[\int_0^T fg\,dt\Big].
\end{aligned}$$

Example 3.1. *Prove directly from the definition of Ito integrals that*

$$\int_0^t s dB(s) = tB(t) - \int_0^t B(s)\,ds.$$

Solution: *Let* $\Pi_n := \{0 = s_0^n < s_1^n < \ldots < s_{m_n}^n = t\}$ *be a partition of* $[0,t]$. *Observe that*

$$s_{j+1}^n B(s_{j+1}^n) - s_j^n B(s_j^n) - s_j^n\big(B(s_{j+1}^n) - B(s_j^n)\big) = B(s_{j+1}^n)(s_{j+1}^n - s_j^n)$$

$$\implies \sum_{j=0}^{m_n-1} s_{j+1}^n B(s_{j+1}^n) - s_j^n B(s_j^n) - \sum_{j=0}^{m_n-1} s_j^n\big(B(s_{j+1}^n) - B(s_j^n)\big) = \sum_{j=0}^{m_n-1} B(s_{j+1}^n)(s_{j+1}^n - s_j^n)$$

$$\implies tB(t) - \sum_{j=0}^{m_n-1} s_j^n\big(B(s_{j+1}^n) - B(s_j^n)\big) = \sum_{j=0}^{m_n-1} B(s_{j+1}^n)(s_{j+1}^n - s_j^n)$$

Taking limit as $n \to \infty$, *we get*

$$tB(t) - \int_0^t s\,dB(s) = \int_0^t B(s)\,ds.$$

3.3. **Ito-formula.** In the previous subsection, we have seen that

$$\frac{1}{2}B^2(t) = \frac{t}{2} + \int_0^t B(s)\, dB(s).$$

Thus, the image of Ito-integral $B(t) = \int_0^t dB(s)$ by the map $g(x) = \frac{1}{2}x^2$ is NOT again an Ito-integral of the form $\int_0^t f(s)\, dB(s)$–but a combination of a $dB(s)$-and ds-integral. It turns out that if we introduce Ito processes as a sum of a $dB(s)$-and ds-integral then the family of integrals is stable under smooth maps.

Definition 3.4 (Ito processes)**.** Let $B(t)$ be a Brownian motion and $\mathcal{F}_t$ be its associated filtration. An Ito process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t a(s)\, ds + \int_0^t g(s)\, dB(s) \tag{3.1}$$

where $X(0)$ is nonrandom, $a(s)$ and $g(s)$ are adapted processes such that integral in the right hand side of (3.1) are well-defined, and the Ito-integral is martingale.

We write the Ito process (3.1) in differential form

$$dX(t) = a(t)dt + g(t)dB(t)\,. \tag{3.2}$$

We first determine the quadratic variation of Ito process.

Lemma 3.6. *The quadratic variation of the Ito process* (3.1) *is*

$$[X, X](t) = \int_0^t g^2(s)\, ds.$$

In differential notation,

$$d[X, X](t) = g^2(t)\, dt.$$

Proof. Let $I_g(t) = \int_0^t g(s)\, dB(s)$ and $I_a(t) = \int_0^t a(s)\, ds$. Let $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = t\}$ be partitions of $[0, t]$. Then

$$\begin{aligned}\sum_{j=0}^{m_n-1} |X(t_{j+1}^n) - X(t_j^n)|^2 &= \sum_{j=0}^{m_n-1} |I_g(t_{j+1}^n) - I_g(t_j^n)|^2 + \sum_{j=0}^{m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)|^2 \\ &\quad + 2\sum_{j=0}^{m_n-1} \big|(I_g(t_{j+1}^n) - I_g(t_j^n))(I_a(t_{j+1}^n) - I_a(t_j^n))\big| \equiv \mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n\,.\end{aligned}$$

Observe that

$$\begin{aligned}\mathcal{B}_n &\le \max_{0\le k\le m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \sum_{j=0}^{m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \\ &\le \max_{0\le k\le m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \sum_{j=0}^{m_n-1} \int_{t_j^n}^{t_{j+1}^n} |a(s)|\, ds \\ &= \max_{0\le k\le m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \int_0^t |a(s)|\, ds \to 0 \quad \text{as } n \to \infty\end{aligned}$$

since $I_a(\cdot)$ is continuous and $\int_0^t |a(s)|\, ds$ is finite for every $t > 0$and a.s. In a similar manner, we can easily show that a.s., $\mathcal{C}_n \to 0$ as $n \to \infty$. In view of Theorem 3.3, we see that a.s., $\mathcal{A}_n \to \int_0^t g^2(s)\,ds$ as $n \to \infty$. Combining these, we get $[X,X](t) = \int_0^t g^2(s)\,ds$. $\square$

We now establish that Ito-process is stable under smooth maps.

Theorem 3.7 (Ito-formula)**.** *Let $X(t)$ be an Ito process given by (3.1) and $f \in C^2([0,\infty)\times\mathbb{R})$. Then $Y(t) := f(t,X(t))$ is again an Ito process and given by its differential form:*

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))\,dt + \frac{\partial f}{\partial x}(t,X(t))\,dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\,d[X,X](t) \\ &= \Big\{\frac{\partial f}{\partial t}(t,X(t)) + \frac{\partial f}{\partial x}(t,X(t))a(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))g^2(t)\Big\}\,dt \\ &\qquad + \frac{\partial f}{\partial x}(t,X(t))g(t)\,dB(t)\,. \end{aligned} \tag{3.3}$$

Proof. We show that $Y(t)$ satisfies the following integral form:

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t \Big(\frac{\partial f}{\partial t}(s,X(s)) + \frac{\partial f}{\partial x}(s,X(s))a(s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,X(s))g^2(s)\Big)\,ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s,X(s))g(s)\,dB(s)\,. \end{aligned} \tag{3.4}$$

We assume that f, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are bounded. For general case, we use approximation arguments: there exist a sequence of C^2-functions $f_n$on $[0,\infty)\times\mathbb{R}$ such that f_n, $\frac{\partial f_n}{\partial t}$, $\frac{\partial f_n}{\partial x}$ and $\frac{\partial^2 f_n}{\partial x^2}$ are bounded for each n and converges uniformly on compact subsets of $[0,\infty)\times\mathbb{R}$ to f, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ respectively. Moreover, we assume that $a(\cdot)$ and $g(\cdot)$ are elementary processes. Let $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = t\}$ be partitions of $[0,t]$. By using Taylor's expansion we have

$$\begin{aligned} f(t,X(t) - f(0,X(0)) &= \sum_{j=0}^{m_n-1} f(t_{j+1}^n, X(t_{j+1}^n)) - f(t_j^n, X(t_j^n)) \\ = \sum_{j=0}^{m_n-1} f_t(t_j^n, X(t_j^n))\Delta t_j^n &+ \sum_{j=0}^{m_n-1} f_x(t_j^n, X(t_j^n))\Delta_n X_j + \frac{1}{2}\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))(\Delta_n X_j)^2 \\ + \sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))\Delta_n X_j \Delta t_j^n &+ \frac{1}{2}\sum_{j=0}^{m_n-1} f_{tt}(t_j^n, X(t_j^n))(\Delta t_j^n)^2 + \sum_{j=0}^{m_n-1} R_j^n\,, \end{aligned}$$

where $\Delta t_j^n := (t_{j+1}^n - t_j^n)$, $\Delta_n X_j = X(t_{j+1}^n) - X(t_j^n)$ and $R_j^n := o\big(|\Delta t_j^n|^2 + |\Delta_n X_j|^2\big)$. One can easily show that

$$\begin{aligned} &\sum_{j=0}^{m_n-1} f_t(t_j^n, X(t_j^n))\Delta t_j^n \overset{n\to\infty}{\longrightarrow} \int_0^t f_t(s,X(s))\,ds \\ &\sum_{j=0}^{m_n-1} f_x(t_j^n, X(t_j^n))\Delta_n X_j \overset{n\to\infty}{\longrightarrow} \int_0^t f_x(s,X(s))\,dX(s) \\ &\qquad\qquad \equiv \int_0^t f_x(s,X(s))a(s)\,ds + \int_0^t f_x(s,X(s))g(s)\,dB(s) \end{aligned}$$

$$\frac{1}{2}\sum_{j=0}^{m_n-1} f_{tt}(t_j^n, X(t_j^n))(\Delta t_j^n)^2 \le \frac{1}{2}\|\Pi_n\|\Big|\sum_{j=0}^{m_n-1} f_{tt}(t_j^n, X(t_j^n))\Delta t_j^n \overset{n\to\infty}{\longrightarrow} 0.$$

Since $a(\cdot)$ and $g(\cdot)$ are elementary functions we have

$$\Delta_n X_j = a(t_j^n)\Delta t_j^n + g(t_j^n)\Delta_n B_j.$$

Thus, we get

$$\begin{aligned}&\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))(\Delta_n X_j)^2\\ &= \underbrace{\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))a^2(t_j^n)(\Delta t_j^n)^2}_{:=\mathbf{A}_1} + 2\underbrace{\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Delta t_j^n\Delta_n B_j}_{:=\mathbf{A}_2}\\ &\qquad + \underbrace{\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))g^2(t_j^n)(\Delta_n B_j)^2}_{:=\mathbf{A}_3}\end{aligned}$$

Since f_{xx} is bounded and $a(\cdot) \in \mathcal{Y}$, $\mathbf{A}_1$ goes to 0 as $n \to \infty$. Moreover, $\mathbf{A}_2 \to 0$ in $L^2(\Omega)$. Indeed, by independent properties of Brownian motion, we have

$$\begin{aligned}\mathbb{E}\big[\mathbf{A}_2^2\big] &= \sum_{j=0}^{m_n-1} \mathbb{E}\Big[\Big(f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Delta t_j^n\Big)^2(\Delta_n B_j)^2\Big]\\ &= \sum_{j=0}^{m_n-1} \mathbb{E}\Big[\Big(f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Delta t_j^n\Big)^2\Big]\mathbb{E}\Big[(\Delta_n B_j)^2\Big]\\ &= \sum_{j=0}^{m_n-1} \mathbb{E}\Big[\Big(f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Big)^2\Big](\Delta t_j^n)^3 \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

We now show that

$$\mathbf{A}_3 \overset{n\to\infty}{\longrightarrow} \int_0^t f_{xx}(s, X(s))g^2(s)\,ds \quad \text{in } L^2(\Omega).$$

Put $\bar{a}(t) = f_{xx}(t, X(t))g^2(t)$. Then

$$\begin{aligned}\mathbb{E}\Big[\big(\mathbf{A}_3 - \sum_{j=0}^{m_n-1} \bar{a}(t_j^n)\Delta t_j^n\big)^2\Big] &= \mathbb{E}\Big[\big(\sum_{j=0}^{m_n-1} \bar{a}(t_j^n)((\Delta_n B_j)^2 - \Delta t_j^n)\big)^2\Big]\\ = \sum_{i,j}\mathbb{E}\Big[\bar{a}(t_j^n)\bar{a}(t_i^n)((\Delta_n B_j)^2 - \Delta t_j^n)((\Delta_n B_i)^2 - \Delta t_i^n)\Big]\end{aligned}$$

If $i < j$, then $\bar{a}(t_j^n)\bar{a}(t_i^n)((\Delta_n B_i)^2 - \Delta t_i^n)$ and $((\Delta_n B_j)^2 - \Delta t_j^n)$ are independent. Thus, we have

$$\mathbb{E}\Big[\big(\mathbf{A}_3 - \sum_{j=0}^{m_n-1} \bar{a}(t_j^n)\Delta t_j^n\big)^2\Big] = \sum_{j=0}^{m_n-1}\mathbb{E}\Big[\bar{a}^2(t_j^n)((\Delta_n B_j)^2 - \Delta t_j^n)^2\Big]$$

$$= \sum_{j=0}^{m_n-1} \mathbb{E}[\bar{a}^2(t_j^n)]\mathbb{E}\Big[(\Delta_n B_j)^4 - 2(\Delta_n B_j)^2 \Delta t_j^n + (\Delta t_j^n)^2\Big] = 2\sum_{j=0}^{m_n-1} \mathbb{E}[\bar{a}^2(t_j^n)](\Delta t_j^n) \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that $\mathbf{A}_3 \overset{n\to\infty}{\longrightarrow} \int_0^t f_{xx}(s, X(s))g^2(s)\, ds$ in $L^2(\Omega)$. Notice that the mixed partial derivative term has no counterpart in Ito formula (3.4), so it needs to go way. Indeed, since $a(\cdot)$ and $g(\cdot)$ are elementary, we have

$$\begin{aligned}
&\sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))\Delta_n X_j \Delta t_j^n \\
&= \sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))a(t_j^n)(\Delta t_j^n)^2 + \sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))g(t_j^n)\Delta t_j^n \Delta_n B_j\,.
\end{aligned}$$

Like in previous arguments, one can easy show that above two terms tend to 0 as $n \to \infty$. Moreover, the argument above also proves that $\sum_{j=0}^{m_n-1} R_j^n \to 0$ as $n \to \infty$. Thus, putting things together, we arrive at Ito formula (3.4). This completes the proof. □

Example 3.2. *Let $B(t)$ be a Brownian motion. Show that*

$$\int_0^t B^2(s)\, dB(s) = \frac{1}{3}B^3(t) - \int_0^t B(s)\, ds.$$

Indeed, applying It-formula for the function $f(x) = x^3$, we have (here Ito process is X(t)=B(t))

$$\begin{aligned}
B^3(t) &= 3\int_0^t B^2(s)\, dB(s) + \frac{1}{2}\int_0^t 6B(s)\, ds \\
\Longrightarrow \int_0^t B^2(s)\, dB(s) &= \frac{1}{3}B^3(t) - \int_0^t B(s)\, ds.
\end{aligned}$$

Example 3.3. *Consider the asset price process given by*

$$S(t) = S(0)\exp\{\int_0^t \sigma(s)\, dB(s) + \int_0^t \big(\alpha(s) - \frac{1}{2}\sigma^2(s)\big)\, ds\}$$

where $S(0)$ is nonrandom and positive and $\alpha(\cdot)$ and $\sigma(\cdot)$ are adapted processes so that integrals are well-defined. Show that $S(t)$ is an Ito process and satisfies the following differential form

$$dS(t) = \alpha(t)S(t)\, dt + \sigma(t)S(t)dB(t).$$

Solution: *First we define an Ito process*

$$X(t) = \int_0^t \sigma(s)\, dB(s) + \int_0^t \big(\alpha(s) - \frac{1}{2}\sigma^2(s)\big)\, ds.$$

Then by given condition $S(t) = S(0)e^{X(t)}$. We may write $S(t)$ as $S(t) = f(X(t))$ where $f(x) = S(0)e^x$. Note that $X(t)$ in the differential form given by

$$dX(t) = \big(\alpha(t) - \frac{1}{2}\sigma^2(t)\big)\, dt + \sigma(t)\, dB(t).$$

By Ito-formula, we have

$$\begin{aligned} dS(t) = df(X(t)) &= \Big(S(0)e^{X(t)}\big(\alpha(t) - \frac{1}{2}\sigma^2(t)\big) + \frac{1}{2}S(0)e^{X(t)}\sigma^2(t)\Big)\, dt + S(0)e^{X(t)}\sigma(t)\, dB(t) \\ &= S(0)e^{X(t)}\alpha(t)\, dt + S(0)e^{X(t)}\sigma(t)\, dB(t) = \alpha(t)S(t)\, dt + \sigma(t)S(t)\, dB(t) \end{aligned}$$

Observe that if $\alpha(t) = 0$, then $S(t)$ is given by $S(t) = S(0) + \int_0^t \sigma(s)S(s)\, dB(s)$. Since the second term in the right hand side is martingale and $S(0)$ is nonrandom, we conclude that $S(t)$ is a martingale provided $\sigma(s)S(s) \in \mathcal{Y}(0,T)$ for each $T > 0$.

Remark 3.1. The following Novikov condition

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T u^2(s)\, ds\Big)\Big] < +\infty$$

is a sufficient to guarantee that the process

$$M(t) := \exp\Big\{\int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u^2(s)\, ds\Big\}$$

is a martingale.

