MTL733

Q1: (i) We know that

$$E[e^{-\alpha \tau_m}] = e^{-\sqrt{2\alpha}\cdot m}$$

differentiating w.r.t α,

$$E[-\tau_m e^{-\alpha \tau_m}] = -\sqrt{2}m \cdot \frac{1}{2\sqrt{\alpha}} \cdot e^{-\sqrt{2\alpha}\cdot m}$$

Differentiating again w.r.t α,

$$E[\tau_m^2 e^{-\alpha\tau_m}] = \left[-\frac{m}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}\, m}\right]'$$

$$= \frac{-m}{\sqrt{2\alpha^3}} \cdot \left(\frac{-1}{2}\right) \cdot e^{-\sqrt{2\alpha}\, m} - \frac{m}{\sqrt{2\alpha}} \cdot \frac{m}{\sqrt{2\alpha}} \cdot e^{-\sqrt{2\alpha}\, m}$$

$$= m\left(\left(\frac{1}{\sqrt{2\alpha}}\right)^3 - \left(\frac{1}{\sqrt{2\alpha}}\right)^2 m\right) e^{-m\sqrt{2\alpha}}$$

as $\alpha \to 0$, this expected value goes to ∞, so $E[\tau_m^2]$ is not finite.

By induction on k, we can show that $E[\tau_m^k e^{-\alpha\tau_m}]$ is a polynomial in $\frac{1}{\sqrt{2\alpha}}$ times an exponential of the form $e^{-m\sqrt{2\alpha}}$. (degree of polynomial $= 2k-1$)

as $\alpha \to 0$, this polynomial blows up to ∞.

Hence $E[\tau_m^k]$ is not finite either for $k \geq 3$.

(ii) I will drop the (t) to make notation more succint.

$$dX = (1-2X)\,dt + 3\sqrt{X}\,dB \qquad X(0) = 3.$$

(a) let $m(t) = E[X(t)]$.

Since the diffusion term $3\sqrt{X}\,dB$ contributes 0 to the expectation (as it is differential of a martingale), we have

$$dm = (1-2m)\,dt$$

Also $m(0) = E[X(0)] = 3$

$$\Rightarrow -2\int_3^m \frac{dm}{1-2m} = -2\int_0^t dt$$

$$\Rightarrow \ln\left(\frac{m-\frac{1}{2}}{\frac{5}{2}}\right) = -2t \Rightarrow \boxed{m(t) = \frac{1}{2} + \frac{5}{2}e^{-2t}}$$

(b). $Y^2 = e^{4t} X^2$

$$d(Y^2) = e^{4t}(2X\,dX + dX\cdot dX) + 4e^{4t}X^2\,dt$$

$$dX\cdot dX = 3\sqrt{X}\cdot 3\sqrt{X}\,dB\,dB = 9X\,dt$$

$$\Rightarrow d(Y^2) = e^{4t}\left[2X(1-\cancel{2X})\,dt + 6X\sqrt{X}\,dB + 9X\,dt\right] + \cancel{4e^{4t}X^2\,dt}$$

$$= e^{4t}\left[2X + 9X\right]dt \quad e^{4t}\,6X\sqrt{X}\,dB$$

$$= e^{4t}\cdot 11X\,dt + \underbrace{e^{4t}\,6X\sqrt{X}\,dB}$$

contributes 0 to mean since its integral is a martingale with expectation zero

Let $M(t) = E[Y^2(t)]$

$$\frac{dM}{dt} = 11e^{4t}E[X] = \frac{11}{2}\left(e^{4t} + 5e^{2t}\right)$$

$$Y^2(0) = e^{4\times 0}X^2(0) = 9$$

$$\Rightarrow M(0) = 9$$

$$\Rightarrow M - M(0) = \frac{11}{2}\left(\frac{e^{4t}-1}{4} + 5\,\frac{e^{2t}-1}{2}\right)$$

$$\Rightarrow \boxed{M(t) = 9 + \frac{11}{2}\left(\frac{e^{4t}-1}{4} + 5\left(\frac{e^{2t}-1}{2}\right)\right)}$$

(c) Second moment of $X(t)$ is $E[X(t)^2] = \frac{1}{e^{4t}}E[Y^2]$

$$= \boxed{9e^{-4t} + \frac{11}{2}\left(\frac{1-e^{-4t}}{4} + \frac{5}{2}\left(e^{-2t} - e^{-4t}\right)\right)}$$

