STOCHASTIC OF FINANCE (MTL733)

ANANTA KUMAR MAJEE

1. Introduction

The theory of stochastic processes turns to be a useful tool in solving problems in various fields such as engineering, genetics, statistics, economics, finance, etc. The word *stochastic* means *random* or *chance*. Stochastic processes can be thought of collection of random variables indexes by some parameter.

Definition 1.1 (**Stochastic process).** A real stochastic process is a collection of random variables $\{X(t) : t \in \mathbf{T}\}$ defined on a common probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with values in $\mathbb{R}$. For each $\omega \in \Omega$, the mapping $t \mapsto X(t, \omega)$ is the corresponding sample path. In general we observe a different sample path.

- $\mathbf{T}$ is called the index set of the process which is usually a subset of $\mathbb{R}$.
- $\{X(t) : t \in \mathbf{T}\}$ is said to be continuous stochastic process if its sample function $X(t, \omega)$ is a continuous function of $t \in \mathbf{T}$ for almost every $\omega \in \Omega$.

Definition 1.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A **filtration** is a family of sub-σ algebras $\{\mathcal{F}_t : t \geq 0\}$ of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.
- A stochastic process $X(t)$ is said to be **adapted to the filtration** $\{\mathcal{F}_t\}$ if for each t, $X(t)$ is $\mathcal{F}_t$-measurable i.e., $\{\omega \in \Omega : X(t, \omega) \in B\} \in \mathcal{F}_t$ for any Borel subset B of $\mathbb{R}$.

We think of $X(t)$ as the price of some asset at time t and $\mathcal{F}_t$ as the information obtained by watching all the prices in the market up to time t.

Example 1.1. *Let $X(\cdot)$ be a real-valued stochastic process. Define*

$$\mathcal{F}_t := \sigma(X(s) : 0 \leq s \leq t)$$

Then $\mathcal{F}_t$ is a filtration. Moreover, $X(t)$ is adapted to the filtration $\mathcal{F}_t$. This filtration is called the history of the process until (and including) time $t \geq 0$. Sometimes it is called **natural filtration** *generated by the process $X(\cdot)$.*

1.1. Conditional expectation.

We now recall definition of conditional expectation and its important properties.

Definition 1.3 (Conditional expectation)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{Y}$ is a sub-σ-algebra of $\mathcal{F}$. For any integrable random variable X, we define conditional expectation of X given the σ-algebra $\mathcal{Y}$, denoted by $\mathbb{E}(X|\mathcal{Y})$ to be a $\mathcal{Y}$-measurable random variable such that

$$\int_A X \, d\mathbb{P} = \int_A \mathbb{E}(X|\mathcal{Y}) \, d\mathbb{P} \quad \forall \, A \in \mathcal{Y}.$$

One can easily check that

i) $\mathbb{E}[\mathbb{E}(X|\mathcal{Y})] = \mathbb{E}[X]$.
ii) $\mathbb{E}(X|\mathcal{W}) = \mathbb{E}[X]$, where $\mathcal{W} = \{\emptyset, \Omega\}$ is the trivial σ-algebra.

Theorem 1.1. *Let X be an integrable random variable. Then for each σ-algebra $\mathcal{Y} \subset \mathcal{F}$, the conditional expectation $\mathbb{E}(X|\mathcal{Y})$ exists and is unique up to $\mathcal{Y}$-measurable sets of probability zero.*

Properties of conditional expectation: We now state some important properties of conditional expectation:

a) For any $\mathcal{Y}$-measurable integrable random variable, we have $\mathbb{E}(X|\mathcal{Y}) = X$ a.s.
b) **Linearity:** for $a, b \in \mathbb{R}$, $\mathbb{E}(aX + bY|\mathcal{Y}) = a\mathbb{E}(X|\mathcal{Y}) + b\mathbb{E}(Y|\mathcal{Y})$ a.s..
c) If X is $\mathcal{Y}$-measurable and XY is integrable, then $\mathbb{E}(XY|\mathcal{Y}) = X\mathbb{E}(Y|\mathcal{Y})$ a.s..
d) **Tower property:** For $\mathcal{W} \subset \mathcal{Y}$, we have
$$\mathbb{E}(X|\mathcal{W}) = \mathbb{E}(\mathbb{E}(X|\mathcal{Y})|\mathcal{W}) = \mathbb{E}(\mathbb{E}(X|\mathcal{W})|\mathcal{Y}) \text{ a.s..}$$
e) **Monotonicity:** $X \le Y$ a.s. implies that $\mathbb{E}(X|\mathcal{Y}) \le \mathbb{E}(Y|\mathcal{Y})$ a.s.

