We denote by K the strike price and by T the expiry date. If the current price of the stock is S, the owner of a call will exercise the option if $K < S$, and the owner of a put will exercise the option if $K > S$.

Based on the exercise of the holder, options can be divided into two types:

a) **European option:** it is an option that can be exercised only at the time of expiry of the contract.
b) **American option:** an option which can be exercised at any time up to the expiry time T.

Remark 5.2. The call or put option described above is called a **simple** or **vanilla** option. An option which is not a vanilla option called an **exotic option**.

Consider the **discounted stock price process** $\tilde{S}(t) := D(t)S(t)$. Then it is given by the formula

$$\tilde{S}(t) = S(0)\exp\Big\{\int_0^t \sigma(s)\,dB(s) + \int_0^t \big(\alpha(s) - r(s) - \frac{1}{2}\sigma^2(s)\big)\,ds\Big\}.$$

Again, by using Ito-formula, one has

$$\begin{aligned} d\tilde{S}(t) &= \big(\alpha(t) - r(t)\big)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\,dB(t) \\ &= \theta(t)\sigma(t)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\,dB(t), \end{aligned}$$

where $\theta(t)$ is the **market price of risk**, defined by

$$\theta(t) = \frac{\alpha(t) - r(t)}{\sigma(t)}.$$

Observe that, compare to undiscounted stock price, the mean rate of return of the discounted stock price $\tilde{S}(t)$ is reduced by the interest rate $r(t)$, i.e., the mean rate of return of $\tilde{S}(t)$ is $\alpha(t) - r(t)$. Note that volatility for $S(t)$ and $\tilde{S}(t)$ remains same.

Definition 5.3. A probability measure Q is said to be risk-neutral if the followings hold:

a) Q and $\mathbb{P}$ are equivalent i.e., for every $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ if and only if $Q(A) = 0$,
b) Under Q, the discounted stock price $\tilde{S}(t)$ is a martingale.

Let Q be a probability measure defined by $Q(A) = \int_A Z(T)\,d\mathbb{P}$ for any $A \in \mathcal{F}$ where the process $Z(t)$ is given by

$$Z(t) = \exp\{-\int_0^t \theta(s)\,dB(s) - \frac{1}{2}\int_0^t \theta^2(s)\,ds\},$$

and $\theta(t)$ is the market price of risk process. Then by Girsanov's theorem

$$\bar{B}(t) = B(t) + \int_0^t \theta(s)\,ds$$

is a Brownian motion under Q. Moreover, the discounted stock price process $\tilde{S}(t)$ can be rewritten in terms of $\bar{B}(t)$ as follows:

$$\begin{aligned} d\tilde{S}(t) &= \theta(t)\sigma(t)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\,dB(t) \\ &= \theta(t)\sigma(t)\tilde{S}(t)\,dt + \sigma(t)\tilde{S}(t)\big\{d\bar{B}(t) - \theta(t)dt\big\} \\ &= \sigma(t)\tilde{S}(t)\,d\bar{B}(t) \end{aligned}$$

Thus, $\tilde{S}(t)$ is a martingale under Q. Moreover, $\mathbb{P}$ and Q are equivalent. Hence the measure Q is a risk-neutral measure.

Remark 5.3. We have the following observation:

a) Mean rate of return of the undiscounted stock price $S(t)$ under the risk-neutral measure Q is equal to the interest rate $r(t)$. Indeed,
$$\begin{aligned} dS(t) &= \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t) \\ &= \alpha(t)S(t)\,dt + \sigma(t)S(t)\big\{d\bar{B}(t) - \theta(t)dt\big\} \\ &= \big(\alpha(t) - \sigma(t)\theta(t)\big)S(t)\,dt + \sigma(t)S(t)\,d\bar{B}(t) \\ &= r(t)S(t)\,dt + \sigma(t)S(t)\,d\bar{B}(t). \end{aligned}$$
b) The volatility of the stock price $S(t)$ does NOT change under the risk-neutral measure.
c) If $\alpha(t) > r(t)$, the change of measure puts more probability on the paths with lower return so that the overall mean rate of return is reduced from $\alpha(t)$ to $r(t)$.

Definition 5.4. We define the followings:

i) A **trading strategy or the portfolio** in the market $X(t) = (S_0(t), S(t))$ is a adapted process $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ such that
$$\int_0^T |\psi_0(t)|\,dt < +\infty; \quad \int_0^T \psi^2(s)\,ds < +\infty \quad \text{a.s.}$$
$\psi_0(t)$ and $\psi(t)$ represents the number of units of shares and bonds respectively.
ii) The **value of the portfolio** at time t is given by
$$V_\Psi(t) = \psi_0(t)S_0(t) + \psi(t)S(t).$$
iii) The portfolio $\Psi(t)$ is called **self-financing** if $V_\Psi(t)$ satisfies the differential form
$$dV_\Psi(t) = \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t)\,. \tag{5.3}$$

In words, the self-financing property means that the investor is not withdrawing any gains from the portfolio for consumption, nor investing additional funds. She starts with an initial investment, and from there on all gains or losses in portfolio value come from price increases or decreases in the stock, or bond. Furthermore, the property tells us that if the investor wants to increase the stock position, say, the funding for this must come from selling bonds .

