Definition 5.5. A self-financing strategy Ψ is called an **arbitrage opportunity** if

$$V_\Psi(0) = 0,\ V_\Psi(T) \geq 0 \text{ a.s.}, \quad \text{and} \quad \mathbb{P}\big(V_\Psi(T) > 0\big) > 0.$$

So, $\Psi(t)$ is an arbitrage if it gives an increase in the value from $t = 0$ to time $t = T$ a.s., and a strict positive increase with positive probability. Hence $\Psi(t)$ generates a profit without any risk of losing money. Such an opportunity exists if and only if there is way to start with positive capital $V_\Psi(0)$ and to beat the money market account. In other words, there exists an arbitrage if and only if there is a way to start with $V_\Psi(0)$ and at a later time T have a portfolio value satisfying

$$\mathbb{P}\Big(V_\Psi(T) \geq \frac{V_\Psi(0)}{D(T)}\Big) = 1, \quad \mathbb{P}\Big(V_\Psi(T) > \frac{V_\Psi(0)}{D(T)}\Big) > 0. \tag{5.5}$$

How can we decide if a given market $(X(t))_{t\in[0,T]}$ allows an arbitrage or not.

Theorem 5.1 (**First fundamental theorem of asset pricing).** *If a market $(X(t))_{t\in[0,T]}$ has a risk-neutral measure, then it does not admit arbitrage.*

Proof. Suppose the given market $(X(t))_{t\in[0,T]}$ has a risk-neutral measure Q. Then under Q, the discounted stock process $\tilde{S}(t)$ is a martingale. We claim that the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$ is a martingale under Q. Indeed, since $\psi(t)$ units of portfolio value $V_\Psi(t)$ is invested in stock, the remainder of the portfolio value $V_\Psi(t) - \psi(t)S(t)$ is invested in the money market account. Thus, the differential of portfolio value is given by

$$\begin{aligned}
dV_\Psi(t) &= \psi(t)\, dS(t) + r(t)\big(V_\Psi(t) - \psi(t)S(t)\big)\, dt \\
&= r(t)V_\Psi(t)\, dt + \psi(t)\Big(dS(t) - r(t)S(t)\, dt\Big) \\
&= r(t)V_\Psi(t)\, dt + \frac{\psi(t)}{D(t)}\Big(D(t)dS(t) - D(t)r(t)S(t)\, dt\Big) \\
&= r(t)V_\Psi(t)\, dt + \frac{\psi(t)}{D(t)} d(D(t)S(t)) \quad (\text{as } d(D(t)) = -r(t)D(t)\, dt) \\
&= r(t)V_\Psi(t)\, dt + \frac{\psi(t)}{D(t)} d\tilde{S}(t)\,.
\end{aligned}$$

By using Ito-product rule, we have

$$\begin{aligned}
d\tilde{V}_\Psi(t) = d\big(D(t)V_\Psi(t)\big) &= -r(t)D(t)V_\Psi(t)\, dt + D(t)\, dV_\Psi(t) \\
&= -r(t)D(t)V_\Psi(t)\, dt + D(t)\Big(r(t)V_\Psi(t)\, dt + \frac{\psi(t)}{D(t)} d\tilde{S}(t)\Big) \\
&= \psi(t)d\tilde{S}(t)\,.
\end{aligned}$$

Since $\tilde{S}(t)$ is a martingale under Q, we conclude from the above differential form that the discounted portfolio value $\tilde{V}_\Psi(t)$ is also a martingale under Q. In particular, for any portfolio value process $V_\Psi(t)$, we have

$$\mathbb{E}_Q[D(T)V_\Psi(T)] = V_\Psi(0).$$

Let $V_\Psi(t)$ be a portfolio value process such that $V_\Psi(0) = 0$ and $\mathbb{P}\big(V_\Psi(T) \geq 0\big) = 1$. Thus we have

$$\mathbb{E}_Q[D(T)V_\Psi(T)] = 0. \tag{5.6}$$

and $\mathbb{P}\big(V_\Psi(T) < 0\big) = 0$. Since $\mathbb{P}$ and Q are equivalent, one has $Q\big(V_\Psi(T) < 0\big) = 0$. Since $D(t) > 0$, we claim that $Q\big(V_\Psi(T) > 0\big) = 0$. If not, then $Q\big(V_\Psi(T) > 0\big) > 0$ and hence $Q\big(D(t)V_\Psi(T) > 0\big) > 0$ which again implies that $\mathbb{E}_Q[D(T)V_\Psi(T)] > 0$— a contradiction to (5.6). Hence $V_\Psi(t)$ is not an arbitrage. In fact there cannot exist an arbitrage since every portfolio value process $V_\Psi(t)$ satisfying $V_\Psi(0) = 0$ cannot be an arbitrage. $\square$

Example 5.2. *Consider a market $X(t) = \big(S_0(t), S(t)\big)$ where*

$$dS_0(t) = 0,\ \ S_0(0) = 1; \quad dS(t) = 2S(t)\,dt + 3S(t)\,dB(t),\ \ S(0) = x > 0.$$

Show that the market has no arbitrage.