Lemma 1.2 (Conditional Jensen's inequality:)**.** *Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is convex with $\mathbb{E}[|\phi(X)|] < +\infty$. Then*
$$\phi(\mathbb{E}(X|\mathcal{Y})) \le \mathbb{E}(\phi(X)|\mathcal{Y}) \text{ a.s..}$$

Definition 1.4 (Martingale)**.** A stochastic process $\{X(t) : t \in \mathbf{T}\}$ is called a **martingale** with respect to a filtration $\{\mathcal{F}_t\}$ if

a) $\{X(t) : t \in \mathbf{T}\}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbf{T}}$
b) $\mathbb{E}[|X(t)|] < +\infty$ for all $t \in \mathbf{T}$.
c) for all $s \le t$, $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s.

- If condition *c*) is replaced by: $\mathbb{E}[X(t)|\mathcal{F}_s] \ge X(s)$ a.s. for all $s \le t$, then it is called **submartingale**.
- If condition *c*) is replaced by: $\mathbb{E}[X(t)|\mathcal{F}_s] \le X(s)$ a.s. for all $s \le t$, then it is called **supermartingale**.

Observe that if $X(t)$ is a martingale, then $\mathbb{E}[X(t)]$ is constant.

Example 1.2. *Let $\{X(t) : t \ge 0\}$ be a stochastic process with stationary and independent increments such that $\mathbb{E}[|X(t)|] < +\infty$ for all $t \ge 0$. Then $\{X(t) : t \ge 0\}$ is a martingale/submartingale/supermartingale with respect to its natural filtration if $\mathbb{E}[X(t)] = 0/\, \mathbb{E}[X(t)] \ge 0/\, \mathbb{E}[X(t)] \le 0$ for all $t \ge 0$ respectively.*

Solution: *For $s \le t$, we have*
$$\begin{aligned}\mathbb{E}[X(t)|\mathcal{F}_s] &= \mathbb{E}[X(t) - X(s) + X(s)|\mathcal{F}_s] = \mathbb{E}[X(t) - X(s)|\mathcal{F}_s] + \mathbb{E}[X(s)|\mathcal{F}_s] \\ &= \mathbb{E}[X(t) - X(s)] + X(s) = \mathbb{E}[X(t-s)] + X(s) \text{ a.s.}\end{aligned}$$
Thus, we have
$$\mathbb{E}[X(t)|\mathcal{F}_s] \begin{cases} = X(s) \text{ a.s.} & \text{if } \mathbb{E}[X(t)] = 0 \\ \ge X(s) \text{ a.s.} & \text{if } \mathbb{E}[X(t)] \ge 0 \\ \le X(s) \text{ a.s.} & \text{if } \mathbb{E}[X(t)] \le 0. \end{cases}$$

Remark 1.1. Poisson process $N(t)$ with respect to its natural filtration is a submartingale .

Definition 1.5 (Stopping time)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space equipped with the filtration $\{\mathcal{F}_t\}$. A random time $\tau : \Omega \mapsto [0, \infty]$ is called a **stopping time** with respect to the filtration $\mathcal{F}_t$ if
$$\{\tau \le t\} \in \mathcal{F}_t \quad \forall\, t \ge 0.$$

Every random time equal to a nonnegative constant is a stopping time.

2. Brownian motion

We will discuss a more useful stochastic process so called Brownian motion and its essential properties. R. Brown in 1826-27 observed the irregular motion of pollen particles suspended in water. He and others noted that

a) the path of a given particle is very irregular, having a tangent at no point, and
b) the motions of two distinct particles appear to be independent.

The mathematical description of the above observation can be given as follows:

Definition 2.1 (Brownian motion)**.** A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a Brownian motion if

a) $B(0) = 0$ a.s.
b) $B(t) - B(s)$ is $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$
c) for all times $0 < t_1 < t_2 < \ldots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent.
d) sample paths are continuous with probability 1.

