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MTL733: Assignment-1

Q.1) Let $B(t) : t \geq 0$ be a Brownian motion. Define a stochastic process

$$\bar{B}(t) := \frac{1}{c} B(c^2 t), \quad c > 0.$$

Show that $\bar{B}(t) : t \geq 0$ is also a Brownian motion.

Q.2) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$, and Y be a random variable such that $\mathbb{E}[|Y|] < +\infty$. Define a stochastic process

$$X(t) := \mathbb{E}\big[Y|\mathcal{F}_t\big].$$

Show that $X(t)$ is a martingale.

Q.3) For a Brownian motion $B(\cdot)$, define the processes, for $t \geq 0$

$$X(t) := B^3(t) - 3tB(t); \quad Y(t) := B^4(t) - 6tB^2(t) + 3t^2.$$

Show that both $X(t)$ and $Y(t)$ are martingale with respect to the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$.

Q.4) Let $X(t) : t \geq 0$ be a martingale with respect to a given filtration. Show that

$$\mathbb{E}[X(t)] = \mathbb{E}[X(0)] \quad \forall\, t \geq 0.$$

Is converse true? Justify your answer.

Q.5) A stochastic process $X(t, \cdot) : \Omega \to \mathbb{R}$ is **continuous in mean square** if $\mathbb{E}[|X(t)|^2] < +\infty$ for all $t \geq 0$ and

$$\lim_{s \to t} \mathbb{E}\big[|X(s) - X(t)|^2\big] = 0 \quad \forall\, t \geq 0.$$

Show that $f(B(t))$ is continuous in mean square where f is Lipschitz continuous function and $B(t)$ is Brownian motion.

Q.6) Let $B(t)$ be Brownian motion and $\mathcal{F}_t$ be its natural filtration. Show that the process $X(t)$ defined by

$$X(t) := 2 \exp\{3B(t) + 4t\}$$

is a Markov process.

Q.7) For any function $f : [0, T] \to \mathbb{R}$, we define the p-th variation of f up to time T as

$$\langle f, f \rangle_p(T) := \lim_{\|\Pi\| \to .0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^p$$

where $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ is a partition of $[0, T]$. Show that p-th variation of Brownian motion equals to zero for all $3 \le p < \infty$ and almost every ω.

Q.8) For $r \in \mathbb{R}^+$, consider the stochastic process

$$X(t) = 3 \exp\{(r - 2)t + 2B(t)\}, \quad t \ge 0$$

where $B(t)$ is Brownian motion. Show that for any $t > 0$,

$$\mathbb{E}\Big[e^{-rt}\big(X(t) - 3\big)^+\Big] = 3N(m_+) - 3e^{-rt}N(m_-)$$

where $N(\cdot)$ is the cumulative standard normal distribution function and $m_\pm = \frac{\sqrt{t}}{2}\big(r \pm 2\big)$.

Q.9) Consider a stochastic process

$$X(t) := 2t + B(t), \quad t \ge 0$$

where $B(t)$ is a Brownian motion.

a) Show that the process

$$Z(t) := \exp\{\sigma X(t) - (2\sigma + \frac{1}{2}\sigma^2)t\}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(B(s) : 0 \le s \le t)$.

b) For $m > 0$, define the first passage time of $X(t)$ in level m

$$\tau_m := \min\{t \ge 0 : X(t) = m\}.$$

Show that

$$\mathbb{E}\Big[\exp\{\sigma m - (2\sigma + \frac{1}{2}\sigma^2)\tau_m\}\mathbf{1}_{\{\tau_m < +\infty\}}\Big] = 1.$$

c) Show that

$$\mathbb{E}\big[e^{-\alpha\tau_m}\big] = \exp\{2m - m\sqrt{2\alpha + 4}\}, \quad \forall\ \alpha > 0.$$

Moreover, prove that $\mathbb{E}[\tau_m]. < +\infty$.

Q.10) Define $X(t) := e^{-t}B(e^{2t})$, where $B(\cdot)$ is a Brownian motion. Show that

$$\mathbb{E}[X(t)X(s)] = e^{-|t-s|}, \quad \forall\ -\infty < s, t < \infty.$$