Example 3.3. *Consider the asset price process given by*

$$S(t) = S(0)\exp\{\int_0^t \sigma(s)\,dB(s) + \int_0^t \big(\alpha(s) - \frac{1}{2}\sigma^2(s)\big)\,ds\}$$

where $S(0)$ is nonrandom and positive and $\alpha(\cdot)$ and $\sigma(\cdot)$ are adapted processes so that integrals are well-defined. Show that $S(t)$ is an Ito process and satisfies the following differential form

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)dB(t).$$

Solution: *First we define an Ito process*

$$X(t) = \int_0^t \sigma(s)\,dB(s) + \int_0^t \big(\alpha(s) - \frac{1}{2}\sigma^2(s)\big)\,ds.$$

Then by given condition $S(t) = S(0)e^{X(t)}$. We may write $S(t)$ as $S(t) = f(X(t))$ where $f(x) = S(0)e^x$. Note that $X(t)$ in the differential form given by

$$dX(t) = \big(\alpha(t) - \frac{1}{2}\sigma^2(t)\big)\,dt + \sigma(t)\,dB(t).$$

By Ito-formula, we have

$$\begin{aligned} dS(t) = df(X(t)) &= \Big(S(0)e^{X(t)}\big(\alpha(t) - \frac{1}{2}\sigma^2(t)\big) + \frac{1}{2}S(0)e^{X(t)}\sigma^2(t)\Big)\,dt + S(0)e^{X(t)}\sigma(t)\,dB(t) \\ &= S(0)e^{X(t)}\alpha(t)\,dt + S(0)e^{X(t)}\sigma(t)\,dB(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t) \end{aligned}$$

Observe that if $\alpha(t) = 0$, then $S(t)$ is given by $S(t) = S(0) + \int_0^t \sigma(s)S(s)\,dB(s)$. Since the second term in the right hand side is martingale and $S(0)$ is nonrandom, we conclude that $S(t)$ is a martingale provided $\sigma(s)S(s) \in \mathcal{Y}(0,T)$ for each $T > 0$.

Remark 3.1. The following Novikov condition

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T u^2(s)\,ds\Big)\Big] < +\infty$$

is a sufficient to guarantee that the process

$$M(t) := \exp\Big\{\int_0^t u(s)\,dB(s) - \frac{1}{2}\int_0^t u^2(s)\,ds\Big\}$$

is a martingale.

Example 3.4. *Let $X(t)$ and $Y(t)$ be real-valued Ito processes. Then show that $X(t)Y(t)$ is again an Ito process and its differential form is given by*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)\cdot dY(t)$$

The above formula is known as Ito product rule. To check this, let $X(t)$ and $Y(t)$ be Ito processes of the form

$$dX(t) = a(t)dt + \sigma(t)dB(t), \quad dY(t) = b(t)dt + \gamma(t)dB(t).$$

Then by applying Ito-formula to the function $f(x) = x^2$, we have

$$\begin{aligned} d[(X(t)+Y(t))^2] &= \{2(X(t)+Y(t))(a(t)+b(t)) + (\sigma(t)+\gamma(t))^2\}dt \\ &\quad + 2(X(t)+Y(t))(\sigma(t)+\gamma(t))\,dB(t) \\ dX^2(t) &= \{2X(t)a(t) + \sigma^2(t)\}dt + 2X(t)\sigma(t)\,dB(t) \end{aligned}$$

$$dY^2(t) = \{2Y(t)b(t) + \gamma^2(t)\}dt + 2Y(t)\gamma(t)\,dB(t)$$

We now use above equations along with the fact that $xy = \frac{1}{2}\big((x+y)^2 - x^2 - y^2\big)$, *we have*

$$\begin{aligned} d[X(t)Y(t)] &= \{X(t)b(t) + Y(t)a(t) + \sigma(t)\gamma(t)\}\,dt + \{X(t)\gamma(t) + Y(t)\sigma(t)\}\,dB(t) \\ &= X(t)\{b(t)dt + \gamma(t)dB(t)\} + Y(t)\{a(t)dt + \gamma(t)dB(t)\} + \sigma(t)\gamma(t)\,dt \\ &= X(t)dY(t) + Y(t)dX(t) + dX(t)\cdot dY(t). \end{aligned}$$

Theorem 3.8 (**Ito integral of a deterministic integrand).** *Let $B(\cdot)$ be a Brownian motion and let $g(s)$ be a nonrandom function of time. Define $I(t) := \int_0^t g(s)\,dB(s)$. For each $t \ge 0$, the random variable $I(t)$ is normally distributed with mean 0 and variance $\int_0^t g^2(s)\,ds$.*

