Q1

(i) $$dX(t) = \frac{-X(t)}{1-t}\,dt + dB(t) \qquad -(1)$$

Let $f(t,x) = \dfrac{x}{1-t} \in C^2([0,1) \times \mathbb{R})$

Using Ito's formula,

$$d(f(t,X(t))) = f_x(t,X(t))\,dX(t) + \frac{1}{2} f_{xx}(t,X(t))\,dX(t)\cdot dX(t) + f_t(t,X(t))\,dt$$

From (1). Note that
$$dX(t)\cdot dX(t) = (g(t)\,dt + dB(t))\,(g(t)\,dt + dB(t)) = dB(t)\cdot dB(t) + 0 = dt \qquad \left(\text{where } g(t) = \frac{-X(t)}{1-t}\right)$$

$$\Rightarrow d\left(\frac{X(t)}{1-t}\right) = \frac{1}{1-t}\,dX(t) + \frac{1}{2}\cdot 0\cdot dX(t)\cdot dX(t) + \frac{X(t)}{(1-t)^2}\,dt$$

$$= \frac{1}{1-t}\cdot\left(dX(t) + \frac{X(t)}{1-t}\,dt\right)$$

$$= \frac{1}{1-t}\,dB(t).$$

So using the integral form, we get

$$\frac{X(t)}{1-t} - \frac{X(0)}{1-0} = \int_0^t \frac{1}{1-s}\,dB(s)$$

$$\Rightarrow X(t) = (1-t)\int_0^t \frac{1}{1-s}\,dB(s).$$

(ii) to check if X is normally distributed, we compute the MGF of this random variable at any time t.

Let $Y(t) = e^{uX(t)}$. We need to find $\mathbb{E}[Y(t)]$

$$dY(t) = u e^{uX(t)}\,dX(t) + \frac{1}{2}u^2 e^{uX(t)}\,dX(t)\cdot dX(t)$$

$$= \left[\frac{1}{2}u^2 + u\left(\frac{-X(t)}{1-t}\right)\right] e^{uX(t)}\,dt - u e^{uX(t)}\,dB(t)$$

integrates to martingale with expectation 0.

$$E[Y(t)] = \frac{1}{2}u^2 E\left[\int_0^t e^{uX(t)}\,dt\right] - \frac{u}{1-t}$$

~~(ii) Since $\frac{1}{1-s}$ is a nonrandom variable, ... integration by parts to get~~

$$\cancel{\int_0^t \frac{1}{1-s}\,dB(s) = \frac{1}{1-t}B(t) - \int_0^t \frac{B(s)}{(1-s)^2}\,ds.}$$

(ii) Since $X(t)$ only has a diffusion ~~component, ... it is a normally distributed process.~~

$$dX(t)\cdot dX(t) = dt \Rightarrow [X,X](t) = t$$

$$X(t) = (1-t)\int_0^t \frac{1}{1-s}\,dB(s) \Rightarrow X \text{ has continuous sample paths a.s.}$$

$$X(0) = 0$$

Since $\frac{1}{1-s}$ is a non-random variable, we can apply integration by parts to get

$$\int_0^t \frac{1}{1-s}\,dB(s) = \frac{1}{1-t}B(t) - \int_0^t \frac{B(s)}{(1-s)^2}\,ds.$$

$$\Rightarrow X(t) = B(t) - \int_0^t \frac{B(s)(1-t)}{(1-s)^2}\,ds.$$

So we only need to show that the second term is martingale (this will give that $X(t) \sim N(0,t)$)

$$E\left[\int_0^t \frac{B(z)(1-t)}{(1-z)^2}\,dz \,\middle|\, \mathcal{F}_s\right]$$

$$= \int_0^s \frac{B(z)}{(1-z)^2}(1-t)\,dz + E\left(\int_s^t \frac{(\overbrace{B(z)-B(s)}^{\text{independent}} + \overbrace{B(s)}^{\text{measurable}})(1-t)}{(1-z)^2}\,dz \,\middle|\, \mathcal{F}_s\right)$$