Example 3.4. *Let $X(t)$ and $Y(t)$ be real-valued Ito processes. Then show that $X(t)Y(t)$ is again an Ito process and its differential form is given by*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)\cdot dY(t)$$

The above formula is known as Ito product rule. To check this, let $X(t)$ and $Y(t)$ be Ito processes of the form

$$dX(t) = a(t)dt + \sigma(t)dB(t), \quad dY(t) = b(t)dt + \gamma(t)dB(t).$$

Then by applying Ito-formula to the function $f(x) = x^2$, we have

$$\begin{aligned} d[(X(t)+Y(t))^2] &= \{2(X(t)+Y(t))(a(t)+b(t)) + (\sigma(t)+\gamma(t))^2\}dt \\ &\quad + 2(X(t)+Y(t))(\sigma(t)+\gamma(t))\, dB(t) \\ dX^2(t) &= \{2X(t)a(t) + \sigma^2(t)\}dt + 2X(t)\sigma(t)\, dB(t) \end{aligned}$$

$$dY^2(t) = \{2Y(t)b(t) + \gamma^2(t)\}dt + 2Y(t)\gamma(t)\,dB(t)$$

We now use above equations along with the fact that $xy = \frac{1}{2}\big((x+y)^2 - x^2 - y^2\big)$, *we have*

$$\begin{aligned} d[X(t)Y(t)] &= \{X(t)b(t) + Y(t)a(t) + \sigma(t)\gamma(t)\}\,dt + \{X(t)\gamma(t) + Y(t)\sigma(t)\}\,dB(t) \\ &= X(t)\{b(t)dt + \gamma(t)dB(t)\} + Y(t)\{a(t)dt + \gamma(t)dB(t)\} + \sigma(t)\gamma(t)\,dt \\ &= X(t)dY(t) + Y(t)dX(t) + dX(t)\cdot dY(t). \end{aligned}$$

Theorem 3.8 (**Ito integral of a deterministic integrand).** *Let* $B(\cdot)$ *be a Brownian motion and let* $g(s)$ *be a nonrandom function of time. Define* $I(t) := \int_0^t g(s)\,dB(s)$. *For each* $t \ge 0$, *the random variable* $I(t)$ *is normally distributed with mean* 0 *and variance* $\int_0^t g^2(s)\,ds$.

Proof. We have seen that $I(t)$ is a martingale and hence $\mathbb{E}[I(t)] = I(0) = 0$. Moreover, thanks to Ito-isometry, we have

$$\mathrm{Var}[I(t)] = \mathbb{E}[I((t))^2] = \int_0^t g^2(s)\,ds.$$

It remains to show that $I(t)$ is normally distributed. To do so, we show that $I(t)$ has moment-generating function of a normal random variable with mean 0 and variance $\int_0^t g^2(s)\,ds$ i.e., we show that

$$\begin{aligned} &\mathbb{E}[e^{uI(t)}] = \exp\{\frac{1}{2}u^2\int_0^t g^2(s)\,ds\} \quad \forall\, u \in \mathbb{R} \\ \iff\ &\mathbb{E}\Big[\exp\{uI(t) - \frac{1}{2}u^2\int_0^t g^2(s)\,ds\}\Big] = 1\,. \end{aligned}$$

This can be written as

$$\mathbb{E}\Big[\exp\{\int_0^t ug(s)\,dB(s) - \frac{1}{2}\int_0^t (ug(s))^2\,ds\}\Big] = 1\,. \tag{3.5}$$

In view of Remark 3.1, the process $Z(t) := \exp\{\int_0^t ug(s)\,dB(s) - \frac{1}{2}\int_0^t (ug(s))^2\,ds\}$ is a martingale and hence we have $\mathbb{E}[Z(t)] = Z(0) = 1$ which gives us (3.5). This completes the proof. □

Example 3.5. *Consider Vasicek model for the interest rate process* $R(t)$ *given by*

$$dR(t) = (\alpha - \beta R(t))\,dt + \sigma\,dB(t)$$

where α, β *and* σ *are positive constants. Show that* $R(t)$ *is normally distributed with mean* $e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ *and variance* $\frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$.

Solution: *The interest rate process* $R(t)$ *is given by (see Assignment-2)*

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dB(s).$$

In view of previous theorem, we see that the random variable $\int_0^t e^{\beta s}\,dB(s)$ *is normally distributed with mean* 0 *and variance* $\int_0^t e^{2\beta s}\,ds = \frac{1}{2\beta}(e^{2\beta t} - 1)$. *Thus, we conclude that* $R(t)$ *is normally distributed with mean* $e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ *and variance* $\sigma^2 e^{-2\beta t}\frac{1}{2\beta}(e^{2\beta t} - 1) = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$.

3.4. **Multivariable Stochastic Calculus:** It is straightforward to extend our definitions to Brownian motions taking values in $\mathbb{R}^n$.

Definition 3.5. An $\mathbb{R}^m$-valued stochastic process $\mathbf{B}(t) = \big(B_1(t), B_2(t), \dots, B_m(t)\big)$ is an m-dimensional Wiener process (or Brownian motion) provided

a) each $B_i(t)$ is a one-dimensional Brownian motion
b) for $i \neq j$, the processes $B_i(t)$ and $B_j(t)$ are independent.

Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0,T]$. For $i \neq j$, define the sample cross variation of B_i and B_j on $[0,T]$, denoted by $[B_i, B_j](T)$, as

$$[B_i, B_j](T) = \lim_{\|\Pi\|\to 0} \sum_{k=0}^{n} \big\{B_i(t_{k+1}) - B_i(t_k)\big\}\big\{B_j(t_{k+1}) - B_j(t_k)\big\} := \lim_{\|\Pi\|\to 0} C_\Pi\,.$$

Observe that, since increments of B_i and B_j are independents and all have mean zero, we get that $\mathbb{E}[C_\Pi] = 0$. Again, by using independent increments of B_i and B_j, we obtain

$$\begin{aligned}
\mathrm{Var}(C_\Pi) = \mathbb{E}[C_\Pi^2] &= \mathbb{E}\Big[\sum_{k=0}^{n-1} \big\{B_i(t_{k+1}) - B_i(t_k)\big\}^2\big\{B_j(t_{k+1}) - B_j(t_k)\big\}^2\Big] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\Big[\big\{B_i(t_{k+1}) - B_i(t_k)\big\}^2\Big]\mathbb{E}\Big[\big\{B_j(t_{k+1}) - B_j(t_k)\big\}^2\Big] \\
&= \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \le \|\Pi\| T \to 0 \text{ as } \|\Pi\| \to 0.
\end{aligned}$$

This implies that the sample cross variation of B_i and B_j is *zero* i.e., $[B_i, B_j](T) = 0$.

Definition 3.6. We define the following:

a) An $M^{n\times m}$-valued stochastic process $\boldsymbol{G} = ((G_{ij}))$ belongs to $\mathcal{Y}_{n\times m}(0,T)$ if

$$G_{ij} \in \mathcal{Y}(0,T) \quad \forall\, 1 \le i \le n,\ 1 \le j \le m.$$

b) An $\mathbb{R}^n$-valued stochastic process $\mathbf{F} = \big(F_1, F_2, \dots, F_n\big)$ belongs to $\mathbb{L}^1_n(0,T)$ if

$$F_i \in \mathbb{L}^1(0,T), \quad 1 \le i \le n.$$

Recall $\mathbb{L}^1(0,T)$ is the space of $\mathcal{F}_t$-adapted, jointly measurable real-valued stochastic process $F(t)$ such that $\mathbb{E}\Big[\int_0^t |F|\,dt\Big] < +\infty$.

Definition 3.7. Let $\mathbf{B} = \big(B_1, B_2, \dots, B_m\big)$ be an m-dimensional Brownian motion. Then for any $\boldsymbol{G} \in \mathcal{Y}_{n\times m}(0,T)$, we define the stochastic integral $\int_0^T \boldsymbol{G}\,d\mathbf{B}$ as an $\mathbb{R}^n$-valued random variable whose i-th component is given by

$$\sum_{j=1}^{m} \int_0^T G_{ij}\,dB_j, \quad 1 \le i \le n.$$

Approximation by step/elementary processes, one can arrive at the following lemma.

Lemma 3.9. *Let* $\mathbf{B} = \big(B_1, B_2, \dots, B_m\big)$ *be an* m*-dimensional Brownian motion and* $\boldsymbol{G} \in \mathcal{Y}_{n\times m}(0,T)$*. Then*

$$\text{i) } \mathbb{E}\Big[\int_0^T \boldsymbol{G}\,d\mathbf{B}\Big] = 0$$

$$\text{ii) } \mathbb{E}\Big[\Big|\int_0^T \boldsymbol{G}\, d\mathbf{B}\Big|^2\Big] = \mathbb{E}\Big[\int_0^T \|\boldsymbol{G}(s)\|^2\, ds\Big]$$

where $\|\boldsymbol{G}(s)\|^2 := \sum_{1\le i\le n,\ 1\le j\le m} |G_{ij}|^2$.

Definition 3.8 ($\mathbb{R}^n$-valued Ito processes)**.** Let $\mathbf{B}(t)$ be an m-dimensional Brownian motion and $\mathcal{F}_t$ be its associated filtration. An $\mathbb{R}^n$-valued Ito process is a stochastic process $\mathbf{X} = \big(X_1, X_2, \ldots, X_n\big)$ of the form

$$\mathbf{X}(r) = \mathbf{X}(s) + \int_0^t \mathbf{F}(s)\, ds + \int_0^t \boldsymbol{G}(s)\, d\mathbf{B}(s)$$

for some $\mathbf{F} = \big(F_1, F_2, \ldots, F_n\big) \in \mathbb{L}^1_n(0,T)$ and $\boldsymbol{G} = ((G_{ij})) \in \mathcal{Y}_{n\times m}(0,T)$ and for all $0 \le s \le r \le T$. We say that $\mathbf{X}(\cdot)$ has the stochastic differential

$$d\mathbf{X} = \mathbf{F}dt + \boldsymbol{G}\, d\mathbf{B}. \tag{3.6}$$

This means that

$$dX_i(t) = F_i\, dt + \sum_{j=1}^m G_{ij}\, dB_j, \quad 1 \le i \le n\,.$$

Like in one-dimensional case, the family of Ito processes are stable under smooth maps.

Theorem 3.10 (Ito's formula in n-dimension)**.** *Suppose that* $\mathbf{X}(\cdot)$ *is a* n*-dimensional Ito process given in* (3.6)*. Let* $\mathbf{u} = (u_1, u_2, \ldots, u_p) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^p$ *be* C^2*-map. Then* $Y(t) = u(\mathbf{X}(t), t)$ *is an Ito process. Moreover, its stochastic differential form is given by the following formula: for* $1 \le k \le p$,

$$du_k(\mathbf{X}(t), t) = \frac{\partial u_k}{\partial t}(\mathbf{X}(t), t)dt + \sum_{i=1}^n \frac{\partial u_k}{\partial x_i}(\mathbf{X}(t), t)\, dX_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 u_k}{\partial x_i \partial x_j}(\mathbf{X}(t), t) dX_i\, dX_j$$

Example 3.6. *Solve the (2-dimensional) stochastic differential equation*

$$dX_1 = X_2(t)dt + \alpha dB_1(t), \quad dX_2(t) = -X_1(t)\,dt + \beta\,dB_2(t)$$

where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion, and α, β are constants.
Solution: *We can re-write the given equation as 2-dimensional Ito process:*

$$d\mathbf{X}(t) = \boldsymbol{A}\mathbf{X}(t)\,dt + \boldsymbol{G}\,d\mathbf{B}(t)$$

where $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\boldsymbol{G} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$ *and* $d\mathbf{B}(t) = \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$. *We apply Ito formula with* $\mathbf{u}(t, x_1, x_2) = \exp(-t\boldsymbol{A}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ *to have*

$$d[\exp(-t\boldsymbol{A})\mathbf{X}(t)] = -\boldsymbol{A}\exp(-t\boldsymbol{A})\mathbf{X}(t)\,dt + \exp(-t\boldsymbol{A})d\mathbf{X}(t)$$

where $\exp(t\boldsymbol{A}) = \sum_{n=0}^{\infty} \frac{t^n \boldsymbol{A}^n}{n!}$. *Again, from the given equation, we have*

$$\exp(-t\boldsymbol{A})d\mathbf{X}(t) = \exp(-t\boldsymbol{A})\boldsymbol{A}\mathbf{X}(t)\,dt + \exp(-t\boldsymbol{A})\boldsymbol{G}\,d\mathbf{B}(t)$$

Combining last two equations, we have

$$\begin{aligned} &d[\exp(-t\boldsymbol{A})\mathbf{X}(t)] = \exp(-t\boldsymbol{A})\boldsymbol{G}\,d\mathbf{B}(t) \\ &\implies \mathbf{X}(t) = \exp(t\boldsymbol{A})\Big[\mathbf{X}(0) + \int_0^t \exp(-s\boldsymbol{A})\boldsymbol{G}\,d\mathbf{B}(s)\Big]. \end{aligned}$$

Since, $\boldsymbol{A}^2 = -\mathbf{I}$, *one has*

$$X_1(t) = X_1(0)\cos(t) + X_2(0)\sin(t) + \alpha\int_0^t \cos(t-s)\,dB_1(s) + \beta\int_0^t \sin(t-s)\,dB_2(s)\,,$$

$$X_2(t) = -X_1(0)\sin(t) + X_2(0)\cos(t) - \alpha\int_0^t \sin(t-s)\,dB_1(s) + \beta\int_0^t \cos(t-s)\,dB_2(s)\,.$$

3.5. **Recognizing a Brownian motion.** We have seen that Brownian motion is a continuous paths martingale and its quadratic variation up to time t is t i.e., $[B, B](t) = t$. These conditions may characterize any stochastic process to be a Brownian motion.

Theorem 3.11 (Levy, one dimensional). *Let $\{M(t : t \geq 0\}$ be a continuous paths martingale relative to a given filtration such that $M(0) = 0$ and $[M, M](t) = t$ for all $t \geq 0$. Then, $M(t)$ is a Brownian motion.*

Proof. We need to show that $M(t)$ is normally distributed with mean 0 and variance t. Since moment generating function uniquely determine the distribution of a stochastic process, we basically show that

$$\mathbb{E}[\exp(uM(t))] = e^{\frac{1}{2}u^2 t}, \quad \forall u \in \mathbb{R}.$$

To do so, we will use Ito formula. Observe that, in the proof of Ito formula for Brownian motion, we have used two important facts of Brownian motion namely it has continuous paths and quadratic variation up to time t is t. Since given stochastic process M has these two properties, the Ito-formula may be applied to M. Hence for any function $f(t, x)$ whose derivatives exist and are continuous, one has

$$df(t, M(t)) = f_t(t, M(t))\,dt + f_x(t, M(t))\,dM(t) + \frac{1}{2}f_{xx}(t, M(t))\,dt$$

$$\implies f(t,M(t)) = f(0,0) + \int_0^t \{f_t(s,M(s)) + \frac{1}{2}f_{xx}(s,M(s))\}\,ds + \int_0^t f_x(s,M(s))\,dM(s)$$

Since M is a martingale, the stochastic integral $\int_0^t f_x(s,M(s))\,dM(s)$ is also a martingale and hence

$$\mathbb{E}\Big[\int_0^t f_x(s,M(s))\,dM(s)\Big] = 0.$$

Thus, we have, after taking expectation in the above integral equation

$$\mathbb{E}\big[f(t,M(t))\big] = f(0,0) + \mathbb{E}\Big[\int_0^t \{f_t(s,M(s)) + \frac{1}{2}f_{xx}(s,M(s))\}\,ds\Big]. \tag{3.7}$$

For any fixed $u \in \mathbb{R}$, consider the function

$$f(t,x) = \exp\big(ux - \frac{1}{2}u^2 t\big).$$

One can easily show that

$$f_t(t,x) + \frac{1}{2}f_{xx}(t,x) = 0 \quad \forall (t,x).$$

Therefore, with this choice of $f(t,x)$, we get

$$\mathbb{E}\Big[\exp\big(uM(t) - \frac{1}{2}u^2 t\big)\Big] = 1 \implies \mathbb{E}[\exp(uM(t))] = e^{\frac{1}{2}u^2 t}.$$

This completes the proof. □

Example 3.7. *Let $\mathbf{B}(\cdot) = \big(B_1(\cdot), B_2(\cdot)\big)$ be a 2-dimensional Brownian motion. Define a stochastic process*

$$M(t) := \rho B_1(t) + \sqrt{1-\rho^2}B_2(t), \quad |\rho| \le 1.$$

Show that $M(t)$ is a Brownian motion.

Solution: *Observe that $M(t)$ is a continuous paths martingale with $M(0) = 0$. If we show that quadratic variation of M is t, then according to one dimensional Levy's theorem $M(t)$ will be a Brownian motion. Now*

$$\begin{aligned} dM(t)dM(t) &= \rho^2\,dB_1(t)\,dB_1(t) + 2\rho\sqrt{1-\rho^2}dB_1(t)\,dB_2(t) + (1-\rho^2)dB_2(t)\,dB_2(t) \\ &= \rho^2\,dt + 0 + (1-\rho^2)\,dt = dt \\ &\implies [M,M](t) = t \quad \forall\, t \ge 0. \end{aligned}$$

Thus, $M(t)$ is a brownian motion.

