INDIAN INSTITUTE OF INFORMATION TECHNOLOGY RANCHI

MATHEMATICS - I

Name : Priyanshu Ranjan

Class : Section A

Reg No : 2022UG2005

ASSIGNMENT - 01

1. Verify Rolle's Theorem for the function $f(x) = e^x(\sin x - \cos x)$ in $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

Soln, For Rolle's Theorem, a function should be:

- Continuous in $[a,b]$.
- Differential in (a,b).
- $f(a) = f(b)$

$\Rightarrow$ e^x, $\sin x$ & $\cos x$ all are continuous function and $\sin x - \cos x$ is also continuous in the given interval. So $f(x)$ is continuous.

$\Rightarrow$ $f(x)$ is differentiable function.

$$f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)$$

$$f'(x) = 2e^x \sin x$$

Now,

$$f(\pi/4) = e^{\pi/4}(\sin \pi/4 - \cos \pi/4)$$

$$= e^{\pi/4}(1/\sqrt{2} - 1/\sqrt{2})$$

$$= 0$$

$$f(5\pi/4) = e^{5\pi/4}(\sin \tfrac{5\pi}{4} - \cos 5\pi/4)$$

$$= e^{5\pi/4}(1/\sqrt{2} - 1/\sqrt{2})$$

$$= 0$$

$$\therefore f(\pi/4) = f(5\pi/4).$$

~~Hence, Rolle's Theorem is verified.~~

So, $f'(c) = 0$

$$2e^c \sin c = 0$$

$$\therefore \sin c = 0$$

$$\Rightarrow \boxed{c = \pi}$$

$\therefore$ Rolle's Theorem is verified.

2. State Rolle's Theorem and verify for $(x-a)^m (x-b)^n$ in the interval $[a,b]$; m, n being positive integers.

Soln: According to Rolle's Theorem, if a function $f(x)$ is

- continuous in closed interval $[a,b]$.
- differential in open interval (a,b).
- $f(a) = f(b)$

then, there exist at least one point $c \in (a,b)$ such that

$$f'(c) = 0$$

For $f(x) = (x-a)^m (x-b)^n$

=> It is continuous in $[a,b]$ as it is a combination of polynomial functions.

=> It is differential in (a,b) as polynomial functions are differential everywhere in its domain.

$$f'(x) = m(x-a)^{m-1} (x-b)^n + n(x-a)^m (x-b)^{n-1}$$

$$= (x-a)^{m-1} (x-b)^{n-1} \left[m(x-b) + n(x-a) \right]$$

$$= (x-a)^{m-1} (x-b)^{n-1} \left[(m+n)x - (mb+na) \right]$$

exists for every value of x in (a,b)

- $f(a) = f(b) = 0$

$$(\cancel{a-a})^m (a-b)^n = (b-a)^m (\cancel{b-b})^n$$

Hence, Rolle's Theorem is verified.

So, $f'(c) = 0$

$$(c-a)^{m-1} (c-b)^{n-1} \{ (m+n)c - (mb+na) \} = 0$$

$$\boxed{c = \frac{mb+na}{m+n}}$$ Ans.

3. Verify Lagrange's Mean Value Theorem for the following functions:

a) $f(x) = \sqrt{x^2-4}$ in $[2, 4]$.

Soln: • $f(x)$ is continuous in the given interval as it is a polynomial function.

• $f(x)$ is differentiable function in $(2, 4)$

$$f'(x) = \frac{1}{2} \times \frac{1}{\sqrt{x^2-4}} \times 2x$$

$$= \frac{x}{\sqrt{x^2-4}}$$

Hence, Lagrange's Mean Value Theorem is ~~verified~~. satisfied.

Now,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sqrt{(4)^2-4} - \sqrt{(2)^2-4}}{4-2}$$

$$= \frac{\sqrt{12} - 0}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\therefore \boxed{f'(c) = \sqrt{3}}$$

b) $f(x) = \log x$ in $[1, e]$.

Soln: $f(x)$ is continuous and differentiable in the given domain.

So, Lagrange's Mean Value Theorem is satisfied.

$$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{\log e - \log 1}{e - 1}$$

$$f'(c) = \frac{\log e}{e - 1}$$

$$\therefore \quad f'(c) = \frac{1}{e-1}$$

4. Prove that for $x > 0$, $x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2(1+x)}$

Soln: Let, $f(x) = \ln(1+x) - \left(x - \frac{x^2}{2}\right)$

$$\Rightarrow f'(x) = \frac{1}{1+x} - (1-x)$$

$$= \frac{x^2}{1+x}$$

$$f'(x) = \frac{x^2}{1+x} > 0 \quad \forall\ x > 0$$

$\therefore$ $f'(x) > 0$, $f(x)$ is increasing function

$$f(x) > f(0) = 0$$

$$f(x) > 0$$

$$\ln(1+x) - \left(x - \frac{x^2}{2}\right) > 0$$

$$\ln(1+x) > x - \frac{x^2}{2} \quad \forall\ x > 0$$

or,

$$x - \frac{x^2}{2} < \ln(1+x) \quad \text{——} \quad (1)$$

Again Let

$$g(x) = x - \frac{x^2}{2(1+x)} - \ln(1+x)$$

$$g'(x) = 1 - \left(\frac{(1+x)\cdot 2x - x^2}{2(1+x)^2}\right) - \frac{1}{1+x}$$

$$= 1 - \frac{2x + x^2}{2(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{x^2}{2(1+x)^2} > 0 \quad \forall\ x > 0$$

$\therefore$ $g'(x) > 0$, $g(x)$ is increasing function

$$g(x) > g(0) = 0$$

$$\therefore g(x) > 0$$

$$x - \frac{x^2}{2(1+x)} - \ln(1+x) > 0$$

$$x - \frac{x^2}{2(1+x)} > \ln(1+x) \quad \forall \; x > 0$$

or,
$$\ln(1+x) < x - \frac{x^2}{2(1+x)} \quad \text{——— (ii)}$$

From (i) & (ii)

