4.1. **Markov property:** Consider the SDE (4.1). We denote by $X^{0,x}(t)$ as solution of (4.1) staring from $t=0$ at position x. By uniqueness of solution one can see that, at least for time-homogeneous drift and diffusion coefficients,

$$X^{0,x}(t+s) = X^{t,X(t)}(s) \quad \forall t, s \geq 0.$$

Let $0 \leq t \leq T$ be given and for any given Borel-measurable function $h(\cdot)$ on $\mathbb{R}^n$, we define the function

$$g(t,x) := \mathbb{E}^{t,x}[h(X(T))]$$

the expectation of $h(X(T))$, where $X(T)$ is a solution of (4.1) at time T with initial condition $X(t) = x$. In other words, $g(t,x) = \mathbb{E}[h(X(T))|X(t) = x]$.

Theorem 4.3. *Let $X(u) : u \geq 0$ be a solution of SDE (4.1) with initial condition at time 0. Then for $0 \leq .t \leq T$,*

$$\mathbb{E}[h(X(T))|\mathcal{F}_t] = g(t, X(t)) \text{ a.s.}$$

In view of Theorem 4.3, and the properties of conditional expectation, we see that $g(t, X(t))$ is a martingale. Indeed, for $0 \leq s \leq t$,

$$\mathbb{E}[g(t,X(t))|\mathcal{F}_s] = \mathbb{E}\Big[\mathbb{E}[h(X(T))|\mathcal{F}_t]\big|\mathcal{F}_s\Big] = \mathbb{E}[h(X(T))|\mathcal{F}_s] = g(s, X(s)) \text{ a.s.}$$

4.2. **Connection with partial differential equation (PDE).** Feynman-Kac theorem below relates SDE and partial differential equation (PDE).

Theorem 4.4 (Feynman-Kac)**.** *Let $0 \leq t \leq T$ be given. For any given Borel-measurable function $h(\cdot)$ on $\mathbb{R}$, we define the function*

$$g(t,x) := \mathbb{E}^{t,x}[h(X(T))]$$

where $X(\cdot)$ is a solution of (4.1). Then $g(t,x)$ satisfies the following backward PDE

$$\begin{cases} g_t(t,x) + a(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0 \\ g(T,x) = h(x), \quad x \in \mathbb{R}\,. \end{cases}$$

Proof. We outline the proof of this theorem. Let $X(t)$ be a solution of SDE (4.1) starting at time 0. First observe that $g(T,x) = h(x)$. Since $g(t, X(t))$ is a martingale, the net dt-term in the differential form of $d(g, X(t))$ must be zero. Now, by Ito formula, we get

$$dg(t,X(t)) = \Big\{g_t(t,X(t)) + a(t,X(t))g_x(t,X(t)) + \frac{1}{2}\sigma^2(t,X(t))g_{xx}(t,X(t))\Big\}\,dt \\ + \sigma(t,X(t))\,dB(t)$$

Setting dt-term to zero, we have

$$g_t(t,X(t)) + a(t,X(t))g_x(t,X(t)) + \frac{1}{2}\sigma^2(t,X(t))g_{xx}(t,X(t)) = 0$$

along every path of $X(t)$. Therefore, we have

$$g_t(t,x) + a(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0$$

at every point (t,x) which can be reached by $(t, X(t))$. This completes the proof. $\square$

The general principle behind the proof of Feynman-Kac theorem is to find martingale, take the differential and then set dt-term equals to zero.

Example 4.3. *Let $Y(t): 0 \leq t \leq T$ be a unique solution of the SDE*

$$dX(t) = r\,dt + \alpha X(t)\,dB(t), \quad r, \alpha \in \mathbb{R}.$$

Then the function $g(t,x) = \mathbb{E}^{t,x}[h(X(T))]$ solves the PDE:

$$\begin{cases} g_t(t,x) + r g_x(t,x) + \frac{1}{2}\alpha^2 x^2 g_{xx}(t,x) = 0 \\ g(T,x) = h(x), \quad x \in \mathbb{R} \end{cases}$$

where h is a Borel-measurable function.