Q2: ⓐ Consider Girsanov's theorem on $\theta(t) \equiv 5$

Then if $Z(t) = \exp\{-5B(t) - \frac{25}{2}t\}$, then the probability measure Q defined by

$$dQ = Z(t)dP$$

has a brownian motion $\tilde{B}(t) = 5t + B(t) \equiv X(t)$

(and is equivalent to P)

$$\Rightarrow \mathbb{P}(\tau \in dt) = \mathbb{E}_P[1_{\tau \in dt}]$$

$$= \mathbb{E}_Q\left[\frac{1}{Z(t)} 1_{\tau \in dt}\right]$$

(due to continuous paths & definition of Q)

$$= \mathbb{E}_Q[\exp(5\underbrace{(\tilde{B}(t) - 5t)}_{B(t)} + \frac{25}{2}t) 1_{\tau \in dt}]$$

$$= \mathbb{E}_Q[\exp(5\tilde{B}(t) - \frac{25}{2}t) 1_{\tau \in dt}]$$

QED

ⓑ Note that $X(t)$ is in fact a Brownian motion under Q.

For an infinitesimal interval dt, $\tilde{B}(t) =$ ~~[illegible]~~ $X(t) = 3$ by definition of τ, & $\tilde{B}$ has continuous paths.

$$\Rightarrow dP = \exp(15 - \frac{25}{2}t) f_{\tau,Q}(t)\,dt$$

But we know τ is the hitting time of Brownian motion $\tilde{B}$ under Q, so

$$f_{\tau,Q}(t) = \frac{1}{\sqrt{2\pi t^3}} \cdot 3 \cdot e^{-9/2t}$$

$$\Rightarrow f_{\tau,P}(t) = \frac{dP}{dt} = \boxed{\frac{3}{\sqrt{2\pi t^3}} \exp\left(15 - \frac{25}{2}t - \frac{9}{2t}\right)}$$

ⓒ $M(t) = \mathbb{E}[B^2(T) \mid \mathcal{F}_t]$.

We know that if $X(t) = B^2(t)$, (omitting the (t) from now on)

$$dX = 2B\,dB + dB \cdot dB$$

$$= 2B\,dB + dt$$

$$\Rightarrow d(X - t) = 2B\,dB \Rightarrow X(t) - t \text{ is a martingale}$$

$$\Rightarrow B^2(t) - t \text{ is a martingale}$$

$$\Rightarrow M(t) = \mathbb{E}[B^2(T) - T + T \mid \mathcal{F}_t]$$

$$= T + B^2(t) - t$$

$$= T + X(t) - t = T + \int_0^t d(X(s) - s)$$

$$\mathbb{E}[M(t)] = \mathbb{E}[\mathbb{E}[M(t)|\mathcal{F}_0]] = T \qquad = \boxed{T + \int_0^t 2B(s)\,dB(s)}$$

which is the required Ito representation

Q3 (a) We will omit the (t) for the sake of convenience wherever it is obvious.

$$dS = tS\,dt + S\,dB \qquad S(0) = 2$$
$$dS_0 = 3S_0\,dt \qquad S_0(0) = 1.$$

From the second eq^n, we have

$$S_0 = e^{3t}$$

We have market price of risk $\theta(t) = \frac{t-3}{1} = t-3$

Consider the probability measure Q defined by

$$dQ = \cancel{\exp}\ Z(T)\,dP$$

where $Z = \exp\left(-\int_0^t (s-3)\,dB(s) - \frac{1}{2}\int_0^t (s-3)^2\,ds\right)$

By Girsanov's theorem,

$$\bar{B}(t) = B(t) + \int_0^t (s-3)\,ds \text{ is a Brownian motion}$$

under Q.

The discounted stock price process is given by

$$\tilde{S} = D \cdot S \quad \text{where } D(t) = e^{-3t}$$

$$d\tilde{S} = D\,dS + S\,dD + \underbrace{dD \cdot dS}_{0}$$ as dD doesn't have dB component

$$= D\,dS - 3DS\,dt$$

$$= DtS\,dt + DS\,dB - 3DS\,dt$$

Now $d\bar{B} = dB + t - 3$

$$\Rightarrow d\tilde{S} = DS\,(d\bar{B} + \cancel{(-t+3)}) + (\cancel{DtS - 3DS})\,dt$$

$$= DS\,d\bar{B}$$

$\Rightarrow \tilde{S}$ is a martingale under Q

Also since $E[Z] = 1$, & $dQ = Z(T)\,dP$ Q is equivalent to P.

$\Rightarrow Q$ is a risk-neutral measure on $\mathcal{F}_T$ for $X(t)$.

(b) Firstly, we will show that $M(t)Z(t)$ is a martingale under P, where Z is as defined before.