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2(1+x)} \; ; \quad x > 0$$

Proved

5. Test the convergence of the following series.

a) $\sum_1^\infty \left(\frac{(n+1)(n+2)}{n^2\sqrt{n}} \right)$

Soln: Let $U_n = \frac{(n+1)(n+2)}{n^2\sqrt{n}}$

$$= \frac{(1+1/n)(1+2/n)}{\sqrt{n}}$$

$$V_n = \frac{1}{n^{1/2}}$$

$\sum_{n=1}^{\infty} V_n$ diverges by p series test as $(p = 1/2) ; (p \leq 1)$

Using limit form test,

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{(1+1/n)(1+2/n)}{\cancel{n^{1/2}}} \times \frac{\cancel{n^{1/2}}}{1}$$

$$= 1 \quad (\text{finite})$$

$\sum_{n=1}^{\infty} U_n$ & $\sum_{n=1}^{\infty} V_n$ behaves similarly.

Hence,

$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \left(\frac{(n+1)(n+2)}{n^2\sqrt{n}} \right)$ also diverges.

b) $\sum_{n=1}^{\infty} \sqrt{\dfrac{3^n - 1}{2^n + 1}}$

Soln: $U_n = \sqrt{\dfrac{3^n - 1}{2^n + 1}}$ $\qquad U_{n+1} = \sqrt{\dfrac{3^{n+1} - 1}{2^{n+1} + 1}}$

Using D' Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \sqrt{\frac{3^{n+1} - 1}{2^{n+1} + 1}} \times \sqrt{\frac{2^n + 1}{3^n - 1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{3^{n+1}\left(1 - \frac{1}{3^{n+1}}\right)}{2^{n+1}\left(1 + \frac{1}{2^{n+1}}\right)} \times \frac{2^n\left(1 + 1/2^n\right)}{3^n\left(1 - 1/3^n\right)}}$$

$$= \sqrt{3/2} \qquad (l > 1)$$

As $l > 1$, then $\sum_{n=1}^{\infty} U_n$ diverges.

c) $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \dfrac{x^6}{5\sqrt{4}} + \cdots\cdots \infty$

Soln: $U_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ $\qquad U_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\frac{U_{n+1}}{U_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n-2}}$$

$$= \frac{x^2 \sqrt{n+1}\sqrt{n}}{n+2}$$

$$= \frac{x^2\sqrt{1 + 1/n}}{1 + 2/n}$$

Using D' Alembert's Ratio Test,

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^2\sqrt{1+1/n}}{1 + 2/n}$$

$$= x^2$$

For, $x^2 < 1$, Series converges

$x^2 > 1$, Series Diverges

$x^2 = 1$, Test fails

At $x^2 = 1$

$$U_n = \frac{1}{(n+1)\sqrt{n}}$$

$$= \frac{1}{n^{3/2}(1 + 1/n)}$$

Let, $V_n = \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} V_n$ converges by p series. test as $p = \frac{3}{2}$ $(p > 1)$

Using Limit form test

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{\cancel{n^{3/2}}(1 + 1/n)} \times \cancel{n^{3/2}}$$

$$= 1 \quad (\text{finite and non-zero})$$

Hence, $\sum_{n=1}^{\infty} U_n$ & $\sum_{n=1}^{\infty} V_n$ behaves similarly.

$\sum_{n=1}^{\infty} U_n$ is convergent at $x^2 = 1$

So, $\sum_{n=1}^{\infty} U_n$, converges when $x^2 \leq 1$
diverges when $x^2 > 1$

d) $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \infty$ for $x > 0$.

Soln: $U_n = \left(\frac{n}{n+1}\right)^{n-1} x^{n-1}$ (Except first term)

$$U_{n+1} = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\frac{U_{n+1}}{U_n} = \left(\frac{n+1}{n+2}\right)^n x^n \times \left(\frac{n+1}{n}\right)^{n-1} \times \frac{1}{x^{n-1}}$$

$$= \left(\frac{1 + 1/n}{1 + 2/n}\right)^n \left(\frac{1 + 1/n}{1}\right)^{n-1} . \, x$$

Using D' Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \left(\frac{1+1/n}{1+2/n}\right)^n \left(1+1/n\right)^{n-1} \cdot x$$

$$= x$$

At, $x > 1$, Diverges

$x < 1$, Converges

$x = 1$, Test Fails

At $x = 1$

$$U_n = \left(\frac{n}{n+1}\right)^{n-1}$$

$$= \left(\frac{1}{1+1/n}\right)^{n-1}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^{n-1} = 1 \quad (\neq 0)$$

Not convergent at $x = 1$.

$$\sum_{n=2}^{\infty} U_n = \sum_{n=2}^{\infty} \left(\frac{n}{n+1}\right)^{n-1} x^{n-1}$$

Converges at $x < 1$

Diverges at $x \geq 1$

e) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Soln:

$$U_n = \left(1 + 1/\sqrt{n}\right)^{-n^{3/2}}$$

Using Cauchy's nth Root Test

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left(\left(1 + 1/\sqrt{n}\right)^{-n^{3/2}}\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{\sqrt{n}+1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \frac{1}{e} \ (1 < 1)$$

So, $\displaystyle\sum_{n=1}^{\infty} U_n$ is convergent.

f) $\frac{1}{2(\ln 2)^p} + \frac{1}{3(\ln 3)^p} + \frac{1}{4(\ln 4)^p} + \cdots\cdots \infty$, $p > 0$.

