

2101-MTL733 Minor Exam

Navneel Singhal

TOTAL POINTS

35 / 40

QUESTION 1

1 Q1 6.5 / 11

✓ + 6 pts Correct i)

+ 3 pts partial_1, part i): writing down the process in appropriate (double integral) form

+ 3 pts partial_2, part i): For Ito product rule and getting final result

+ 2 pts correct ii)

+ 3 pts correct iii)

+ 2 pts partial_1, part iii): writing correct expression for second moment

+ 1 pts partial_2, part iii): correct final answer

+ 0 pts Incorrect/ Not attempting

+ 0.5 Point adjustment

💬 For the effort towards iii), I am giving 0.5 marks

QUESTION 2

2 Q2 10 / 10

+ 3 pts correct i)

✓ + 1 pts partial_1, part i): choosing appropriate Ito process

✓ + 2 pts partial_2, part i): Application of Ito formula and for final result

✓ + 4 pts correct ii)

+ 3 pts partial_1, part ii): deriving the equation for $e(t)$

+ 1 pts partial_2, part ii): finding appropriate condition to show the required limit

✓ + 3 pts correct iii)

+ 0 pts Incorrect/not attempting

+ 0.5 pts Click here to replace this description.

- 0.5 pts Reason is not given

+ 1 pts Click here to replace this description.

+ 0 pts Click here to replace this description.

QUESTION 3

3 Q3 11 / 11

✓ + 3 pts Correct a)

+ 1 pts partial_1, a): considering appropriate Ito process

+ 2 pts partial_2, a): for Ito-formula and final result

✓ + 4 pts correct b)

✓ + 4 pts correct c)

+ 1.5 pts partial_1, c): showing $A_1 = A_2$

+ 1.5 pts partial_2, c): for Ito-isometry

+ 1 pts partial_3, c): showing $f=g$

+ 0 pts Incorrect/Not attempting

QUESTION 4

4 Q4 7.5 / 8

✓ + 4 pts Correct a)

+ 2 pts partial_1, part a):

✓ + 2 pts Correct b)

+ 1 pts partial_1, part b): finding quadratic variation correctly

✓ + 2 pts correct c)

+ 0 pts Incorrect/ Not attempting

+ 1 pts Click here to replace this description.

✓ - 0.5 pts Click here to replace this description.

Q1:

$$(i) \quad dX(t) = \frac{-X(t)}{1-t} dt + dB(t) \quad - (1)$$

$$\text{let } f(t, x) = \frac{x}{1-t} \in C^2([0, 1) \times \mathbb{R})$$

Using Ito's formula,

$$d(f(t, X(t))) = f_x(t, X(t)) dX(t) + \frac{1}{2} f_{xx}(t, X(t)) dX(t) \cdot dX(t) + f_t(t, X(t)) dt$$

$$\begin{aligned} \text{From (1), note that } dX(t) \cdot dX(t) &= (g(t) dt + dB(t)) (g(t) dt + dB(t)) \\ &= dB(t) \cdot dB(t) + 0 \\ &= dt \quad (\text{where } g(t) = \frac{-X(t)}{1-t}) \end{aligned}$$

$$\begin{aligned} \Rightarrow d\left(\frac{X(t)}{1-t}\right) &= \frac{1}{1-t} dX(t) + \frac{1}{2} \cdot 0 \cdot dX(t) \cdot dX(t) + \frac{X(t)}{(1-t)^2} dt \\ &= \frac{1}{1-t} \left(dX(t) + \frac{X(t)}{1-t} dt \right) \\ &= \frac{1}{1-t} dB(t) \end{aligned}$$

So using the integral form, we get

$$\frac{X(t)}{1-t} - \frac{X(0)}{1-0} = \int_0^t \frac{1}{1-s} dB(s)$$

$$\Rightarrow X(t) = (1-t) \int_0^t \frac{1}{1-s} dB(s)$$

(ii) to check if x is normally distributed, we compute the MGF of this random variable at any time t .

