For any elementary process $G \in \mathcal{Y}$, we define indefinite Ito integral of G as

$$I_G(t) := \int_0^t G\, dB(s) = \sum_{j=0}^{m-1} G_j \big(B(t_{j+1} \wedge t) - B(t_j \wedge t)\big), \quad 0 \le t \le T.$$

Theorem 3.2. *$I_G(t)$ is a martingale.*

Proof. Let $0 \le s \le t \le T$ be given. Assume that s and t are in different sub-interval of the partition Π.. i.e., there exist partition points t_l and t_k with $t_l < t_k$ such that $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. Then, we have

$$\begin{aligned} I_G(t) &= \underbrace{\sum_{j=0}^{l-1} G_j(B(t_{j+1}) - B(t_j))}_{:=\mathcal{A}} + \underbrace{G_l(B(t_{l+1}) - B(t_l))}_{:=\mathcal{B}} + \underbrace{\sum_{j=l+1}^{k-1} G_j(B(t_{j+1}) - B(t_j))}_{:=\mathcal{C}} \\ &\qquad + \underbrace{G_k\big(B(t) - B(t_k)\big)}_{:=\mathcal{D}} \\ I_G(s) &= \sum_{j=0}^{l-1} G_j(B(t_{j+1}) - B(t_j)) + G_l(B(s) - B(t_l)). \end{aligned}$$

We must show that $\mathbb{E}\big[I_G(t)|\mathcal{F}_s\big] = I_G(s)$ a.s. Observe that $\mathcal{A}$ is $\mathcal{F}_s$-measurable and hence $\mathbb{E}\big[\mathcal{A}|\mathcal{F}_s\big] = \mathcal{A}$. Since $B(\cdot)$ is martingale and G_l is $\mathcal{F}_s$-measurable, we have

$$\mathbb{E}\big[\mathcal{B}|\mathcal{F}_s\big] = G_l \mathbb{E}\big[(B(t_{l+1}) - B(t_l))|\mathcal{F}_s\big] = G_l\big(\mathbb{E}\big[B(t_{l+1})|\mathcal{F}_s\big] - B(t_l)\big) = G_l(B(s) - B(t_l))\,.$$

By using tower property of conditional expectation and the fact that $t_j \ge t_{l+1} > s$, we get

$$\begin{aligned} \mathbb{E}\big[\mathcal{C}|\mathcal{F}_s\big] &= \sum_{j=l+1}^{k-1} \mathbb{E}\big[G_j(B(t_{j+1}) - B(t_j))|\mathcal{F}_s\big] \\ &= \sum_{j=l+1}^{k-1} \mathbb{E}\Big[\Big(\mathbb{E}\big[G_j(B(t_{j+1}) - B(t_j))|\mathcal{F}_{t_j}\big]\Big)|\mathcal{F}_s\Big] \\ &= \sum_{j=l+1}^{k-1} \mathbb{E}\Big[\Big(G_j \mathbb{E}\big[(B(t_{j+1}) - B(t_j))|\mathcal{F}_{t_j}\big]\Big)|\mathcal{F}_s\Big] \\ &= \sum_{j=l+1}^{k-1} \mathbb{E}\Big[G_j \underbrace{\mathbb{E}\big[B(t_{j+1}) - B(t_j))\big]}_{=0} |\mathcal{F}_s\Big] = 0. \end{aligned}$$

Similar argument reveals that $\mathbb{E}\big[\mathcal{D}|\mathcal{F}_s\big] = 0$. Combining all these, we have

$$\begin{aligned} \mathbb{E}\big[I_G(t)|\mathcal{F}_s\big] &= \mathbb{E}\big[\mathcal{A}|\mathcal{F}_s\big] + \mathbb{E}\big[\mathcal{B}|\mathcal{F}_s\big] + \mathbb{E}\big[\mathcal{C}|\mathcal{F}_s\big] + \mathbb{E}\big[\mathcal{D}|\mathcal{F}_s\big] \\ &= \sum_{j=0}^{l-1} G_j(B(t_{j+1}) - B(t_j)) + G_l(B(s) - B(t_l)) + 0 + 0 = I_G(s) \quad \text{a.s.} \end{aligned}$$

$\square$

Since $I_G(t)$ is a martingale and $I_G(0)=0$, we have $\mathbb{E}\big[I_G(t)\big]=0$ for all $0\le t\le T$. Moreover, by using Ito-isometry, one can easily see that

$$\mathrm{Var}(I_G(t))=\mathbb{E}\big[I_G^2(t)\big]=\mathbb{E}\Big[\int_0^t G^2(r)\,dr\Big].$$

Next we turn to. the quadratic variation of the Ito-integral $I_G(t)$.