Soln:

$$\sum_{n=2}^{\infty} U_n = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

$$\int_2^{\infty} U(x) = \int_2^{\infty} \frac{1}{x(\ln x)^p}$$

U_x is integrable, non-negative as $x \geq 2$ and monotonically decreasing as $U_{n+1} < U_n$. Then, using Cauchy's Integral Test

$$\int_2^{\infty} U(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^p}$$

Let,

$\ln x = t$ $\qquad x \to 2, \quad t \to \ln 2$

$\frac{1}{x}\,dx = dt$ $\qquad x \to \infty, \quad t \to \infty$

$$= \int_{\ln 2}^{\infty} (t)^{-p}\,dt$$

$$= \left[\frac{t^{-p+1}}{-p+1}\right]_{\ln 2}^{\infty}$$

$$= 0 - \frac{(\ln 2)^{-p+1}}{-p+1}$$

$$= \frac{(\ln 2)^{1-p}}{p-1} \quad (\text{finite}) \quad p > 0$$

$\sum_{n=2}^{\infty} U_n$ is convergent.

g) $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \dots + \frac{1}{1+n^2} + \dots \infty$

Soln: $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{1}{1+n^2}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2(1+1/n^2)}$$

Let, $V_n = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} V_n$ converges by P-series test as $p=2$ $(p>1)$.

Using limit form test

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n^2}{n^2(1+1/n^2)}$$

$$= 1 \quad (\text{finite and non-zero})$$

So, $\sum_{n=1}^{\infty} U_n$ & $\sum_{n=1}^{\infty} V_n$ behaves similarly.

Hence, $\sum_{n=1}^{\infty} U_n$ also converges.

h) $x^2 + \frac{2^2}{3.4} x^4 + \frac{2^2.4^2}{3.4.5.6} x^6 + \frac{2^2 4^2 6^2}{3.4.5.6.7.8} x^8 + \dots$

Soln. Except 1st term

$$\sum_{n=2}^{\infty} U_{n+1} = \frac{2^2 4^2 6^2 \dots (2n)^2}{3.4.5.6.7.8 \dots (2n+1)(2n+2)} . x^{2n+2}$$

$$\sum_{n=2}^{\infty} U_n = \frac{2^2 4^2 6^2 \dots (2n-2)^2}{3.4.5.6.7.8 \dots (2n)} x^{2n}$$

$$\frac{U_{n+1}}{U_n} = \frac{(2n)^2}{(2n+1)(2n+2)} x^2$$

$$= \frac{2}{(2+1/n)(1+1/n)} x^2$$

Using D'Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{2}{(2+1/n)(1+1/n)} x^2$$

$$= x^2$$

At, $x^2 < 1$
or $|x| < 1$, Converges
$|x| > 1$, Diverges
$|x| = 1$, Test fails

At $x^2 = 1$

Using Rabbe's Test

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = n\left\{\frac{(2n+1)(2n+2) - (2n)^2}{(2n)^2}\right\}$$

$$= \frac{3n+1}{2n}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{3n+1}{2n}$$

$$= \lim_{n\to\infty} \frac{3+1/n}{2}$$

$$= \frac{3}{2} \quad (l > 1)$$

As $l > 1$ at $x^2 = 1$, series is convergent.

Given series converges when $|x| \leq 1$

Given series diverges when $|x| > 1$

i) $1 + \frac{2}{2!}x + \frac{3^2}{3!}x^2 + \frac{4^3}{4!}x^3 + \frac{5^4}{5!}x^4 + \dots \quad x > 0$

Sol^n: $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{(n)^{n-1}}{n!} x^{n-1}$

$$U_{n+1} = \frac{(n+1)^n}{(n+1)!} x^n$$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^n x^n}{(n+1)!\, x^{n-1}} \cdot \frac{n!}{(n)^{n-1}}$$

$$= \left(\frac{n+1}{n}\right)^{n-1} x$$

Using D' Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n-1} x$$

$$= e x$$

At, $ex < 1$ or $x < 1/e$, converges

$x > 1/e$, Diverges

$x = 1/e$, Test fails

At $x = 1/e$

Using Logarithmic Test

$$\ln\left(\frac{U_n}{U_{n+1}}\right) = \ln\left\{\left(\frac{n}{n+1}\right)^{n-1} \cdot e\right\}$$

$$= \ln\left(\frac{n}{n+1}\right)^{n-1} + \ln e$$

$$= 1 + (n-1) \ln\left(\frac{n}{n+1}\right)$$

$$= 1 + (n-1) \ln\left(\frac{n+1}{n}\right)^{-1}$$

$$= 1 - (n-1) \ln\left(1 + \frac{1}{n}\right)$$

$$= 1 - n \ln\left(1 + \frac{1}{n}\right) + \ln(1 + 1/n)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right] + \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$= \left[1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \cdots\right] + \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$\lim_{n\to\infty} n \ln\left(\frac{U_n}{U_{n+1}}\right)$$

$$= \lim_{n\to\infty} n\left[\left(\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right) + \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= \lim_{n\to\infty} \left(\left(\frac{1}{2} - \frac{1}{3n} + \cdots\right) + \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots\right)\right)$$

$$= \frac{1}{2} + 1 = \frac{3}{2} \quad (l > 1)$$

At $x = 1/e$, it is convergent.

Given Series,

converges when $x \leq 1/e$

diverges when $x > 1/e$

j) $1 + \frac{(1!)^2}{2!}x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots \quad x > 0$

Soln: Except 1st term

$$\sum_{n+1}^{\infty} U_n = \frac{(n!)^2}{(2n)!}x^n$$

$$U_{n+1} = \frac{((n+1)!)^2}{(2n+2)!}x^{n+1}$$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^2}{(2n+2)(2n+1)}x$$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^2}{2(n+1)(2n+1)}\cdot x$$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)}{2(2n+1)} \cdot x$$

Using D' Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)}{2(2n+1)} x$$

$$= \lim_{n\to\infty} \frac{1+1/n}{2(2+1/n)} \cdot x$$

$$= x/4$$

At $\frac{x}{4} < 1$

or

$x < 4$, converges

$x > 4$, diverges

$x = 4$, Test fails

At $x = 4$

Using Raabe's Test

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n+2}{4(n+1)} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{-2}{4(n+1)}\right)$$

$$= \lim_{n\to\infty} -\frac{1}{2(1+1/n)} = -\frac{1}{2} \quad (l < 1)$$

As $l = -1/2$ at $x = 4$, series ~~converges~~. diverges.