Notice in particular that

$$\mathbb{E}[B(t)] = 0, \quad \mathbb{E}[B^2(t)] = t \text{ for each time } t \geq 0.$$

Again, from the definition, we see that for all $t > 0$ and $a \geq b$

$$\mathbb{P}(a \leq B(t) \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}}\, dx\,.$$

Moreover, we can find the joint probability as follows: for $0 < t_1 < t_2 < \ldots < t_n$ and $a_i \leq b_i$ $i = 1, 2, \ldots, n$, the joint probability is given by

$$\mathbb{P}\Big(a_1 \leq B(t_1) \leq b_1, a_2 \leq B(t_2) \leq b_2, \ldots, a_n \leq B(t_n) \leq b_n\Big)$$
$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_n}^{b_n} g(x_1, t_1|0) g(x_2, t_2 - t_1|x_1) \ldots g(x_n, t_n - t_{n-1}|x_{n-1}) dx_n\, dx_{n-1} \ldots dx_2\, dx_1$$

where $g(x, t|y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$.

Remark 2.1. In view of Example 1.2, it is easy to see that Brownian motion is a martingale with respect to its natural filtration.

Theorem 2.1. *Let $B(t)$ be a Brownian motion. Then the processes $X(t) := B^2(t) - t$ and $M(t) := \exp(\sigma B(t) - \frac{1}{2}\sigma^2 t)$, $\sigma \in \mathbb{R}$ are martingale with respect to its natural filtration $\mathcal{F}_t$. $M(t)$ is called* **exponential martingale***.*

Proof. First we prove that $X(t)$ is a martingale. For $t \geq s \geq 0$, we have

$$\begin{aligned}
\mathbb{E}[X(t)|\mathcal{F}_s] &= \mathbb{E}[B^2(t) - B^2(s) + B^2(s) - t|\mathcal{F}_s] = \mathbb{E}[(B^2(t) - B^2(s))|\mathcal{F}_s] + B^2(s) - t \\
&= \mathbb{E}[\{(B(t) - B(s))^2 + 2B(s)(B(t) - B(s))\}|\mathcal{F}_s] + B^2(s) - t \\
&= \mathbb{E}[\{(B(t) - B(s))^2|\mathcal{F}_s] + 2\mathbb{E}[B(s)(B(t) - B(s))|\mathcal{F}_s] + B^2(s) - t
\end{aligned}$$

Since $B(t) - B(s)$ is $\mathcal{F}_s$ independent and $B(s)$ is $\mathcal{F}_s$-measurable, by using properties of conditional expectation, we have

$$\mathbb{E}[\{(B(t) - B(s))^2|\mathcal{F}_s] = \mathbb{E}[(B(t) - B(s))^2] = t - s$$
$$\mathbb{E}[B(s)(B(t) - B(s))|\mathcal{F}_s] = B(s)\mathbb{E}[(B(t) - B(s))|\mathcal{F}_s] = B(s)\mathbb{E}[B(t) - B(s)] = 0.$$

Thus, we obtain

$$\mathbb{E}[X(t)|\mathcal{F}_s] = (t - s) + B^2(s) - t = X(s) \text{ a.s.}$$

This shows that $X(t)$ is a martingale.
We now show that $M(t)$ is a martingale. First of all, $M(t)$ is integrable. Indeed,

$$\mathbb{E}[e^{\sigma B(t)}] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x} e^{-\frac{x^2}{2t}}\, dx = e^{\frac{1}{2}\sigma^2 t} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma t)^2}{2t}}\, dx = e^{\frac{1}{2}\sigma^2 t}$$

$$\implies \mathbb{E}[M(t)] = 1.$$

Since for any $t \ge s \ge 0$, $B(t) - B(s) \sim \mathcal{N}(0, t-s)$, by using similar calculation, we see that

$$\mathbb{E}\Big[\exp\big\{\sigma(B(t) - B(s))\big\}\Big] = \exp\big\{\frac{1}{2}\sigma^2(t-s)\big\}.$$