Theorem 3.12 (Levy, n-dimensional). *Let $\mathbf{M}(t) = \big(M_1(t), M_2(t), \dots, M_n(t)\big)$ be a n-dimensional martingale relative to a filtration. Assume that for $i = 1,2,\dots,n$, $M_i(0) = 0$ and $M_i(\cdot)$ has continuous paths and $[M_i, M_j](t) = \delta_{ij}t$, $1 \le i,j \le n$. Then $\mathbf{M}(t) = \big(M_1(t), M_2(t), \dots, M_n(t)\big)$ is a n-dimensional Brownian motion.*

Example 3.8. *Suppose $B_1(t)$ and $B_2(t)$ are Brownian motions such that*

$$dB_1(t)\,dB_2(t) = \rho(t)\,dt$$

where $\rho(\cdot)$ is a stochastic process with values in $(-1,1)$. Define two stochastic processes $M_1(t)$ and $M_2(t)$ as follows:

$$M_1(t) = B_1(t), \quad B_2(t) = \int_0^t \rho(s)\,dM_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\,dM_2(s)$$

Show that $\mathbf{M}(t) = \big(M_1(t), M_2(t)\big)$ *is a 2-dimensional Brownian motion.*

Solution: *Assume that* $\big(M_1(t), M_2(t)\big)$ *is a martingales given by the differential form*

$$dM_1(t) = dB_1(t), \quad dM_2(t) = \alpha(t)\, dB_1(t) + \beta(t)\, dB_2(t)$$

where $\alpha(\cdot)$ *and* $\beta(\cdot)$ *are adapted processes such that* $[M_i, M_j](t) = \delta_{ij} t$ *for* $1 \le i, j \le 2$. *That is we need to choose* $\alpha(t)$ *and* $\beta(t)$ *such that*

$$\begin{aligned} &\alpha^2(t) + \beta^2(t) + 2\alpha(t)\beta(t)\rho(t) = 1 \\ &\alpha(t) + \beta(t)\rho(t) = 0 \end{aligned}$$

Solving these two equation, we obtain

$$\alpha(t) = -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}, \quad \beta(t) = \frac{1}{\sqrt{1-\rho^2(t)}}.$$

Thus, by Levy's theorem 3.12, the process $\mathbf{M}(t) = \big(M_1(t), M_2(t)\big)$ *is a 2-dimensional Brownian motion. It remains to show that*

$$B_2(t) = \int_0^t \rho(s)\, dM_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dM_2(s).$$

Indeed, we have

$$\begin{aligned} dM_2(t) &= -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}\, dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}}\, dB_2(t) \\ \Longrightarrow\ dB_2(t) &= \rho(t) dB_1(t) + \sqrt{1-\rho^2(t)}\, dM_2(t) \\ \Longrightarrow\ B_2(t) &= \int_0^t \rho(s) dB_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dM_2(s) \\ &= \int_0^t \rho(s)\, dM_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dM_2(s). \end{aligned}$$

3.6. **Martingale representation theorem.** We have seen that $X(t) = X(0) + \int_0^t v dB(s)$ is a martingale. We wish to know about its converse, i.e., any martingale can be represented as an Ito integral. This result is known as **Martingale representation theorem** . To do so, we need a technical lemma, which we are stating without proof.

Lemma 3.13. *The linear span of the random variables of the type*

$$\exp\Big\{ \int_0^T h(t)\, dB(t) - \frac{1}{2} \int_0^T h^2(t)\, dt \Big\}$$

is dense in $L^2(\mathcal{F}_T, \mathbb{P})$, where h is a deterministic function with $h \in L^2[0,T]$.

Theorem 3.14 (**The Ito Representation Theorem**)**.** *Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$. Then there exists a unique stochastic process $f \in \mathcal{Y}(0,T)$ such that*

$$F = \mathbb{E}[F] + \int_0^T f(t)\, dB(t).$$

Proof. First assume that F has the form

$$F = \exp\Big\{ \int_0^T h(t)\, dB(t) - \frac{1}{2} \int_0^T h^2(t)\, dt \Big\} \tag{3.8}$$

for some deterministic function $h \in L^2(0,T)$. Define

$$Y(t) = \exp\Big\{ \int_0^t h(s)\, dB(s) - \frac{1}{2} \int_0^t h^2(s)\, ds \Big\}, \quad 0 \le t \le T.$$

Take $X(t) = \int_0^t h(s)\, dB(s) - \frac{1}{2} \int_0^t h^2(s)\, ds$. Then $Y(t) = f(X(t))$, where $f(x) = e^x$. By applying Ito formula, we have

$$\begin{aligned} dY(t) = Y(t)h(t)\, dB(t) &\implies Y(t) = 1 + \int_0^t Y(s)h(s)\, dB(s) \\ \implies F = 1 + \int_0^T Y(s)h(s)\, dB(s) &\implies \mathbb{E}[F] = 1 \end{aligned}$$

Hence for the above $F \in L^2(\mathcal{F}_T, \mathbb{P})$, we have the representation

$$F = \mathbb{E}[F] + \int_0^T f(t)\, dB(t).$$

If $F \in L^2(\mathcal{F}_T, \mathbb{P})$ is arbitrary, then by Lemma 3.13 we can approximate F in $L^2(\mathcal{F}_T, \mathbb{P})$ by linear combination of F_n of the functions of the form (3.8). Thus, for each n, we have

$$F_n = \mathbb{E}[F_n] + \int_0^T f_n(t)\, dB(t), \quad f_n \in \mathcal{Y}(0,T).$$

In view of Ito-isometry, we observe that

$$\mathbb{E}[(F_n - F_m)^2] = \big(\mathbb{E}[F_n - F_m]\big)^2 + \mathbb{E}\Big[\int_0^T (f_n - f_m)^2\, ds \Big] \longrightarrow 0 \quad \text{as } n, m \to \infty.$$

This shows that $\{f_n\}$ is a Cauchy sequence in $L^2((0,T) \times \Omega)$, and hence there exists $f \in L^2((0,T) \times \Omega)$ such that $f_n \to f$ in $L^2((0,T) \times \Omega)$. Moreover, since $f_n \in \mathcal{Y}(0,T)$, we see that $f \in \mathcal{Y}(0,T)$. Furthermore, $f(t,\omega)$ is $\mathcal{F}_t$-adapted. Hence

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty} \Big(\mathbb{E}[F_n] + \int_0^T f_n(t)\,dB(t)\Big) = \mathbb{E}[F] + \int_0^T f(t)\,dB(t)$$

where the limit is taken in $L^2(\mathcal{F}_T,\mathbb{P})$. We now prove uniqueness. Suppose the exist $f_1, f_2 \in \mathcal{Y}(0,T)$. such that

$$F = \mathbb{E}[F] + \int_0^T f_1\,dB(t) = \mathbb{E}[F] + \int_0^T f_2\,dB(t).$$

Then, in view of Ito-isometry, we get that

$$0 = \mathbb{E}\Big[\int_0^T (f_1 - f_2)^2\,dt\Big] \implies f_1(t,\omega) = f_2(t,\omega) \text{ for a.e. } (t,\omega) \in [0,T]\times\Omega.$$

This completes the proof. □

Example 3.9. *Find $f \in \mathcal{Y}(0,T)$ such that $F = \sin(B(T))$ can be written as $F = \mathbb{E}[F] + \int_0^T f(t)\,dB(t)$.*

Solution: *Observe that $\sin(B(T)) \in L^2(\mathcal{F}_T,\mathbb{P})$. In view of Ito formula and the Ito product rule, we have*

$$d(e^{\frac{t}{2}}\sin(B(t))) = (e^{\frac{t}{2}}\cos(B(t))\,dB(t) \implies \sin(B(T)) = \int_0^T e^{\frac{t-T}{2}}\cos(B(t))\,dB(t)\,.$$

Hence $\mathbb{E}[\sin(B(T))] = 0$, and therefore, we get the required representation as

$$\sin(B(T)) = \mathbb{E}[\sin(B(T))] + \int_0^T f(t)\,dB(t)$$

where $f(t,\omega) = e^{\frac{t-T}{2}}\cos(B(t)) \in \mathcal{Y}(0,T)$.

Example 3.10. *Find $f \in \mathcal{Y}(0,T)$ such that $F = B^3(T)$ can be written as $F = \mathbb{E}[F] + \int_0^T f(t)\,dB(t)$.*

Solution: *Observe that $B^3(T) \in L^2(\mathcal{F}_T,\mathbb{P})$, and $\mathbb{E}[B^3(T)] = 0$. We know that*

$$B^3(T) = 3\int_0^T B^2(s)\,dB(s) + 3\int_0^T B(s)\,ds$$
$$\int_0^T B(s)\,ds = TB(T) - \int_0^T s\,dB(s)\,.$$

Combining these two relation, we get

$$B^3(T) = 3\int_0^T B^2(s)\,dB(s) + 3T\int_0^T dB(s) - 3\int_0^T s\,dB(s) = \int_0^T 3(B^2(s) - T + s)\,dB(s)$$
$$= \mathbb{E}[B^3(T)] + \int_0^T f(t)\,dB(t) \quad \text{where } f(s,\omega) = 3(B^2(s) - T + s) \in \mathcal{Y}(0,T).$$

Theorem 3.15 (Martingale Representation Theorem). *Let $M(t) : t \geq 0$ be a square integrable martingale with respect to a filtration generated only by Brownian motion. Then there exists a unique stochastic process $g \in \mathcal{Y}(0,t)$ for all $t \geq 0$ such that a.s., there holds*

$$M(t) = \mathbb{E}[M(0)] + \int_0^t g(s)\,dB(s) \quad \forall t \geq 0.$$

Proof. By Ito representation theorem, there exists $f^{(t)}(s) \in L^2(\mathcal{F}_t, \mathbb{P})$ such that

$$M(t) = \mathbb{E}[M(t)] + \int_0^t f^{(t)}(s)\, dB(s).$$

Now assume that $0 \le t_1 < t_2$. Then

$$\begin{aligned} M(t_1) = \mathbb{E}\big[M(t_2)|\mathcal{F}_{t_1}\big] &= \mathbb{E}[M(0)] + \mathbb{E}\Big[\int_0^{t_2} f^{(t_2)}(s)\, dB(s)\big|\mathcal{F}_{t_1}\Big] \\ &= \mathbb{E}[M(0)] + \int_0^{t_1} f^{(t_2)}(s)\, dB(s) \end{aligned}$$

But, we already have

$$M(t_1) = \mathbb{E}[M(0)] + \int_0^{t_1} f^{(t_1)}(s)\, dB(s),$$

and therefore by using Ito-isometry, we get

$$\begin{aligned} &\mathbb{E}\Big[\int_0^{t_1} (f^{(t_1)} - f^{(t_2)})^2\, ds\Big] = 0 \\ &\implies f^{(t_2)}(s,\omega) = f^{(t_1)}(s,\omega) \text{ for a.e. } (s,\omega) \in [0,t_1] \times \Omega. \end{aligned}$$

So, we can define $f(s,\omega)$ for a.e. $(s,\omega) \in [0,\infty) \times \Omega$ by setting

$$f(s,\omega) = f^{(N)}(s,\omega), \quad s \in [0,N].$$

Thus, we obtain

$$M(t) = \mathbb{E}[M(t)] + \int_0^t f^{(t)}(s)\, dB(s) = \mathbb{E}[M(t)] + \int_0^t f(s)\, dB(s), \quad \forall\, t \ge 0.$$

□

Example 3.11. *Write down the corresponding form of Ito representation theorem for*

$$M(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big], 0 \le t \le T.$$

Solution: *Observe that $M(t)$ is a square integrable martingale with respect to Brownian filtration $\mathcal{F}_t$, and $\mathbb{E}[M(t)] = \mathbb{E}[B^2(T)] = T$. Now by using martingale property of Brownian motion, we get*

$$\begin{aligned} M(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big] &= \mathbb{E}\big[(B(T)-B(t))^2|\mathcal{F}_t\big] + B^2(t) + 2\mathbb{E}\big[B(t)(B(T)-B(t))|\mathcal{F}_t\big] \\ &= \mathbb{E}\big[(B(T)-B(t))^2\big] + B^2(t) = T + B^2(t) - t = \mathbb{E}[M(0)] + B^2(t) - t \end{aligned}$$

From Ito formula, we know that $B^2(t) - t = 2\int_0^t B(s)\, dB(s)$. Thus, we get

$$\begin{aligned} M(t) &= \mathbb{E}[M(0)] + \int_0^t 2B(s)\, dB(s) \\ &= \mathbb{E}[M(0)] + \int_0^t f(s)\, dB(s), \quad f(s) = 2B(s) \in \mathcal{Y}(0,T). \end{aligned}$$

Example 3.12. *Write down the corresponding form of Ito representation theorem for*

$$N(t) = \mathbb{E}\big[\exp(\sigma B(T))|\mathcal{F}_t\big], 0 \le t \le T.$$

Solution: *We know that* $Y(t) := \exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\}$ *is a* $\mathcal{F}_t$*-martingale and therefore*

$$\mathbb{E}\Big[\exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\}\Big] = 1 \implies \mathbb{E}[N(t)] = \mathbb{E}\big[\exp(\sigma B(T))\big] = e^{\frac{1}{2}\sigma^2 T}.$$

Rewriting $N(t)$*, we have*

$$N(t) = e^{\frac{1}{2}\sigma^2 T}\mathbb{E}\Big[\exp\{\sigma B(T) - \frac{1}{2}\sigma^2 T\}|\mathcal{F}_t\Big] = e^{\frac{1}{2}\sigma^2 T}\exp\{\sigma B(t) - \frac{1}{2}\sigma^2 t\} = e^{\frac{1}{2}\sigma^2 T}Y(t)$$

Moreover, $Y(t)$ *satisfies the differential equation*

$$dY(t) = \sigma Y(t)\,dB(t).$$

In other words, we have $Y(t) = 1 + \sigma\int_0^t Y(s)\,dB(s)$*. Thus,*

$$\begin{aligned}
N(t) &= e^{\frac{1}{2}\sigma^2 T}\ \Big(1 + \sigma\int_0^t Y(s)\,dB(s)\Big) \\
&= e^{\frac{1}{2}\sigma^2 T} + \int_0^t \sigma e^{\frac{1}{2}\sigma^2 T}Y(s)\,dB(s) \\
&= \mathbb{E}[N(0)] + \int_0^t f(s)\,dB(s),\ \textit{where } f(t,\omega) = \sigma e^{\frac{1}{2}\sigma^2 T}Y(t) \in \mathcal{Y}(0,T).
\end{aligned}$$

3.7. **Girsanov's theorem.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and Z be a non-negative random variable with $\mathbb{E}[Z] = 1$. Then we can define another probability measure Q on $(\Omega, \mathcal{F})$ such that $Q << \mathbb{P}$ (read as Q is absolutely continuous with respect to $\mathbb{P}$ i.e., for any measurable set A, $\mathbb{P}(A) = 0$ implies $Q(A) = 0$) by

$$Q(A) = \int_A Z d\mathbb{P}.$$

In this case, we say that Z is the Radon-Nikodyme derivative of Q with respect to $\mathbb{P}$ and denoted by

$$Z = \frac{dQ}{d\mathbb{P}}.$$

For any random variable X, we now have two expectation; one with respect to original probability measure $\mathbb{P}$ and another is with respect to new probability measure Q–denoted it by $\mathbb{E}_Q(\cdot)$.

Lemma 3.16. *For any random variable X, one has $\mathbb{E}_Q[X] = \mathbb{E}[ZX]$. In addition, if $\mathbb{P}(Z > 0) = 1$, then $\mathbb{P}$ and Q are equivalent i.e., $Q << \mathbb{P}$ and $\mathbb{P} << Q$.*

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and Z is as above. Define the Radon-Nikodyme derivative process

$$Z(t) := \mathbb{E}[Z|\mathcal{F}_t], \quad t \geq 0.$$

Then clearly $Z(t)$ is a $\mathcal{F}_t$-martingale.

Lemma 3.17. *Let Y be $\mathcal{F}_t$-measurable random variable and $Z(t)$ is the Radon-Nikodyme derivative process . Then*

a) $\mathbb{E}_Q[Y] = \mathbb{E}[YZ(t)]$.
b) For $0 \leq s \leq t$, $\mathbb{E}_Q[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$.

Proof. In view of previous lemma, properties of conditional expectation, and definition of $Z(t)$, we have

$$\mathbb{E}_Q[Y] = \mathbb{E}[YZ] = \mathbb{E}[\mathbb{E}[YZ|\mathcal{F}_t]] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}_t]] = \mathbb{E}[YZ(t)].$$

To prove b), we need to show

$$\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\, dQ = \int_A Y dQ, \quad \forall\, A \in \mathcal{F}_s.$$

We have

$$\begin{aligned} \int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\, dQ &= \mathbb{E}_Q\Big[\mathbf{1}_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\Big] = \mathbb{E}\Big[\mathbf{1}_A \mathbb{E}[YZ(t)|\mathcal{F}_s]\Big] \\ &= \mathbb{E}\big[\mathbb{E}[\mathbf{1}_A YZ(t)|\mathcal{F}_s]\big] = \mathbb{E}\big[\mathbf{1}_A YZ(t)\big] = \mathbb{E}_Q[\mathbf{1}_A Y] = \int_A Y dQ\,. \end{aligned}$$

□

We now state Girsanov's theorem for one dimensional Brownian motion.