Let $Y(t) = e^{uX(t)}$. We need to find $E[Y(t)]$

$$\begin{aligned} dY(t) &= u e^{uX(t)} dX(t) + \frac{1}{2} u^2 e^{uX(t)} dX(t) \cdot dX(t) \\ &= \left[\frac{1}{2} u^2 + u \left(\frac{-X(t)}{1-t} \right) \right] e^{uX(t)} dt - u e^{uX(t)} dB(t) \end{aligned}$$

$$E[Y(t)] = \frac{1}{2} u^2 \int_0^t E[e^{uX(s)}] ds - \frac{u}{1-t}$$

integrates to martingale with expectation 1.

(i) Since $\frac{1}{1-s}$ is a non-random variable, we can apply integration by parts to get

$$\int_0^t \frac{1}{1-s} dB(s) = \frac{1}{1-t} B(t) - \int_0^t \frac{B(s)}{(1-s)^2} ds.$$

(ii) Since $X(t)$ only has a diffusion component, ~~it is~~ it is a normally distributed process.

$$dX(t) \cdot dX(t) = dt \Rightarrow [X, X](t) = t.$$

$$X(t) = (1-t) \int_0^t \frac{1}{1-s} dB(s) \Rightarrow X(t) \text{ has continuous sample paths a.s.}$$

$$X(0) = 0$$

Since $\frac{1}{1-s}$ is a non-random variable, we can apply integration by parts to get

$$\int_0^t \frac{1}{1-s} dB(s) = \frac{1}{1-t} B(t) - \int_0^t \frac{B(s)}{(1-s)^2} ds.$$

$$\Rightarrow X(t) = B(t) - \int_0^t \frac{B(s)(1-t)}{(1-s)^2} ds.$$

So we only need to show that the second term is a martingale (this will give that $X(t) \sim N(0, t)$)

$$\begin{aligned} & E \left[\int_0^t \frac{B(z)(1-t)}{(1-z)^2} dz \mid \mathcal{F}_s \right] \\ &= \int_0^s \frac{B(z)(1-t)}{(1-z)^2} dz + E \left[\int_s^t \frac{\overbrace{(B(z) - B(s) + B(s))}^{\text{independent}} \overbrace{(1-t)}^{\text{measurable}}}{(1-z)^2} dz \mid \mathcal{F}_s \right] \\ &= \int_0^s \frac{B(z)(1-t)}{(1-z)^2} dz + B(s) \int_s^t \frac{1}{(1-z)^2} dz \\ &= \int_0^s \frac{(1-t)B(z)}{(1-z)^2} dz + B(s)(1-t) \left(\frac{1}{1-t} - \frac{1}{1-s} \right) \\ &= \int_0^s \frac{B(z)(1-s)}{(1-z)^2} dz \end{aligned}$$

$\Rightarrow X(t)$ is a martingale. so we are done.

$$(iii). d(X^2(t)) = 2X dX + \cancel{dt} \cdot dX \cdot dX.$$

$$= \frac{-2X^2}{1-t} dt + \cancel{dt} + \underbrace{dB}_0$$

taking expectations, if $E[X^2(t)] = f(t)$, we have

$$f' = -\frac{2f}{1-t} + 1$$

$$X(0) = 0$$

$$\Rightarrow f(0) = 0.$$

~~f = ...~~

$$(1-t) \frac{df}{dt} = (-2f + 1) dt$$

$$(1-t)^2 df + (2f) (1-t) dt = (1-t) dt$$

$$\frac{f'}{(1-t)^2} + \frac{2f}{(1-t)^3} = \frac{1}{(1-t)^2}$$

$$\Rightarrow \frac{f(t)}{(1-t)^2} - \frac{f(0)}{(1-0)^2} = \left(\frac{1}{(1-t)^2} - 1 \right)$$

$$\Rightarrow f(t) = (1-t)^2 f(0) + 1 - (1-t)^2$$

$$\text{as } t \rightarrow 1, f(t) \rightarrow 1$$

$$\Rightarrow \boxed{\lim_{t \rightarrow 1^-} E[X^2(t)] = 1}$$

1 Q1 6.5 / 11

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💬 For the effort towards iii), I am giving 0.5 marks