We have

$$\begin{aligned}
\mathbb{E}\Big[M(t)|\mathcal{F}_s\Big] &= \mathbb{E}\Big[\exp\{\sigma(B(t) - B(s))\}\exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}|\mathcal{F}_s\Big] \\
&= \exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}\mathbb{E}\Big[\exp\{\sigma(B(t) - B(s))\}|\mathcal{F}_s\Big] \\
&= \exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}\mathbb{E}\big[\exp\{\sigma(B(t) - B(s))\}\big] \\
&= \exp\{\sigma B(s) - \frac{1}{2}\sigma^2 t\}\exp\big\{\frac{1}{2}\sigma^2(t-s)\big\} = M(s) \text{ a.s.}
\end{aligned}$$

Thus, $M(t)$ is a martingale. □

Lemma 2.2. *Let $B(\cdot)$ be a one-dimensional Brownian motion. Then*

$$\mathbb{E}[B(t)B(s)] = t \wedge s = \min\{t, s\} \quad t, s \ge 0.$$

Proof. Assume that $t \ge s \ge 0$. Then

$$\begin{aligned}
\mathbb{E}[B(t)B(s)] &= \mathbb{E}[(B(s) + B(t) - B(s))B(s)] = \mathbb{E}[B^2(s)] + \mathbb{E}[(B(t) - B(s))B(s)] \\
&= s + \mathbb{E}[B(t) - B(s)]\mathbb{E}[B(s)] = s = t \wedge s.
\end{aligned}$$

In the above, we have used the fact that $B(s)$ is normally distributed with mean zero and variance s and $B(t) - B(s)$ is independent of $B(s)$. □

Example 2.1. *Let $B(t)$ be a Brownian motion . Define a stochastic process*

$$X(t) = \begin{cases} 0, & t = 0 \\ tB(\frac{1}{t}), & t > 0. \end{cases}$$

Then $X(t)$ is a Brownian motion. Indeed, for $t > s$, we have

$$X(t) - X(s) = (t-s)B(\frac{1}{t}) + s\big(B(\frac{1}{t}) - B(\frac{1}{s})\big)$$

Observe that

$$s\big(B(\frac{1}{t}) - B(\frac{1}{s})\big) \sim \mathcal{N}\big(0, s^2(\frac{1}{s} - \frac{1}{t})\big), \quad (t-s)B(\frac{1}{t}) \sim \mathcal{N}\big(0, \frac{(t-s)^2}{t}\big).$$

Moreover, $s\big(B(\frac{1}{t}) - B(\frac{1}{s})\big)$ and $(t-s)B(\frac{1}{t})$ are independent. Hence $X(t) - X(s)$ is a normally distributed random variable with mean zero and variance $s^2\big(\frac{1}{s} - \frac{1}{t}\big) + \frac{(t-s)^2}{t} = t - s$. In other words, the increments $X(t) - X(s)$ is $\mathcal{N}(0, t-s)$ for all $t \ge s > 0$. Next we show that it has independent increments. Note that

$$\begin{aligned}
\mathrm{Cov}(X(t), X(s)) &= \mathbb{E}[X(t)X(s)] = st\mathbb{E}[B(\frac{1}{t})B(\frac{1}{s})] = st\mathrm{Cov}(B(\frac{1}{t}), B(\frac{1}{s})) \\
&= st\min(\frac{1}{t}, \frac{1}{s}) = \min(s, t).
\end{aligned}$$

Let $s < t$. Then

$$\mathrm{Cov}(X(s), X(t) - X(s)) = \mathrm{Cov}(X(s), X(t)) - \mathrm{Cov}(X(s), X(s)) = \min(s, t) - s = 0\,.$$

Since $X(s)$ and $X(t) - X(s)$ are normal random variables, and $\mathrm{Cov}(X(s), X(t) - X(s)) = 0$, we conclude that they are independent. It remains to show that sample paths are continuous with *probability* 1. *Observe that* $X(t)$ *is continuous for* $t > 0$. *We show that* $X(t)$ *is continuous at* $t = 0$.

$$\lim_{t\to 0} X(t) = \lim_{t\to 0} tB(\frac{1}{t}) = \lim_{n\to\infty} \frac{1}{n} B(n) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \{B(i) - B(i-1)\}$$

Note that $B(i) - B(i-1)$ *are* **i.i.d** *sequence of random variables with mean zero. Hence by SLLN, we get*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \{B(i) - B(i-1)\} = 0 \quad a.s.$$

This completes the proof.

