3.3. **Ito-formula.** In the previous subsection, we have seen that

$$\frac{1}{2}B^2(t) = \frac{t}{2} + \int_0^t B(s)\,dB(s).$$

Thus, the image of Ito-integral $B(t) = \int_0^t dB(s)$ by the map $g(x) = \frac{1}{2}x^2$ is NOT again an Ito-integral of the form $\int_0^t f(s)\,dB(s)$–but a combination of a $dB(s)$-and ds-integral. It turns out that if we introduce Ito processes as a sum of a $dB(s)$-and ds-integral then the family of integrals is stable under smooth maps.

Definition 3.4 (Ito processes)**.** Let $B(t)$ be a Brownian motion and $\mathcal{F}_t$ be its associated filtration. An Ito process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t a(s)\,ds + \int_0^t g(s)\,dB(s) \tag{3.1}$$

where $X(0)$ is nonrandom, $a(s)$ and $g(s)$ are adapted processes such that integral in the right hand side of (3.1) are well-defined, and the Ito-integral is martingale.

We write the Ito process (3.1) in differential form

$$dX(t) = a(t)dt + g(t)dB(t)\,. \tag{3.2}$$

We first determine the quadratic variation of Ito process.

Lemma 3.6. *The quadratic variation of the Ito process* (3.1) *is*

$$[X,X](t) = \int_0^t g^2(s)\,ds.$$

In differential notation,

$$d[X,X](t) = g^2(t)\,dt.$$

Proof. Let $I_g(t) = \int_0^t g(s)\,dB(s)$ and $I_a(t) = \int_0^t a(s)\,ds$. Let $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = t\}$ be partitions of $[0,t]$. Then

$$\begin{aligned}\sum_{j=0}^{m_n-1} |X(t_{j+1}^n) - X(t_j^n)|^2 &= \sum_{j=0}^{m_n-1} |I_g(t_{j+1}^n) - I_g(t_j^n)|^2 + \sum_{j=0}^{m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)|^2 \\ &\quad + 2\sum_{j=0}^{m_n-1} \left|(I_g(t_{j+1}^n) - I_g(t_j^n))(I_a(t_{j+1}^n) - I_a(t_j^n))\right| \equiv \mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n\,.\end{aligned}$$

Observe that

$$\begin{aligned}\mathcal{B}_n &\le \max_{0\le k\le m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \sum_{j=0}^{m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \\ &\le \max_{0\le k\le m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \sum_{j=0}^{m_n-1} \int_{t_j^n}^{t_{j+1}^n} |a(s)|\,ds \\ &= \max_{0\le k\le m_n-1} |I_a(t_{j+1}^n) - I_a(t_j^n)| \int_0^t |a(s)|\,ds \to 0 \quad \text{as } n\to\infty\end{aligned}$$

since $I_a(\cdot)$ is continuous and $\int_0^t |a(s)|\,ds$ is finite for every $t > 0$and a.s. In a similar manner, we can easily show that a.s., $\mathcal{C}_n \to 0$ as $n \to \infty$. In view of Theorem 3.3, we see that a.s., $\mathcal{A}_n \to \int_0^t g^2(s)\,ds$ as $n \to \infty$. Combining these, we get $[X,X](t) = \int_0^t g^2(s)\,ds$. □

We now establish that Ito-process is stable under smooth maps.

Theorem 3.7 (Ito-formula)**.** *Let $X(t)$ be an Ito process given by (3.1) and $f \in C^2([0,\infty)\times\mathbb{R})$. Then $Y(t) := f(t, X(t))$ is again an Ito process and given by its differential form:*

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))\,dt + \frac{\partial f}{\partial x}(t,X(t))\,dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\,d[X,X](t) \\ &= \Big\{\frac{\partial f}{\partial t}(t,X(t)) + \frac{\partial f}{\partial x}(t,X(t))a(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))g^2(t)\Big\}\,dt \\ &\qquad + \frac{\partial f}{\partial x}(t,X(t))g(t)\,dB(t)\,. \end{aligned} \tag{3.3}$$

Proof. We show that $Y(t)$ satisfies the following integral form:

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t \Big(\frac{\partial f}{\partial t}(s,X(s)) + \frac{\partial f}{\partial x}(s,X(s))a(s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,X(s))g^2(s)\Big)\,ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s,X(s))g(s)\,dB(s)\,. \end{aligned} \tag{3.4}$$