Theorem 3.18 (Girsanov's Theorem). *Let $B(t) : 0 \le t \le T$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with Brownian filtration. Let $\theta(t)$ be a adapted process such that*

$$Z(t) := \exp\Big\{ - \int_0^t \theta(s)\, dB(s) - \frac{1}{2}\int_0^t \theta^2(s)\, ds\Big\}$$

becomes a martingale. Then the process

$$\bar{B}(t) = \int_0^t \theta(s)\, ds + B(t), \quad 0 \le t \le T$$

is a Brownian motion with respect to the new probability measure Q where

$$dQ(\omega) = Z(T)\, d\mathbb{P}(\omega).$$

Proof. Note that, since $Z(t)$ is a martingale, we have $\mathbb{E}[Z(T)] = Z(0) = 1$ and $Z(T)$ is $\mathcal{F}_T$-adapted random variable such that it is positive a.s. Thus, Q is a probability measure. We use Levy's theorem to show that $\bar{B}(t)$ is a Brownian motion. Observe that $\bar{B}(0) = 0$ and the quadratic variation of $\bar{B}$ is same as quadratic variation of Brownian motion. Hence it remains to show that $\bar{B}$ is a martingale under Q. Since $Z(t)$ is martingale, we see that $Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t]$ is the Radon-Nikodyme derivative process. Next we claim that $M(t) := \bar{B}(t)Z(t)$ is a martingale under $\mathbb{P}$ (see Assignment 2) . Let $0 \le s \le t$. Then by Lemma 3.17, we have

$$\mathbb{E}_Q[\bar{B}(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[\bar{B}(t)Z(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[M(t)|\mathcal{F}_s] = \frac{M(s)}{Z(s)} = \bar{B}(s).$$

In other words, $\bar{B}(t)$ is martingale under Q. This completes the proof. □

Corollary 3.19. *Let $Y(t); 0 \le t \le T$ be a Ito process*

$$dY(t) = \beta(t)\, dt + \theta(t)dB(t)$$

and there exist adapted processes $u(\cdot)$ and $\alpha(\cdot)$ such that

$$Z(t) := \exp\Big\{ - \int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u^2(s)\, ds\Big\}$$

becomes a martingale, and

$$\theta(t)u(t) = \beta(t) - \alpha(t).$$

Then $\bar{B}(t) = B(t) + \int_0^t u(s)\, ds$ is a Brownian motion under the new probability measure Q given by

$$dQ(\omega) = Z(T)\, d\mathbb{P}(\omega).$$

Moreover, in terms of $\bar{B}(\cdot)$, the process $Y(t)$ has the stochastic integral representation

$$Y(t) = Y(0) + \int_0^t \alpha(s)\, ds + \int_0^t \theta(s)\, d\bar{B}(s).$$

Proof. From Girsanov's theorem (cf. Theorem 3.18), it follows that $\bar{B}(t) : 0 \le t \le T$ is a Brownian motion under the new probability measure Q. Moreover, we have

$$\begin{aligned} dY(t) &= \beta(t)\, dt + \theta(t)dB(t) = \beta(t)\, dt + \theta(t)\{d\bar{B}(t) - u(t)\, dt\} \\ &= \{\beta(t) - \theta(t)u(t)\}\, dt + \theta(t)\, d\bar{B}(t) = \alpha(t)\, dt + \theta(t)\, d\bar{B}(t) \end{aligned}$$

$$\implies Y(t) = Y(0) + \int_0^t \alpha(s)\, ds + \int_0^t \theta(s)\, d\bar{B}(s)\,.$$

□

Example 3.13. *Let $Y(t) = t + B(t) : t \geq 0$. For each $T > 0$, find a probability measure Q_T on $\mathcal{F}_T$ such that $Y(t) : 0 \leq t \leq T$ is a Brownian motion under Q_T. Show that there exists a probability measure Q on $\mathcal{F}_\infty$ such that*

$$Q\Big|_{\mathcal{F}_T} = Q_T \ \forall\ T > 0.$$

Solution: *Taking $\theta(t) = 1$ in the Girsanov's theorem (cf. Theorem 3.18) , we see that $Y(t) = t + B(t)$ is a Brownian motion under the probability measure Q_T on $\mathcal{F}_T$, where Q_T is given by*

$$dQ_T(\omega) = Z(T)\, d\mathbb{P}(\omega); \quad Z(t) = e^{-B(t) - \frac{t}{2}}.$$

Note here that $Z(t)$ is a martingale under $\mathbb{P}$. To prove the second part, we first show that

$$Z(T)\, d\mathbb{P} = Z(t)\, d\mathbb{P} \ \text{on}\ \mathcal{F}_t,\ \ t \leq T.$$

Indeed, for any bounded $\mathcal{F}_t$- measurable function, one has

$$\begin{aligned}\int_\Omega f Z(T)\, d\mathbb{P} &= \mathbb{E}[fZ(T)] = \mathbb{E}\big[\mathbb{E}[fZ(T)|\mathcal{F}_t]\big] \\ &= \mathbb{E}\big[f\mathbb{E}[Z(T)|\mathcal{F}_t]\big] = \mathbb{E}[fZ(t)] = \int_\Omega f Z(t)\, d\mathbb{P}\,.\end{aligned}$$

Thus, $Q_T = Q_S$ on $\mathcal{F}_t$ for all $t \leq \min\{T, S\}$. Hence there exists Q on $\mathcal{F}_\infty$ such that $Q = Q_T$ on $\mathcal{F}_T$ for all $T < \infty$. Hence the result follows.

Example 3.14. *Find a probability measure Q on $\mathcal{F}_T$ such that the process $Y(t) : 0 \leq t \leq T$ given by*

$$dY(t) = t\, dt + (2t+1)\, dB(t)$$

becomes a martingale under Q.

Solution: *Observe that $s \mapsto \frac{s^2}{(2s+1)^2}$ is continuous and therefore $\int_0^T \frac{s^2}{(2s+1)^2}\, ds < +\infty$. Hence the stochastic process*

$$Z(t) = \exp\Big\{ \int_0^t \frac{s}{2s+1}\, dB(s) - \frac{1}{2}\int_0^t \frac{s^2}{(2s+1)^2}\, ds \Big\}$$

is a martingale under $\mathbb{P}$ and $\mathbb{E}[Z(T)] = 1$. Define $dQ(\omega) = Z(T)\, d\mathbb{P}(\omega)$. Then Q is a probability measure and by Girsanov's theorem (cf. Theorem 3.18), the process

$$\bar{B}(t) = B(t) + \int_0^t \frac{s}{2s+1}\, ds : 0 \leq t \leq T$$

is a Brownian motion under Q. The process $Y(t)$ can be expressed in terms of $\bar{B}(t)$ as

$$\begin{aligned}dY(t) &= t\, dt + (2t+1)\, dB(t) = t\, dt + (2t+1)\{d\bar{B}(t) - \frac{t}{2t+1}\, dt\} = (2t+1)\, d\bar{B}(t) \\ &\implies Y(t) = Y(0) + \int_0^t (2s+1)\, d\bar{B}(s)\,.\end{aligned}$$

Hence $Y(t) : 0 \leq t \leq T$ is a martingale under the probability measure Q.

By using n-dimensional Levy's theorem i.e., Theorem 3.12 we arrive at the following multi-dimensional Girsanov's theorem.

Theorem 3.20 (Girsanov's theorem, n-dimensional). *Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \ldots, B_n(t)\big) : 0 \le t \le T$ be a n-dimensional Brownian motion associated with a given filtration $\{\mathcal{F}_t\}$. Let $\boldsymbol{\Theta}(t) = \big(\theta_1(t), \ldots, \theta_n(t)\big)$ be a n-dimensional adapted process such that*

$$Z(t) = \exp\Big\{ - \int_0^t \sum_{i=1}^n \theta_i(s)\, dB_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^d \theta_i^2(s)\, ds \Big\}$$

becomes a martingale. Then the stochastic process, defined by

$$\bar{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \boldsymbol{\Theta}(s)\, ds \quad 0 \le t \le T$$

is a n-dimensional Brownian motion under the probability measure $\tilde{\mathbb{P}}$, where

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

We have n-dimensional version of Corollary 3.19.

Corollary 3.21. *Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \ldots, B_m(t)\big) : 0 \le t \le T$ be a m-dimensional Brownian motion relative to a filtration $\{\mathcal{F}_t\}$, and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{L}_n^1[0,T]$, $\boldsymbol{\Theta}(t) = ((\theta_{ij})) \in \mathcal{Y}_{n\times m}(0,T)$. Let $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t), \ldots, Y_n(t)\big)$ be a n-dimensional Ito process of the form*

$$d\mathbf{Y}(t) = \mathbf{b}(t)\, dt + \boldsymbol{\Theta}(t)\, d\mathbf{B}(t), \quad 0 \le t \le T.$$

Suppose there exist $\boldsymbol{u} = \big(u_1, u_2, \ldots, u_m\big) \in \mathcal{Y}_{1\times m}(0,T)$ and $\mathbf{a} \in \mathbb{L}_n^1(0,T)$ such that

$$\boldsymbol{\Theta}(t)\boldsymbol{u}(t) = \mathbf{b}(t) - \mathbf{a}(t)$$

and

$$Z(t) = \exp\Big\{ - \int_0^t \sum_{i=1}^m u_i(s)\, dB_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^m u_i^2(s)\, ds \Big\}$$

becomes a martingale. Then the stochastic process, defined by

$$\bar{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \mathbf{u}(s)\, ds \quad 0 \le t \le T$$

is a m-dimensional Brownian motion under the probability measure $\tilde{\mathbb{P}}$, where

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

Moreover, in terms of $\bar{\mathbf{B}}(t)$, the n-dimensional process $\mathbf{Y}(t)$ has the following representation:

$$d\mathbf{Y}(t) = \mathbf{a}(t)\, dt + \boldsymbol{\Theta}(t)\, d\bar{\mathbf{B}}(t).$$

Example 3.15. *Consider a 2-dimensional Ito process $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t)\big)$ given by*

$$dY_1(t) = 2dt + dB_1(t) + dB_2(t), \quad dY_2(t) = 4dt + +dB_1(t) - dB_2(t)$$

where $\mathbf{B}(t) = \big(B_1(t), B_2(t)\big)$ is a 2-dimensional Brownian motion. Find a probability measure $\bar{\mathbb{P}}$ such that $\mathbf{Y}(t)$ is a martingale with respect to $\bar{\mathbb{P}}$.

Solution: *Given Ito process can be written in the form*

$$d\mathbf{Y}(t) = \mathbf{b}(t)\,dt + \mathbf{\Theta}(t)\,d\mathbf{B}(t), \quad 0 \le t \le T.$$

where $\mathbf{b} = (2,4) \in \mathbb{L}_2^1[0,T]$ *and* $\mathbf{\Theta}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathcal{Y}_{2\times 2}(0,T)$. *Let* $\boldsymbol{u} = \big(u_1, u_2\big) \in \mathcal{Y}_{1\times 2}(0,T)$ *such that* $\mathbf{\Theta}(t)\boldsymbol{u}(t) = \mathbf{b}(t)$. *Then* $u_1(t) = 3$ *and* $u_2(t) = -1$. *Note that the process*

$$Z(t) = \exp\Big\{ -\int_0^t \sum_{i=1}^2 u_i(s)\,dB_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^2 u_i^2(s)\,ds \Big\}$$

becomes a martingale (Novikov condition holds trivially). Define a probability measure $\bar{\mathbb{P}}$ *as*

$$\bar{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

Then by Corollary 3.21, the stochastic process

$$\bar{\mathbf{B}}(t) = \mathbf{B}(t) + \int_0^t \mathbf{u}(s)\,ds \quad 0 \le t \le T$$

is a 2*-dimensional Brownian motion under the probability measure* $\bar{\mathbb{P}}$. *Moreover,* $\mathbf{Y}(t)$ *can be written as*

$$d\mathbf{Y}(t) = \mathbf{\Theta}(t)\,d\bar{\mathbf{B}}(t).$$

This shows that $\mathbf{Y}(t)$ *is a martingale with respect to the probability measure* $\bar{\mathbb{P}}$.

4. Stochastic Differential Equation and connection with PDEs

Let $B(t)$ be a m-dimensional Brownian motion and Z be a random variable independent of $B(\cdot)$. Consider a stochastic differential equation (**SDE**) of the type

$$\begin{cases} dX(t) = a(t, X(t))\,dt + \sigma(t, X(t))\,dB(t) \\ X(0) = Z \end{cases} \tag{4.1}$$

where $a : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given functions. We now discuss existence and uniqueness of solution of the SDE. Let $\mathcal{F}_t$ be the filtration generated by Z and $B(t)$ i.e., $\mathcal{F}_t = \sigma(Z, B(s) : 0 \le s \le t)$. Let us first define the solution concept.

Definition 4.1. Let $B(t)$ be a m-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Z be a random variable on it, independent of $B(\cdot)$. An $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ on $(\Omega, \mathcal{F})$ is a (strong) solution of the SDE (4.1) if

i) $X(t)$ is $\mathcal{F}_t$-adapted stochastic process where $\mathcal{F}_t = \sigma(Z, B(s) : 0 \le s \le t)$.

ii) For all $0 \le t \le T$, there holds

$$X(t) = Z + \int_0^t a(s, X(s))\,ds + \int_0^t \sigma(s, X(s))\,dB(s)$$

where the integrals on the right hand side are well-defined.

We now prove a technical lemma so called **Gronwall's lemma** which is useful in our uniqueness proof.

Lemma 4.1. *Let $\phi(\cdot)$ and $f(\cdot)$ are nonnegative continuous function on $[0,T]$ and $C \geq 0$ be a constant. If $\phi(t) \leq C + \int_0^t f(s)\phi(s)\,ds$ for all $t \in [0,T]$, then*

$$\phi(t) \leq Ce^{\int_0^t f(s)\,ds}, \quad t \in [0,T].$$

Proof. Let $h(t) = C + \int_0^t f(s)\phi(s)\,ds$. Then by given condition, $\phi(t) \leq h(t)$ and hence, since f is nonnegative, $f(t)\phi(t) \leq f(t)h(t)$. We have

$$\begin{aligned}
&h'(t) = f(t)\phi(t) \leq f(t)h(t) \\
&\implies \big(h'(t) - h(t)f(t)\big)e^{-\int_0^t f(s)\,ds} \leq 0 \\
&\implies \Big(h(t)e^{-\int_0^t f(s)\,ds}\Big)' \leq 0 \\
&\implies h(t)e^{-\int_0^t f(s)\,ds} \leq h(0)e^{-\int_0^0 f(s)\,ds} = C \\
&\implies h(t) \leq Ce^{\int_0^t f(s)\,ds} \\
&\implies \phi(t) \leq Ce^{\int_0^t f(s)\,ds}.
\end{aligned}$$

This completes the proof. □

We now prove existence and uniqueness theorem for SDE.

Theorem 4.2. *Let $a : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|a(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^d,\ t \in [0,T] \tag{4.2}$$

$$|a(t,x) - a(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|, \quad x,y \in \mathbb{R}^n,\ t \in [0,T] \tag{4.3}$$

for some constants C and D and $Z \in L^2(\Omega)$. Then the SDE (4.1) has a unique continuous solution $X(t)$ in the sense of Definition 4.1. Moreover, there exists $K = K(C,T)$ such that

$$\mathbb{E}[|X(t)|^2] \leq K(1 + \mathbb{E}[Z^2])e^{Kt}.$$

Proof. We first prove its uniqueness. Suppose there exist two solutions X and Y of (4.1) with continuous paths a.s. Then for all $t \in [0,T]$, we have

$$X(t) - Y(t) = \int_0^t \big(a(s,X(s)) - b(s,Y(s))\big)\,ds + \int_0^t \big(\sigma(s,X(s)) - \sigma(s,Y(s))\big)\,dB(s)\,.$$

We now use Ito-isometry, Cauchy-Schwartz inequality, together with the assumption (4.3) to have

$$\begin{aligned}
&\mathbb{E}\Big[|X(t) - Y(t)|^2\Big] \\
&\leq 2\mathbb{E}\Big[\Big(\int_0^t \big(a(s,X(s)) - a(s,Y(s))\big)\,ds\Big)^2 + \Big(\int_0^t \big(\sigma(s,X(s)) - \sigma(s,Y(s))\big)\,dB(s)\Big)^2\Big] \\
&\leq 2T\mathbb{E}\Big[\int_0^t \big(a(s,X(s)) - a(s,Y(s))\big)^2\,ds\Big] + 2\mathbb{E}\Big[\int_0^t \big(\sigma(s,X(s)) - \sigma(s,Y(s))\big)^2\,ds\Big] \\
&\leq 2D^2(T+1)\int_0^t \mathbb{E}\Big[|X(s) - Y(s)|^2\Big]\,ds
\end{aligned}$$

Hence by Grownwall's lemma, $\mathbb{E}\Big[|(X(t)-Y(t)|^2\Big] = 0$ for all $t \in [0,T]$. Hence

$$\mathbb{P}\Big(|X(t)-Y(t)| = 0 \text{ for all } t \in \mathbb{Q} \cap [0,T]\Big) = 1$$

where $\mathbb{Q}$ denotes the rational numbers. By continuity $t \mapsto |X(t)-Y(t)|$, it follows that

$$\mathbb{P}\Big(X(t) = Y(t) \text{ for all } t \in [0,T]\Big) = 1.$$

This completes the uniqueness proof.

