5.3.2. **Chooser option:** A chooser option on a stock is a derivative that gives holders the right to choose at a contracted time $t < T$ if they want to call or put option.

If the exercise time of the chooser option is T with strike price K, the holder will either have a call or a put at time T depending on the choice made at the earlier time t. We are going to derive the arbitrage-free price at time 0 for this option.

Let $V_c(t)$ and $V_p(t)$ be the price at time t for a call and put option respectively with strike price K and exercise time T. The holder of the chooser option will take the call option at time t if $V_c(t) \geq V_p(t)$, and the put option otherwise. The payoff function at time T is

$$X := \max\{0, S(T) - K\}\mathbf{1}_{\{V_c(t) \geq V_p(t)\}} + \max\{0, K - S(T)\}\mathbf{1}_{\{V_c(t) < V_p(t)\}}\,.$$

From the previous theory, we find the arbitrage-free price $V(0)$ at time of entering the chooser option to be

$$V(0) = e^{-rT}\mathbb{E}_Q[X],$$

where Q is the risk-neutral measure. Our task is to calculate this price. Observe that X can be re-written as

$$X = \max\{0, S(T) - K\} + (K - S(T))\mathbf{1}_{\{V_c(t) < V_p(t)\}}.$$

Thus, we have

$$\begin{aligned}
V(0) &= e^{-rT}\mathbb{E}_Q\big[\max\{0, S(T) - K\}\big] + e^{-rT}\mathbb{E}_Q\big[(K - S(T))\mathbf{1}_{\{V_c(t) < V_p(t)\}}\big] \\
&= e^{-rT}\mathbb{E}_Q\big[\max\{0, S(T) - K\}\big] + \mathbb{E}_Q\big[e^{-rT}(K - S(T))\mathbf{1}_{\{V_c(t) < V_p(t)\}}\big] \\
&= e^{-rT}\mathbb{E}_Q\big[\max\{0, S(T) - K\}\big] + \mathbb{E}_Q\Big[\mathbb{E}_Q\big[e^{-rT}(K - S(T))\mathbf{1}_{\{V_c(t) < V_p(t)\}}\big|\mathcal{F}_t\big]\Big] \\
&\equiv \mathcal{A}_1 + \mathcal{A}_2\,.
\end{aligned}$$

We recognize $\mathcal{A}_1$ as the price of a call option with strike price K at exercise time T, denoted it by $V_c(0; K, T)$. We need to calculate $\mathcal{A}_2$. From the Black-Scholes-Merton formula, we see that $V_c(t)$ is $\mathcal{F}_t$-adapted. By using Put-Call-Parity formula, we can say that $V_p(t)$ is $\mathcal{F}_t$-adapted. In conclusion, the random variable $\mathbf{1}_{\{V_c(t) < V_p(t)\}}$ is only depends on the stock price at time t, and therefore $\mathcal{F}_t$-adapted. Thus, $\mathcal{A}_2$ can be written as

$$\begin{aligned}
\mathcal{A}_2 &= \mathbb{E}_Q\Big[\mathbf{1}_{\{V_c(t) < V_p(t)\}}\mathbb{E}_Q\big[e^{-rT}(K - S(T))\big|\mathcal{F}_t\big]\Big] \\
&= \mathbb{E}_Q\Big[\mathbf{1}_{\{V_c(t) < V_p(t)\}}\big\{Ke^{-rT} - \mathbb{E}_Q\big[D(T)S(T)\big|\mathcal{F}_t\big]\big\}\Big] \\
&= \mathbb{E}_Q\Big[\mathbf{1}_{\{V_c(t) < V_p(t)\}}\big\{Ke^{-rT} - e^{-rt}S(t)\big\}\Big] \\
&= e^{-rt}\mathbb{E}_Q\Big[\big(Ke^{-r(T-t)} - S(t)\big)\mathbf{1}_{\{V_c(t) < V_p(t)\}}\Big]\,.
\end{aligned}$$

Let $\omega \in \Omega$ such that $V_c(t, \omega) < V_p(t, \omega)$. Then from the Put-Call-Parity formula, $S(t, \omega) < Ke^{-r(T-t)}$. Moreover, the two events

$$\{\omega \in \Omega : V_c(t, \omega) < V_p(t, \omega)\}, \quad \{\omega \in \Omega : S(t, \omega) < Ke^{-r(T-t)}\}$$

are identical. Therefore, we get

$$\begin{aligned}
\mathcal{A}_2 &= e^{-rt}\mathbb{E}_Q\Big[\big(Ke^{-r(T-t)} - S(t)\big)\mathbf{1}_{\{S(t) < Ke^{-r(T-t)}\}}\Big] \\
&= e^{-rt}\mathbb{E}_Q\Big[\max\{0, Ke^{-r(T-t)} - S(t)\}\Big]\,.
\end{aligned}$$