Suppose the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing. Therefore, one has
$$\begin{aligned} &\psi_0(t)S_0(t) + \psi(t)S(t) = V_\Psi(0) + \int_0^t \psi_0(u)\,dS_0(u) + \int_0^t \psi(u)\,dS(u) \\ \implies\ &\psi_0(t)S_0(t) = V_\Psi(0) + \int_0^t \psi_0(u)\,dS_0(u) + \int_0^t \psi(u)\,dS(u) - \psi(t)S(t). \end{aligned}$$
Set
$$Y_0(t) = \psi_0(t)S_0(t), \quad A(t) = \int_0^t \psi(u)\,dS(u) - \psi(t)S(t).$$
Then one has
$$dY_0(t) = \psi_0(t)\,dS_0(t) + dA(t) = r(t)Y_0(t)\,dt + dA(t).$$

Consider the discount process $D(t)$. Observe that

$$D(t)Y_0(t) = D(t)S_0(t)\psi_0(t) = \psi_0(t).$$

By Ito product rule, we get

$$\begin{aligned}
& d(D(t)Y_0(t)) = dD(t)Y_0(t) + D(t)\,dY_0(t) \\
& = -D(t)r(t)Y_0(t)\,dt + D(t)\{r(t)Y_0(t)\,dt + dA(t)\} \\
& = D(t)\,dA(t) \\
& \implies \psi_0(t) = \psi_0(0) + \int_0^t D(u)\,dA(u) \\
& \implies \psi_0(t) = \psi_0(0) - A(0) + D(t)A(t) + \int_0^t r(s)A(s)D(s)\,ds\,, \qquad (5.4)
\end{aligned}$$

where in the last line we have used the integration by parts formula. This argument goes both the ways in the sense that if we define $\psi_0(t)$ by (5.4), then we get (5.3). Indeed, from (5.4) and the definition of A, we get

$$d\psi_0(t) = D(t)\,dA(t); \quad dA(t) = \psi(t)\,dS(t) - S(t)\,d\psi(t) - \psi(t)\,dS(t) - d[\psi(t), S(t)]\,.$$

Hence, we obtain

$$\begin{aligned}
dV_\psi(t) &= d\big(\psi_0(t)S_0(t) + \psi(t)S(t)\big) \\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t) + S_0(t)\,d\psi_0(t) + S(t)\,d\psi(t) + d[\psi(t), S(t)] \\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t) + S_0(t)D(t)\,dA(t) + S(t)\,d\psi(t) + d[\psi(t), S(t)] \\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t) + dA(t) + S(t)\,d\psi(t) + d[\psi(t), S(t)] \\
&= \psi_0(t)\,dS_0(t) + \psi(t)\,dS(t)\,.
\end{aligned}$$

Therefore, if $\psi(t)$ is chosen, we can always make the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ self-financing by choosing $\psi_0(t)$ according to (5.4) . Moreover, we are free to choose the initial value $V_\Psi(0)$ of the portfolio.

Remark 5.4. Consider the discounted portfolio value process $\tilde{V}_\Psi(t) := D(t)V_\Psi(t)$. A trading strategy $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing if and only if $\tilde{V}_\Psi(t)$ can be expressed for all $t \in [0, T]$ as

$$\tilde{V}_\Psi(t) = V_\Psi(0) + \int_0^t \psi(u)\,d\tilde{S}(u)$$

where $\tilde{S}(t)$ is the discounted stock price process.

Example 5.1. *Consider a market* $X(t) = \big(S_0(t), S(t)\big)$ *given by*

$$dS_0(t) = 2S_0(t)\,dt,\ \ S_0(0) = 1; \quad dS(t) = S(t)\,dt + 2S(t)\,dB(t),\ \ S(0) = x > 0.$$

Show that the portfolio $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$, *given by*

$$\psi_0(t) = -\int_0^t e^{-2u}S^3(u)\,du, \quad \psi(t) = \int_0^t S^2(u)\,du,$$

is self-financing.

Solution: *Note that $S_0(t) = e^{2t}$ and hence the corresponding discounted process $D(t) = e^{-2t}$. Moreover,*

$$d\psi_0(t) = -S^3(t)D(t)\,dt = -\frac{S^3(t)}{S_0(t)}; \quad d\psi(t) = S^2(t)\,dt\,.$$

The value of the portfolio $V_\Psi(t)$ is given by

$$V_\Psi(t) = \psi_0(t)S_0(t) + \psi(t)S(t).$$

To show that $\Psi(t) = \big(\psi_0(t), \psi(t)\big)$ is self-financing, we need to show that

$$S_0(t)\,d\psi_0(t) + S(t)\,d\psi(t) = 0.$$

Indeed, we have

$$S_0(t)\,d\psi_0(t) + S(t)\,d\psi(t) = -S_0(t)\frac{S^3(t)}{S_0(t)}\,dt + S(t)S^2(t)\,dt = 0.$$