Existence proof: It is similar to the existence proof for ODE. Consider the Picard type iteration as follows:

$$\begin{aligned}
Y^{(0)}(t) &= Z, \\
Y^{(k+1)}(t) &= Z + \int_0^t a(s, Y^{(k)}(s))\, ds + \int_0^t \sigma(s, Y^{(k)}(s))\, dB(s).
\end{aligned}$$

Then for $k \geq 1$ and $t \in [0, T]$, we have similar to uniqueness proof

$$\mathbb{E}\big[|Y^{(k+1)}(t) - Y^{(k)}(t)|^2\big] \leq 2D^2(1+T) \int_0^t \mathbb{E}\big[|Y^{(k)}(s) - Y^{(k-1)}(s)|^2\big]\, ds.$$

Observe that

$$\mathbb{E}\big[|Y^{(1)}(t) - Y^{(0)}(t)|^2\big] \leq 2\mathbb{E}\Big[\Big(\int_0^t a(s, Z)\, ds\Big)^2\Big] + 2\mathbb{E}\Big[\int_0^t \sigma^2(s, Z)\, ds\Big] \leq tA_1$$

where the constant A_1 only depends on C, T and $\mathbb{E}[Z^2]$. Hence by induction on k, we get

$$\mathbb{E}\big[|Y^{(k+1)}(t) - Y^{(k)}(t)|^2\big] \leq \frac{A_2^{k+1} t^{k+1}}{(k+1)!}, \quad k \geq 0,\ t \in [0, T]$$

where $A_2 \equiv A_2(C, D, T, \mathbb{E}[Z^2])$. Thus, for $m > n \geq 0$, we get

$$\begin{aligned}
&\big\|Y^{(m)}(t) - Y^{(n)}(t)\big\|_{L^2([0,T]\times\Omega)} = \big\|\sum_{k=n}^{m-1} Y^{(k+1)}(t) - Y^{(k)}(t)\big\|_{L^2([0,T]\times\Omega)} \\
&\leq \sum_{k=n}^{m-1} \big\|Y^{(k+1)}(t) - Y^{(k)}(t)\big\|_{L^2([0,T]\times\Omega)} \leq \sum_{k=n}^{m-1} \Big(\int_0^T \mathbb{E}\big[|Y^{(1)}(t) - Y^{(0)}(t)|^2\big]\, dt\Big)^{\frac{1}{2}} \\
&\leq \sum_{k=n}^{m-1} \Big(\int_0^T \frac{A_2^{k+1} t^{k+1}}{(k+1)!}\, dt\Big)^{\frac{1}{2}} = \sum_{k=n}^{m-1} \Big(\frac{A_2^{k+1} T^{k+2}}{(k+2)!}\Big)^{\frac{1}{2}} \to 0 \quad \text{as } m, n \to \infty.
\end{aligned}$$

Thus, $\{Y^{(n)}(t)\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$ and hence it is convergent. Thus, there exists a $\mathcal{F}_t$-adapted stochastic process $X(t)$ such that

$$X(t) := \lim_{n\to\infty} Y^{(n)}(t) \quad \text{in } L^2([0,T]\times\Omega).$$

By using (4.3), Cauchy-Schwartz and Ito-isometry along with the fact that $X(t) := \lim_{n\to\infty} Y^{(n)}(t)$, we get, for all $t \in [0, T]$

$$\begin{aligned}
\int_0^t a(s, Y^{(n)}(s))\, ds &\to \int_0^t a(s, X(s))\, ds, \\
\int_0^t \sigma(s, Y^{(n)}(s))\, dB(s) &\to \int_0^t \sigma(s, X(s))\, dB(s)
\end{aligned}$$

in $L^2(\Omega)$. Passing to the limit as $k \to \infty$ in Picard iteration, we get

$$X(t) = Z + \int_0^t a(s, X(s)) + \int_0^t \sigma(s, X(s))\, dB(s). \tag{4.4}$$

It remains to show that $X(t)$ can be chosen to be continuous. Note that the right hand side of (4.4) has a continuous version. Denote it by $\tilde{X}(t)$. Then $\tilde{X}(t)$ is continuous and

$$\begin{aligned}\tilde{X}(t) &= Z + \int_0^t a(s, X(s)) + \int_0^t \sigma(s, X(s))\, dB(s) \text{ a.s.} \\ &= Z + \int_0^t a(s, \tilde{X}(s)) + \int_0^t \sigma(s, \tilde{X}(s))\, dB(s) \text{ a.s.}\end{aligned}$$

In view of the assumption (4.2) and Ito-isometry, we see that

$$\begin{aligned}\mathbb{E}[|X(t)|^2] &\le 3\Big\{\mathbb{E}[Z^2] + TC^2\int_0^t (1+\mathbb{E}[|X(s)|^2])\, ds + 2C^2\int_0^t (1+\mathbb{E}[|X(s)|^2])\, ds\Big\} \\ &\le K(1+\mathbb{E}[Z^2]) + K\int_0^t \mathbb{E}[|X(s)|^2]\, ds\end{aligned}$$

for some constant $K = K(C,T)$. Hence from Gronwall's lemma, we get

$$\mathbb{E}[|X(t)|^2] \le K(1+\mathbb{E}[Z^2])e^{Kt}.$$

□

Example 4.1. *Solve the stochastic differential equation:*

$$dX(t) = (m - X(t))\, dt + \sigma dB(t), \quad X(0) = Z$$

where Z is non-random and $m, \sigma \in \mathbb{R}$. Show that variance of the solution $X(t)$ tends to $\frac{\sigma^2}{2}$ as $t \to \infty$.

Solution: *The SDE can be written as in the form of (4.1) with $a(t,x) = m - x$ and $\sigma(t,x) = \sigma$. It is easy to see that the functions $a(t,x)$ and $\sigma(t,x)$ satisfy the conditions (4.2) and (4.3). Hence by existence and uniqueness theorem, given SDE has a unique strong solution $X(t)$. By applying Ito product rule, we have*

$$\begin{aligned} d(X(t)e^t) &= e^t dX(t) + X(t)e^t\, dt = me^t\, dt + \sigma e^t\, dB(t) \\ \implies X(t)e^t &= Z + \int_0^t me^s\, ds + \int_0^t \sigma e^s\, dB(s) \\ \implies X(t) &= m(1-e^{-t}) + Ze^{-t} + \sigma\int_0^t e^{-(t-s)}\, dB(s).\end{aligned}$$

Thus the unique solution is given by

$$X(t) = m(1-e^{-t}) + Ze^{-t} + \sigma\int_0^t e^{-(t-s)}\, dB(s).$$

Since Ito-integral is martingale, we have

$$\mathbb{E}[X(t)] = m(1-e^{-t}) + Ze^{-t}.$$

By using Ito-isometry, we have

$$\begin{aligned}\mathrm{Var}(X(t)) &= \mathbb{E}[(X(t))^2] - (\mathbb{E}[X(t)])^2 = \mathbb{E}\Big[\Big(\sigma\int_0^t e^{-(t-s)}\, dB(s)\Big)^2\Big] \\ &= \sigma^2\int_0^t e^{2(s-t)}\, ds = \frac{\sigma^2}{2}(1-e^{-2t}) \to \frac{\sigma^2}{2} \ \textit{as } t\to\infty.\end{aligned}$$

Example 4.2. *Solve the SDE: for $r, \alpha \in \mathbb{R}$*

$$dX(t) = r\,dt + \alpha X(t)\,dB(t), \quad X(0) = Z,\ \mathbb{E}[Z^2] < \infty$$

Solution: *The SDE can be written as in the form of (4.1) with $a(t,x) = r$ and $\sigma(t,x) = \alpha x$. It is easy to see that the functions $a(t,x)$ and $\sigma(t,x)$ satisfy the conditions (4.2) and (4.3). Hence by existence and uniqueness theorem, given SDE has a unique strong solution $X(t)$. We now find explicit solution of the given SDE. Consider a stochastic process given by*

$$Y(t) = \exp\Big\{ -\alpha B(t) + \frac{1}{2}\alpha^2 t \Big\}.$$

Then $Y(t)$ satisfies the differential form

$$dY(t) = \alpha^2 Y(t)\,dt - \alpha Y(t)\,dB(t).$$

By using Ito product rule, we get

$$\begin{aligned} d(X(t)Y(t)) &= X(t)dY(t) + Y(t)dX(t) - \alpha^2 Y(t)X(t)\,dt \\ &= X(t)\Big(\alpha^2 Y(t)\,dt - \alpha Y(t)\,dB(t)\Big) + Y(t)\Big(r\,dt + \alpha X(t)\,dB(t)\Big) - \alpha^2 Y(t)X(t)\,dt \\ &= rY(t)\,dt \\ &\implies X(t) = \frac{1}{Y(t)}\Big(Z + r\int_0^t Y(s)\,ds\Big). \end{aligned}$$

Thus the unique solution of the given SDE is given by

$$X(t) = \exp\Big\{\alpha B(t) - \frac{1}{2}\alpha^2 t\Big\} Z + \int_0^t \exp\Big\{\alpha(B(t) - B(s)) - \frac{1}{2}\alpha^2(t-s)\Big\}\,ds.$$

4.1. **Markov property:** Consider the SDE (4.1). We denote by $X^{0,x}(t)$ as solution of (4.1) staring from $t=0$ at position x. By uniqueness of solution one can see that, at least for time-homogeneous drift and diffusion coefficients,

$$X^{0,x}(t+s) = X^{t,X(t)}(s) \quad \forall t, s \geq 0.$$

Let $0 \leq t \leq T$ be given and for any given Borel-measurable function $h(\cdot)$ on $\mathbb{R}^n$, we define the function

$$g(t,x) := \mathbb{E}^{t,x}[h(X(T))]$$

the expectation of $h(X(T))$, where $X(T)$ is a solution of (4.1) at time T with initial condition $X(t)=x$. In other words, $g(t,x) = \mathbb{E}[h(X(T))|X(t)=x]$.

Theorem 4.3. *Let $X(u) : u \geq 0$ be a solution of SDE (4.1) with initial condition at time 0. Then for $0 \leq .t \leq T$,*

$$\mathbb{E}[h(X(T))|\mathcal{F}_t] = g(t, X(t)) \text{ a.s.}$$

In view of Theorem 4.3, and the properties of conditional expectation, we see that $g(t,X(t))$ is a martingale. Indeed, for $0 \leq s \leq t$,

$$\mathbb{E}[g(t,X(t))|\mathcal{F}_s] = \mathbb{E}\Big[\mathbb{E}[h(X(T))|\mathcal{F}_t]\big|\mathcal{F}_s\Big] = \mathbb{E}[h(X(T))|\mathcal{F}_s] = g(s,X(s)) \text{ a.s.}$$

4.2. **Connection with partial differential equation (PDE).** Feynman-Kac theorem below relates SDE and partial differential equation (PDE).

Theorem 4.4 (Feynman-Kac)**.** *Let $0 \leq t \leq T$ be given. For any given Borel-measurable function $h(\cdot)$ on $\mathbb{R}$, we define the function*

$$g(t,x) := \mathbb{E}^{t,x}[h(X(T))]$$

where $X(\cdot)$ is a solution of (4.1). Then $g(t,x)$ satisfies the following backward PDE

$$\begin{cases} g_t(t,x) + a(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0 \\ g(T,x) = h(x), \quad x \in \mathbb{R}\,. \end{cases}$$

Proof. We outline the proof of this theorem. Let $X(t)$ be a solution of SDE (4.1) starting at time 0. First observe that $g(T,x) = h(x)$. Since $g(t,X(t))$ is a martingale, the net dt-term in the differential form of $d(g,X(t))$ must be zero. Now, by Ito formula, we get

$$\begin{aligned} dg(t,X(t)) = &\Big\{g_t(t,X(t)) + a(t,X(t))g_x(t,X(t)) + \frac{1}{2}\sigma^2(t,X(t))g_{xx}(t,X(t))\Big\}\,dt \\ &+ \sigma(t,X(t))\,dB(t) \end{aligned}$$

Setting dt-term to zero, we have

$$g_t(t,X(t)) + a(t,X(t))g_x(t,X(t)) + \frac{1}{2}\sigma^2(t,X(t))g_{xx}(t,X(t)) = 0$$

along every path of $X(t)$. Therefore, we have

$$g_t(t,x) + a(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0$$

at every point (t,x) which can be reached by $(t,X(t))$. This completes the proof. $\square$

The general principle behind the proof of Feynman-Kac theorem is to find martingale, take the differential and then set dt-term equals to zero.

Example 4.3. *Let $Y(t) : 0 \le t \le T$ be a unique solution of the SDE*

$$dX(t) = r\,dt + \alpha X(t)\,dB(t), \quad r, \alpha \in \mathbb{R}.$$

Then the function $g(t,x) = \mathbb{E}^{t,x}[h(X(T))]$ solves the PDE:

$$\begin{cases} g_t(t,x) + r g_x(t,x) + \frac{1}{2}\alpha^2 x^2 g_{xx}(t,x) = 0 \\ g(T,x) = h(x), \quad x \in \mathbb{R} \end{cases}$$

where h is a Borel-measurable function.

Example 4.4. *Let Let $X(t) : 0 \le t \le T$ be a solution of SDE (4.1) starting at time 0. For any Borel-measurable function h, define*

$$f(t,x) := \mathbb{E}^{t,x}[e^{-r(T-t)} h(X(T))], \quad r \in \mathbb{R}^*.$$

Then $f(t,x)$ solves the PDE

$$\begin{cases} f_t(t,x) + a(t,x) f_x(t,x) + \frac{1}{2}\sigma^2(t,x) f_{xx}(t,x) = r f(t,x) \\ f(T,x) = h(x), \quad x \in \mathbb{R}. \end{cases}$$

Observe that $f(t,X(t)) = \mathbb{E}[e^{-r(T-t)} h(X(T)) | \mathcal{F}_t]$, and hence $f(t,X(t))$ is NOT a martingale. But $e^{-rt} f(t,X(t))$ is a martingale. Hence applying Ito product rule, we get

$$\begin{aligned} d\big(e^{-rt} f(t,X(t))\big) = e^{-rt}\Big(&- r f(t,X(t)) + f_t(t,X(t)) + a(t,X(t)) f_x(t,X(t)) \\ &+ \frac{1}{2}\sigma^2(t,X(t)) f_{xx}(t,X(t))\Big)\,dt + e^{-rt}\sigma(t,X(t))\,dB(t) \end{aligned}$$

Setting dt-term to zero, we get the desired result.

5. Application to Finance

Mathematical finance is the study of financial markets and is one of the rapidly growing subjects in applied mathematics. Suppose that two assets are traded: one risk free and one risk security. The risk-free asset can be thought of as a bank deposit or a bond issued by a government. The risky security will typically be some stock.

Consider a money market account with variable interest rate $r(t)$. Let the price of money market account at time t is $S_0(t)$. Assume that $S_0(t)$ is determined by the differential equation

$$dS_0(t) = r(t) S_0(t)\,dt, \quad S_0(0) = 1. \tag{5.1}$$

Then $S_0(t)$ is given by

$$S_0(t) = \exp\Big\{ \int_0^t r(s)\,ds \Big\}.$$

Consider the risky asset. The price of stock at time t will be denoted by $S(t)$. The future price $S(t)$ for $t > 0$ remains unknown in general. Mathematically $S(t)$ can be represented as a positive random variable on a probability space Ω i.e.,

$$S(t) : \Omega \to (0, \infty).$$

The probability space Ω consists of all feasible price movement scenarios $\omega \in \Omega$. The behaviour of the stock price is determined by the stochastic differential equation

$$dS(t) = \alpha(t) S(t) dt + \sigma(t) S(t)\,dB(t); \quad S(0) = x > 0 \tag{5.2}$$

where $B(t)$ is one-dimensional Brownian motion, $\alpha(t)$ and $\sigma(t)$ are adapted processes. The asset price $S(t)$ has *instantaneous* mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. The word *instantaneous* signifies that $\alpha(t)$ and $\sigma(t)$ depends on the time and sample paths.

Remark 5.1. If α and σ are constants, we have the usual geometric Brownian motion model and the distribution of $S(t)$ is log-normal. In general $S(t)$ does not need to be log-normal because α and σ are allowed to be time-varying and random.

Suppose we have an adapted interest rate process $r(t)$. We define the discount process

$$D(t) = \exp\{-\int_0^t r(s)\,ds\}.$$

One can easily check, by applying Ito-formula, that

$$dD(t) = -r(t)D(t)\,dt.$$

Observe that, because of smoothness, $D(t)$ has zero quadratic variation. Note also that $S_0(t) = \frac{1}{D(t)}$.

Definition 5.1. The Ito-process $X(t) = \big(S_0(t), S(t)\big)$ where $S_0(t)$ and $S(t)$ satisfies (5.1) and (5.2) respectively is called a **market**.

The financial derivatives or financial securities are financial contracts whose value is derived from some underlying assets. In general financial derivatives can be grouped into three groups: **options, forwards, and futures**. We will be mainly discussing the options. The options constitutes an important building block for pricing financial derivatives.

Definition 5.2. An option is a financial contract that gives the holder the right (but not the obligation) to buy or sell some underlying asset at a specific price (called **strike price**) and specific date (called **expiry date**). There are two main types of option contract.

i) **Call option:** it gives the holder the right to buy a stock at a strike price within the expiry date.
ii) **Put option:** it gives the holder the right to sell some asset at a strike price within the expiry date.