Theorem 3.3. *The quadratic variation of $I_G(\cdot)$ up to time t is given by*

$$[I_G,I_G](t)=\int_0^t G^2(u)\,du.$$

Proof. Suppose $t_k\le t<t_{k+1}$ where $\{t_k\}_{k=0}^m$ is a partition points of $[0,T]$ such that $G=G_k$ on $[t_k,t_{k+1})$. We first compute the quadratic variation on one of the sub-interval $[t_j,t_{j+1}]$. Let $\Pi_j=\{t_j=s_0<s_1<s_2<\ldots<s_{\bar m}=t_{j+1}\}$ is a partition of $[t_j,t_{j+1}]$. Then

$$\sum_{i=0}^{\bar m-1}|I_G(s_{i+1})-I_G(s_i)|^2=\sum_{i=0}^{\bar m-1}G_j^2\big(B(s_{i+1})-B(s_i)\big)^2=G_j^2\sum_{i=0}^{\bar m-1}\big(B(s_{i+1})-B(s_i)\big)^2$$
$$\to G_j^2(t_{j+1})-t_j)=\int_{t_j}^{t_{j+1}}G^2(u)\,du$$

as $\bar m\to\infty$. Analogously, the quadratic variation accumulated by the Ito integral between times t_k and t is $\int_{t_k}^t G^2(u)\,du$. Adding up all these pieces, we obtain

$$[I_G,I_G](t)=\int_0^t G^2(u)\,du.$$

□

3.2. **Ito integral for general integrands.** We now use Ito isometry to extend the definition of Ito integral from elementary functions to functions in $\mathcal{Y}$. We will do this several steps.

Step 1: Let $g\in\mathcal{Y}$ be and a.s. bounded and continuous process. Then there exist elementary functions $g_n\in\mathcal{Y}$ such that $\mathbb{E}\Big[\int_0^T(g-g_n)^2\,dt\Big]\to 0\quad$ as $n\to\infty$. Indeed, by fixing a partition $\Pi_n=\{t_j^n\}$ of $[0,T]$ such that $\Delta_n=\max\{(t_{j+1}^n-t_j^n)\}$ tends to zero as $n\to\infty$, we define

$$g_n(t)=g(t_j^n),\quad t_j^n\le t<t_{j+1}^n.$$

Since a.s., $t\mapsto g(t,\omega)$ is continuous, $g_n(t)\to g(t)$ a.s. Since g is bounded a.s., by bounded convergence theorem, we get $\mathbb{E}\Big[\int_0^T(g-g_n)^2\,dt\Big]\to 0\quad$ as $n\to\infty$.

Step 2: Let $g\in\mathcal{Y}$ be bounded a.s. Then there exists a sequence of a.s. bounded continuous process $g_n\in\mathcal{Y}$ such that $\mathbb{E}\Big[\int_0^T(g-g_n)^2\Big]\to 0\quad$ as $n\to\infty$. To show this, we define

$$g_n(t):=n\int_{t-\frac{1}{n}}^t g(r)\,dr.$$

Since g is a.s. bounded, g_n also a.s. bounded. Moreover, for any $t,s\in[0,T]$, we have $|g_n(t)-g_n(s)|\le 2n|t-s|M$, where $M>0$ such that $|g|\le M$ a.s. Since $g\in\mathcal{Y}$, $g_n\in\mathcal{Y}$.

Furthermore, $g_n(t) \to g(t)$ a.s. Hence, by bounded convergence theorem, we conclude that $\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$.

Step 3: For any $g \in \mathcal{Y}$, there exist a sequence of a.s. bounded process $g_n \in \mathcal{Y}$ such that $\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$. Indeed, by defining

$$g_n(t) = \begin{cases} -n, & g(t) < -n \\ g(t), & -n \le g(t) \le n \\ n, & g(t) > n \end{cases}$$

we see that $g_n \in \mathcal{Y}$ and a.s. bounded. Moreover, $g_n \to g$ a.e. (t,ω). Observe that, since $|g_n(t)| \le |g(t)|$,

$$\int_0^T (g_n - g)^2\,dt \le 2\int_0^T g_n^2\,dt + 2\int_0^T g^2\,dt = 4\int_0^T g^2\,dt.$$

Since $\mathbb{E}\Big[\int_0^T g^2(s)\,ds\Big] < +\infty$, by dominated convergence theorem, we conclude that $\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty$.

