Tutorial-3 (unit-4)

1. Find the images of the $|z+1| = 1$ under the map $w = 1/z$

Ans $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x+iy = \frac{1}{u+iv} \Rightarrow x+iy = \frac{u-iv}{u^2+v^2}$

$\therefore x = \frac{u}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2}$

Now, $|z+1| = 1$

$\Rightarrow |x+iy+1| = 1$

$\Rightarrow |(x+1)+iy| = 1 \Rightarrow (x+1)^2 + y^2 = 1$

$\Rightarrow \cancel{\left(\frac{u}{u^2+v^2}+1\right)^2 + \frac{v^2}{(u^2+v^2)^2} = 1}$

$\Rightarrow x^2+2x+\cancel{1}+y^2 = \cancel{1}$

$\Rightarrow x^2+2x+y^2 = 0$

$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{2u}{(u^2+v^2)} + \frac{v^2}{(u^2+v^2)^2} = 0$

$\Rightarrow u^2+v^2 + 2u(u^2+v^2) = 0$

$\Rightarrow (u^2+v^2)(1+2u) = 0$

$\because u^2+v^2 \neq 0 \quad \therefore 1+2u = 0 \Rightarrow \boxed{u = -\frac{1}{2}}$

2. Find the images of the $|z-2i| = 2$ where the map $w = 1/z$

Ans: Same problem as 1. (Already done in the class)

3. Describe the transformation $w = 1/z$.

Ans This transformation represents inversion.

Now if $z = x+iy$ and $w = u+iv$ then $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$\therefore x+iy = \frac{1}{u+iv} = \frac{u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2}$

$\therefore\ x = \dfrac{u}{u^2+v^2}$ & $y = \dfrac{-v}{u^2+v^2}$ (Ans).

4. Define Bilinear Transformation :-

The transformation $w = \dfrac{az+b}{cz+d}$ is said to be Bilinear transformation

Where a, b, c, d are complex no.s and $ad-bc \neq 0$.

If $ad-bc=0$ then all points z becomes the critical points of this mapping.

5. Find the bilinear map which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$

Ans let $z_1 = 1,\ z_2 = i,\ z_3 = -1,\ z_4 = z$

$w_1 = i,\ w_2 = 0,\ w_3 = -i,\ w_4 = w$

$\because$ Bilinear map preserves the cross ratio so,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$$

$$\Rightarrow \frac{(1-i)(-1-z)}{(i+1)(z-1)} = \frac{(i-0)(-i-w)}{(0+i)(w-i)}$$

$$\Rightarrow \left(\frac{i-1}{i+1}\right)\left(\frac{z+1}{z-1}\right) = -\frac{(w+i)}{(w-i)}$$

$$\Rightarrow \frac{(i-1)^2}{-1-1}\left(\frac{z+1}{z-1}\right) = -\frac{w+i}{w-i}$$

$$\Rightarrow \frac{-\cancel{1}-2i+\cancel{1}}{-2}\left(\frac{z+1}{z-1}\right) = -\frac{w+i}{w-i} \Rightarrow i\left(\frac{z+1}{z-1}\right) = \cancel{-}\frac{w+i}{w-i}$$

$$i\,\frac{z+1}{z-1} = i^2\,\frac{w+i}{w-i}$$

$$\Rightarrow \frac{1}{i}\,\frac{(z+1)}{(z-1)} = \frac{w+i}{w-i}$$

Now, by C/D method.

$$\frac{(z+1)+i(z-1)}{(z+1)-i(z-1)} = \frac{(w+\cancel{i})+(w-\cancel{i})}{(\cancel{w}+i)-(\cancel{w}-i)}$$

$$\Rightarrow \frac{z(1+i)+(1-i)}{z(1-i)+(1+i)} = \frac{w}{i}$$

$$\Rightarrow \frac{z(1+i)^2+(1-i)(1+i)}{z(1+i)(1-i)+(1+i)^2} = \frac{w}{i}$$

$$\Rightarrow \frac{z[\cancel{1}+2i-\cancel{1}]+2}{z[2]+(\cancel{1}+2i-\cancel{1})} = \frac{w}{i}$$

$$\Rightarrow \frac{2iz+2}{2z+2i} = \frac{w}{i}$$

$$\therefore\ w = i\left(\frac{iz+1}{z+i}\right) = \frac{-z+i}{z+i}\ \text{(Ans)}.$$

6. Find the bilinear map which maps the points $z=\infty, i, 0$ into the points $w=0, -i, \infty$.

Ans $z_1=\infty,\ z_2=i,\ z_3=0,\ z_4=z$

$w_1=0,\ w_2=-i,\ w_3=\infty,\ w_4=w$

$$\therefore \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\Rightarrow \frac{(0+i)(\infty-w)}{(-i-\infty)(w-0)} = \frac{(\infty-i)(0-z)}{(i-0)(z-\infty)}$$

$$\Rightarrow \frac{i\left(1-\frac{w}{\infty}\right)}{\left(-\frac{i}{\infty}-1\right)w} = \frac{\left(1-\frac{i}{\infty}\right)(0-z)}{i\left(z/\infty-1\right)}$$

$$\Rightarrow \frac{i}{-w} = \frac{\cancel{-}z}{\cancel{-}i} \Rightarrow \frac{i^2}{-w} = z \Rightarrow \boxed{w = \frac{1}{z}}$$

7) Same as problem 6.