Q2:

(i) According to the multi-dimensional Ito formula,
we have, if $X(t) = f(t, B_1(t), B_2(t), \dots, B_m(t))$

$$\begin{aligned} dX(t) &= \frac{\partial f}{\partial t}(t, B_1(t), \dots, B_m(t)) dt \\ &+ \sum_{i=1}^m \frac{\partial f}{\partial x_i}(t, B_1(t), \dots, B_m(t)) dB_i(t) \\ &+ \sum_{i=1}^m \frac{1}{2} \frac{\partial^2 f}{\partial x_i^2}(t, B_1(t), \dots, B_m(t)) dB_i(t) \cdot dB_i(t). \end{aligned}$$

Here

$$f(t, x_1, \dots, x_m) = e^{ct + \sum_{j=1}^m \alpha_j x_j}$$

$$\frac{\partial f}{\partial t} = c \cdot e^{ct + \sum_{j=1}^m \alpha_j x_j} = cf$$

$$\frac{\partial f}{\partial x_i} = \alpha_i \cdot e^{ct + \sum_{j=1}^m \alpha_j x_j} = \alpha_i f$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i} (\alpha_i f) = \alpha_i \frac{\partial f}{\partial x_i} = \alpha_i^2 f$$

$$\begin{aligned} \Rightarrow dX(t) &= c X(t) dt + \sum_{i=1}^m \alpha_i X(t) dB_i(t) \\ &+ \sum_{i=1}^m \frac{1}{2} \alpha_i^2 X(t) dt \\ &= \left(c + \frac{1}{2} \sum_{j=1}^m \alpha_j^2 \right) X(t) dt + X(t) \left(\sum_{j=1}^m \alpha_j dB_j(t) \right) \end{aligned}$$

(ii) upon integrating this equation (writing it in an integral form, to be precise), the second part integrates to an Ito integral, which is a martingale and has expectation 0.

$$E[X(t) - X(0)] = E \left[\int_0^t \underbrace{\left(c + \frac{1}{2} \sum_{j=1}^m \alpha_j^2 \right) X(s)}_{A \text{ (say)}} ds \right]$$

$X(0) = 1$ a.s. (by defⁿ of B_j 's).

$$\Rightarrow E[X(t)] = 1 + E \left[\int_0^t A X(s) ds \right] = 1 + A \int_0^t E[X(s)] ds$$

$$\Rightarrow e(t) = 1 + A \int_0^t e(s) ds$$

differentiating, we get $e'(t) = A e(t)$

$$\Rightarrow e(t) = k e^{At} \text{ for some } k > 0 \quad (e(0) = 1, \text{ so } k = 1)$$

So for $\lim_{t \rightarrow \infty} c(t) = 0$, we need $A < 0$,

i.e.,
$$c < -\frac{1}{2} \sum_{j=1}^m \alpha_j^2$$

(iii). $X(t) = UB(t)$ $U = \begin{pmatrix} \cos(x_0) & \sin(x_0) \\ -\sin(x_0) & \cos(x_0) \end{pmatrix}$

~~$X_i(t) = \cos(x_0) B_1(t) + \sin(x_0) B_2(t)$~~

Let $\theta_i = x_0 + \frac{\pi}{2} i$

Then $X_i(t) = \cos(\theta_i) B_1(t) + \sin(\theta_i) B_2(t)$

1) Note that $X_i(t)$ is a martingale since it is a linear combination (with const. coeff) of $B_1(t)$ & $B_2(t)$ (both of which are martingales).

2) We have $X_i(t)$ having continuous sample paths a.s. since both $B_1(t)$ & $B_2(t)$ have continuous sample paths.