We denote by K the strike price and by T the expiry date. If the current price of the stock is S, the owner of a call will exercise the option if $K < S$, and the owner of a put will exercise the option if $K > S$.

Based on the exercise of the holder, options can be divided into two types:

a) **European option:** it is an option that can be exercised only at the time of expiry of the contract.
b) **American option:** an option which can be exercised at any time up to the expiry time T.

Remark 5.2. The call or put option described above is called a **simple** or **vanilla** option. An option which is not a vanilla option called an **exotic option**.

Consider the **discounted stock price process** $\tilde{S}(t) := D(t)S(t)$. Then it is given by the formula

$$\tilde{S}(t) = S(0)\exp\Big\{\int_0^t \sigma(s)\,dB(s) + \int_0^t \big(\alpha(s) - r(s) - \frac{1}{2}\sigma^2(s)\big)\,ds\Big\}.$$

Again, by using Ito-formula, one has

$$\begin{aligned} d\tilde{S}(t) &= \big(\alpha(t) - r(t)\big)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\,dB(t) \\ &= \theta(t)\sigma(t)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\,dB(t), \end{aligned}$$

where $\theta(t)$ is the **market price of risk**, defined by

$$\theta(t) = \frac{\alpha(t) - r(t)}{\sigma(t)}.$$

Observe that, compare to undiscounted stock price, the mean rate of return of the discounted stock price $\tilde{S}(t)$ is reduced by the interest rate $r(t)$, i.e., the mean rate of return of $\tilde{S}(t)$ is $\alpha(t) - r(t)$. Note that volatility for $S(t)$ and $\tilde{S}(t)$ remains same.

Definition 5.3. A probability measure Q is said to be risk-neutral if the followings hold:

a) Q and $\mathbb{P}$ are equivalent i.e., for every $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ if and only if $Q(A) = 0$,
b) Under Q, the discounted stock price $\tilde{S}(t)$ is a martingale.

Let Q be a probability measure defined by $Q(A) = \int_A Z(T)\,d\mathbb{P}$ for any $A \in \mathcal{F}$ where the process $Z(t)$ is given by

$$Z(t) = \exp\{-\int_0^t \theta(s)\,dB(s) - \frac{1}{2}\int_0^t \theta^2(s)\,ds\},$$

and $\theta(t)$ is the market price of risk process. Then by Girsanov's theorem

$$\bar{B}(t) = B(t) + \int_0^t \theta(s)\,ds$$

is a Brownian motion under Q. Moreover, the discounted stock price process $\tilde{S}(t)$ can be rewritten in terms of $\bar{B}(t)$ as follows:

$$\begin{aligned} d\tilde{S}(t) &= \theta(t)\sigma(t)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\,dB(t) \\ &= \theta(t)\sigma(t)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\big\{d\bar{B}(t) - \theta(t)dt\big\} \\ &= \sigma(t)\tilde{S}(t)\,d\bar{B}(t) \end{aligned}$$

Thus, $\tilde{S}(t)$ is a martingale under Q. Moreover, $\mathbb{P}$ and Q are equivalent. Hence the measure Q is a risk-neutral measure.

Remark 5.3. We have the following observation:

a) Mean rate of return of the undiscounted stock price $S(t)$ under the risk-neutral measure Q is equal to the interest rate $r(t)$. Indeed,
$$\begin{aligned} dS(t) &= \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t) \\ &= \alpha(t)S(t)\,dt + \sigma(t)S(t)\big\{d\bar{B}(t) - \theta(t)dt\big\} \\ &= \big(\alpha(t) - \sigma(t)\theta(t)\big)S(t)\,dt + \sigma(t)S(t)\,d\bar{B}(t) \\ &= r(t)S(t)\,dt + \sigma(t)S(t)\,d\bar{B}(t). \end{aligned}$$
b) The volatility of the stock price $S(t)$ does NOT change under the risk-neutral measure.
c) If $\alpha(t) > r(t)$, the change of measure puts more probability on the paths with lower return so that the overall mean rate of return is reduced from $\alpha(t)$ to $r(t)$.

Definition 5.4. We define the followings:

i) A **trading strategy or the portfolio** in the market $X(t) = (S_0(t), S(t))$ is a adapted process $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ such that
$$\int_0^T |\psi_0(t)|\,dt < +\infty; \quad \int_0^T \psi^2(s)\,ds < +\infty \quad \text{a.s.}$$
$\psi_0(t)$ and $\psi(t)$ represents the number of units of shares and bonds respectively.
ii) The **value of the portfolio** at time t is given by
$$V_\Psi(t) = \psi_0(t)S_0(t) + \psi(t)S(t).$$
iii) The portfolio $\Psi(t)$ is called **self-financing** if $V_\Psi(t)$ satisfies the differential form
$$dV_\Psi(t) = \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t)\,. \tag{5.3}$$

In words, the self-financing property means that the investor is not withdrawing any gains from the portfolio for consumption, nor investing additional funds. She starts with an initial investment, and from there on all gains or losses in portfolio value come from price increases or decreases in the stock, or bond. Furthermore, the property tells us that if the investor wants to increase the stock position, say, the funding for this must come from selling bonds .

Suppose the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing. Therefore, one has
$$\begin{aligned} &\psi_0(t)S_0(t) + \psi(t)S(t) = V_\Psi(0) + \int_0^t \psi_0(u)\,dS_0(u) + \int_0^t \psi(u)\,dS(u) \\ \implies\ &\psi_0(t)S_0(t) = V_\Psi(0) + \int_0^t \psi_0(u)\,dS_0(u) + \int_0^t \psi(u)\,dS(u) - \psi(t)S(t). \end{aligned}$$
Set
$$Y_0(t) = \psi_0(t)S_0(t), \quad A(t) = \int_0^t \psi(u)\,dS(u) - \psi(t)S(t).$$
Then one has
$$dY_0(t) = \psi_0(t)\,dS_0(t) + dA(t) = r(t)Y_0(t)\,dt + dA(t).$$

Consider the discount process $D(t)$. Observe that

$$D(t)Y_0(t) = D(t)S_0(t)\psi_0(t) = \psi_0(t).$$

By Ito product rule, we get

$$\begin{aligned}
&d(D(t)Y_0(t)) = dD(t)Y_0(t) + D(t)\,dY_0(t)\\
&= -D(t)r(t)Y_0(t)\,dt + D(t)\{r(t)Y_0(t)\,dt + dA(t)\}\\
&= D(t)\,dA(t)\\
&\implies \psi_0(t) = \psi_0(0) + \int_0^t D(u)\,dA(u)\\
&\implies \psi_0(t) = \psi_0(0) - A(0) + D(t)A(t) + \int_0^t r(s)A(s)D(s)\,ds\,, \qquad (5.4)
\end{aligned}$$

where in the last line we have used the integration by parts formula. This argument goes both the ways in the sense that if we define $\psi_0(t)$ by (5.4), then we get (5.3). Indeed, from (5.4) and the definition of A, we get

$$d\psi_0(t) = D(t)\,dA(t); \quad dA(t) = \psi(t)\,dS(t) - S(t)\,d\psi(t) - \psi(t)\,dS(t) - d[\psi(t), S(t)]\,.$$

Hence, we obtain

$$\begin{aligned}
dV_\psi(t) &= d\big(\psi_0(t)S_0(t) + \psi(t)S(t)\big)\\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t) + S_0(t)\,d\psi_0(t) + S(t)\,d\psi(t) + d[\psi(t), S(t)]\\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t) + S_0(t)D(t)\,dA(t) + S(t)\,d\psi(t) + d[\psi(t), S(t)]\\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t) + dA(t) + S(t)\,d\psi(t) + d[\psi(t), S(t)]\\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t)\,.
\end{aligned}$$

Therefore, if $\psi(t)$ is chosen, we can always make the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ self-financing by choosing $\psi_0(t)$ according to (5.4) . Moreover, we are free to choose the initial value $V_\Psi(0)$ of the portfolio.

Remark 5.4. Consider the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$. A trading strategy $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing if and only if $\tilde{V}_\Psi(t)$ can be expressed for all $t \in [0, T]$ as

$$\tilde{V}_\Psi(t) = V_\Psi(0) + \int_0^t \psi(u)\,d\tilde{S}(u)$$

where $\tilde{S}(t)$ is the discounted stock price process.

Example 5.1. *Consider a market $X(t) = \big(S_0(t), S(t)\big)$ given by*

$$dS_0(t) = 2S_0(t)\,dt,\;\; S_0(0) = 1; \quad dS(t) = S(t)\,dt + 2S(t)\,dB(t),\;\; S(0) = x > 0.$$

Show that the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$, given by

$$\psi_0(t) = -\int_0^t e^{-2u}S^3(u)\,du, \quad \psi(t) = \int_0^t S^2(u)\,du,$$

is self-financing.

Solution: *Note that* $S_0(t) = e^{2t}$ *and hence the corresponding discounted process* $D(t) = e^{-2t}$*. Moreover,*

$$d\psi_0(t) = -S^3(t)D(t)\,dt = -\frac{S^3(t)}{S_0(t)}; \quad d\psi(t) = S^2(t)\,dt\,.$$

The value of the portfolio $V_\Psi(t)$ *is given by*

$$V_\Psi(t) = \psi_0(t)S_0(t) + \psi(t)S(t).$$

To show that $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ *is self-financing, we need to show that*

$$S_0(t)\,d\psi_0(t) + S(t)\,d\psi(t) = 0.$$

Indeed, we have

$$S_0(t)\,d\psi_0(t) + S(t)\,d\psi(t) = -S_0(t)\frac{S^3(t)}{S_0(t)}\,dt + S(t)S^2(t)\,dt = 0.$$

Definition 5.5. A self-financing strategy Ψ is called an **arbitrage opportunity** if
$$V_\Psi(0)=0,\ V_\Psi(T)\geq 0 \text{ a.s.,}\quad \text{and}\quad \mathbb{P}\big(V_\Psi(T)>0\big)>0.$$

So, $\Psi(t)$ is an arbitrage if it gives an increase in the value from $t=0$ to time $t=T$ a.s., and a strict positive increase with positive probability. Hence $\Psi(t)$ generates a profit without any risk of losing money. Such an opportunity exists if and only if there is way to start with positive capital $V_\Psi(0)$ and to beat the money market account. In other words, there exists an arbitrage if and only if there is a way to start with $V_\Psi(0)$ and at a later time T have a portfolio value satisfying
$$\mathbb{P}\Big(V_\Psi(T)\geq \frac{V_\Psi(0)}{D(T)}\Big)=1,\quad \mathbb{P}\Big(V_\Psi(T)> \frac{V_\Psi(0)}{D(T)}\Big)>0. \tag{5.5}$$
How can we decide if a given market $(X(t))_{t\in[0,T]}$ allows an arbitrage or not.

Theorem 5.1 (**First fundamental theorem of asset pricing).** *If a market $(X(t))_{t\in[0,T]}$ has a risk-neutral measure, then it does not admit arbitrage.*

Proof. Suppose the given market $(X(t))_{t\in[0,T]}$ has a risk-neutral measure Q. Then under Q, the discounted stock process $\tilde{S}(t)$ is a martingale. We claim that the discounted portfolio value process $\tilde{V_\Psi}(t):=D(t)V_\Psi(t)$ is a martingale under Q. Indeed, since $\psi(t)$ units of portfolio value $V_\Psi(t)$ is invested in stock, the remainder of the portfolio value $V_\Psi(t)-\psi(t)S(t)$ is invested in the money market account. Thus, the differential of portfolio value is given by
$$\begin{aligned}
dV_\Psi(t)&=\psi(t)\,dS(t)+r(t)\big(V_\Psi(t)-\psi(t)S(t)\big)\,dt\\
&=r(t)V_\Psi(t)\,dt+\psi(t)\Big(dS(t)-r(t)S(t)\,dt\Big)\\
&=r(t)V_\Psi(t)\,dt+\frac{\psi(t)}{D(t)}\Big(D(t)dS(t)-D(t)r(t)S(t)\,dt\Big)\\
&=r(t)V_\Psi(t)\,dt+\frac{\psi(t)}{D(t)}d(D(t)S(t))\quad(\text{as } d(D(t))=-r(t)D(t)\,dt)\\
&=r(t)V_\Psi(t)\,dt+\frac{\psi(t)}{D(t)}d\tilde{S}(t)\,.
\end{aligned}$$
By using Ito-product rule, we have
$$\begin{aligned}
d\tilde{V_\Psi}(t)&=d\big(D(t)V_\Psi(t)\big)=-r(t)D(t)V_\Psi(t)\,dt+D(t)\,dV_\Psi(t)\\
&=-r(t)D(t)V_\Psi(t)\,dt+D(t)\Big(r(t)V_\Psi(t)\,dt+\frac{\psi(t)}{D(t)}d\tilde{S}(t)\Big)\\
&=\psi(t)d\tilde{S}(t)\,.
\end{aligned}$$
Since $\tilde{S}(t)$ is a martingale under Q, we conclude from the above differential form that the discounted portfolio value $\tilde{V_\Psi}(t)$ is also a martingale under Q. In particular, for any portfolio value process $V_\Psi(t)$, we have
$$\mathbb{E}_Q[D(T)V_\Psi(T)]=V_\Psi(0).$$
Let $V_\Psi(t)$ be a portfolio value process such that $V_\Psi(0)=0$ and $\mathbb{P}\big(V_\Psi(T)\geq 0\big)=1$. Thus we have
$$\mathbb{E}_Q[D(T)V_\Psi(T)]=0. \tag{5.6}$$

and $\mathbb{P}\big(V_\Psi(T) < 0\big) = 0$. Since $\mathbb{P}$ and Q are equivalent, one has $Q\big(V_\Psi(T) < 0\big) = 0$. Since $D(t) > 0$, we claim that $Q\big(V_\Psi(T) > 0\big) = 0$. If not, then $Q\big(V_\Psi(T) > 0\big) > 0$ and hence $Q\big(D(t)V_\Psi(T) > 0\big) > 0$ which again implies that $\mathbb{E}_Q[D(T)V_\Psi(T)] > 0$— a contradiction to (5.6). Hence $V_\Psi(t)$ is not an arbitrage. In fact there cannot exist an arbitrage since every portfolio value process $V_\Psi(t)$ satisfying $V_\Psi(0) = 0$ cannot be an arbitrage. □

Example 5.2. *Consider a market* $X(t) = \big(S_0(t), S(t)\big)$ *where*

$$dS_0(t) = 0,\ S_0(0) = 1; \quad dS(t) = 2S(t)\,dt + 3S(t)\,dB(t),\ S(0) = x > 0.$$

Show that the market has no arbitrage.

