3.7. **Girsanov's theorem.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and Z be a non-negative random variable with $\mathbb{E}[Z] = 1$. Then we can define another probability measure Q on $(\Omega, \mathcal{F})$ such that $Q << \mathbb{P}$ (read as Q is absolutely continuous with respect to $\mathbb{P}$ i.e., for any measurable set A, $\mathbb{P}(A) = 0$ implies $Q(A) = 0$) by

$$Q(A) = \int_A Z d\mathbb{P}.$$

In this case, we say that Z is the Radon-Nikodyme derivative of Q with respect to $\mathbb{P}$ and denoted by

$$Z = \frac{dQ}{d\mathbb{P}}.$$

For any random variable X, we now have two expectation; one with respect to original probability measure $\mathbb{P}$ and another is with respect to new probability measure Q–denoted it by $\mathbb{E}_Q(\cdot)$.

Lemma 3.16. *For any random variable X, one has $\mathbb{E}_Q[X] = \mathbb{E}[ZX]$. In addition, if $\mathbb{P}(Z > 0) = 1$, then $\mathbb{P}$ and Q are equivalent i.e., $Q << \mathbb{P}$ and $\mathbb{P} << Q$.*

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and Z is as above. Define the Radon-Nikodyme derivative process

$$Z(t) := \mathbb{E}[Z|\mathcal{F}_t], \quad t \geq 0.$$

Then clearly $Z(t)$ is a $\mathcal{F}_t$-martingale.

Lemma 3.17. *Let Y be $\mathcal{F}_t$-measurable random variable and $Z(t)$ is the Radon-Nikodyme derivative process . Then*

a) $\mathbb{E}_Q[Y] = \mathbb{E}[YZ(t)]$.
b) For $0 \leq s \leq t$, $\mathbb{E}_Q[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$.

Proof. In view of previous lemma, properties of conditional expectation, and definition of $Z(t)$, we have

$$\mathbb{E}_Q[Y] = \mathbb{E}[YZ] = \mathbb{E}[\mathbb{E}[YZ|\mathcal{F}_t]] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}_t]] = \mathbb{E}[YZ(t)].$$

To prove b), we need to show

$$\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\, dQ = \int_A Y dQ, \quad \forall\, A \in \mathcal{F}_s.$$

We have

$$\begin{aligned}\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\, dQ &= \mathbb{E}_Q\Big[\mathbf{1}_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\Big] = \mathbb{E}\Big[\mathbf{1}_A\mathbb{E}[YZ(t)|\mathcal{F}_s]\Big] \\ &= \mathbb{E}\big[\mathbb{E}[\mathbf{1}_A YZ(t)|\mathcal{F}_s]\big] = \mathbb{E}\big[\mathbf{1}_A YZ(t)\big] = \mathbb{E}_Q[\mathbf{1}_A Y] = \int_A Y dQ\,.\end{aligned}$$

$\square$

We now state Girsanov's theorem for one dimensional Brownian motion.

Theorem 3.18 (Girsanov's Theorem). *Let $B(t) : 0 \le t \le T$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with Brownian filtration. Let $\theta(t)$ be a adapted process such that*

$$Z(t) := \exp\Big\{ -\int_0^t \theta(s)\, dB(s) - \frac{1}{2}\int_0^t \theta^2(s)\, ds \Big\}$$

becomes a martingale. Then the process

$$\bar{B}(t) = \int_0^t \theta(s)\, ds + B(t), \quad 0 \le t \le T$$

is a Brownian motion with respect to the new probability measure Q where

$$dQ(\omega) = Z(T)\, d\mathbb{P}(\omega).$$

Proof. Note that, since $Z(t)$ is a martingale, we have $\mathbb{E}[Z(T)] = Z(0) = 1$ and $Z(T)$ is $\mathcal{F}_T$-adapted random variable such that it is positive a.s. Thus, Q is a probability measure. We use Levy's theorem to show that $\bar{B}(t)$ is a Brownian motion. Observe that $\bar{B}(0) = 0$ and the quadratic variation of $\bar{B}$ is same as quadratic variation of Brownian motion. Hence it remains to show that $\bar{B}$ is a martingale under Q. Since $Z(t)$ is martingale, we see that $Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t]$ is the Radon-Nikodyme derivative process. Next we claim that $M(t) := \bar{B}(t)Z(t)$ is a martingale under $\mathbb{P}$ (see Assignment 2) . Let $0 \le s \le t$. Then by Lemma 3.17, we have

$$\mathbb{E}_Q[\bar{B}(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[\bar{B}(t)Z(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[M(t)|\mathcal{F}_s] = \frac{M(s)}{Z(s)} = \bar{B}(s).$$