$$= \int_0^s \frac{B(z)(1-t)}{(1-z)^2}\,dz + B(s)\int_s^t \frac{(1-t)}{(1-z)^2}\,dz.$$

$$= \int_0^s \frac{(1-t)B(z)}{(1-z)^2}\,dz + B(s)(1-t)\left(\frac{1}{1-t} - \frac{1}{1-s}\right).$$

$$= \int_0^s \frac{B(z)(1-s)}{(1-z)^2}\,dz$$

$\Rightarrow X(t)$ is a martingale. So we are done.

(iii). $d(X^2(t)) = 2X\,dX + dX \cdot dX$

$$= \frac{-2X^2}{1-t}\,dt + dt + \underbrace{dB}_{0}$$

taking expectations, if $E[X^2(t)] = f(t)$, we have

$$f' = -\frac{2f}{1-t} + 1$$

$X(0) = 0$
$\Rightarrow f(0) = 0$

$$\frac{f'}{(1-t)^2} + \frac{2f}{(1-t)^3} = \frac{1}{(1-t)^2}$$

$$\Rightarrow \frac{f(t)}{(1-t)^2} - \frac{f(0)}{(1-0)^2} = \left(\frac{1}{(1-t)^2} - 1\right)$$

$$\Rightarrow f(t) = (1-t)^2 f(0) + 1 - (1-t)^2$$

as $t \to 1$, $f(t) \to 1$

$$\Rightarrow \boxed{\lim_{t \to 1^-} E[X^2(t)] = 1}$$

Q2:

(i) According to the multi-dimensional Ito formula, we have, if $X(t) = f(t, B_1(t), B_2(t), \dots, B_m(t))$

$$dX(t) = \frac{\partial f}{\partial t}(t, B_1(t), \dots, B_m(t))\,dt$$
$$+ \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(t, B_1(t), \dots, B_m(t))\, dB_i(t)$$
$$+ \sum_{i=1}^{m} \frac{1}{2} \frac{\partial^2 f}{\partial x_i^2}(t, B_1(t), \dots, B_m(t))\, dB_i(t) \cdot dB_i(t).$$

Here

$$f(t, x_1, \dots, x_m) = e^{ct + \sum_{j=1}^{m} \alpha_j x_j}$$

$$\frac{\partial f}{\partial t} = c \cdot e^{ct + \sum_{j=1}^{m} \alpha_j x_j} = cf$$

$$\frac{\partial f}{\partial x_i} = \alpha_j \cdot e^{ct + \sum_{j=1}^{m} \alpha_j x_j} = \alpha_j \cdot f$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}(\alpha_j f) = \alpha_j \frac{\partial f}{\partial x_i} = \alpha_j^2 f.$$

$$\Rightarrow dX(t) = c\, X(t)\, dt + \sum_{i=1}^{m} \alpha_i X(t)\, dB_i(t) + \sum_{i=1}^{m} \frac{1}{2} \alpha_i^2 X(t)\, dt$$

$$= \left(c + \frac{1}{2}\sum_{j=1}^{m} \alpha_j^2\right) X(t)\, dt + X(t)\left(\sum_{j=1}^{m} \alpha_j\, dB_j(t)\right)$$

(ii) upon integrating this equation (writing it in an integral form, to be precise), the second part integrates to an Ito integral, which is a martingale and has expectation 0. So,

$$E[X(t) - X(0)] = E \int_0^t \underbrace{\left(c + \frac{1}{2}\sum_{j=1}^{m} \alpha_j^2\right)}_{A \text{ (say)}} X(t)\, dt$$

$X(0) = 1$ a.s. (by def^n of B_i's).