In view of **Steps** 1-3, we arrive at the following lemma.

Lemma 3.4 (Approximation by step processes)**.** *For any $g \in \mathcal{Y}$, there exists a sequence of a.s bounded elementary processes $g_n \in \mathcal{Y}$ such that*

$$\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty.$$

Thanks to Ito-isometry, we see that

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T (g_n - g_m)\,dB(t)\Big)^2\Big] &= \mathbb{E}\Big[\int_0^T (g_n-g_m)^2\,dt\Big] \\ &\le 2\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] + 2\mathbb{E}\Big[\int_0^T (g-g_m)^2\,dt\Big] \to 0 \quad \text{as } n,m \to \infty.\end{aligned}$$

Hence $\{\int_0^T g_n\,dB(t)\}_n$ is a Cauchy sequence in $L^2(\Omega)$. Hence $\lim_{n\to\infty}\int_0^T g_n\,dB(t)$ exists.

Definition 3.3 (The Ito integral)**.** For any $g \in \mathcal{Y}$, the Ito integral of g is defined by

$$\int_0^t g\,dB(t) := \lim_{n\to\infty}\int_0^T g_n\,dB(t) \quad (\text{limit in } L^2(\Omega))$$

where $\{g_n\}$ is a sequence of elementary functions such that

$$\mathbb{E}\Big[\int_0^T (g-g_n)^2\,dt\Big] \to 0 \quad \text{as } n \to \infty.$$

In view of Definition 3.3, Lemma 3.1 and Theorem 3.2 , we arrive at the following:

Theorem 3.5. *For any $g \in \mathcal{Y}$,*

$$\mathbb{E}\Big[\Big(\int_0^T g\,dB(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T g^2(t)\,dt\Big] \quad (\textit{ Ito-isometry})$$

Moreover, the process $I_g(t) := \int_0^t g(s)\,dB(s),\ 0 \le t \le T$ is a martingale.

In view of Ito-isometry and the identity $2ab = (a+b)^2 - a^2 - b^2$, we see that for $f, g \in \mathcal{Y}$

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T f\,dB(t)\int_0^T g\,dB(t)\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[\Big(\int_0^T (f+g)\,dB(t)\Big)^2 - \Big(\int_0^T f\,dB(t)\Big)^2 - \Big(\int_0^T g\,dB(t)\Big)^2\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[\int_0^T (f+g)^2\,dt - \int_0^T f^2\,dt - \int_0^T g^2\,dt\Big] = \mathbb{E}\Big[\int_0^T fg\,dt\Big]\,.
\end{aligned}$$

Example 3.1. *Prove directly from the definition of Ito integrals that*

$$\int_0^t s dB(s) = tB(t) - \int_0^t B(s)\,ds.$$

Solution: *Let* $\Pi_n := \{0 = s_0^n < s_1^n < \ldots < s_{m_n}^n = t\}$ *be a partition of* $[0,t]$. *Observe that*

$$s_{j+1}^n B(s_{j+1}^n) - s_j^n B(s_j^n) - s_j^n\big(B(s_{j+1}^n) - B(s_j^n)\big) = B(s_{j+1}^n)(s_{j+1}^n - s_j^n)$$

$$\Longrightarrow \sum_{j=0}^{m_n-1} s_{j+1}^n B(s_{j+1}^n) - s_j^n B(s_j^n) - \sum_{j=0}^{m_n-1} s_j^n\big(B(s_{j+1}^n) - B(s_j^n)\big) = \sum_{j=0}^{m_n-1} B(s_{j+1}^n)(s_{j+1}^n - s_j^n)$$

$$\Longrightarrow tB(t) - \sum_{j=0}^{m_n-1} s_j^n\big(B(s_{j+1}^n) - B(s_j^n)\big) = \sum_{j=0}^{m_n-1} B(s_{j+1}^n)(s_{j+1}^n - s_j^n)$$

Taking limit as $n \to \infty$, *we get*

$$tB(t) - \int_0^t s\,dB(s) = \int_0^t B(s)\,ds\,.$$