2.1. **First and Quadratic Variation.** Let f be a function defined on $[0, T]$ and $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = T\}$ be a partition of $[0, T]$. We define **first-variation** of f up to time T as

$$\mathrm{FV}_T(f) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \big|f(t_{j+1}) - f(t_j)\big|.$$

The **quadratic variation** of f up to time T is defined by

$$[f, f](T) := \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \big(f(t_{j+1}) - f(t_j)\big)^2.$$

Observe that if $f : [0, T] \to \mathbb{R}$ such that $|f'|$ is Riemann integrable on $[0, T]$, then

$$\mathrm{FV}_T(f) = \int_0^T |f'(t)|\, dt.$$

Indeed, by MVT, there exists $t_j^* \in (t_j, t_{j+1})$ such that

$$f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j).$$

This implies that

$$\begin{aligned}
&\sum_{j=0}^{n-1} \big|f(t_{j+1}) - f(t_j)\big| = \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j) \\
&\implies \mathrm{FV}_T(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \big|f(t_{j+1}) - f(t_j)\big| = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j) = \int_0^T |f'(t)|\, dt\,.
\end{aligned}$$

Notice that if f has continuous derivative then $[f, f](T) = 0$. Indeed, by using MVT we have

$$\sum_{j=0}^{n-1} \big(f(t_{j+1}) - f(t_j)\big)^2 = \sum_{j=0}^{n-1} |f'(t_j^*)|^2 (t_{j+1} - t_j)^2 \le \|\Pi\| \sum_{j=0}^{n-1} \|f'(t_j^*)|^2 (t_{j+1} - t_j)$$

$$\begin{aligned}
\implies [f, f](T) &\le \lim_{\|\Pi\|\to 0} \Big\{\|\Pi\| \sum_{j=0}^{n-1} \|f'(t_j^*)|^2 (t_{j+1} - t_j)\Big\} = \lim_{\|\Pi\|\to 0} \|\Pi\| \cdot \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \|f'(t_j^*)|^2 (t_{j+1} - t_j) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \int_0^T \|f'(t)|^2\, dt\,.
\end{aligned}$$

Since f' is continuous, $\int_0^T |f'(t)|^2\, dt$ is finite and hence we obtain that $[f, f](T) = 0$.

Theorem 2.3. *Let* $B(\cdot)$ *be a one-dimensional Brownian motion. Then* $[B, B](T) = T$ *for all* $T \ge 0$ *a.s.*

Proof. Let $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = T\}$ be a partition of $[0,T]$. Consider the random variable $Q_T^n := \sum_{j=0}^{n-1}(B(t_{j+1}) - B(t_j))^2$. Then

$$\begin{aligned}\mathbb{E}\big[(Q_T^n - T)^2\big] &= \mathbb{E}\Big[\Big(\sum_{j=0}^{n-1}(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\Big)^2\Big] \\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\mathbb{E}\Big[\Big((B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\Big)\Big((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)\Big)\Big].\end{aligned}$$

Thanks to independent increments and the fact that $B(t) - B(s) \sim \mathcal{N}(0, t-s)$ for all $t \ge s \ge 0$, we have for $k \neq j$

$$\mathbb{E}\Big[\Big((B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\Big)\Big((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)\Big)\Big] = 0.$$

Thus, we get that

$$\mathbb{E}\big[(Q_T^n - T)^2\big] = \sum_{k=0}^{n-1}\mathbb{E}\Big[(Y_k^2 - 1)^2(t_{k+1} - t_k)^2\Big]$$

where

$$Y_k := \frac{B(t_{k+1}) - B(t_k)}{\sqrt{t_{k+1} - t_k}} \sim \mathcal{N}(0,1).$$

Hence, for some constant $C > 0$, we have

$$\mathbb{E}\big[(Q_T^n - T)^2\big] \le C\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \implies \lim_{\|\Pi\|\to 0}\mathbb{E}(Q_T^n - T)^2 = 0.$$

Therefore, there exists a sub-sequence along which the convergence is almost surely. Thus, $[B,B](T) = T$ for all $T \ge 0$ a.s. □

Remark 2.2. Brownian motion accumulates $b - a$ units of quadratic variation over the interval $[a,b]$.