Solution: *Observe that* $S_0(t) = 1$ *and hence the discount process* $D(t) = 1$. *If we show that there exists a probability measure* Q *such that* $\mathbb{P}$ *and* Q *are equivalent and the discounted stock price* $\tilde{S}(t)$ *is a martingale under* Q, *then from Theorem 5.1, we conclude that given market has no arbitrage. Consider an adapted stochastic process* $u(t)$ *such that*

$$3S(t)u(t) = 2S(t) \implies u(t) = \frac{2}{3}.$$

Then by Girsanov's theorem, $\bar{B}(t) = B(t) + \frac{2}{3}t$ *is a Brownian motion under the new probability measure* Q *where*

$$dQ(\omega) = Z(T)d\,\mathbb{P}(\omega) \ \textit{with}\ Z(t) = \exp\Big\{-\frac{2}{3}B(t) - \frac{2}{9}t\Big\}.$$

One can check easily that Q *and* $\mathbb{P}$ *are equivalent. Moreover,* $S(t)$ *can be rewritten as*

$$dS(t) = 3S(t)\,d\bar{B}(t).$$

Since $D(t) = 1$, *we see that the discounted stock price process is a martingale under* Q. *Thus, the given market has no arbitrage.*

5.1. **Black-Scholes-Metron equation:** Consider a European call option with strike price K and expiry time T. Let $C(t,x)$ denote the value of the call at time t if the stock price at time t is $S(t) = x$. The value of the option is random and it is the stochastic process $C(t,S(t))$. Suppose the stock is geometric Brownian motion and the rate of interest if constant i.e., $\alpha(t) = \alpha$, $\sigma(t) = \sigma$ and $r(t) = r$. By Ito-formula, we have

$$\begin{aligned} dC(t,S(t)) &= C_t(t,S(t))dt + C_x(t,S(t))\,dS(t) + \frac{1}{2}C_{xx}(t,S(t))\,dS(t)\,dS(t) \\ &= \Big\{C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big\}\,dt \\ &\qquad + \sigma S(t)C_x(t,S(t))\,dB(t)\,. \end{aligned}$$

Next we calculate the discounted option price $D(t)C(t,S(t))$. Observe that, here $D(t) = e^{-rt}$. Thus, by Ito product rule, we get

$$\begin{aligned} d\big(e^{-rt}C(t,S(t))\big) &= -re^{-rt}C(t,S(t))\,dt + e^{-rt}dC(t,S(t)) \\ &= e^{-rt}\Big[-rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) \\ &\qquad + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big]\,dt + e^{-rt}\sigma S(t)C_x(t,S(t))\,dB(t)\,. \end{aligned} \tag{5.7}$$

A **hedge** is an investment that reduces the risk in an existing position. A (short option) hedging portfolio starts with some initial capital $V_\Psi(0)$ and invest in the stock and money market account so that the portfolio value $V_\Psi(t)$ at each time $t \in [0,T]$ agrees with the option price $C(t, S(t))$. This will happen if and only if

$$d(e^{-rt}V_\Psi(t)) = d\big(e^{-rt}C(t,S(t))\big), \quad \forall\, t \in [0,T). \tag{5.8}$$

If $\tilde{S}(t)$ is the discounted stock price process, we have seen that

$$d(e^{-rt}V_\Psi(t)) = \psi(t)\, d\tilde{S}(t) = \psi(t)e^{-rt}\big((\alpha - r)S(t)\, dt + \sigma S(t)\, dB(t)\big)\,.$$

Thus, keeping in mind (5.7), we see that (5.8) holds if and only if

$$\begin{aligned}
&\psi(t)\big((\alpha - r)S(t)\, dt + \sigma S(t)\, dB(t)\big)\\
&= \Big[-rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t))\\
&\qquad + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big]\, dt + \sigma S(t)C_x(t,S(t))\, dB(t)\,.
\end{aligned} \tag{5.9}$$

We examine what is required in order for (5.9) to hold. Equating $dB(t)$ terms in (5.9), we have

$$\psi(t) = C_x(t,S(t)).$$

This is called the **delta-hedging rule**. The quantity $C_x(t,S(t))$ is called the **delta** of the option. Again, equating dt-terms in (5.9), and then putting the value of $\psi(t)$, we obtain

$$\begin{aligned}
&C_x(t,S(t))(\alpha - r)S(t)\\
&= -rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big]\\
&\implies C_t(t,S(t)) + rS(t)C_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{xx}(t,S(t)) = rC(t,S(t))\,.
\end{aligned}$$

Therefore, we should seek a continuous function $C(t,x)$ that is a solution of the PDE

$$C_t(t,x) + rxC_x(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) = rC(t,x)\ t \in [0,T),\ x \geq 0, \tag{5.10}$$

with the terminal condition

$$C(T,x) = \max\{x - K, 0\}. \tag{5.11}$$

Suppose we have found this function. If an investor starts with initial capital $V_\Psi(0) = C(0,S(0))$, and uses the hedge $\psi(t) = C_x(t,S(t))$, then the equation (5.9) holds and hence we see that $V_\Psi(t) = C(t,S(t))$. Taking the limit as $t \to T$, and using the fact that $V_\Psi(t)$ and $C(t,S(t))$ is continuous, we have $V_\Psi(T) = C(T,S(T))$. The European call option with strike price K and expiry date T has the pay-off value $C(T,S(T)) = \max\{S(T) - K, 0\}$. Thus, we have $V_\Psi(T) = \max\{S(T) - K, 0\}$. This means that the short position has been successfully hedged. Equations (5.10)-(5.11) is known as **Black-Scholes-Merton** equation.

Boundary conditions: In order to determine the solution, one needs boundary conditions at $x = 0$ and $x = \infty$. Substituting $x = 0$ in (5.10), we have

$$C_t(t,0) = rC(t,0) - -- \text{ an ordinary differential equation.}$$

Solution to this ODE is given by $C(t,0) = e^{rt}C(0,0)$. Again, by substituting $t = T$ into this equation, and the fact that $C(T,0) = \max\{0 - K, 0\}$, we have

$$0 = e^{rT}C(0,0) \implies C(0,0) = 0.$$

Hence one boundary condition is given by

$$C(t,0) = 0 \quad \forall\, t \in [0,T]. \tag{5.12}$$

As $x \to \infty$, the function $C(t,x)$ grows without bound. In such case, we give boundary condition at $x = \infty$ by specifying the rate of growth. One such way to specify a boundary condition at $x = \infty$ for the European call is

$$\lim_{x\to\infty}\Big[C(t,x) - e^{-r(T-t)}K\Big] = 0 \quad \forall\, t \in [0,T]\,. \tag{5.13}$$

Solution of Black-Scholes-Merton equation: Let $h(x) = \max\{0, x-K\}$. Then by discounted Feynman-Kac formula, we see that the function

$$C(t,x) = e^{-r(T-t)}\mathbb{E}[h(Z^{t,x}(T))]$$

solves the PDE (5.10) with terminal condition (5.11), where the stochastic process $Z^{t,x}(s)$ is given by: for $s \ge t$

$$Z^{t,x}(s) = x + r\int_t^s Z^{t,x}(u)\,du + \sigma\int_t^s Z^{t,x}(u)\,dB(u).$$

Observe that $Z^{t,x}(s)$ is a geometric Brownian motion starting at x and time t and we have

$$Z^{t,x}(s) = x\exp\Big\{\big(r - \frac{\sigma^2}{2}\big)(s-t) + \sigma\big(B(s) - B(t)\big)\Big\}.$$

Observe that $\sigma\big(B(T) - B(t)\big) \sim \mathcal{N}(0, \sigma^2(T-t))$ and therefore

$$\sigma\big(B(T) - B(t)\big) = \sigma\sqrt{T-t}Y, \quad Y := \frac{B(T) - B(t)}{\sqrt{T-t}} \sim \mathcal{N}(0,1).$$

Thus,

$$\mathbb{E}[h(Z^{t,x}(T))] = \mathbb{E}\Big[\max\{0, \exp\{\log(x) + \big(r - \frac{\sigma^2}{2}\big)(T-t) + \sigma\sqrt{T-t}Y\} - K\Big]$$

Note that the random variable inside the expectation is zero when Y is such that

$$\log(x) + \big(r - \frac{\sigma^2}{2}\big)(T-t) + \sigma\sqrt{T-t}Y < \log(K) \Leftrightarrow Y < -d_2,$$

where

$$d_2 := \frac{\log(\frac{x}{K}) + \big(r - \frac{\sigma^2}{2}\big)(T-t)}{\sigma\sqrt{T-t}}\,.$$

Thus, we get

$$\mathbb{E}[h(Z^{t,x}(T))] = \int_{-d_2}^{\infty}\Big[\exp\{\log(x) + \big(r - \frac{\sigma^2}{2}\big)(T-t) + \sigma\sqrt{T-t}y\} - K\Big]\phi(y)\,dy$$

$$= xe^{r(T-t)} \int_{-d_2}^{\infty} \exp\{-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}y\}\phi(y)\,dy - K\int_{-d_2}^{\infty} \phi(y)\,dy\,,$$

where ϕ is the probability density function of Y. Since ϕ is symmetry around *zero*, one has

$$K\int_{-d_2}^{\infty} \phi(y)\,dy = KN(d_2),$$

where N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz.$$

By using change of variable $z = y - \sigma\sqrt{T-t}$, we see that

$$\begin{aligned}
&\int_{-d_2}^{\infty} \exp\{-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}y\}\phi(y)\,dy \\
&= \int_{-d^2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(y-\sigma\sqrt{T-t}\right)^2}{2}}\,dy = N(d_2 + \sigma\sqrt{T-t}) = N(d_1)\,,
\end{aligned}$$

where

$$d_1 = d_2 + \sigma\sqrt{T-t}.$$

Therefore, the solution of the Black-Scholes-Merton equation (5.10) with the terminal conditions (5.11) is given by the formula

$$\begin{aligned}
C(t,x) &= e^{-r(T-t)}\mathbb{E}[h(Z^{t,x}(T))] \\
&= e^{-r(T-t)}\Big\{xe^{r(T-t)}N(d_2 + \sigma\sqrt{T-t}) - KN(d_2)\Big\} \\
&= xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big)\,,
\end{aligned} \tag{5.14}$$

where

$$d_\pm(\tau,x) := \frac{1}{\sigma\sqrt{\tau}}\Big[\log(\frac{x}{K}) + \big(r \pm \frac{\sigma^2}{2}\big)\tau\Big].$$

Example 5.3. *Consider a European call option with strike price* \$70 *and 2 years to expiry. The stock price. is* \$50 *and risk-free interest rate is* 8% *per year, and the volatility is* 20% *annually. What is the value of call option.*

Solution: *Here the parameter values are*

$$K = 70,\ r = 0.08,\ T = 2,\ S = 50,\ \sigma = 0.2\,.$$

Thus

$$\begin{aligned}
d_1 &:= \frac{\log(\frac{50}{70}) + \big(0.08 - \frac{(0.02)^2}{2}\big)(2)}{0.2\sqrt{2}} = -0.4825 \\
&\Longrightarrow N(d_1) = 1 - N(-d_1) = 1 - 0.685 = 0.315. \\
N(d_2) &= N(d_1 - 0.2\sqrt{2}) = N(-0.765) = 0.22\,.
\end{aligned}$$

Therefore, the value of call option is given by

$$C = 50N(d_1) - 70e^{-0.16}N(d_2) = 2.63.$$

Greeks: The option pricing formula depends on five parameters namely S, K, T, r and σ. It is important to analyze the change of option price with respect to these parameters. These variations are known as **Greeks**.

Hedging/ delta of a call option: The *delta* of a European call option is the rate of change of its value with respect to the underlying asset price. The number of shares in the hedge is $\psi_H(t) = \partial_x C(t, S(t))$, and thus, we need to calculate $\partial_x C(t,x)$. Now

$$\partial_x C(t,x) = \frac{\partial}{\partial x}\Big[e^{-r(T-t)}\mathbb{E}[h(Z^{t,x}(T))]\Big] = e^{-r(T-t)}\mathbb{E}\Big[h'(Z^{t,x}(T))\frac{\partial}{\partial x}Z^{t,x}(T)\Big].$$

Observe that

$$\frac{\partial}{\partial x}Z^{t,x}(T) = \frac{\partial}{\partial x}\Big[x\exp\Big\{\big(r-\frac{\sigma^2}{2}\big)(s-t)+\sigma\big(B(s)-B(t)\big)\Big\}\Big] = Z^{t,1}(T).$$

Since we are replication a call option, a.e $x \in \mathbb{R}^*$

$$h'(x) = \begin{cases} 1, & \text{if } x > K \\ 0, & \text{if } x < K. \end{cases}$$

Thus,

$$\begin{aligned}
\partial_x C(t,x) &= e^{-r(T-t)}\mathbb{E}\Big[h'(Z^{t,x}(T))Z^{t,1}(T)\Big] \\
&= e^{-r(T-t)}\mathbb{E}\Big[Z^{t,1}(T)\mathbf{1}_{\{Z^{t,x}(T)>K\}}\Big] \\
&= e^{-r(T-t)}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{\big(r-\frac{\sigma^2}{2}\big)(T-t)+\sigma\sqrt{T-t}y-\frac{y^2}{2}\}\,dy \\
&= \int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy = N(d_1).
\end{aligned}$$

Thus, we have proved the following theorem.

Theorem 5.2. *The number of shares of the underlying stock in the hedging portfolio of a call option with strike price K at expiry date T is given by*

$$\psi_H(t) = N(d_1).$$

Remark 5.5. Since $N(\cdot)$ is the cumulative standard normal distribution, and the delta of a European call option $\psi_H(t) = \partial_x C(t, S(t)) = N(d_1)$, we see that

$$0 < \psi_H(t) < 1.$$

Therefore, the value of a European call option is always increasing as the underlying asset price increases. The delta of the put option is also given by the option's first derivative with respect to the underlying asset price. The delta of the put option is given by $\psi_H(t) - 1 < 0$.

Example 5.4. *Consider a European call option as described in the Example 5.3. Compute the delta for this option.*

Solution: *In the Example 5.3, we have seen that $N(d_1) = 0.315$, and hence the delta for this option is 0.315.*

Theta: the time decay factor The theta (Θ) of a European claim with value function $C(t, S(t))$ is the rate of change of option price with respect to the real time i.e.,

$$\Theta = \frac{\partial C}{\partial t}.$$

Thus we have

$$\Theta = xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}.$$

Observe that

$$\begin{aligned} Ke^{-r(T-t)}N'(d_2) &= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} = Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1-\sigma.\sqrt{T-t})^2}{2}} \\ &= Ke^{-r(T-t)}e^{\sigma d_1\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \\ &= Ke^{-r(T-t)}e^{\sigma d_1\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_1). \end{aligned}$$

In view of the definition of d_1 i.e.,

$$d_1 = \frac{\log(\frac{x}{K}) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

we get that

$$e^{\sigma d_1\sqrt{T-t}} = \frac{x}{K}e^{(r+\frac{\sigma^2}{2})(T-t)}$$

and hence we have

$$\begin{aligned} \Theta &= xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{x}{K}e^{(r+\frac{\sigma^2}{2})(T-t)}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_1)\frac{\partial d_2}{\partial t} \\ &= xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - xN'(d_1)\frac{\partial d_2}{\partial t} \\ &= xN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - xN'(d_1)\frac{\partial}{\partial t}\left[d_1 - \sigma\sqrt{T-t}\right] \\ &= -\frac{\sigma x}{2\sqrt{T-t}}N'(d_1) - rKe^{-r(T-t)}N(d_2). \end{aligned}$$

Since both $N(\cdot)$ and $N'(\cdot)$ are positive, theta is always negative and therefore the value of a European call option is a decreasing function of time. Theta is not the same type of hedge parameter as delta. This is because although there is some uncertainty about the future stock price there is no uncertainty about the passage of time. It does not make sense to hedge against the effect of the passage of time on an option portfolio.

Gamma: the convexity factor The gamma (Γ) of a European call option is the sensitivity of delta with respect to asset price. Thus

$$\Gamma = \frac{\partial^2}{\partial x^2}C(t,x) = \frac{\partial}{\partial x}N(d_1) = N'(d_1)\frac{\partial}{\partial x}d_1 = \frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}}.$$

Note that $\Gamma > 0$. If gamma is small, then delta changes only slowly and adjustments in the hedge ration need only be made infrequently. However, if gamma is large, then the hedge ration delta is highly sensitive to changes in the price of the underlying security.

Rho: the interest rate factor It is the rate of change of the value of the financial derivative with respect to the interest rate. For a European call option, rho is given by

$$\rho := \frac{\partial C}{\partial r} = xN'(d_1)\frac{\partial d_1}{\partial r} + K(T-t)e^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r}\,.$$

Since

$$Ke^{-r(T-t)}N'd_2 = xN'(d_1), \quad \frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r},$$

we have

$$\rho = K(T-t)e^{-r(T-t)}N(d_2)\,.$$

ρ is always positive.

Vega: the volatility factor. This is the rate of change of value of the derivative with respect to the volatility of the underlying asset. For a European call option, the vega is given by

$$\begin{aligned}\nu : = \frac{\partial C}{\partial \sigma} &= xN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma} \\ &= xN'(d_1)\Big\{\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\Big\} = xN'(d_1)\sqrt{T-t} \\ &= x\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}.\end{aligned}$$

Vega is always positive. An increase in the volatility will lead to an increase in the call option value.

5.2. **Pricing under the risk-neutral measure:** So far, we have derived the Black-Scholes-Merton equation for the value of a European call by asking what initial capital $V_\Psi(0)$ and portfolio process $\psi(t)$ an agent would need in order to hedge a short position in the call i.e., in order to have

$$V_\Psi(T) = \max\{0, S(T) - K\}.$$

We now ask the following general question: *what initial capital $V_\Psi(0)$ and portfolio process $\psi(t), 0 \le t \le T$, an agent would need in order to hedge a short position to have*

$$V_\Psi(T) = V(T) \ a.s. \tag{5.15}$$

where $V(T)$ is a $\mathcal{F}_T$-measurable given random variable.