$$\Rightarrow E[X(t)] = 1 + E\left[\int_0^t A X(t)\, dt\right] = A \int_0^t E[X(t)]\, dt$$

$$\Rightarrow e(t) = 1 + A \int_0^t e(t)\, dt$$

differentiating, we get: $e'(t) = A\, e(t)$

$\Rightarrow e(t) = k e^{At}$ for some $k > 0$ ($e(0) = 0$, so $k = 1$)

So for $\lim_{t\to\infty} e(t) = 0$, we need $A < 0$,

i.e., $\boxed{c < -\frac{1}{2}\sum_{j=1}^{m}\alpha_j^2}$

(iii). $X(t) = UB(t)$ $\quad U = \begin{pmatrix} \cos(x_0) & \sin(x_0) \\ -\sin(x_0) & \cos(x_0) \end{pmatrix}$

~~$X_1(t) = \cos(x_0)B_1(t) + \sin(x_0)$~~

Let $\theta_j = x_0 + \frac{\pi}{2}j$

Then $X_i(t) = \cos(\theta_i)B_1(t) + \sin(\theta_i)B_2(t)$

1) Note that $X_i(t)$ is a martingale since it is a linear combination (with const. coeff) of $B_1(t)$ & $B_2(t)$ (both of which are martingales.

2). we have $X_i(t)$ having continuous sample paths a.s. since both $B_1(t)$ & $B_2(t)$ have continuous sample paths.

3). $X_i(0) = \cos(\theta_i)B_1(0) + \sin(\theta_i)B_2(0) = 0$

4). $dX_i(t)\, dX_i(t) = \cos^2(\theta_i)\, dB_1(t)\cdot dB_1(t)$
$+ \sin^2(\theta_i)\, dB_2(t)\cdot dB_2(t)$
$+ \sin\theta_i\cos\theta_i\, dB_1(t)\, dB_2(t)$
$+ \cos\theta_i\sin\theta_i\, dB_2(t)\, dB_1(t).$

$= \cos^2(\theta_i)\,dt + \sin^2(\theta_i)\,dt + 0$

$= dt. \quad \forall t.$

$\Rightarrow [X_i, X_i](t) = \int_0^t 1\, ds = t.$

5) $dX_1(t)\cdot dX_2(t) = (\cos(x_0)\,dB_1(t) + \sin(x_0)\,dB_2(t))$
$\cdot(-\sin(x_0)\,dB_1(t) + \cos(x_0)\,dB_2(t))$

$= -\sin(x_0)\cos(x_0)\,dB_1(t)\cdot dB_1(t) + \sin(x_0)\cos(x_0)\,dB_2(t)\cdot dB_2(t)$
$+ \cos^2(x_0)\,dB_1(t)\,dB_2(t)$
$- \sin^2(x_0)\,dB_2(t)\,dB_1(t)$

$= -\sin(x_0)\cos(x_0)\,dt + \sin(x_0)\cos(x_0)\,dt + 0$

$= 0$

$\Rightarrow [X_1, X_2](t) = 0 \quad \forall t.$

Similarly $[X_2, X_1](t) = 0 \quad \forall t$

Hence this is a 2-D brownian motion by Levy's Characterisation

Q3: (a) $dY(t) = \alpha Y(t)\,dt + \sigma Y(t)\,dB(t)$

$$\xi(t) = e^{-\theta B(t) - (r+\frac{1}{2}\theta^2)t} \qquad \theta = \frac{\alpha - r}{\sigma}$$

let $f(t,x) = e^{-\theta x - (r+\frac{1}{2}\theta^2)t}$

By Ito's formula, we have

$$d\xi(t) = f_t(t, B(t))\,dt + f_x(t, B(t))\,dB(t) + \frac{1}{2} f_{xx}(t, B(t))\,\underbrace{dB(t)\,dB(t)}_{dt}$$

$$f_t(t,x) = -\left(r+\frac{1}{2}\theta^2\right) e^{-\theta x - (r+\frac{1}{2}\theta^2)t} = -\left(r+\frac{1}{2}\theta^2\right) f(t,x)$$

$$f_x(t,x) = -\theta f(t,x)$$

$$f_{xx}(t,x) = \theta^2 f(t,x).$$

$$\Rightarrow d\xi(t) = -\left(r+\frac{1}{2}\theta^2\right)\xi(t)\,dt - \theta\,\xi(t)\,dB(t) + \frac{1}{2}\theta^2\xi(t)\,dt.$$

$$= -r\xi(t)\,dt - \theta\,\xi(t)\,dB(t)$$ as needed.