Lemma 2.4. *For any $T > 0$, the first variation of Brownian motion $B(\cdot)$ up to time T is infinite almost surely.*

Proof. Let $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = T\}$ be a partition of $[0,T]$. Observe that

$$\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)^2 \le \sup_{0\le k\le n-1}|B(t_{k+1}) - B(t_k)|\sum_{j=0}^{n-1}|B(t_{j+1}) - B(t_j)|$$

We know that $\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)^2 \to T$ a.s. Pick an ω for which this holds and the sample paths of Brownian motion is continuous. Then

$$0 < T \le \lim_{\|\Pi\|\to 0}\sup_{0\le k\le n-1}|B(t_{k+1}) - B(t_k)|\sum_{j=0}^{n-1}|B(t_{j+1}) - B(t_j)|$$

Since $\lim_{\|\Pi\|\to 0}\sup_{0\le k\le n-1}|B(t_{k+1}) - B(t_k)| = 0$, we see that the sample paths have infinite variation with probability one:

$$\mathrm{FV}_T(B) := \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\big|B(t_{j+1}) - B(t_j)\big| = \infty.$$

□

We now show how to use the quadratic variation of Brownian motion to identify the volatility σ in asset-price model used in the Black-Scholes-Merton option-pricing formula. Consider the **geometric Brownian motion**

$$S(t) = S(0)\exp\{\sigma B(t) + (\alpha - \frac{1}{2}\sigma^2)t\}$$

where α and $\sigma > 0$ are constant. Let $0 \le T_1 < T_2$ be given and suppose we observe geometric Brownian motion $S(t)$ for $T_1 \le t \le T_2$. Let $\Pi = \{T_1 = t_0 < t_1 < \ldots < t_n = T_2\}$ be a partition of $[T_1, T_2]$. Then we have

$$\begin{aligned}
\log\frac{S(t_{j+1})}{S(t_j)} &= \sigma\big(B(t_{j+1}) - B(t_j)\big) + (\alpha - \frac{1}{2}\sigma^2)(t_{j+1} - t_j) \\
\Longrightarrow \sum_{j=0}^{n-1}\Big(\log\frac{S(t_{j+1})}{S(t_j)}\Big)^2 &= \sigma^2\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)^2 + (\alpha - \frac{1}{2}\sigma^2)^2\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2 \\
&\quad + 2\sigma(\alpha - \frac{1}{2}\sigma^2)\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)(t_{j+1} - t_j) \\
&\equiv \sigma^2\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \qquad (2.1)
\end{aligned}$$

Observe that

$$\lim_{\|\Pi\|\to 0}\mathcal{A}_1 = T_2 - T_1, \qquad \lim_{\|\Pi\|\to 0}\mathcal{A}_2 = 0.$$

We claim that $\lim_{\|\Pi\|\to 0}\mathcal{A}_3 = 0$. Indeed,

$$\begin{aligned}
\big|\big(B(t_{j+1}) - B(t_j)\big)(t_{j+1} - t_j)\big| &\le \max_{0\le k\le n-1}\big|B(t_{k+1}) - B(t_k)\big|(t_{j+1} - t_j) \\
\Longrightarrow \Big|\sum_{j=0}^{n-1}\big(B(t_{j+1}) - B(t_j)\big)(t_{j+1} - t_j)\Big| &\le \max_{0\le k\le n-1}\big|B(t_{k+1}) - B(t_k)\big|\sum_{j=0}^{n-1}(t_{j+1} - t_j) \\
&= (T_2 - T_1)\max_{0\le k\le n-1}\big|B(t_{k+1}) - B(t_k)\big|
\end{aligned}$$

Since $B(\cdot)$ is continuous on $[T_1, T_2]$, we see that $\max_{0\le k\le n-1}\big|B(t_{k+1}) - B(t_k)\big|$ converges to zero as $\|\Pi\| \to 0$. Thus $\lim_{\|\Pi\|\to 0}\mathcal{A}_3 = 0$. Hence, from (2.1) we conclude that when the maximum step size $\|\Pi\|$ is small, we approximate the volatility σ as

$$\sigma^2 \approx \frac{1}{T_2 - T_1}\sum_{j=0}^{n-1}\Big(\log\frac{S(t_{j+1})}{S(t_j)}\Big)^2.$$