We will revert to using $\theta(t) = t-3$.

$$Z = \exp\left(-\int_0^t (s-3)\,dB(s) - \frac{1}{2}\int_0^t (s-3)^2\,ds\right)$$

$$= \exp\left(-\int_0^t \theta\,dB - \frac{1}{2}\int_0^t \theta^2\,ds\right)$$

$$\cancel{W = -\int_0^t \theta\,dB - \frac{1}{2}\int_0^t \theta^2\,ds}$$

$$\cancel{\Rightarrow dW = -\theta\,dB - \frac{1}{2}\theta^2\,ds}$$

$$\cancel{d(e^W) = e^W dW + \frac{1}{2}e^W\,dW\cdot dW}$$

$$\cancel{=}$$

We know that M is a martingale under Q.

$$\Rightarrow E_Q[M(t) \mid F_s] = M(s)$$

Now we also know

$$E_Q[M(t) \mid F_s] = \frac{1}{Z(s)} E_P[M(t)Z(t) \mid F_s]$$ from properties of equivalent probability measures with Radon Nikodym derivative.

$$\Rightarrow E_P[M(t)Z(t) \mid F_s] = Z(s)\,E_Q[M(t) \mid F_s] = M(s)Z(s)$$

$\Rightarrow$ MZ is a martingale under P.

Now, $M = (MZ)\cdot\frac{1}{Z}$.

$$Y(t) = (1/Z(t)) = \exp\left(\int_0^t \theta\,dB + \frac{1}{2}\int_0^t \theta^2\,ds\right)$$

Let $W(t) = \int_0^t \theta\,dB + \frac{1}{2}\int_0^t \theta^2\,ds$

$$\Rightarrow dW = \theta\,dB + \frac{1}{2}\theta^2\,dt$$

$$dY = d(e^W) = e^W dW + \frac{1}{2}e^W\,dW\cdot dW$$

$$= Y\left(\theta\,dB + \frac{1}{2}\theta^2\,dt + \frac{1}{2}\theta\cdot\theta\,dt\right)$$

$$= Y(\theta\,dB + \theta^2\,dt) = \frac{\theta\,dB + \theta^2\,dt}{Z}$$

Now note that MZ is a martingale under P

$\Rightarrow$ $\exists$ an adapted process $f(t)$ such that

$$M(t)\,Z(t) = M(0)\,Z(0) + \int_0^t f(s)\,dB(s)$$

$$\Rightarrow \quad d(MZ) = f\,dB.$$

$$\Rightarrow \quad dM = d\left(MZ \cdot \frac{1}{Z}\right) = d(MZ)\cdot\frac{1}{Z} + MZ\cdot d\left(\frac{1}{Z}\right) + d(MZ)\,d\left(\frac{1}{Z}\right)$$

$$= \frac{f\,dB}{Z} + M\cancel{Z}\cdot\left(\frac{\theta\,dB + \theta^2\,dt}{\cancel{Z}}\right) + \left(\frac{f\,\theta}{Z}\right)dt$$

Note that $d\bar{B} = dB + \theta\,dt$

$$\Rightarrow \quad dM = \left(\frac{f}{Z} + M\theta\right)dB + \left(M\theta^2 + \frac{f\,\theta}{Z}\right)dt$$

$$= \underbrace{\left(\frac{f}{Z} + M\theta\right)}_{\text{adapted process to the filtration generated by } B} d\bar{B}$$

So if we set $\bar{f} = \frac{f}{Z} + M\theta$, $\quad dM = \bar{f}\,d\bar{B}$

$$\Rightarrow \quad M(t) = E[M(0)] + \int_0^t \bar{f}\,d\bar{B}$$

(the fact that the constant of integration is $E[M(0)]$ can be seen by taking expectation of both sides of the eqn).

Since $\bar{f}$ is an adapted process (to the filtration generated by B), we are done.

Q4 (a) ~~X = e ln S~~

let $Y = \ln S$

$$dY = d(\ln S) = \frac{dS}{S} + \frac{1}{2}\cdot\left(\frac{-1}{S^2}\right) dS\, dS$$

$$= t\,dt + t^2 d\tilde{B} - \frac{1}{2S^2}\left(t^4 S^2 d\tilde{B}\right)$$

$$\cancel{= \left(t\,dt + \left(t^2 - \frac{t^4}{2}\right) d\tilde{B}\right)}$$

$$\cancel{Y(t) = Y(0) + \frac{t^2}{2} + \int_0^t \left(s^2 - \frac{s^4}{2}\right) d\tilde{B}}$$

$$= \left(t - \frac{t^4}{2}\right) dt + t^2 d\tilde{B}$$

$$\rightarrow Y(t) = Y(0) + \frac{t^2}{10}(5 - t^3) + \underbrace{\int_0^t s^2\, d\tilde{B}(s)}_{I}$$

Note that I is an Ito integral of a non random variable, so it is normally distributed with mean 0 & variance $\int_0^t s^4\, ds = \frac{t^5}{5}$

$$\Rightarrow Y(t) = Y(0) + \frac{t^2}{10}(5 - t^3) + I \quad \text{where } I \sim N\left(0, \frac{t^5}{5}\right)$$

$$\Rightarrow S(t) = S(0)\, e^{X(t)}$$

where $X = \frac{t^2}{10}(5 - t^3) + I$ where $I \sim N\left(0, \frac{t^5}{5}\right)$

setting $t = T$,

$X(T)$ is a normal random variable with mean $\frac{T^2}{10}(5 - T^3)$ & variance $\frac{T^5}{5}$,

so $X \sim N\left(\frac{T^2}{10}(5 - T^3), \frac{T^5}{5}\right)$ as needed.