Proof. We have seen that $I(t)$ is a martingale and hence $\mathbb{E}[I(t)] = I(0) = 0$. Moreover, thanks to Ito-isometry, we have

$$\mathrm{Var}[I(t)] = \mathbb{E}[I((t))^2] = \int_0^t g^2(s)\,ds.$$

It remains to show that $I(t)$ is normally distributed. To do so, we show that $I(t)$ has moment-generating function of a normal random variable with mean 0 and variance $\int_0^t g^2(s)\,ds$ i.e., we show that

$$\begin{aligned} &\mathbb{E}[e^{uI(t)}] = \exp\{\frac{1}{2}u^2 \int_0^t g^2(s)\,ds\} \quad \forall\, u \in \mathbb{R} \\ \iff\ &\mathbb{E}\Big[\exp\{uI(t) - \frac{1}{2}u^2 \int_0^t g^2(s)\,ds\}\Big] = 1\,. \end{aligned}$$

This can be written as

$$\mathbb{E}\Big[\exp\{\int_0^t ug(s)\,dB(s) - \frac{1}{2}\int_0^t (ug(s))^2\,ds\}\Big] = 1\,. \tag{3.5}$$

In view of Remark 3.1, the process $Z(t) := \exp\{\int_0^t ug(s)\,dB(s) - \frac{1}{2}\int_0^t (ug(s))^2\,ds\}$ is a martingale and hence we have $\mathbb{E}[Z(t)] = Z(0) = 1$ which gives us (3.5). This completes the proof. $\square$

Example 3.5. *Consider Vasicek model for the interest rate process $R(t)$ given by*

$$dR(t) = (\alpha - \beta R(t))\,dt + \sigma\,dB(t)$$

where α, β and σ are positive constants. Show that $R(t)$ is normally distributed with mean $e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ and variance $\frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$.

Solution: *The interest rate process $R(t)$ is given by (see Assignment-2)*

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dB(s).$$

In view of previous theorem, we see that the random variable $\int_0^t e^{\beta s}\,dB(s)$ is normally distributed with mean 0 and variance $\int_0^t e^{2\beta s}\,ds = \frac{1}{2\beta}(e^{2\beta t} - 1)$. Thus, we conclude that $R(t)$ is normally distributed with mean $e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ and variance $\sigma^2 e^{-2\beta t}\frac{1}{2\beta}(e^{2\beta t} - 1) = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$.

3.4. **Multivariable Stochastic Calculus:** It is straightforward to extend our definitions to Brownian motions taking values in $\mathbb{R}^n$.

Definition 3.5. An $\mathbb{R}^m$-valued stochastic process $\mathbf{B}(t) = \big(B_1(t), B_2(t), \dots, B_m(t)\big)$ is an m-dimensional Wiener process (or Brownian motion) provided

a) each $B_i(t)$ is a one-dimensional Brownian motion
b) for $i \neq j$, the processes $B_i(t)$ and $B_j(t)$ are independent.

Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0,T]$. For $i \neq j$, define the sample cross variation of B_i and B_j on $[0,T]$, denoted by $[B_i, B_j](T)$, as

$$[B_i, B_j](T) = \lim_{\|\Pi\| \to 0} \sum_{k=0}^{n} \big\{B_i(t_{k+1}) - B_i(t_k)\big\}\big\{B_j(t_{k+1}) - B_j(t_k)\big\} := \lim_{\|\Pi\| \to 0} C_\Pi \,.$$

Observe that, since increments of B_i and B_j are independents and all have mean zero, we get that $\mathbb{E}[C_\Pi] = 0$. Again, by using independent increments of B_i and B_j, we obtain

$$\begin{aligned}
\mathrm{Var}(C_\Pi) = \mathbb{E}[C_\Pi^2] &= \mathbb{E}\Big[\sum_{k=0}^{n-1} \big\{B_i(t_{k+1}) - B_i(t_k)\big\}^2 \big\{B_j(t_{k+1}) - B_j(t_k)\big\}^2\Big] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\Big[\big\{B_i(t_{k+1}) - B_i(t_k)\big\}^2\Big] \mathbb{E}\Big[\big\{B_j(t_{k+1}) - B_j(t_k)\big\}^2\Big] \\
&= \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \le \|\Pi\| T \to 0 \text{ as } \|\Pi\| \to 0.
\end{aligned}$$

This implies that the sample cross variation of B_i and B_j is *zero* i.e., $[B_i, B_j](T) = 0$.