Suppose our agent wishes to choose initial capital $V_\Psi(0)$ and portfolio strategy $\psi(t), 0 \le t \le T$ such that (5.15) holds. Let Q be the risk-neutral measure. Then the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$ is a martingale under Q i.e,

$$D(t)V_\Psi(t) = \mathbb{E}_Q\big[D(T)V_\Psi(T)\big|\mathcal{F}_t\big] = \mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_t\big], \tag{5.16}$$

where in the last equality, we have used (5.15). The value $V_\Psi(t)$ of the hedging portfolio in (5.16) is the capital required at time t in order to successfully complete the hedge of the short position in the derivative security with payoff $V(T)$. We call this price of the derivative security at time by $V(t)$. Thus we have

$$D(t)V(t) = \mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_t\big], \ 0 \le t \le T.$$

Since $D(t)$ is $\mathcal{F}_t$-measurable and bounded, by using the properties of conditional expectation and the definition of discounted process $D(t)$, we get

$$V(t) = \mathbb{E}_Q\Big[\exp\{-\int_t^T r(s)\,ds\}V(T)\Big|\mathcal{F}_t\Big]. \tag{5.17}$$

Equation (5.17) is known as **risk-neutral pricing formula**. Therefore, we determine the *correct initial capital* to be

$$V(0) = \mathbb{E}_Q\big[D(T)V(T)\big]$$

and the value of the hedging portfolio at every time t, $0 \le t \le T$, to be $V(t)$ given in the risk-neutral pricing formula (5.17). It remains to find the portfolio value process $\psi(t)$. **To do so, we consider the following assumption: the filtration $\{\mathcal{F}_t\}$ is generated by the Brownian motion $B(t) : 0 \le t \le T$.** We first claim that the process $D(t)V(t)$ is a Q-martingale. Indeed, for $0 \le s \le t \le T$, one has

$$\mathbb{E}_Q\big[D(t)V(t)\big|\mathcal{F}_s\big] = \mathbb{E}_Q\Big[\mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_t\big]\Big|\mathcal{F}_s\Big] = \mathbb{E}_Q\big[D(T)V(T)\big|\mathcal{F}_s\big] = D(s)V(s).$$

Observe that the filtration $\{\mathcal{F}_t\}$ is generated by $B(t) : 0 \le t \le T$, not the Q-Brownian motion $\tilde{B}(t) : 0 \le t \le T$. Still the martingale representation theorem holds for $D(t)V(t)$ (one needs to work to show this, see e.g. Assignment-4). Hence there exists a adapted process $\tilde{f}(t) : 0 \le t \le T$ such that

$$D(t)V(t) = D(0)V(0) + \int_0^t \tilde{f}(d)\,d\tilde{B}(s), \quad 0 \le t \le T.$$

Since $D(0)=1$, we have

$$D(t)V(t) = V(0) + \int_0^t \tilde{f}(s)\, d\tilde{B}(s), \quad 0 \le t \le T\,. \tag{5.18}$$

Again, we know that

$$\tilde{V}_\Psi(t) := D(t)V(t) = V_\Psi(0) + \int_0^t \psi(s)\sigma(s)D(s)S(s)\, d\tilde{B}(s)\,. \tag{5.19}$$

Let us assume that the volatility $\sigma(t)$ does not vanishes. In order to have $V_\Psi(t) = V(t)$ for all t, we should have, comparing to (5.18) and (5.19)

$$V_\Psi(0) = V(0)$$

and choose the $\psi(t)$ to satisfy

$$\begin{aligned} \psi(s)\sigma(s)D(s)S(s) &= \tilde{f}(s) \\ \implies \psi(s) &= \frac{\tilde{f}(s)}{\sigma(s)D(s)S(s)}, \quad 0 \le s \le T. \end{aligned}$$

With these choices, we have a hedge for a short position in the derivative security with payoff $V(T)$ at time T.

Remark 5.6. i): Our key assumption is that the volatility $\sigma(t)$ is not zero. If the volatility vanishes, then the randomness of the Brownian motion does not enter the stock, although it may still enter in the payoff $V(T)$ of the derivative security. In this case, the stock is no longer an effective hedging instrument.
ii): There is no randomness in the derivative security apart from the Brownian motion randomness, which can be hedged by trading the stock.

Under these two assumptions, every $\mathcal{F}_T$-measurable derivative security can be hedged. Such a model is said to be **complete**.

5.3. **Put-Call Parity and pricing formula for European put option:** Consider a model with a unique risk-neutral measure Q and constant interest rate r. According to the risk-neutral pricing formula, the price at time t of a European call expiring at time T is

$$C(t) = \mathbb{E}_Q\Big[e^{-r(T-t)}\max\{0, S(T)-K\}\big|\mathcal{F}_t\Big]$$

where $S(T)$ is the underlying asset price at time T and K is the strike price of the call. Similarly, the price at time t of a European put expiring at time T is

$$P(t) = \mathbb{E}_Q\Big[e^{-r(T-t)}\max\{0, K-S(T)\}\big|\mathcal{F}_t\Big]\,.$$

Since

$$x - K = \max\{0, x-K\} - \max\{0, K-x\},$$

one has

$$\begin{aligned} C(t) &= e^{-r(T-t)}\mathbb{E}_Q\Big[(S(T)-K)\big|\mathcal{F}_t\Big] + e^{-r(T-t)}\mathbb{E}_Q\Big[\max\{0, K-S(T)\}\big|\mathcal{F}_t\Big] \\ &= e^{rt}\mathbb{E}_Q\Big[(D(T)S(T)\big|\mathcal{F}_t\Big] - Ke^{-r(T-t)} + P(t)\,. \end{aligned}$$

Since the discounted stock price $\tilde{S}(t) := D(t)S(t)$ is a martingale under Q, one has $\mathbb{E}_Q\Big[(D(T)S(T)\big|\mathcal{F}_t\Big] = e^{-rt}S(t)$ and hence

$$C(t) - P(t) = S(t) - Ke^{-r(T-t)}.$$

A forward contract with delivery price K obligates its holder to buy one share of the stock at expiration time T in exchange for payment K. At the expiration, the value of forward contract is $S(T) - K$. Let $f(t,x)$ denotes the value of forward contract at earlier time $t \in [0,T]$ if the stock price at time t is $S(t) = x$. Then

$$f(t,x) = x - Ke^{-r(T-t)}.$$

Since the value at time T of the forward contact agrees with the value of the portfolio that is long one at call and short one put, we must have

$$f(t,x) = C(t,x) - P(t,x), \quad x \geq 0, \quad 0 \leq t \leq T. \tag{5.20}$$

The relation (5.20) is called **Put-Call-Parity** relation.

5.3.1. **Black-Scholes-Merton put formula:** We have deduced Black-Scholes-Merton call formula for a European call option. By using Put-Call-Parity formula, we now ready to deduce the Black-Scholes-Merton put formula. Indeed, we have

$$\begin{aligned} P(t,x) &= C(t,x) - f(t,x) \\ &= xN(d_1) - Ke^{-r(T-t)}N(d_2) - x + Ke^{-r(T-t)} \\ &= -x\big\{1 - N(d_1)\big\} + Ke^{-r(T-t)}\big\{1 - N(d_2)\big\} \\ &= -xN(-d_1) + Ke^{-r(T-t)}N(-d_2)\,. \end{aligned} \tag{5.21}$$

Hedging/ delta of a put option: From the Put-Call-Parity formula, we see that

$$P_x(t,x) = C_x(t,x) - 1 = N(d_1) - 1 < 0.$$

Therefore, the value of a European put option is decreasing as the underlying asset price increases.

Gamma of a put option: The gamma of a European put option, denoted by (Γ_P), is given by

$$\Gamma_P := P_{xx}(t,x) = C_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_1).$$

Theta of a put option: The theta of a European put option, denoted by (θ_P), is given by

$$\begin{aligned} \theta_P : &= P_t(t,x) = C_t(t,x) + rKe^{-r(T-t)} \\ &= -\frac{\sigma x}{2\sqrt{T-t}}N'(d_1) - rKe^{-r(T-t)}N(d_1) + rKe^{-r(T-t)} \\ &= rKe^{-r(T-t)}N(-d_2) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_1)\,. \end{aligned}$$

Rho of a put option: The rho of a European put option ρ_P is given by

$$\rho_P = P_r(t,x) = C_r(t,x) = K(T-t)e^{-r(T-t)}N(d_2) > 0.$$

Vega of a put option: The vega of a European put option ν_P is given by

$$\nu_P = P_\sigma(t,x) = C_\sigma(t,x) = x\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}.$$

Vega is always positive. An increase in the volatility will lead to an increase in the put option value.

Remark 5.7. The value function $f(t,x)$ of a forward contract satisfies the same Black-Scholes-Merton PDE satisfied by $C(t,x)$. Indeed, since $f(t,x) = x - Ke^{-r(T-t)}$, one has

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\Big) f(t,x)\\ &= rKe^{-r(T-t)} + \frac{1}{2}\sigma^2 x^2 \cdot 0 + rx \cdot 1 - r\big(x - Ke^{-r(T-t)}\big) = 0\,.\end{aligned}$$

By the Put-Call-Parity formula, we also conclude that the value function of a European put option $P(t,x)$ satisfies the same Black-Scholes-Merton PDE satisfied by $C(t,x)$, European call option.

5.3.2. **Chooser option:** A chooser option on a stock is a derivative that gives holders the right to choose at a contracted time $t < T$ if they want to call or put option.

If the exercise time of the chooser option is T with strike price K, the holder will either have a call or a put at time T depending on the choice made at the earlier time t. We are going to derive the arbitrage-free price at time 0 for this option.

Let $V_c(t)$ and $V_p(t)$ be the price at time t for a call and put option respectively with strike price K and exercise time T. The holder of the chooser option will take the call option at time t if $V_c(t) \geq V_p(t)$, and the put option otherwise. The payoff function at time T is

$$X := \max\{0, S(T) - K\}\mathbf{1}_{\{V_c(t)\geq V_p(t)\}} + \max\{0, K - S(T)\}\mathbf{1}_{\{V_c(t)<V_p(t)\}}\,.$$

From the previous theory, we find the arbitrage-free price $V(0)$ at time of entering the chooser option to be

$$V(0) = e^{-rT}\mathbb{E}_Q[X],$$

where Q is the risk-neutral measure. Our task is to calculate this price. Observe that X can be re-written as

$$X = \max\{0, S(T) - K\} + (K - S(T))\mathbf{1}_{\{V_c(t)<V_p(t)\}}.$$

Thus, we have

$$\begin{aligned}
V(0) &= e^{-rT}\mathbb{E}_Q\big[\max\{0, S(T) - K\}\big] + e^{-rT}\mathbb{E}_Q\big[(K - S(T))\mathbf{1}_{\{V_c(t)<V_p(t)\}}\big] \\
&= e^{-rT}\mathbb{E}_Q\big[\max\{0, S(T) - K\}\big] + \mathbb{E}_Q\big[e^{-rT}(K - S(T))\mathbf{1}_{\{V_c(t)<V_p(t)\}}\big] \\
&= e^{-rT}\mathbb{E}_Q\big[\max\{0, S(T) - K\}\big] + \mathbb{E}_Q\Big[\mathbb{E}_Q\big[e^{-rT}(K - S(T))\mathbf{1}_{\{V_c(t)<V_p(t)\}}\big|\mathcal{F}_t\big]\Big] \\
&\equiv \mathcal{A}_1 + \mathcal{A}_2\,.
\end{aligned}$$

We recognize $\mathcal{A}_1$ as the price of a call option with strike price K at exercise time T, denoted it by $V_c(0; K, T)$. We need to calculate $\mathcal{A}_2$. From the Black-Scholes-Merton formula, we see that $V_c(t)$ is $\mathcal{F}_t$-adapted. By using Put-Call-Parity formula, we can say that $V_p(t)$ is $\mathcal{F}_t$-adapted. In conclusion, the random variable $\mathbf{1}_{\{V_c(t)<V_p(t)\}}$ is only depends on the stock price at time t, and therefore $\mathcal{F}_t$-adapted. Thus, $\mathcal{A}_2$ can be written as

$$\begin{aligned}
\mathcal{A}_2 &= \mathbb{E}_Q\Big[\mathbf{1}_{\{V_c(t)<V_p(t)\}}\mathbb{E}_Q\big[e^{-rT}(K - S(T))\big|\mathcal{F}_t\big]\Big] \\
&= \mathbb{E}_Q\Big[\mathbf{1}_{\{V_c(t)<V_p(t)\}}\big\{Ke^{-rT} - \mathbb{E}_Q\big[D(T)S(T)\big|\mathcal{F}_t\big]\big\}\Big] \\
&= \mathbb{E}_Q\Big[\mathbf{1}_{\{V_c(t)<V_p(t)\}}\big\{Ke^{-rT} - e^{-rt}S(t)\big\}\Big] \\
&= e^{-rt}\mathbb{E}_Q\Big[\big(Ke^{-r(T-t)} - S(t)\big)\mathbf{1}_{\{V_c(t)<V_p(t)\}}\Big]\,.
\end{aligned}$$

Let $\omega \in \Omega$ such that $V_c(t,\omega) < V_p(t,\omega)$. Then from the Put-Call-Parity formula, $S(t,\omega) < Ke^{-r(T-t)}$. Moreover, the two events

$$\{\omega \in \Omega : V_c(t,\omega) < V_p(t,\omega)\}, \quad \{\omega \in \Omega : S(t,\omega) < Ke^{-r(T-t)}\}$$

are identical. Therefore, we get

$$\begin{aligned}
\mathcal{A}_2 &= e^{-rt}\mathbb{E}_Q\Big[\big(Ke^{-r(T-t)} - S(t)\big)\mathbf{1}_{\{S(t)<Ke^{-r(T-t)}\}}\Big] \\
&= e^{-rt}\mathbb{E}_Q\Big[\max\{0, Ke^{-r(T-t)} - S(t)\}\Big]\,.
\end{aligned}$$

We recognize this term as the price of a put option with exercise time t and strike price $Ke^{-r(T-t)}$, denoted it by $V_p(0; Ke^{-r(T-t)}, t)$. Therefore, we have

$$V(0) = V_c(0; K, T) + V_p(0; Ke^{-r(T-t)}, t)\,.$$

Thus we arrive at the following conclusion: *the chooser option. has an arbitrage-free price which is a sum of a call option with exercise K at time T and a put option with exercise time t and strike price $Ke^{-r(T-t)}$. Moreover, one can derive the Black-Scholes formula for this option. Indeed, by Put-Call-Parity formula, one has*

$$V_p(0; Ke^{-r(T-t)}, t) = V_c(0; Ke^{-r(T-t)}, t) - S(0) + Ke^{-rT}.$$

Hence, we have

$$V(0) = V_c(0; K, T) + V_c(0; Ke^{-r(T-t)}, t) - S(0) + Ke^{-rT}.$$

The price of the two call options $V_c(0; K, T)$ and $V_c(0; Ke^{-r(T-t)}, t)$ can be expressed by the Black-Scholes-Merton formula. In particular, one has

$$V_c(0; Ke^{-r(T-t)}, t) = S(0)N(\bar{d}_1) - Ke^{-rT}N(\bar{d}_2),$$

where

$$\bar{d}_2 := \frac{\log(\frac{S(0)}{Ke^{-r(T-t)}}) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \quad \bar{d}_1 = \bar{d}_2 + \sigma\sqrt{t}\,.$$

Remark 5.8. The payoff function of the chooser option can be written as

$$X = \max\{0, K - S(T)\} + (S(T) - K)\mathbf{1}_{\{V_c(t) \geq V_p(t)\}}.$$

Hence, a similar argument reveals that

$$V(0) = V_p(0) + e^{-rt}\mathbb{E}_Q\Big[\max\{0, S(t) - Ke^{-r(T-t)}\}\Big].$$

The last term is the price of a call option at time 0 with strike $Ke^{-r(T-t)}$ and exercise time t. In other words,

$$V(0) = V_p(0) + V_c(0; Ke^{-r(T-t)}, t).$$

5.4. **Implied Volatility:** *We have discussed how to estimate the volatility σ from the historical stock prices namely*

$$\sigma^2 \approx \frac{1}{T_2 - T_1}\sum_{j=0}^{n-1}\Big(\log\big(\frac{S(t_{j+1})}{S(t_j)}\big)\Big)^2$$

when we observe geometric Brownian motion $S(t)$ for $0 \leq T_1 \leq t \leq T_2$ and $\Pi = \{T_1 = t_0 < t_1 < \ldots < t_n = T_2\}$ is a partition of $[T_1, T_2]$. Choosing an appropriate value for n is not easy because σ does change over time and data that are too old may not be relevant for the present or the future.

The volatility is the only parameter which is unknown to us when pricing a call option contract. Suppose we know that a call option with strike price K and time to exercise T is traded for a price p in the market. At the same time, we read of from the stock exchange monitor that the underlying stock is traded for price s. Hence from the Black-Scholes-Metron equation, we have

$$p = sN(\mathrm{d}_1) - Ke^{-rT}N(\mathrm{d}_2)$$

where

$$\mathrm{d}_1 = \mathrm{d}_2 + \sigma\sqrt{T}, \quad \mathrm{d}_2 := \frac{\log(\frac{s}{K})) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Since only unknown is σ, we can solve for this and find the volatility used by the market. Ideally, this should coincide (at least approximately) with the historical volatility, but this is rarely the case. Since this volatility is derived from the actual option prices, we call it the **Implied volatility**. *Since explicit formula for σ is not possible, one needs to use numerical method to solve for σ. Such an efficient method is the* Newton-Rapshon method, *an iterative method to solve the equation*

$$F(\sigma, s, K, r, T) - p = 0$$

where $F(\cdot)$ is the pricing model that depends on σ. From the initial guess of σ_0, the iteration function is

$$\sigma_{i+1} = \sigma_i F(\sigma_i)\frac{\partial F}{\partial \sigma}(\sigma_i).$$

The derivative $\frac{\partial F}{\partial \sigma}$ is known as vega and therefore for European option

$$\frac{\partial F}{\partial \sigma}(\sigma_i) = s\sqrt{T}e^{-rT}N(\mathrm{d}_1^i)$$

where d_1^i is given by

$$\mathrm{d}_1^i := \frac{\log(\frac{s}{K})) + (r + \frac{1}{2}\sigma_i^2)T}{\sigma\sqrt{T}}.$$

With the help of software such as MATLAB or Mathematica, one can solve the above equation.

Remark 5.9. In general volatility need not be constant and could depend on the price of the underlying stock. These models are known as **Stochastic Modelling Approach** for mathematical finance.