We assume that f, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are bounded. For general case, we use approximation arguments: there exist a sequence of C^2-functions $f_n$on $[0,\infty)\times\mathbb{R}$ such that f_n, $\frac{\partial f_n}{\partial t}$, $\frac{\partial f_n}{\partial x}$ and $\frac{\partial^2 f_n}{\partial x^2}$ are bounded for each n and converges uniformly on compact subsets of $[0,\infty)\times\mathbb{R}$ to f, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ respectively. Moreover, we assume that $a(\cdot)$ and $g(\cdot)$ are elementary processes. Let $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = t\}$ be partitions of $[0,t]$. By using Taylor's expansion we have

$$\begin{aligned} f(t,X(t) - f(0,X(0)) &= \sum_{j=0}^{m_n-1} f(t_{j+1}^n, X(t_{j+1}^n)) - f(t_j^n, X(t_j^n)) \\ = \sum_{j=0}^{m_n-1} f_t(t_j^n, X(t_j^n))\Delta t_j^n &+ \sum_{j=0}^{m_n-1} f_x(t_j^n, X(t_j^n))\Delta_n X_j + \frac{1}{2}\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))(\Delta_n X_j)^2 \\ + \sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))\Delta_n X_j \Delta t_j^n &+ \frac{1}{2}\sum_{j=0}^{m_n-1} f_{tt}(t_j^n, X(t_j^n))(\Delta t_j^n)^2 + \sum_{j=0}^{m_n-1} R_j^n\,, \end{aligned}$$

where $\Delta t_j^n := (t_{j+1}^n - t_j^n)$, $\Delta_n X_j = X(t_{j+1}^n) - X(t_j^n)$ and $R_j^n := o\big(|\Delta t_j^n|^2 + |\Delta_n X_j|^2\big)$. One can easily show that

$$\begin{aligned} &\sum_{j=0}^{m_n-1} f_t(t_j^n, X(t_j^n))\Delta t_j^n \stackrel{n\to\infty}{\longrightarrow} \int_0^t f_t(s,X(s))\,ds \\ &\sum_{j=0}^{m_n-1} f_x(t_j^n, X(t_j^n))\Delta_n X_j \stackrel{n\to\infty}{\longrightarrow} \int_0^t f_x(s,X(s))\,dX(s) \\ &\qquad\qquad \equiv \int_0^t f_x(s,X(s))a(s)\,ds + \int_0^t f_x(s,X(s))g(s)\,dB(s) \end{aligned}$$

$$\frac{1}{2}\sum_{j=0}^{m_n-1} f_{tt}(t_j^n, X(t_j^n))(\Delta t_j^n)^2 \le \frac{1}{2}\|\Pi_n\|\Big|\sum_{j=0}^{m_n-1} f_{tt}(t_j^n, X(t_j^n))\Delta t_j^n \overset{n\to\infty}{\longrightarrow} 0.$$

Since $a(\cdot)$ and $g(\cdot)$ are elementary functions we have

$$\Delta_n X_j = a(t_j^n)\Delta t_j^n + g(t_j^n)\Delta_n B_j.$$

Thus, we get

$$\begin{aligned}
&\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))(\Delta_n X_j)^2\\
&= \underbrace{\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))a^2(t_j^n)(\Delta t_j^n)^2}_{:=\mathbf{A}_1} + 2\underbrace{\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Delta t_j^n\Delta_n B_j}_{:=\mathbf{A}_2}\\
&\qquad + \underbrace{\sum_{j=0}^{m_n-1} f_{xx}(t_j^n, X(t_j^n))g^2(t_j^n)(\Delta_n B_j)^2}_{:=\mathbf{A}_3}
\end{aligned}$$

Since f_{xx} is bounded and $a(\cdot) \in \mathcal{Y}$, $\mathbf{A}_1$ goes to 0 as $n \to \infty$. Moreover, $\mathbf{A}_2 \to 0$ in $L^2(\Omega)$. Indeed, by independent properties of Brownian motion, we have

$$\begin{aligned}
\mathbb{E}\big[\mathbf{A}_2^2\big] &= \sum_{j=0}^{m_n-1} \mathbb{E}\Big[\Big(f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Delta t_j^n\Big)^2(\Delta_n B_j)^2\Big]\\
&= \sum_{j=0}^{m_n-1} \mathbb{E}\Big[\Big(f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Delta t_j^n\Big)^2\Big]\mathbb{E}\Big[(\Delta_n B_j)^2\Big]\\
&= \sum_{j=0}^{m_n-1} \mathbb{E}\Big[\Big(f_{xx}(t_j^n, X(t_j^n))a(t_j^n)g(t_j^n)\Big)^2\Big](\Delta t_j^n)^3 \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