3). $X_i(0) = \cos(\theta_i) B_1(0) + \sin(\theta_i) B_2(0) = 0$

4).
$$\begin{aligned} dX_i(t) \cdot dX_i(t) &= \cos^2(\theta_i) dB_1(t) \cdot dB_1(t) \\ &\quad + \sin^2(\theta_i) dB_2(t) \cdot dB_2(t) \\ &\quad + 2 \sin \theta_i \cos \theta_i dB_1(t) dB_2(t) \\ &\quad + \cos \theta_i \sin \theta_i dB_2(t) dB_1(t) \\ &= \cos^2(\theta_i) dt + \sin^2(\theta_i) dt + 0 \\ &= dt \end{aligned}$$

$\Rightarrow [X_i, X_i](t) = \int_0^t 1 ds = t \quad \forall t$

5)
$$\begin{aligned} dX_1(t) \cdot dX_2(t) &= (\cos(x_0) dB_1(t) + \sin(x_0) dB_2(t)) \\ &\quad \cdot (-\sin(x_0) dB_1(t) + \cos(x_0) dB_2(t)) \\ &= -\sin(x_0) \cos(x_0) dB_1(t) \cdot dB_1(t) + \sin(x_0) \cos(x_0) dB_2(t) \cdot dB_2(t) \\ &\quad + \cos^2(x_0) dB_1(t) dB_2(t) - \sin^2(x_0) dB_2(t) dB_1(t) \\ &= -\sin(x_0) \cos(x_0) dt + \sin(x_0) \cos(x_0) dt + 0 \\ &= 0 \end{aligned}$$

$\Rightarrow [X_1, X_2](t) = 0 \quad \forall t$

Similarly $[X_2, X_1](t) = 0 \quad \forall t$

Hence this is a 2-D brownian motion by Levy's Characterisation

2 Q2 10 / 10

- + 3 pts correct i)
- ✓ + 1 pts partial_1, part i): choosing appropriate Ito process
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- ✓ + 3 pts correct iii)
 - + 0 pts Incorrect/not attempting
 - + 0.5 pts [Click here to replace this description.](#)
 - 0.5 pts Reason is not given
 - + 1 pts [Click here to replace this description.](#)
 - + 0 pts [Click here to replace this description.](#)

Q3: (a) $dY(t) = \alpha Y(t) dt + \sigma Y(t) dB(t)$

$$\xi(t) = e^{-\theta B(t) - (r + \frac{1}{2}\theta^2)t}$$

$$\theta = \frac{\alpha - r}{\sigma}$$

let $f(t, x) = e^{-\theta x - (r + \frac{1}{2}\theta^2)t}$

By Ito's formula, we have

$$d\xi(t) = f_t(t, B(t)) dt + f_x(t, B(t)) dB(t) + \frac{1}{2} f_{xx}(t, B(t)) \underbrace{dB(t) dB(t)}_{dt}$$

$$f_t(t, x) = -\left(r + \frac{1}{2}\theta^2\right) e^{-\theta x - (r + \frac{1}{2}\theta^2)t}$$

$$= -\left(r + \frac{1}{2}\theta^2\right) f(t, x)$$

$$f_x(t, x) = -\theta f(t, x)$$

$$f_{xx}(t, x) = \theta^2 f(t, x)$$

$$\Rightarrow d\xi(t) = -\left(r + \frac{1}{2}\theta^2\right) \xi(t) dt - \theta \xi(t) dB(t) + \frac{1}{2} \theta^2 \xi(t) dt$$

$$= -r \xi(t) dt - \theta \xi(t) dB(t) \quad \text{as needed}$$

(b): $dX(t) = rX(t) dt + \gamma(t)(\alpha - r)Y(t) dt + \sigma\gamma(t)Y(t) dB(t)$