(b): $dX(t) = rX(t)\,dt + \gamma(t)(\alpha - r)Y(t)\,dt + \sigma\gamma(t)Y(t)\,dB(t)$

$Z(t) = \xi(t)X(t).$

For the sake of convenience, I will drop the (t) here

$$dX = rX\,dt + \gamma(\alpha - r)Y\,dt + \sigma\gamma Y\,dB$$

$$Z = \xi X$$

By product rule, we have

$$dZ = \xi\,dX + X\,d\xi + d\xi \cdot dX$$

$$\rightarrow d\xi \cdot dX = (-r\xi\,dt - \underline{\theta\xi\,dB}) \cdot (rX\,dt + \gamma(\alpha-r)Y\,dt + \underline{\sigma\gamma Y\,dB})$$

$$= -\theta\xi\gamma\sigma Y\,dt$$ (the rest of the terms are 0).

$$\Rightarrow dZ = \underline{(\xi r X + \xi\gamma(\alpha - r)Y)}\,dt + \sigma\xi\gamma Y\,dB - \underline{Xr\xi\,dt} - X\theta\xi\,dB - \underline{\theta\xi\gamma\sigma Y\,dt}$$

Collecting all coefficients of dt, we have

$$dZ = a\,dt + g\,dB(t)$$

where $a = \xi\left[\cancel{zX} + \gamma(\alpha - r)Y - \cancel{Xr} - \theta\gamma\sigma Y\right]$

Now $\theta\sigma = \alpha - r$, so

$$a = \xi\gamma Y\left[\alpha - r - \theta\sigma\right] = 0$$

$\Rightarrow$ $dZ = g\,dB(t)$ for some g. $(= (\sigma\xi\gamma Y - X\sigma\xi))$

$$Z(t) = Z(0) + \int_0^t g\,dB(s)$$

Since this is an Ito integral, it is a martingale

$\Rightarrow Z(t)$ is a martingale.

(c). We have

$$A_2 - A_1 = \int_0^T f(t)\,dB(t) - \int_0^T g(t)\,dB(t)$$

$$= \int_0^T [f(t) - g(t)]\,dB(t). \quad \mathbb{P}\text{-a.s.}$$

Using Ito isometry, we have

$$(A_2 - A_1)^2 = \mathbb{E}\left[(A_2 - A_1)^2\right]$$

$$= \mathbb{E}\left[\left(\int_0^T (f(t) - g(t))\,dB(t)\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^T [f(t) - g(t)]^2\,ds\right]$$

So showing $A_1 = A_2$ is equivalent to showing that $f(t,w) = g(t,w)$ for a.a. $(t,w) \in [0,T] \times \Omega$.

~~Taking derivative w.r.t. T, we~~

~~Suppose $f(t,w) \neq g(t,w)$ for a.a. $(t,w) \in [0,T]\times\Omega$.~~
~~then consider the set of (t,w) where~~

RHS is an Ito integral evaluated at T, so expectation should be 0. since it is a martingale & integral at $t=0$ is 0.

So $E[A_2 - A] = 0 \Rightarrow A_1 = A_2$.

$\Rightarrow$ We are done.