Definition 3.6. We define the following:

a) An $M^{n\times m}$-valued stochastic process $\boldsymbol{G} = ((G_{ij}))$ belongs to $\mathcal{Y}_{n\times m}(0,T)$ if
$$G_{ij} \in \mathcal{Y}(0,T) \quad \forall\, 1 \le i \le n,\ 1 \le j \le m.$$
b) An $\mathbb{R}^n$-valued stochastic process $\mathbf{F} = \big(F_1, F_2, \dots, F_n\big)$ belongs to $\mathbb{L}^1_n(0,T)$ if
$$F_i \in \mathbb{L}^1(0,T), \quad 1 \le i \le n.$$
Recall $\mathbb{L}^1(0,T)$ is the space of $\mathcal{F}_t$-adapted, jointly measurable real-valued stochastic process $F(t)$ such that $\mathbb{E}\Big[\int_0^t |F|\, dt\Big] < +\infty$.

Definition 3.7. Let $\mathbf{B} = \big(B_1, B_2, \dots, B_m\big)$ be an m-dimensional Brownian motion. Then for any $\boldsymbol{G} \in \mathcal{Y}_{n\times m}(0,T)$, we define the stochastic integral $\int_0^T \boldsymbol{G}\, d\mathbf{B}$ as an $\mathbb{R}^n$-valued random variable whose i-th component is given by

$$\sum_{j=1}^{m} \int_0^T G_{ij}\, dB_j, \quad 1 \le i \le n.$$

Approximation by step/elementary processes, one can arrive at the following lemma.

Lemma 3.9. *Let* $\mathbf{B} = \big(B_1, B_2, \dots, B_m\big)$ *be an* m*-dimensional Brownian motion and* $\boldsymbol{G} \in \mathcal{Y}_{n\times m}(0,T)$*. Then*

$$\text{i) } \mathbb{E}\Big[\int_0^T \boldsymbol{G}\, d\mathbf{B}\Big] = 0$$

$$\text{ii) } \mathbb{E}\Big[\big|\int_0^T \boldsymbol{G}\, d\mathbf{B}\big|^2\Big] = \mathbb{E}\Big[\int_0^T \|\boldsymbol{G}(s)\|^2\, ds\Big]$$

where $\|\boldsymbol{G}(s)\|^2 := \sum_{1\le i\le n,\ 1\le j\le m} |G_{ij}|^2$.

Definition 3.8 ($\mathbb{R}^n$-valued Ito processes)**.** Let $\mathbf{B}(t)$ be an m-dimensional Brownian motion and $\mathcal{F}_t$ be its associated filtration. An $\mathbb{R}^n$-valued Ito process is a stochastic process $\mathbf{X} = \big(X_1, X_2, \ldots, X_n\big)$ of the form

$$\mathbf{X}(r) = \mathbf{X}(s) + \int_0^t \mathbf{F}(s)\, ds + \int_0^t \boldsymbol{G}(s)\, d\mathbf{B}(s)$$

for some $\mathbf{F} = \big(F_1, F_2, \ldots, F_n\big) \in \mathbb{L}^1_n(0,T)$ and $\boldsymbol{G} = ((G_{ij})) \in \mathcal{Y}_{n\times m}(0,T)$ and for all $0 \le s \le r \le T$. We say that $\mathbf{X}(\cdot)$ has the stochastic differential

$$d\mathbf{X} = \mathbf{F} dt + \boldsymbol{G}\, d\mathbf{B}. \tag{3.6}$$

This means that

$$dX_i(t) = F_i\, dt + \sum_{j=1}^m G_{ij}\, dB_j, \quad 1 \le i \le n\,.$$

Like in one-dimensional case, the family of Ito processes are stable under smooth maps.

Theorem 3.10 (Ito's formula in n-dimension)**.** *Suppose that* $\mathbf{X}(\cdot)$ *is a* n*-dimensional Ito process given in* (3.6)*. Let* $\mathbf{u} = (u_1, u_2, \ldots, u_p) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^p$ *be* C^2*-map. Then* $Y(t) = u(\mathbf{X}(t), t)$ *is an Ito process. Moreover, its stochastic differential form is given by the following formula: for* $1 \le k \le p$,

$$du_k(\mathbf{X}(t), t) = \frac{\partial u_k}{\partial t}(\mathbf{X}(t), t)dt + \sum_{i=1}^n \frac{\partial u_k}{\partial x_i}(\mathbf{X}(t), t)\, dX_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 u_k}{\partial x_i \partial x_j}(\mathbf{X}(t), t) dX_i\, dX_j$$