Given series

converges when $x < 4$

Diverges when $x \geq 4$

k) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots \infty$

Soln: $\sum_{n=1}^{\infty} (-1)^{n-1} U_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\sqrt{n}}$

By Leibnitz test

Rule: i) $U_{n+1} \leq U_n$

$(n+1)\sqrt{n+1} \geq n\sqrt{n}$

$\frac{1}{(n+1)\sqrt{n+1}} \leq \frac{1}{n\sqrt{n}}$

ii) $\lim_{n\to\infty} U_n = 0$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}}$

$= 0$

The Alternating series is convergent.

6. Find the expansion of $\tan(x + \pi/4)$ in ascending powers of x upto terms in x^4 and find approximately the value of $\tan(43)^\circ$.

Soln: By Taylor's Expansion

$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f''''(a) + \dots$

$a = \pi/4$, $h = x$

$f(x + \frac{\pi}{4}) = f(\pi/4) + x f'(\pi/4) + \frac{x^2}{2!} f''(\pi/4) + \dots$ —— ①

Let, $f(x + \pi/4) = \tan(x + \frac{\pi}{4})$; $f(\pi/4) = \tan \pi/4 = 1$

- $f'(\pi/4) = \frac{d}{dx}(\tan x)$

$= \sec^2 x = \sec^2(\frac{\pi}{4}) = 2$

- $f''(\pi/4) = 2\sec^2 x \tan x$

$= 2 \sec^2(\pi/4)\tan(\pi/4)$

$= 4$

- $f'''(\pi/4) = 2[\sec^4 x + 2\tan^2 x \sec^2 x]$

$= 2[\sec^4(\pi/4) + 2\tan^2(\pi/4)\sec^2(\pi/4)]$

$= 2[4 + 2\times 2]$

$= 16$

- $f''''(\pi/4) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$= 80$

Putting all the values in ①

$$\tan(x + \pi/4) = 1 + x \cdot 2 + \frac{x^2}{2!} \cdot 4 + \frac{x^3}{3!} \cdot 16 + \frac{x^4}{4!} \cdot 80 + \dots$$

$$\tan(x + \pi/4) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$$

For finding $\tan(43°)$,

$= \tan(45 + (-2))°$

$45° = \frac{45}{180} \times \pi$ $\qquad$ $2° = \frac{2}{180} \times \pi$

$= \frac{\pi}{4}$ $\qquad$ $= \frac{\pi}{90}$

$x = \left(-\pi/90\right)$. So,

$$\tan\left(\frac{\pi}{4} + \left(-\frac{\pi}{90}\right)\right) = 1 + 2\left(-\frac{\pi}{90}\right) + 2\left(\frac{-\pi}{90}\right)^2 + \frac{8}{3}\left(\frac{-\pi}{90}\right)^3 + \frac{10}{3}\left(-\frac{\pi}{90}\right)^4 + \cdots$$

$$\tan 43^\circ = 1 + 2t + 2t^2 + \frac{8}{3}t^3 + \frac{10}{3}t^4 + \cdots$$

Here $\left(t = -\frac{\pi}{90}\right)$

After Calculation,

$$\tan(43)^\circ = 1 - 0.0697 + 0.0024 - 0.0001 + \cdots$$

$$= 0.9326$$

7. If $u = \ln(x^2 + xy + y^2)$, then show that $x\frac{du}{dx} + y\frac{du}{dy} = 2$.

Soln:

$$\frac{du}{dx} = \frac{1}{x^2 + xy + y^2}(2x + y)$$

$$x\frac{du}{dx} = \frac{2x^2 + xy}{x^2 + xy + y^2} \quad \text{——— (I)}$$

$$\frac{du}{dy} = \frac{1}{x^2 + xy + y^2}(x + 2y)$$

$$y\frac{du}{dy} = \frac{xy + 2y^2}{x^2 + xy + y^2} \quad \text{——— (II)}$$

Adding (I) & (II)

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{2x^2 + xy + 2y^2 + xy}{x^2 + xy + y^2}$$

$$= \frac{2\left(\cancel{x^2 + xy + y^2}\right)}{\left(\cancel{x^2 + xy + y^2}\right)}$$

$$x\frac{du}{dx} + y\frac{du}{dy} = 2.$$

Hence Proved

8. Verify Euler's Theorem for the following functions :

1) $u = \ln\left(\frac{x^2+y^2}{xy}\right)$

Soln:

$$u = \ln\left\{\frac{(1+(y/x)^2)\, x^2}{x\cdot(y/x)\cdot x}\right\}$$

$$u = \ln\left\{\frac{1+(y/x)^2}{(y/x)}\right\}$$

Homogeneous function of order $n=0$.

According to Euler's Theorem,

$$x\frac{du}{dx} + y\frac{du}{dy} = nu$$

Solving RHS

$$nu = 0 \quad (\text{at } n=0)$$

Solving LHS

$$\frac{du}{dx} = \frac{\frac{1}{y} - \frac{y}{x^2}}{\frac{x^2+y^2}{xy}}$$

$$x\frac{du}{dx} = \frac{x^2-y^2}{x^2+y^2} \quad \text{--- (i)}$$

Similarly

$$y\frac{du}{dy} = \frac{y^2-x^2}{x^2+y^2} \quad \text{--- (ii)}$$

$$x\frac{du}{dx} + y\frac{du}{dx} = \frac{\cancel{x^2} - \cancel{y^2} + \cancel{y^2} - \cancel{x^2}}{x^2+y^2}$$

$$= 0$$

$\therefore$ LHS = RHS

Euler's Theorem is verified.

