# Tutorial Sheet - 2 (Unit-4)

1. Find the image of the circle $|z| = 3$ under the transformation $w = 2z$

Ans $w = 2z \Rightarrow z = \frac{w}{2}$

$$\Rightarrow |z| = \left|\frac{w}{2}\right| \Rightarrow 3 = \left|\frac{w}{2}\right| \Rightarrow |w| = 6$$

if $w = u + iv$ then, $|w| = 6 \Rightarrow u^2 + v^2 = 6^2$

i.e. the image of the circle $|z| = 3$ is again a circle $|w| = 6$.

2. Find a fun $w$ such that $w = u + iv$ is analytic if $u = e^x \sin y$

Ans here $u$ is given so, we can apply Milne-Thomson method

$$\frac{\partial u}{\partial x} = e^x \sin y = \phi_1(x,y) \qquad \frac{\partial u}{\partial y} = e^x \cos y = \phi_2(x,y)$$

$$\therefore f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\} dz = \int (0 - ie^z) dz = -ie^z + c \text{ (Ans).}$$

3. Determine the analytic fun $u + iv$ whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Ans $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \phi_1(x,y) \qquad \frac{\partial u}{\partial y} = -6xy - 6y = \phi_2(x,y)$$

$$\therefore f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\} dz$$

$$= \int (3z^2 + 6z) - i(0)\, dz = z^3 + 3z^2 + c \text{ (Ans).}$$

4. Find the analytic fun. $f(z) = u + iv$ if $u - v = e^x(\cos y - \sin y)$

At $u - v = e^x(\cos y - \sin y)$

$\therefore u_x - v_x = e^x(\cos y - \sin y)$

$u_y - v_y = e^x(-\sin y - \cos y)$

$\because f(z)$ is analytic so, $u_x = v_y$ & $u_y = -v_x$

$$\therefore u_x + u_y = e^x(\cos y - \sin y)$$

$$u_y - u_x = e^x(-\sin y - \cos y)$$

$$+ \quad \overline{2u_y = e^x(\cancel{\cos y} - \sin y - \sin y - \cancel{\cos y})}$$

$$\Rightarrow u_y = -e^x \sin y = \phi_2(x, y)$$

$$\therefore u_x = e^x(\cos y - \sin y) - u_y = e^x(\cos y - \cancel{\sin y} + \cancel{\sin y})$$

$$= e^x \cos y = \phi_1(x, y)$$

$$\therefore f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\} dz$$

$$= \int e^z dz = \boxed{e^z + c} \text{ (Ans).}$$

5. Find the analytic fun. $f(z) = u + iv$ if $u - v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

At $$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{2\cosh 2y \cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x - 2}{(\cosh 2y - \cos 2x)^2} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\text{---(1)}$$

Similarly $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cosh 2y - \cos 2x)\, 0 - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$

$$\Rightarrow \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \quad \text{———(II)}$$

$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\therefore$ (I) & (II) takes the form

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2\cosh 2y \cos 2x - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = -\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

adding we have $2\frac{\partial u}{\partial y} = \frac{2\cosh 2y \cos 2x - 2\sin 2x \sinh 2y - 2}{(\cosh 2y - \cos 2x)^2}$

$$\therefore \frac{\partial u}{\partial y} = \frac{\cosh 2y \cos 2x - \sinh 2y \sin 2x - 1}{(\cosh 2y - \cos 2x)^2} = \phi_2(x,y)$$

Subtracting $\frac{\partial u}{\partial x} = \frac{\cosh 2y \cos 2x + \sinh 2y \sin 2x - 1}{(\cosh 2y - \cos 2x)^2} = \phi_1(x,y)$

$$\therefore f(z) = \int \left(\phi_1(z,0) - i\phi_2(z,0)\right) dz = \int \left\{ \frac{\cos 2z - 1}{(1-\cos 2z)^2} - i\,\frac{\cos 2z - 1}{(1-\cos 2z)^2} \right\} dz$$

$$= \int \frac{-1}{1-\cos 2z} + i\,\frac{1}{1-\cos 2z}\, dz$$

$$= -\int \frac{dz}{2\sin^2 z} + i\int \frac{dz}{2\sin^2 z}$$

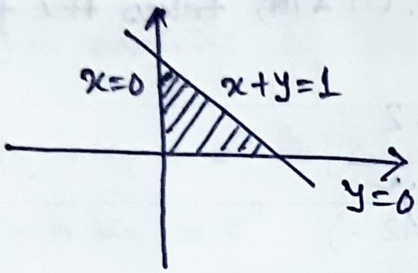
$$= -\frac{1}{2}\int \operatorname{cosec}^2 z\, dz + \frac{i}{2}\int \operatorname{cosec}^2 z\, dz$$

$$= -\frac{1}{2}\cot z + \frac{i}{2}\cot z + C$$

$$= \frac{1}{2}(i-1)\cot z + C$$ (Ans).

6. Determine the region D of the w-plane into which the triangular region D enclosed by the lines $x=0, y=0, x+y=1$ is transformed under the transformation $w=2z$.

Ans




z-plane

$$w = 2z = 2(x+iy)$$

$$\therefore u+iv = w = 2x + i\,2y$$

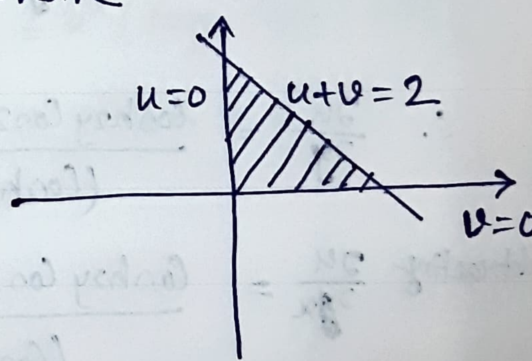
$$\therefore u = 2x \quad v = 2y$$

Now, $x=0 \Rightarrow u=0$

$y=0 \Rightarrow v=0$

$x+y=1 \Rightarrow u+v = 2(x+y) = 2$

$\therefore$ we have




w-plane

7. Find an analytic fun $f(z) = u+iv$ given $2u+3v = \dfrac{\sin 2x}{\cosh 2y - \cos x}$

This is same as problem-5