$$Z(t) = \xi(t)X(t)$$

For the sake of convenience, I will drop the (t) here

$$dX = rX dt + \gamma(\alpha - r)Y dt + \sigma\gamma Y dB$$

$$Z = \xi X$$

By product rule, we have

$$dZ = \xi dX + X d\xi + d\xi \cdot dX$$

$$\rightarrow d\xi \cdot dX = (-r\xi dt - \theta\xi dB) \cdot (rX dt + \gamma(\alpha - r)Y dt + \sigma\gamma Y dB)$$

$$= -\theta\xi\gamma\sigma Y dt \quad (\text{the rest of the terms are 0})$$

$$\Rightarrow dZ = \underbrace{(\xi rX + \xi \gamma(\alpha - r)Y)}_{\text{rest of terms are 0}} dt + \sigma\xi\gamma Y dB - \underbrace{X r \xi dt + X \theta \xi dB + \theta \xi \gamma \sigma Y dt}_{\text{rest of terms are 0}}$$

Collecting all coefficients of dt , we have

$$dz = a dt + g dB(t)$$

$$\text{where } a = \xi \left[\cancel{g} + \gamma(\alpha - \lambda)Y \cancel{a} - \cancel{X} - \theta \gamma \sigma Y \right]$$

Now $\theta \sigma = \alpha - \lambda$, so

$$a = \xi \gamma Y [\alpha - \lambda - \theta \sigma] = 0$$

$$\Rightarrow dz = g dB(t) \text{ for some } g. (= (\sigma \xi \gamma Y - X \sigma \xi))$$

$$Z(t) = Z(0) + \int_0^t g dB(s)$$

Since $\int_0^t g dB(s)$ is an Ito integral, it is a martingale

$\Rightarrow Z(t)$ is a martingale.

(c). We have

$$\begin{aligned} A_2 - A_1 &= \int_0^T f(t) dB(t) - \int_0^T g(t) dB(t) \\ &= \int_0^T [f(t) - g(t)] dB(t). \quad \text{P-a.s.} \end{aligned}$$

Using Ito isometry, we have

$$\begin{aligned} (A_2 - A_1)^2 &= E[(A_2 - A_1)^2] \\ &= E\left[\left(\int_0^T [f(t) - g(t)] dB(t)\right)^2\right] \\ &= E\left[\int_0^T [f(t) - g(t)]^2 dt\right] \end{aligned}$$

So showing $A_1 = A_2$ is equivalent to showing that $f(t, \omega) = g(t, \omega)$ for a.a. $(t, \omega) \in [0, T] \times \Omega$.

~~Using derivative at $t=T$~~

Suppose $f(t, \omega) \neq g(t, \omega)$ for a.a. $(t, \omega) \in [0, T] \times \Omega$.
then ~~consider the set of (t, ω) where~~

RHS is an Ito integral evaluated at T , so expectation should be 0. since it is a martingale & integral at $t=0$ is 0.

$$\text{So } E[A_2 - A_1] = 0 \Rightarrow A_1 = A_2.$$

\Rightarrow We are done.

3 Q3 11 / 11

✓ + 3 pts Correct a)

+ 1 pts partial_1, a): considering appropriate Ito process

+ 2 pts partial_2, a): for Ito-formula and final result

✓ + 4 pts correct b)

✓ + 4 pts correct c)

+ 1.5 pts partial_1, c): showing $A_1=A_2$

+ 1.5 pts partial_2,c): for Ito-isometry

+ 1 pts partial_3, c): showing $f=g$

+ 0 pts Incorrect/Not attempting

Q4 (a) For a martingale, we need

① ~~M(t)~~ $X(t)$ adapted to filtration F_t

this is true ~~also~~ by definition of 2D Brownian motion

③ $E[X(t) | F_s] = X(s)$

$$E[B_1(t) B_2(t) | F_s]$$

$$= E[(B_1(t) - B_1(s) + B_1(s))(B_2(t) - B_2(s) + B_2(s)) | F_s]$$

$$= E[(B_1(t) - B_1(s))(B_2(t) - B_2(s)) + B_1(s)(B_2(t) - B_2(s)) + B_2(s)(B_1(t) - B_1(s)) + B_1(s)B_2(s) | F_s]$$

In the first term, both are independent wrt F_s ,
in the second term, $B_1(s)$ is F_s measurable & $B_2(t) - B_2(s)$ is F_s -independent

similar for 3rd term

2nd last term has $B_1(s) B_2(s)$ F_s measurable.