11) $u = \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

Soln,

$$u = \frac{x^{1/3}\left(1 + (y/x)^{1/3}\right)}{x^{1/2}\left(1 + (y/x)^{1/2}\right)}$$

$$u = x^{-1/6}\left\{\frac{1 + (y/x)^{1/3}}{1 + (y/x)^{1/2}}\right\}$$

Homogeneous function of ~~equation~~ order $n = -1/6$

According to Euler's theorem,

$$x\frac{du}{dx} + y\frac{du}{dy} = nu$$

Solving RHS

$$nu = -\frac{1}{6}u$$

Solving LHS

$$x\frac{du}{dx} = x\left\{\frac{\frac{1}{3}x^{-2/3}(x^{1/2} + y^{1/2}) - \frac{1}{2}x^{-1/2}(x^{1/3} + y^{1/3})}{(x^{1/2} + y^{1/2})^2}\right\} \quad \text{—(I)}$$

Similarly,

$$y\frac{du}{dy} = y\left\{\frac{\frac{1}{3}y^{-2/3}(x^{1/2} + y^{1/2}) - \frac{1}{2}y^{-1/2}(x^{1/3} + y^{1/3})}{(x^{1/2} + y^{1/2})^2}\right\} \quad \text{—(II)}$$

Adding (I) & (II)

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{2(x^{1/3} + y^{1/3})(x^{1/2} + y^{1/2}) - 3(x^{1/3} + y^{1/3})(x^{1/2} + y^{1/2})}{6(x^{1/2} + y^{1/2})}$$

$$= -\frac{1}{6}\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$$

$$= -\frac{1}{6}u$$

LHS = RHS

Euler Theorem is verified.

9. If $u = f(r)$, where $r^2 = x^2 + y^2$, Prove that $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f''(r) + \frac{1}{r} f'(r)$.

Soln: $u = f(r)$; $r = \sqrt{x^2+y^2}$

$$\frac{du}{dr} = f'(r) \quad ; \quad \frac{d^2u}{dr^2} = f''(r)$$

Solving LHS

$$\frac{du}{dx} = \frac{du}{dr} \times \frac{dr}{dx}$$

$$= f'(r) \times \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}}$$

$$= f'(r) \times \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{d^2u}{dx^2} = f''(r) \frac{dr}{dx} \times \left(\frac{x}{\sqrt{x^2+y^2}}\right) + f'(r) \left\{ \frac{\sqrt{x^2+y^2} - \frac{x \times 2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} \right\}$$

$$= f''(r) \left(\frac{x^2}{x^2+y^2}\right) + f'(r) * \left(\frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}\right)$$

$$\frac{d^2u}{dx^2} = f''(r) \left(\frac{x^2}{r^2}\right) + f'(r) \left(\frac{y^2}{r^2 \cdot r}\right) \text{ --- (I)}$$

Similarly

$$\frac{d^2u}{dy^2} = f''(r)\left(\frac{y^2}{r^2}\right) + f'(r) \cdot \left(\frac{x^3}{r^3}\right) \text{ --- (II)}$$

Adding (I) & (II)

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f''(r)\left(\frac{x^2}{r^2}\right) + f'(r)\left(\frac{y^2}{r^3}\right) + f''(r)\left(\frac{y^2}{r^2}\right) + f'(r)\left(\frac{x^2}{r^3}\right)$$

$$= f''(r)\left(\frac{x^2+y^2}{r^2}\right) + f'(r)\left(\frac{x^2+y^2}{r^3}\right)$$

$$= f''(r) + \frac{1}{r} f'(r) \qquad (r^2 = x^2 + y^2)$$

Hence,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f''(r) + \frac{1}{r} f'(r)$$

Proved

10. If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$, Prove that $x\frac{du}{dx} + y\frac{du}{dy} = 2\cot u$.

Also evaluate value for $x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dxdy} + y^2\frac{d^2u}{dy^2}$.

Soln: Since u is not homogeneous.

Assume,

$$z = \sec u = \frac{x^3 - y^3}{x+y} = x^2\left(\frac{1-(y/x)^3}{1+(y/x)}\right)$$

$$z = x^2 \phi(y/x)$$

So, z is homogeneous function of order $n = 2$ and $z = f(u) = \sec u$.

i) By Euler's I deduction, we have

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{n f(u)}{f'(u)}$$

$n = 2$; $f(u) = \sec u$; $f'(u) = \sec u . \tan u$

$$= \frac{2\sec u}{\sec u . \tan u}$$

$$= 2\cot u$$

So,

$$x\frac{du}{dx} + y\frac{du}{dy} = 2\cot u$$

Proved

ii) By Euler's II deduction, we have

$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dxdy} + y^2\frac{d^2u}{dy^2} = g(u)\left[g'(u) - 1\right]$$

where $g(u) = \frac{n f(u)}{f'(u)} = 2 \cot u$

$$g'(u) = -2 \cdot \operatorname{cosec}^2 u$$

So,

$$x^2 \frac{d^2u}{dx^2} + 2xy \frac{d^2u}{dxdy} + y^2 \frac{d^2u}{dy^2} = 2 \cot u \left[-2 \operatorname{cosec}^2 u - 1 \right]$$

$$= -2 \cot u \left[1 + 2 \operatorname{cosec}^2 u \right]$$

11. If $u = f(2x-3y, 3y-4z, 4z-2x)$, prove that

$$6 \frac{du}{dx} + 4 \frac{du}{dy} + 3 \frac{du}{dz} = 0 .$$

Soln,

$$u = f(2x-3y, 3y-4z, 4z-2x)$$

Let, $u = f(r, s, t)$

where, $r = 2x-3y$; $s = 3y-4z$; $t = 4z-2x$

$$\frac{du}{dx} = \frac{du}{dr} \cdot \frac{dr}{dx} + \frac{du}{ds} \cdot \frac{ds}{dx} + \frac{du}{dt} \cdot \frac{dt}{dx}$$

