Department of Mathematics, IIT Delhi

MTL733: Assignment-3

Q.1) Find the process $f(t,\omega) \in \mathcal{Y}(0,T)$ such that $F = \mathbb{E}[F] + \int_0^T f(t,\omega)\, dB(t)$ for $F = B^2(T)$ and $F = e^{B(T)}$.

Q.2) Find the Ito representation form for the martingales:

i) $X(t) := B^3(t) - 3tB(t),\ t \geq 0$
ii) $Y(t) := B^4(t) - 6tB^2(t) + 3t^2,\ t \geq 0$
iii) $Z(t) = \mathbb{E}\big[B^2(T)|\mathcal{F}_t\big],\ 0 \leq t \leq T.$

Q.3) Let X be a standard normal random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Find a probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that the random variable

$$Y = X + \theta, \quad 0 \neq \theta \in \mathbb{R}$$

becomes a standard normal under the measure $\bar{\mathbb{P}}$.

Q.4) Consider a 2-dimensional Ito process $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t)\big)$ given by

$$dY_1(t) = dB_1(t) + 3dB_2(t), \quad dY_2(t) = dt - dB_1(t) - 2dB_2(t)$$

where $\mathbf{B}(t) = \big(B_1(t), B_2(t)\big)$ is a 2-dimensional Brownian motion. Find a probability measure $\bar{\mathbb{P}}$ such that $\mathbb{P}$ and $\bar{\mathbb{P}}$ are equivalent, and $\mathbf{Y}(t)$ is a martingale with respect to $\bar{\mathbb{P}}$.

Q.5) Suppose $\mathbf{Y}(t) = \big(Y_1(t), Y_2(t)\big) \in \mathbb{R}^2$ is given by

$$dY_1(t) = \beta_1(t)\, dt + dB_1(t) + 2dB_2(t) + 3dB_3(t)$$
$$dY_2(t) = \beta_2(t)\, dt + dB_1(t) + 2dB_2(t) + 2dB_3(t)$$

where β_1, β_2 are bounded adapted processes and $\mathbf{B}(t) = \big(B_1(t), B_2(t), B_3(t)\big)$ is 3-dimensional Brownian motion. Show that there are infinitely many equivalent martingale measures Q for $\mathbf{Y}(t)$.

Q.6) Let $B(t)$ be a 1-dimensional Brownian motion. Use Girsanov's theorem to evaluate

$$\mathbb{E}\Big[(B^2(T) - T)\exp\{-\int_0^T s^2\, dB(s)\}\Big], \quad \text{for any } T > 0.$$

Q.7) Let $\mathbf{B}(t) := \big(B_1(t), B_2(t)\big) : 0 \le t \le T$ be a 2-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that there exists a probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that the stochastic process $\bar{\mathbf{B}}(t) = \big(\bar{B}_1(t), \bar{B}_2(t)\big) : 0 \le t \le T$ given by

$$\bar{B}_1(t) = B_1(t), \quad \bar{B}_2(t) = B_2(t) + \int_0^t B_1(s)\, ds$$

is a 2-dimensional Brownian motion under $\bar{\mathbb{P}}$. Show that

$$\bar{\mathrm{Cov}}(B_1(T), B_2(T)) \neq \mathrm{Cov}(B_1(T), B_2(T))$$

Q.8) Show that solution of the SDE

$$dX(t) = \kappa(\alpha - \log(X(t)))X(t)\, dt + \sigma X(t)\, dB(t); \quad X(0) = x > 0$$

is given by the formula

$$X(t) = \exp\Big\{e^{-\kappa t}\ln(x) + \big(\alpha - \frac{\sigma^2}{2\kappa}\big)\big(1 - e^{-\kappa t}\big) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\, dB(s)\Big\},$$

where $\sigma, \kappa, \alpha, x$ are positive constant. Find the mean of $X(t)$.

Q.9) Consider a nonlinear SDE of the form

$$dX(t) = f(t, X(t))\, dt + \alpha X(t)\, dB(t), \quad X(0) = x \tag{0.1}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given continuous deterministic function, and $\alpha \in \mathbb{R}$ is a constant.

a) Show that
$$d(F(t)X(t)) = F(t)f(t, X(t))\, dt,$$
where the process $F(t)$ is given by $F(t) = \exp\{-\alpha B(t) + \frac{\alpha^2 t}{2}\}$.

b) Define the process $Y(t) = F(t)X(t)$ so that $X(t) = (F(t))^{-1}Y(t)$. Deduce that $Y(t)$ satisfies a deterministic differential equation in the function $t \mapsto Y(t, \omega)$ for each $\omega \in \Omega$.

Q.10) Use **Q. 9)** to solve the following SDEs:

i) $dX(t) = \frac{1}{X(t)}\, dt + \alpha X(t)\, dB(t); \quad X(0) = x > 0$, where α is a constant.

ii) $dX(t) = X^\gamma(t)\, dt + 4X(t)\, dB(t); \quad X(0) = x > 0$, where γ is a constant.

Q.11) For any positive, smooth function f, show that the process

$$M(t) := f(B(t))\exp\{-\frac{1}{2}\int_0^t f''(B(s))\, ds\}$$

is a martingale.