Solution: *Observe that $S_0(t) = 1$ and hence the discount process $D(t) = 1$. If we show that there exists a probability measure Q such that $\mathbb{P}$ and Q are equivalent and the discounted stock price $\tilde{S}(t)$ is a martingale under Q, then from Theorem 5.1, we conclude that given market has no arbitrage. Consider an adapted stochastic process $u(t)$ such that*

$$3S(t)u(t) = 2S(t) \implies u(t) = \frac{2}{3}.$$

Then by Girsanov's theorem, $\bar{B}(t) = B(t) + \frac{2}{3}t$ is a Brownian motion under the new probability measure Q where

$$dQ(\omega) = Z(T)d\,\mathbb{P}(\omega) \ \ \textit{with}\ Z(t) = \exp\Big\{ -\frac{2}{3}B(t) - \frac{2}{9}t\Big\}.$$

One can check easily that Q and $\mathbb{P}$ are equivalent. Moreover, $S(t)$ can be rewritten as

$$dS(t) = 3S(t)\,d\bar{B}(t).$$

Since $D(t) = 1$, we see that the discounted stock price process is a martingale under Q. Thus, the given market has no arbitrage.

5.1. **Black-Scholes-Metron equation:** Consider a European call option with strike price K and expiry time T. Let $C(t,x)$ denote the value of the call at time t if the stock price at time t is $S(t) = x$. The value of the option is random and it is the stochastic process $C(t, S(t))$. Suppose the stock is geometric Brownian motion and the rate of interest if constant i.e., $\alpha(t) = \alpha$, $\sigma(t) = \sigma$ and $r(t) = r$. By Ito-formula, we have

$$\begin{aligned} dC(t,S(t)) &= C_t(t,S(t))dt + C_x(t,S(t))\,dS(t) + \frac{1}{2}C_{xx}(t,S(t))\,dS(t)\,dS(t) \\ &= \Big\{C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big\}\,dt \\ &\qquad + \sigma S(t)C_x(t,S(t))\,dB(t)\,. \end{aligned}$$

Next we calculate the discounted option price $D(t)C(t,S(t))$. Observe that, here $D(t) = e^{-rt}$. Thus, by Ito product rule, we get

$$\begin{aligned} d\big(e^{-rt}C(t,S(t))\big) &= -re^{-rt}C(t,S(t))\,dt + e^{-rt}dC(t,S(t)) \\ &= e^{-rt}\Big[-rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) \\ &\qquad + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big]\,dt + e^{-rt}\sigma S(t)C_x(t,S(t))\,dB(t)\,. \end{aligned} \tag{5.7}$$

A **hedge** is an investment that reduces the risk in an existing position. A (short option) hedging portfolio starts with some initial capital $V_\Psi(0)$ and invest in the stock and money market account so that the portfolio value $V_\Psi(t)$ at each time $t \in [0,T]$ agrees with the option price $C(t,S(t))$. This will happen if and only if

$$d(e^{-rt}V_\Psi(t)) = d\big(e^{-rt}C(t,S(t))\big), \quad \forall\, t \in [0,T). \tag{5.8}$$

If $\tilde{S}(t)$ is the discounted stock price process, we have seen that

$$d(e^{-rt}V_\Psi(t)) = \psi(t)\,d\tilde{S}(t) = \psi(t)e^{-rt}\big((\alpha-r)S(t)\,dt + \sigma S(t)\,dB(t)\big)\,.$$

Thus, keeping in mind (5.7), we see that (5.8) holds if and only if

$$\begin{aligned}
&\psi(t)\big((\alpha-r)S(t)\,dt + \sigma S(t)\,dB(t)\big)\\
&= \Big[-rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t))\\
&\qquad + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big]\,dt + \sigma S(t)C_x(t,S(t))\,dB(t)\,.
\end{aligned} \tag{5.9}$$

We examine what is required in order for (5.9) to hold. Equating $dB(t)$ terms in (5.9), we have

$$\psi(t) = C_x(t,S(t)).$$

This is called the **delta-hedging rule**. The quantity $C_x(t,S(t))$ is called the **delta** of the option. Again, equating dt-terms in (5.9), and then putting the value of $\psi(t)$, we obtain

$$\begin{aligned}
&C_x(t,S(t))(\alpha-r)S(t)\\
&= -rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) + \frac{1}{2}C_{xx}(t,S(t))\sigma^2 S^2(t)\Big]\\
&\implies C_t(t,S(t)) + rS(t)C_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{xx}(t,S(t)) = rC(t,S(t))\,.
\end{aligned}$$

Therefore, we should seek a continuous function $C(t,x)$ that is a solution of the PDE

$$C_t(t,x) + rxC_x(t,x) + \frac{1}{2}\sigma^2x^2C_{xx}(t,x) = rC(t,x)\ t \in [0,T),\ x \geq 0, \tag{5.10}$$

with the terminal condition

$$C(T,x) = \max\{x-K, 0\}. \tag{5.11}$$

Suppose we have found this function. If an investor starts with initial capital $V_\Psi(0) = C(0,S(0))$, and uses the hedge $\psi(t) = C_x(t,S(t))$, then the equation (5.9) holds and hence we see that $V_\Psi(t) = C(t,S(t))$. Taking the limit as $t \to T$, and using the fact that $V_\Psi(t)$ and $C(t,S(t))$ is continuous, we have $V_\Psi(T) = C(T,S(T))$. The European call option with strike price K and expiry date T has the pay-off value $C(T,S(T)) = \max\{S(T)-K, 0\}$. Thus, we have $V_\Psi(T) = \max\{S(T)-K,0\}$. This means that the short position has been successfully hedged. Equations (5.10)-(5.11) is known as **Black-Scholes-Merton** equation.