$$\frac{du}{dx} = 2 \frac{du}{dr} + 0 - 2 \frac{du}{dt}$$

$$6 \frac{du}{dx} = 12 \frac{du}{dr} - 12 \frac{du}{dt} \quad \text{—— (I)}$$

$$\frac{du}{dy} = \frac{du}{dr} \cdot \frac{dr}{dy} + \frac{du}{ds} \cdot \frac{ds}{dy} + \frac{du}{dt} \cdot \frac{dt}{dy}$$

$$= -3 \frac{du}{dr} + 3 \frac{du}{ds} + 0$$

$$4 \frac{du}{dy} = -12 \frac{du}{dr} + 12 \frac{du}{ds} \quad \text{—— (II)}$$

$$\frac{du}{dz} = \frac{du}{dr}\cdot\frac{dr}{dz} + \frac{du}{ds}\cdot\frac{ds}{dz} + \frac{du}{dt}\cdot\frac{dt}{dz}$$

$$= 0 - 4\frac{du}{ds} + 4\frac{du}{dt}$$

$$3\frac{du}{dz} = -12\frac{du}{ds} + 12\frac{du}{dt} \quad \text{—— (iii)}$$

Adding (i) (ii) & (iii)

$$6\frac{du}{dx} + 4\frac{du}{dy} + 3\frac{du}{dz} = 0$$

Hence, Proved

12. If $\omega = f(x, y)$, where $x = e^u \cos v$, $y = e^u \sin v$, show that

$$y\frac{d\omega}{du} + x\frac{d\omega}{dv} = e^{2u}\frac{d\omega}{dy}$$

Soln: We have $x = e^u \cos v$, $y = e^u \sin v$

$$\frac{\partial\omega}{du} = \frac{d\omega}{dx}\cdot\frac{dx}{du} + \frac{d\omega}{dy}\cdot\frac{dy}{du}$$

$$= (e^u \cos v)\frac{d\omega}{dx} + (e^v \sin v)\frac{d\omega}{dy}$$

$$y\frac{d\omega}{du} = xy\frac{d\omega}{dx} + y^2\frac{d\omega}{dy} \quad \text{—— (i)}$$

and,

$$\frac{d\omega}{dv} = \frac{d\omega}{dx}\cdot\frac{dx}{dv} + \frac{d\omega}{dy}\cdot\frac{dy}{dv}$$

$$= [e^u(-\sin v)]\frac{d\omega}{dx} + [e^u(\cos v)]\frac{d\omega}{dy}$$

$$x\frac{d\omega}{dv} = -xy\frac{d\omega}{dx} + x^2\frac{d\omega}{dy} \quad \text{—— (ii)}$$

Adding (i) & (ii)

$$y\frac{d\omega}{du} + x\frac{d\omega}{dv} = (x^2+y^2)\frac{d\omega}{dy}$$

$$x^2+y^2 = (e^u\cos v)^2 + (e^u \sin v)^2$$

$$= e^{2u}$$

So,

$$y\frac{dw}{du} + x\frac{dw}{dv} = e^{2u}\frac{dw}{dy}$$

Hence Shown

13. If $w = \sqrt{x^2+y^2+z^2}$ and $x = u\cos v$, $y = u\sin v$, $z = uv$, then prove that $u\frac{dw}{du} - v\frac{dw}{dv} = \frac{u}{\sqrt{1+v^2}}$.

Soln, $x = u\cos v$; $y = u\sin v$; $z = uv$

$$\frac{dw}{du} = \frac{dw}{dx}\cdot\frac{dx}{du} + \frac{dw}{dy}\cdot\frac{dy}{du} + \frac{dw}{dz}\cdot\frac{dz}{du}$$

$$= \left(\frac{x}{w}\right)(\cos v) + \left(\frac{y}{w}\right)(\sin v) + \left(\frac{z}{w}\right)v$$

$$u\frac{dw}{du} = (u\cos v)\frac{x}{w} + (u\sin v)\frac{y}{w} + (uv)\frac{z}{w}$$

$$= \frac{x^2+y^2+z^2}{w}$$

$$u\frac{dw}{du} = w \quad \text{—— (1)}$$

Similarly for $\frac{dw}{dv}$,

$$\frac{dw}{dv} = \frac{dw}{dx}\cdot\frac{dx}{dv} + \frac{dw}{dy}\cdot\frac{dy}{dv} + \frac{dw}{dz}\cdot\frac{dz}{dv}$$

$$= \left(\frac{x}{w}\right)(u(-\sin v)) + \left(\frac{y}{w}\right)(u\cos v) + u\left(\frac{z}{w}\right)$$

$$= \frac{-\cancel{vxy} + \cancel{vxy} + z^2}{w}$$

$$v\frac{dw}{dv} = \frac{z^2}{w} \quad \text{—— (ii)}$$

Adding (i) & (ii)

$$u\frac{dw}{du} - v\frac{dw}{dv} = w - \frac{z^2}{w}$$

$$= \frac{w^2 - z^2}{w}$$

$$= \frac{x^2+y^2+z^2-z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{(u^2\cos^2 v + u^2\sin^2 v)}{\sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2 v^2}}$$

$$= \frac{u^2}{\sqrt{u^2(1+v^2)}}$$

$$= \frac{u}{\sqrt{1+v^2}}$$

So,

$$u\frac{dw}{du} - v\frac{dw}{dv} = \frac{u}{\sqrt{1+v^2}}$$ Hence Proved

14. Show that $u = x+y+z$, $v = x^2+y^2+z^2$, $w = x^3+y^3+z^3-3xyz$. Prove that u, v & w are not independent and hence find the relation between them.