Example 4.4. *Let Let $X(t): 0 \leq t \leq T$ be a solution of SDE (4.1) starting at time 0. For any Borel-measurable function h, define*

$$f(t,x) := \mathbb{E}^{t,x}[e^{-r(T-t)}h(X(T))], \quad r \in \mathbb{R}^*.$$

Then $f(t,x)$ solves the PDE

$$\begin{cases} f_t(t,x) + a(t,x) f_x(t,x) + \frac{1}{2}\sigma^2(t,x) f_{xx}(t,x) = r f(t,x) \\ f(T,x) = h(x), \quad x \in \mathbb{R}\,. \end{cases}$$

Observe that $f(t,X(t)) = \mathbb{E}[e^{-r(T-t)}h(X(T))|\mathcal{F}_t]$, and hence $f(t,X(t))$ is NOT a martingale. But $e^{-rt}f(t,X(t))$ is a martingale. Hence applying Ito product rule, we get

$$d\big(e^{-rt}f(t,X(t))\big) = e^{-rt}\Big(-rf(t,X(t)) + f_t(t,X(t)) + a(t,X(t))f_x(t,X(t)) \\ + \frac{1}{2}\sigma^2(t,X(t))f_{xx}(t,X(t))\Big)\,dt + e^{-rt}\sigma(t,X(t))\,dB(t)$$

Setting dt-term to zero, we get the desired result.

5. Application to Finance

Mathematical finance is the study of financial markets and is one of the rapidly growing subjects in applied mathematics. Suppose that two assets are traded: one risk free and one risk security. The risk-free asset can be thought of as a bank deposit or a bond issued by a government. The risky security will typically be some stock.

Consider a money market account with variable interest rate $r(t)$. Let the price of money market account at time t is $S_0(t)$. Assume that $S_0(t)$ is determined by the differential equation

$$dS_0(t) = r(t)S_0(t)\,dt, \quad S_0(0) = 1. \tag{5.1}$$

Then $S_0(t)$ is given by

$$S_0(t) = \exp\Big\{\int_0^t r(s)\,ds\Big\}.$$

Consider the risky asset. The price of stock at time t will be denoted by $S(t)$. The future price $S(t)$ for $t > 0$ remains unknown in general. Mathematically $S(t)$ can be represented as a positive random variable on a probability space Ω i.e.,

$$S(t) : \Omega \to (0, \infty).$$

The probability space Ω consists of all feasible price movement scenarios $\omega \in \Omega$. The behaviour of the stock price is determined by the stochastic differential equation

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)\,dB(t); \quad S(0) = x > 0 \tag{5.2}$$

where $B(t)$ is one-dimensional Brownian motion, $\alpha(t)$ and $\sigma(t)$ are adapted processes. The asset price $S(t)$ has *instantaneous* mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. The word *instantaneous* signifies that $\alpha(t)$ and $\sigma(t)$ depends on the time and sample paths.

Remark 5.1. If α and σ are constants, we have the usual geometric Brownian motion model and the distribution of $S(t)$ is log-normal. In general $S(t)$ does not need to be log-normal because α and σ are allowed to be time-varying and random.

Suppose we have an adapted interest rate process $r(t)$. We define the discount process

$$D(t) = \exp\{-\int_0^t r(s)\, ds\}.$$

One can easily check, by applying Ito-formula, that

$$dD(t) = -r(t)D(t)\, dt.$$

Observe that, because of smoothness, $D(t)$ has zero quadratic variation. Note also that $S_0(t) = \frac{1}{D(t)}$.

Definition 5.1. The Ito-process $X(t) = \big(S_0(t), S(t)\big)$ where $S_0(t)$ and $S(t)$ satisfies (5.1) and (5.2) respectively is called a **market**.

The financial derivatives or financial securities are financial contracts whose value is derived from some underlying assets. In general financial derivatives can be grouped into three groups: **options, forwards, and futures**. We will be mainly discussing the options. The options constitutes an important building block for pricing financial derivatives.

Definition 5.2. An option is a financial contract that gives the holder the right (but not the obligation) to buy or sell some underlying asset at a specific price (called **strike price**) and specific date (called **expiry date**). There are two main types of option contract.

i) **Call option:** it gives the holder the right to buy a stock at a strike price within the expiry date.
ii) **Put option:** it gives the holder the right to sell some asset at a strike price within the expiry date.