So the first term contributes $0 \times 0 = 0$

" 2nd " " $E[B_1(s)] \cdot 0 = 0$

" 3rd " " $E[B_2(s)] \cdot 0 = 0$

" 4th " " $E[B_1(s) B_2(s) | F_s] = B_1(s) B_2(s)$

$$\Rightarrow E[B_1(t) B_2(t) | F_s] = B_1(s) B_2(s)$$

② $E[|B_1(t) B_2(t)|] < +\infty \quad \forall t$

$$|B_1(t) B_2(t)| \leq \frac{1}{2} (|B_1(t)|^2 + |B_2(t)|^2)$$

$$\Rightarrow E[\quad] \leq \frac{1}{2} E[B_1(t)^2] + \frac{1}{2} E[B_2(t)^2] < +\infty$$

(b) For 1D Brownian motion we need:

① Continuous ^{sample} paths a.s. ~~a.s.~~ This is true as $B_1(t)$ & $B_2(t)$ they are linear comb. of which have the same property.

② ~~should~~ be martingales: Same reason as above.

$$③ M(0) = 0 : M_1(0) = \frac{1}{\sqrt{5}} B_1(0) + \frac{2}{\sqrt{5}} B_2(0) = 0$$

$$M_2(0) = \frac{3}{5} B_1(0) + \frac{4}{5} B_2(0) = 0$$

④ $[M, M](t) = t \quad \forall 0 \leq t \leq T$

$$dM_1(t) dM_1(t) = \left(\frac{1}{\sqrt{5}} dB_1 + \frac{2}{\sqrt{5}} dB_2 \right) \cdot \left(\frac{1}{\sqrt{5}} dB_1 + \frac{2}{\sqrt{5}} dB_2 \right)$$

$$= \frac{1}{5} dB_1 \cdot dB_1 + \frac{4}{5} dB_2 \cdot dB_2 + \frac{4}{5} dB_1 \cdot dB_2$$

$$= \frac{1}{5} dt + \frac{4}{5} dt + 0 = dt$$

Similarly

$$dM_2(t) \cdot dM_2(t) = \frac{9}{25} dt + \frac{16}{25} dt + 0 = dt.$$

$$\Rightarrow [M_1, M_1](t) = [M_2, M_2](t) = t$$

So both of these are 1-D Brownian motions

why?

c): No.

$$\begin{aligned} dM_1(t) \cdot dM_2(t) &= \frac{1}{\sqrt{5}} \cdot \frac{2}{5} dB_1(t) \cdot dB_1(t) \\ &\quad + \frac{2}{\sqrt{5}} \cdot \frac{4}{5} dB_2(t) \cdot dB_2(t) \\ &\quad + \left(\frac{1}{\sqrt{5}} \cdot \frac{4}{5} + \frac{2}{\sqrt{5}} \cdot \frac{2}{5} \right) dB_1(t) \cdot dB_2(t) \\ &= \left(\frac{3}{5\sqrt{5}} + \frac{6}{5\sqrt{5}} \right) dt. \end{aligned}$$

$$\Rightarrow [M_1, M_2](t) = \frac{9}{5\sqrt{5}} t \neq 0.$$

So cross variation is non-zero, hence this is not a 2D Brownian motion.

4 Q4 7.5 / 8

✓ + 4 pts Correct a)

+ 2 pts partial_1, part a):

✓ + 2 pts Correct b)

+ 1 pts partial_1, part b): finding quadratic variation correctly

✓ + 2 pts correct c)

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+ 1 pts [Click here to replace this description.](#)

✓ - 0.5 pts [Click here to replace this description.](#)