Soln: We have $u = x+y+z$, $v = x^2+y^2+z^2$ and $w = x^3+y^3+z^3-3xyz$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} du/dx & du/dy & du/dz \\ dv/dx & dv/dy & dv/dz \\ dw/dx & dw/dy & dw/dz \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$$= 6\Big[(yz^2 - xy^2 - y^2z + xz^2) - (xz^2 - x^2y - x^2z + yz^2) + (xy^2 - x^2z - x^2y + y^2z)\Big]$$

$$= 0$$

So, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Hence, functional relation exist between u, v & w and they are dependent.

As we know,

$(a^3 + b^3 + c^3 - 3abc) = (a+b+c)(a^2+b^2+c^2 - ab - bc - ac)$

And,

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

So,

$w = u(v - (xy + yz + xz))$ —— (I)

and

$u^2 = v + 2(xy + yz + xz)$ —— (II)

Putting value of $(xy + yz + xz)$ from (II) to (I)

$$w = u\left(v - \frac{(u^2-v)}{2}\right)$$

$$w = u\left(\frac{2v - u^2 + v}{2}\right)$$

$$\therefore \boxed{2w = u(3v - u^2)}$$

15. If $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$, then show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 .$$

Soln:

Let,

$$f_1 = x - v^2 - w^2$$
$$f_2 = y - w^2 - u^2$$
$$f_3 = z - u^2 - v^2$$

Now,

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \partial f_1/\partial u & \partial f_1/\partial v & \partial f_1/\partial w \\ \partial f_2/\partial u & \partial f_2/\partial v & \partial f_2/\partial w \\ \partial f_3/\partial u & \partial f_3/\partial v & \partial f_3/\partial w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2v & -2w \\ -2u & 0 & -2w \\ -2u & -2v & 0 \end{vmatrix}$$

$$= -16\, uvw$$

Now,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \; \frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} \Big/ \frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}$$

$$= \frac{1}{16\, uvw}$$

Similarly,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 16\, uvw$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1 .$$

16. If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$ and $y = r\sin\theta$, then show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Soln: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2+y^2) = 4r^2$$

and
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

Since,
$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= 4r^2 \cdot r$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$ Proved

17. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, then show that
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}.$$

Soln: Let, $f_1 = u^3 + v^3 - x - y$
$f_2 = u^2 + v^2 - x^3 - y^3$

Now, $$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= 6uv \begin{vmatrix} u & v \\ 1 & 1 \end{vmatrix} = 6uv(u-v)$$

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 \\ x^2 & y^2 \end{vmatrix}$$

$$= 3(y^2 - x^2)$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} \Big/ \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$= \frac{3(y^2 - x^2)}{6uv(u-v)}$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u-v)}$$ Hence Shown

18. Find the values of x and y for which $x^2+y^2+6x=12$ has minimum value and find this minimum.

Sol^n: Let,

$$f(x, y) = x^2 + y^2 + 6x - 12 = 0$$

$$\frac{\partial f}{\partial x} = 2x + 6 = 0$$

$$x = -3$$

$$\frac{\partial f}{\partial y} = 2y = 0$$

$$y = 0$$

Stationary point $(-3, 0)$

It is given point $(x, y) = (-3, 0)$ has minimum value at it.

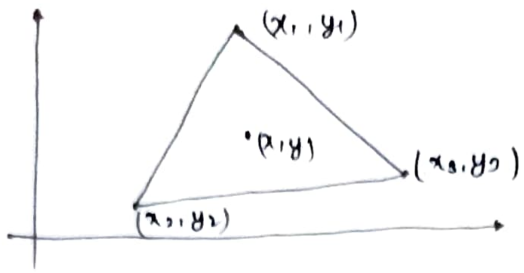
$$f(-3, 0) = (-3)^2 + 0 + 6(-3) - 12$$

$$= 9 - 18 - 12$$

$$f(-3, 0) = -21$$

Minima at $(-3, 0)$ Minimum value is -21.

19. Find a point within a triangle such that the sum of the square of its distance from the three angular points is a minimum.



Sol^n: Let the point inside the triangle from where the sum of square of its distances from the three angular point is a minimum is (x, y).

Three angular points $= (x_1, y_1)$, (x_2, y_2), (x_3, y_3)

$$D_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

$$D_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2}$$

$$D_3 = \sqrt{(x-x_3)^2 + (y-y_3)^2}$$

As $D_1, D_2, D_3 > 0$

So, $D_1^2, D_2^2, D_3^2 > 0$

Assume,

$$f(x,y) = D_1^2 + D_2^2 + D_3^2$$

$$= (x-x_1)^2 + (y-y_1)^2 + (x-x_2)^2 + (y-y_2)^2 + (x-x_3)^2 + (y-y_3)^2$$

$$\frac{\partial f}{\partial x} = 2(x-x_1) + 2(x-x_2) + 2(x-x_3) = 0$$

$$x = \frac{x_1 + x_2 + x_3}{3}$$

$$\frac{\partial f}{\partial y} = 2(y-y_1) + 2(y-y_2) + 2(y-y_3) = 0$$

$$y = \frac{y_1 + y_2 + y_3}{3}$$

$$r = \frac{d^2 f}{dx^2} = 6 \quad (r > 0)$$

$$s = \frac{d^2 f}{dx\,dy} = 0$$

$$t = \frac{d^2 f}{dy^2} = 6$$

Stationary points $(x, y) \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$

As we can see

$$rt - s^2 > 0 \qquad (rt - s^2 = 36)$$

and

$$r > 0 \qquad (r = 6)$$

At point (x, y) there is minima for $f(x, y)$.

So, $f(x, y)$ will give minimum value for $(x, y) \equiv$

$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$ which is centroid of a triangle.