Q4 (a) For a martingale, we need

(1) $X(t)$ adapted to filtration F_t

This is true by definition of 2D Brownian motion

(3) $\mathbb{E}[X(t) | F_s] = X(s)$

$$\mathbb{E}[B_1(t) B_2(t) | F_s]$$

$$= \mathbb{E}[(B_1(t) - B_1(s) + B_1(s))(B_2(t) - B_2(s) + B_2(s)) | F_s]$$

$$= \mathbb{E}[(B_1(t) - B_1(s))(B_2(t) - B_2(s)) + B_1(s)(B_2(t) - B_2(s)) + B_2(s)(B_1(t) - B_1(s)) + B_1(s) B_2(s) | F_s]$$

In the first term, both are independent wrt F_s.
in the second term, $B_1(s)$ in F_s measurable & $B_2(t) - B_2(s)$ is F_s-independent

similar for 3rd term
& last term has $B_1(s) B_2(s)$ F_s measurable.

So the first term contributes $0 \times 0 = 0$
" 2nd " " $E[B_1(s)|F_s] \cdot 0 = 0$
" 3rd " " $E[B_2(s)|F_s] \cdot 0 = 0$
" 4th " " $E[B_1(s) B_2(s) | F_s] = B_1(s) B_2(s)$

$$\Rightarrow \mathbb{E}[B_1(t) B_2(t) | F_s] = B_1(s) B_2(s).$$

(2) $E[|B_1(t) B_2(t)|] < +\infty \quad \forall t$.

$$|B_1(t) B_2(t)| \le \frac{1}{2}(|B_1(t)|^2 + |B_2(t)|^2)$$

$$\Rightarrow E[\quad " \quad] \le \frac{1}{2} E[B_1(t)^2] + \frac{1}{2} E[B_2(t)^2] < +\infty$$

(b): For 1D Brownian motion we need:

(1) Continuous sample paths a.s.: This is true as they are linear comb. of $B_1(t)$ & $B_2(t)$ which have the same property.

(2) should be martingales: Same reason as above.

(3) $M(0) = 0$: $M_1(0) = \frac{1}{\sqrt{5}} B_1(0) + \frac{2}{\sqrt{5}} B_2(0) = 0$

$$M_2(0) = \frac{3}{5} B_1(0) + \frac{4}{5} B_2(0) = 0$$

(4) $[M, M](t) = t \quad \forall\, 0 \le t \le T$

$$dM_1(t)\, dM_1(t) = \left(\frac{1}{\sqrt{5}} dB_1 + \frac{2}{\sqrt{5}} dB_2\right) \cdot \left(\frac{1}{\sqrt{5}} dB_1 + \frac{2}{\sqrt{5}} dB_2\right)$$

$$= \frac{1}{5} dB_1 \cdot dB_1 + \frac{4}{5} dB_2 \cdot dB_2 + \frac{4}{\sqrt{5}} dB_1 \cdot dB_2$$

$= \frac{1}{5} dt + \frac{4}{5} dt + 0 = dt$

Similarly

$dM_2(t) \cdot dM_2(t) = \frac{9}{25} dt + \frac{16}{25} dt + 0 = dt.$

$\rightarrow [M_1, M_1](t) = [M_2, M_2](t) = t.$

So both of these are 1-D Brownian motions.

c): No.

$dM_1(t)\, dM_2(t) = \frac{1}{\sqrt{5}} \cdot \frac{3}{5}\, dB_1(t)\, dB_1(t)$

$+ \frac{2}{\sqrt{5}} \cdot \frac{4}{5}\, dB_2(t) \cdot dB_2(t)$

$+ \left(\frac{1}{\sqrt{5}} \cdot \frac{4}{5} + \frac{2}{\sqrt{5}} \cdot \frac{3}{5}\right) dB_1(t)\, dB_2(t)$

$= \left(\frac{3}{5\sqrt{5}} + \frac{6}{5\sqrt{5}}\right) dt.$

$\Rightarrow [M_1, M_2](t) = \frac{9}{5\sqrt{5}}\, t \neq 0.$

So cross variation is non-zero, hence this is not a 2D Brownian motion.