(b): ~~Please~~ on the next page

$$C(0, S(0)) = E_Q\left[e^{-\int_0^T s\,ds} (S(T)-K)^+\right]$$

$$= E_Q\left[e^{-T^2/2} (S(T)-K)^+\right]$$

$$= E_Q\left[e^{-T^2/2} (S(T)-K)\, 1_{S(T)>K}\right]$$

$S(T) = S(0)\, e^{X}$ where $X \sim N\left(\frac{T^2}{10}(5-T^3), \frac{T^5}{5}\right)$

$S(T) > K \Leftrightarrow X > \ln\left(\frac{K}{S(0)}\right)$

Define $X = \frac{T^2}{10}(5-T^3) + \sqrt{\frac{T^5}{5}}\, Y$

where $Y \sim N(0,1)$

Then

$$S(T) > K \Leftrightarrow Y > \frac{\ln\left(\frac{K}{S(0)}\right) - \frac{T^2}{10}(5-T^3)}{\sqrt{\frac{T^5}{5}}} = d$$

$$\Rightarrow C(0, S(0)) = \int_d^\infty \left(\cancel{e^{-T^2/2}}\, S(0)\, e^{\cancel{\frac{T^2}{2}} - T^5/10 + \sqrt{\frac{T^5}{5}}\, y} - e^{-T^2/2} K\right) e^{-\frac{y^2}{2}}\, dy$$

$$= \int_d^\infty S(0)\, e^{-\frac{T^5}{10} + \sqrt{\frac{T^5}{5}}\, y - \frac{y^2}{2}}\, dy$$

$$- \int_d^\infty \left(K e^{-T^2/2}\right) e^{-y^2/2}\, dy$$

$$= A_1 - A_2$$

$$-\frac{T^5}{10} + \sqrt{\frac{T^5}{5}}\, y - \frac{y^2}{2} = -\frac{1}{2}\left(y - \sqrt{\frac{T^5}{5}}\right)^2$$

$$\Rightarrow A_1 = \int_d^\infty S(0)\, e^{-\frac{1}{2}\left(y - \sqrt{T^5/5}\right)^2}\, dy$$

$$= \int_{d - \sqrt{\frac{T^5}{5}}}^\infty S(0)\, e^{-\frac{1}{2}(y)^2}\, dy$$

$$= \int_{-\infty}^{\sqrt{\frac{T^5}{5}} - d} S(0)\, e^{-\frac{1}{2}y^2}\, dy = S(0)\, N\left(\sqrt{\frac{T^5}{5}} - d\right)$$

$$A_2 = K e^{-T^2/2} \int_{-\infty}^{-d} e^{-y^2/2}\, dy$$

$$= K e^{-T^2/2}\, N(-d)$$

$$\Rightarrow\ c(0, S(0)) = A_1 - A_2$$

$$= S(0)\, N\left(\sqrt{\tfrac{TT}{5}} - d\right) - K e^{-T^2/2} N(-d)$$

where $d = \dfrac{\log\left(\frac{K}{S(0)}\right) - \frac{T^2}{10}(5 - T^3)}{\sqrt{\frac{TS}{5}}}$.

Note that if $r = \frac{T}{2}$, $x = S(0)$, $\sigma = \frac{T^2}{\sqrt{5}}$, we have

$$d = -d_2$$

$$\sqrt{\tfrac{TS}{5}} - d = d_2 + \sigma\sqrt{T} = d_1$$

$$S(0) = x$$

$$-T^2/2 = -rT$$

$$\Rightarrow\ c(0, S(0)) = BSM\left(T, S(0); K, \frac{T}{2}, \frac{T^2}{\sqrt{5}}\right)$$ as needed.

(c) Put-call parity formula is as follows:

$$c(t, x) - p(t, x) = x - K e^{-r(T-t)}$$

We know $c(t, x) = 6$

$p(t, x) = 5$

$x = 100$

$K = 100$

$T = 1$, ($t = 0$, implied)

$$\Rightarrow\ 1 = 100(1 - e^{-r}) \Rightarrow r = \ln \frac{100}{99} \text{ per year}$$

$= 0.01005$ per year

$= 1.005\%$ per year

So the risk free interest rate is 1.005% per year.