We recognize this term as the price of a put option with exercise time t and strike price $Ke^{-r(T-t)}$, denoted it by $V_p(0; Ke^{-r(T-t)}, t)$. Therefore, we have

$$V(0) = V_c(0; K, T) + V_p(0; Ke^{-r(T-t)}, t)\,.$$

Thus we arrive at the following conclusion: *the chooser option. has an arbitrage-free price which is a sum of a call option with exercise K at time T and a put option with exercise time t and strike price $Ke^{-r(T-t)}$. Moreover, one can derive the Black-Scholes formula for this option. Indeed, by Put-Call-Parity formula, one has*

$$V_p(0; Ke^{-r(T-t)}, t) = V_c(0; Ke^{-r(T-t)}, t) - S(0) + Ke^{-rT}.$$

Hence, we have

$$V(0) = V_c(0; K, T) + V_c(0; Ke^{-r(T-t)}, t) - S(0) + Ke^{-rT}.$$

The price of the two call options $V_c(0; K, T)$ and $V_c(0; Ke^{-r(T-t)}, t)$ can be expressed by the Black-Scholes-Merton formula. In particular, one has

$$V_c(0; Ke^{-r(T-t)}, t) = S(0)N(\bar{d}_1) - Ke^{-rT}N(\bar{d}_2),$$

where

$$\bar{d}_2 := \frac{\log(\frac{S(0)}{Ke^{-r(T-t)}}) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \quad \bar{d}_1 = \bar{d}_2 + \sigma\sqrt{t}\,.$$

Remark 5.8. The payoff function of the chooser option can be written as

$$X = \max\{0, K - S(T)\} + (S(T) - K)\mathbf{1}_{\{V_c(t) \ge V_p(t)\}}.$$

Hence, a similar argument reveals that

$$V(0) = V_p(0) + e^{-rt}\mathbb{E}_Q\Big[\max\{0, S(t) - Ke^{-r(T-t)}\}\Big].$$

The last term is the price of a call option at time 0 with strike $Ke^{-r(T-t)}$ and exercise time t. In other words,

$$V(0) = V_p(0) + V_c(0; Ke^{-r(T-t)}, t).$$

5.4. **Implied Volatility:** *We have discussed how to estimate the volatility σ from the historical stock prices namely*

$$\sigma^2 \approx \frac{1}{T_2 - T_1}\sum_{j=0}^{n-1}\Big(\log\big(\frac{S(t_{j+1})}{S(t_j)}\big)\Big)^2$$

when we observe geometric Brownian motion $S(t)$ for $0 \le T_1 \le t \le T_2$ and $\Pi = \{T_1 = t_0 < t_1 < \ldots < t_n = T_2\}$ is a partition of $[T_1, T_2]$. Choosing an appropriate value for n is not easy because σ does change over time and data that are too old may not be relevant for the present or the future.

The volatility is the only parameter which is unknown to us when pricing a call option contract. Suppose we know that a call option with strike price K and time to exercise T is traded for a price p in the market. At the same time, we read of from the stock exchange monitor that the underlying stock is traded for price s. Hence from the Black-Scholes-Metron equation, we have

$$p = sN(\mathrm{d}_1) - Ke^{-rT}N(\mathrm{d}_2)$$

where

$$\mathrm{d}_1 = \mathrm{d}_2 + \sigma\sqrt{T}, \quad \mathrm{d}_2 := \frac{\log(\frac{s}{K})) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,.$$

Since only unknown is σ, we can solve for this and find the volatility used by the market. Ideally, this should coincide (at least approximately) with the historical volatility, but this is rarely the case. Since this volatility is derived from the actual option prices, we call it the **Implied volatility**. *Since explicit formula for σ is not possible, one needs to use numerical method to solve for σ. Such an efficient method is the* Newton-Rapshon method, *an iterative method to solve the equation*

$$F(\sigma, s, K, r, T) - p = 0$$

where $F(\cdot)$ is the pricing model that depends on σ. From the initial guess of σ_0, the iteration function is

$$\sigma_{i+1} = \sigma_i F(\sigma_i)\frac{\partial F}{\partial \sigma}(\sigma_i)\,.$$

The derivative $\frac{\partial F}{\partial \sigma}$ is known as vega and therefore for European option

$$\frac{\partial F}{\partial \sigma}(\sigma_i) = s\sqrt{T}e^{-rT}N(\mathrm{d}_1^i)$$

where d_1^i is given by

$$\mathrm{d}_1^i := \frac{\log(\frac{s}{K})) + (r + \frac{1}{2}\sigma_i^2)T}{\sigma\sqrt{T}}.$$

With the help of software such as MATLAB or Mathematica, one can solve the above equation.

Remark 5.9. In general volatility need not be constant and could depend on the price of the underlying stock. These models are known as **Stochastic Modelling Approach** for mathematical finance.