We now show that

$$\mathbf{A}_3 \overset{n\to\infty}{\longrightarrow} \int_0^t f_{xx}(s, X(s))g^2(s)\,ds \quad \text{in } L^2(\Omega).$$

Put $\bar{a}(t) = f_{xx}(t, X(t))g^2(t)$. Then

$$\begin{aligned}
&\mathbb{E}\Big[\big(\mathbf{A}_3 - \sum_{j=0}^{m_n-1}\bar{a}(t_j^n)\Delta t_j^n\big)^2\Big] = \mathbb{E}\Big[\big(\sum_{j=0}^{m_n-1}\bar{a}(t_j^n)((\Delta_n B_j)^2 - \Delta t_j^n)\big)^2\Big]\\
&= \sum_{i,j}\mathbb{E}\Big[\bar{a}(t_j^n)\bar{a}(t_i^n)((\Delta_n B_j)^2 - \Delta t_j^n)((\Delta_n B_i)^2 - \Delta t_i^n)\Big]
\end{aligned}$$

If $i < j$, then $\bar{a}(t_j^n)\bar{a}(t_i^n)((\Delta_n B_i)^2 - \Delta t_i^n)$ and $((\Delta_n B_j)^2 - \Delta t_j^n)$ are independent. Thus, we have

$$\mathbb{E}\Big[\big(\mathbf{A}_3 - \sum_{j=0}^{m_n-1}\bar{a}(t_j^n)\Delta t_j^n\big)^2\Big] = \sum_{j=0}^{m_n-1}\mathbb{E}\Big[\bar{a}^2(t_j^n)((\Delta_n B_j)^2 - \Delta t_j^n)^2\Big]$$

$$= \sum_{j=0}^{m_n-1} \mathbb{E}[\bar{a}^2(t_j^n)]\mathbb{E}\Big[(\Delta_n B_j)^4 - 2(\Delta_n B_j)^2\Delta t_j^n + (\Delta t_j^n)^2\Big] = 2\sum_{j=0}^{m_n-1} \mathbb{E}[\bar{a}^2(t_j^n)](\Delta t_j^n) \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that $\mathbf{A}_3 \overset{n\to\infty}{\longrightarrow} \int_0^t f_{xx}(s, X(s))g^2(s)\,ds$ in $L^2(\Omega)$. Notice that the mixed partial derivative term has no counterpart in Ito formula (3.4), so it needs to go way. Indeed, since $a(\cdot)$ and $g(\cdot)$ are elementary, we have

$$\begin{aligned}
&\sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))\Delta_n X_j \Delta t_j^n \\
&= \sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))a(t_j^n)(\Delta t_j^n)^2 + \sum_{j=0}^{m_n-1} f_{tx}(t_j^n, X(t_j^n))g(t_j^n)\Delta t_j^n \Delta_n B_j\,.
\end{aligned}$$

Like in previous arguments, one can easy show that above two terms tend to 0 as $n \to \infty$. Moreover, the argument above also proves that $\sum_{j=0}^{m_n-1} R_j^n \to 0$ as $n \to \infty$. Thus, putting things together, we arrive at Ito formula (3.4). This completes the proof. □

Example 3.2. *Let $B(t)$ be a Brownian motion. Show that*

$$\int_0^t B^2(s)\,dB(s) = \frac{1}{3}B^3(t) - \int_0^t B(s)\,ds.$$

Indeed, applying It-formula for the function $f(x) = x^3$, we have (here Ito process is X(t)=B(t))

$$\begin{aligned}
B^3(t) &= 3\int_0^t B^2(s)\,dB(s) + \frac{1}{2}\int_0^t 6B(s)\,ds \\
\Longrightarrow \int_0^t B^2(s)\,dB(s) &= \frac{1}{3}B^3(t) - \int_0^t B(s)\,ds.
\end{aligned}$$