20. Decompose a positive number 'a' into three parts so that its product is maximum.

Soln: Let three parts in which 'a' is divided for product to be maximum is x, y, z.

So,

$$x + y + z = a$$

$$f(x, y, z) = xyz$$

$$\phi(x, y, z) = 0 = a - (x + y + z)$$

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

↳ By Lagrange's Method

$$f(x,y,z) = \cancel{f(x,y)} \; xyz + \lambda(a-(x+y+z))$$

$$\frac{df}{dx} = yz - \lambda = 0 \quad \text{—— (I)}$$

$$\frac{df}{dy} = xz - \lambda = 0 \quad \text{—— (II)}$$

$$\frac{df}{dz} = xy - \lambda = 0 \quad \text{—— (III)}$$

$x \times$ (I), $y \times$ (II), $z \times$ (III), we get

$$xyz - \lambda x = 0$$

$$xyz - \lambda y = 0$$

$$xyz - \lambda z = 0$$

$$x = y = z = \frac{xyz}{\lambda}$$

Putting value of (x,y,z) in $\phi(x,y,z) = 0$.

$$a - \frac{3xyz}{\lambda} = 0$$

$$\lambda = \frac{3xyz}{a}$$

So,

$$x = y = z = \frac{xyz}{3xyz} \times a$$

$$\boxed{x = y = z = \frac{a}{3}}$$

Q1. Using the Lagrange's method of undetermined multipliers, find the largest product of the numbers x, y, z when $x^2+y^2+z^2 = 9$.

Soln. Let $f(x,y,z) = xyz$

$$\phi = (x,y,z) = 0 = x^2+y^2+z^2-9$$

By Lagrange's method of undetermined multiplier.

$$f(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$= xyz + \lambda(x^2+y^2+z^2-9)$$

$$\frac{df}{dx} = 0 = yz + 2\lambda x \quad \text{— (I)}$$

$$\frac{df}{dx} = 0 = xz + 2\lambda y \quad \text{— (II)}$$

$$\frac{df}{dz} = 0 = xy + 2\lambda z \quad \text{— (III)}$$

$x \times$ (I), $y \times$ (II), $z \times$ (III), we get

$$xyz + 2\lambda x^2 = 0$$

$$xyz + 2\lambda y^2 = 0$$

$$xyz + 2\lambda z^2 = 0$$

$$x^2 = y^2 = z^2 = -\frac{xyz}{2\lambda}$$

Putting values of x^2, y^2 & z^2 in $\phi(x,y,z)=0$

$$x^2+y^2+z^2-9 = 0$$

$$-\frac{3xyz}{2\lambda} = 9$$

$$\lambda = -\frac{xyz}{6}$$

$\cancel{x^2+y^2+z}$ $x^2 = y^2 = z^2 = \frac{-xyz \times 6}{2(-xyz)}$

$$= 3$$

$$x = y = z = \sqrt{3}$$

Largest product $xyz = 3\sqrt{3}$

22. Use the Lagrange's method of undetermined multipliers to find the minimum value of $x^2+y^2+z^2$ subject to the conditions $x+y+z=1$ and $xyz+1=0$.

Soln:
$$f(x,y,z) = x^2+y^2+z^2$$
$$\phi_1(x,y,z) = 0 = x+y+z-1$$
$$\phi_2(x,y,z) = 0 = xyz+1$$

Using Lagrange's method of undetermined multipliers.

$$f(x,y,z) = f(x,y,z) + \lambda_1\phi_1(x,y,z) + \lambda_2\phi_2(x,y,z)$$
$$= x^2+y^2+z^2 + \lambda_1(x+y+z-1) + \lambda_2(xyz+1)$$

$$\frac{df}{dx} = 2x + \lambda_1 + yz\lambda_2 = 0 \quad \text{——— (i)}$$

$$\frac{df}{dy} = 2y + \lambda_1 + xz\lambda_2 = 0 \quad \text{——— (ii)}$$

$$\frac{df}{dz} = 2z + \lambda_1 + xy\lambda_2 = 0 \quad \text{——— (iii)}$$

By (i) & (ii)

$$\lambda_1 = -(2x+yz\lambda_2) = -(2y+xz\lambda_2)$$

$$2x-2y = \lambda_2 z(x-y)$$

$$2(x-y) = \lambda_2 z(x-y)$$

$$(x=y) \quad \text{or} \quad (\lambda_2 z = 2)$$

Similarly, we can show

$$y=z \quad \text{or} \quad \lambda_2 x = 2$$

and

$$x=z \quad \text{or} \quad \lambda_2 y = 2$$

So, $\lambda_2 x = \lambda_2 y = \lambda_2 z = 2$

$$x = y = z = \frac{2}{\lambda_2}$$

This would contradict the equality constrains, so all $x = y = z$ is not possible.

Hence
$x = y$ or $y = z$ or $x = z$

So if $x = y$

For z, putting value in $\phi_2(x, y, z) = 0$

$$xyz = -1$$

$$z = \frac{-1}{x^2}$$

Putting values of x, y & z in $\phi_1(x, y, z) = 0$ in terms of x.

$$x + y + z = 1$$

$$x + x - \frac{1}{x^2} = 1$$

$$2x^3 - x^2 - 1 = 0$$

$$(x-1)(2x^2 + x + 1) = 0$$

$x = 1$ or $x = \frac{-1 \pm \sqrt{-7}}{4}$

So $x = y = 1$

$$z = -\frac{1}{x^2} = -1$$

Similarly

If $y = z$, we get

$$y = z = 1$$

$$x = -1$$

And

If $x = z$, we get

$$x = z = 1$$

$$y = -1$$

So, $(x, y, z) = (1, 1, -1)$ or
$(1, -1, 1)$ or
$(-1, 1, 1)$

then,
$[x^2 + y^2 + z^2 = 3]$ which is the only value (min) we get that also satisfies the given equality constraints.