In other words, $\bar{B}(t)$ is martingale under Q. This completes the proof. □

Corollary 3.19. *Let $Y(t); 0 \le t \le T$ be a Ito process*

$$dY(t) = \beta(t)\, dt + \theta(t) dB(t)$$

and there exist adapted processes $u(\cdot)$ and $\alpha(\cdot)$ such that

$$Z(t) := \exp\Big\{ -\int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u^2(s)\, ds \Big\}$$

becomes a martingale, and

$$\theta(t)u(t) = \beta(t) - \alpha(t).$$

Then $\bar{B}(t) = B(t) + \int_0^t u(s)\, ds$ is a Brownian motion under the new probability measure Q given by

$$dQ(\omega) = Z(T)\, d\mathbb{P}(\omega).$$

Moreover, in terms of $\bar{B}(\cdot)$, the process $Y(t)$ has the stochastic integral representation

$$Y(t) = Y(0) + \int_0^t \alpha(s)\, ds + \int_0^t \theta(s)\, d\bar{B}(s).$$

Proof. From Girsanov's theorem (cf. Theorem 3.18), it follows that $\bar{B}(t) : 0 \le t \le T$ is a Brownian motion under the new probability measure Q. Moreover, we have

$$\begin{aligned} dY(t) &= \beta(t)\, dt + \theta(t) dB(t) = \beta(t)\, dt + \theta(t)\{d\bar{B}(t) - u(t)\, dt\} \\ &= \{\beta(t) - \theta(t)u(t)\}\, dt + \theta(t)\, d\bar{B}(t) = \alpha(t)\, dt + \theta(t)\, d\bar{B}(t) \end{aligned}$$

$$\implies Y(t) = Y(0) + \int_0^t \alpha(s)\,ds + \int_0^t \theta(s)\,d\bar{B}(s)\,.$$

$\square$

Example 3.13. *Let $Y(t) = t + B(t) : t \ge 0$. For each $T > 0$, find a probability measure Q_T on $\mathcal{F}_T$ such that $Y(t) : 0 \le t \le T$ is a Brownian motion under Q_T. Show that there exists a probability measure Q on $\mathcal{F}_\infty$ such that*

$$Q\Big|_{\mathcal{F}_T} = Q_T \ \forall\ T > 0.$$

Solution: *Taking $\theta(t) = 1$ in the Girsanov's theorem (cf. Theorem 3.18) , we see that $Y(t) = t + B(t)$ is a Brownian motion under the probability measure Q_T on $\mathcal{F}_T$, where Q_T is given by*

$$dQ_T(\omega) = Z(T)\,d\mathbb{P}(\omega); \quad Z(t) = e^{-B(t)-\frac{t}{2}}.$$

Note here that $Z(t)$ is a martingale under $\mathbb{P}$. To prove the second part, we first show that

$$Z(T)\,d\mathbb{P} = Z(t)\,d\mathbb{P} \text{ on } \mathcal{F}_t,\ t \le T.$$

Indeed, for any bounded $\mathcal{F}_t$- measurable function, one has

$$\begin{aligned}\int_\Omega f Z(T)\,d\mathbb{P} &= \mathbb{E}[fZ(T)] = \mathbb{E}\big[\mathbb{E}[fZ(T)|\mathcal{F}_t]\big] \\ &= \mathbb{E}\big[f\mathbb{E}[Z(T)|\mathcal{F}_t]\big] = \mathbb{E}[fZ(t)] = \int_\Omega fZ(t)\,d\mathbb{P}\,.\end{aligned}$$

Thus, $Q_T = Q_S$ on $\mathcal{F}_t$ for all $t \le \min\{T, S\}$. Hence there exists Q on $\mathcal{F}_\infty$ such that $Q = Q_T$ on $\mathcal{F}_T$ for all $T < \infty$. Hence the result follows.

Example 3.14. *Find a probability measure Q on $\mathcal{F}_T$ such that the process $Y(t) : 0 \le t \le T$ given by*

$$dY(t) = t\,dt + (2t+1)\,dB(t)$$

becomes a martingale under Q.

Solution: *Observe that $s \mapsto \frac{s^2}{(2s+1)^2}$ is continuous and therefore $\int_0^T \frac{s^2}{(2s+1)^2}\,ds < +\infty$. Hence the stochastic process*

$$Z(t) = \exp\Big\{ \int_0^t \frac{s}{2s+1}\,dB(s) - \frac{1}{2}\int_0^t \frac{s^2}{(2s+1)^2}\,ds \Big\}$$

is a martingale under $\mathbb{P}$ and $\mathbb{E}[Z(T)] = 1$. Define $dQ(\omega) = Z(T)\,d\mathbb{P}(\omega)$. Then Q is a probability measure and by Girsanov's theorem (cf. Theorem 3.18), the process

$$\bar{B}(t) = B(t) + \int_0^t \frac{s}{2s+1}\,ds : 0 \le t \le T$$

is a Brownian motion under Q. The process $Y(t)$ can be expressed in terms of $\bar{B}(t)$ as

$$\begin{aligned} dY(t) &= t\,dt + (2t+1)\,dB(t) = t\,dt + (2t+1)\{d\bar{B}(t) - \frac{t}{2t+1}\,dt\} = (2t+1)\,d\bar{B}(t) \\ &\implies Y(t) = Y(0) + \int_0^t (2s+1)\,d\bar{B}(s)\,. \end{aligned}$$

Hence $Y(t) : 0 \le t \le T$ is a martingale under the probability measure Q.