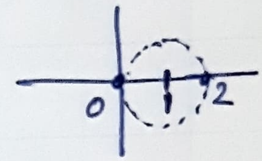
Tutorial-2 (Unit-5)

1. State Taylor's and Laurent's theorem:

Ans Done in class

2. Obtain the Taylor's series of $f(z) = \frac{z-1}{z^2}$ in power of $z-1$.

Ans Here $f(z)$ is analytic at $z=1$ and over the circle centered at $|z-1|<1$ is the region of convergence of $f(z)$.



Let $u = z-1 \Rightarrow z = u+1$

$$\therefore f(z) = \frac{z-1}{z^2} = \frac{u}{(u+1)^2} = u(1+u)^{-2}$$
$$= u(1-2u+3u^2-4u^3+\cdots)$$
$$= u - 2u^2 + 3u^3 - 4u^4 + \cdots$$
$$= (z-1) - 2(z-1)^2 + 3(z-1)^3 - 4(z-1)^4 + \cdots$$

(Ans).

3. Find the Laurent's series expansion of $f(z) = \frac{1}{z(z-1)}$ in $|z|<1$ and $|z|>1$

Ans $f(z) = \frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$

now, for $|z|<1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z} = -\frac{1}{(1-z)} - \frac{1}{z} = -(1-z)^{-1} - \frac{1}{z}$$
$$= -[1+z+z^2+\cdots] - \frac{1}{z}$$
$$= -\sum_{n=0}^{\infty} z^n - \frac{1}{z}$$

for $|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$

$$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z} = \frac{1}{z(1-1/z)} - \frac{1}{z}$$

$$= \frac{1}{z}(1-1/z)^{-1} - 1/z$$

$$= \frac{1}{z}\left[1+1/z+\frac{1}{z^2}+\cdots\right] - \frac{1}{z}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}$$

4. Find the residue of $f(z)$ at $z=0$

i) $f(z) = e^{1/z}$

ii) $f(z) = \frac{\sin z}{z^4}$

Ans i) $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$

here, the principal part of $f(z)$ contains infinite no. of terms about $z=0$.

$\therefore$ $z=0$ is an essential singularity

Now, the co-efficient of $\frac{1}{z}$ is 1 $\therefore$ $\text{Res}(z=0)=1$.

ii) $f(z) = \frac{\sin z}{z^4}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$f(z) = \frac{z - z^3/3! + z^5/5! - \cdots}{z^4}$$

$$= \frac{z(1 - z^2/3! + z^4/5! - \cdots)}{z^4} = \frac{1}{z^3}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)$$

$= \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \cdots$ here $z=0$ is a pole of order 3.

The co-efficient of $\frac{1}{z} = -\frac{1}{3!} = -1/6$

$\therefore$ Res $(z=0) = -1/6$

5. Find the residue of $\frac{e^z}{z^8}$.

At $z=0$ is a pole of order 8.

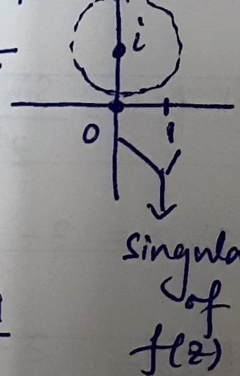
$\therefore \text{Res}(z=0) = \frac{1}{7!} \lim_{z\to 0}\left[\frac{d^7}{dz^7}\left(z^8 \cdot \frac{e^z}{z^8}\right)\right]$

$= \frac{1}{7!} \lim_{z\to 0}\left[\frac{d^7}{dz^7}(e^z)\right]$

$= \frac{1}{7!} \lim_{z\to 0} e^z = \frac{1}{7!}$ (Ans).

6. Find the Taylor's expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point $z=i$

As the fun^n $f(z) = \frac{2z^3+1}{z^2+z} = \frac{2z^3+1}{z(z+1)}$ is not analytic at $z=0$ & $z=-1$ so, we can find the Taylor's expansion about $z=i$. So, we can consider a circle centered at $z=i$ & radius less than 1. $C: |z-i|<1$



$f(z) = \frac{2z^3+1}{z^2+z} =$ $\qquad f(i) = \frac{2i^3+1}{i^2+i} = \frac{-2i+1}{-1+i}$

$f'(z) = \frac{(z^2+z)(6z^2)-(2z^3+1)(2z+1)}{(z^2+z)^2}$

$= \frac{6z^4+6z^3-4z^4-2z-2z^3-1}{(z^2+z)^2} = \frac{2z^4+4z^3-2z-1}{(z^2+z)^2}$

$$f'(i) = \frac{2i^4+4i^3-2i-1}{(i^2+i)^2} = \frac{2-4i-2i-1}{(-1+i)^2}$$

$$= \frac{1-6i}{\cancel{1}-2i-\cancel{1}} = \frac{6i-1}{2i}$$

$$\therefore f(z) = f(i) + (z-i)f'(i) + \frac{(z-i)^2}{2!}f''(i) + \cdots$$

$$= \frac{1-2i}{i-1} + (z-i)\left(\frac{6i-1}{2i}\right) + \cdots \quad \text{(Ans)}$$

7. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region

i) $|z|<1$

ii) $1<|z|<2$

iii) $|z|>2$

iv) $0<|z-1|<1$

Ans $f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$

i) $|z|<1$ $\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$

$$= \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z/2} \quad \because |z|<1 \quad \therefore |z/2|<1$$

$$= (1-z)^{-1} - \frac{1}{2}(1-z/2)^{-1}$$

$$= [1+z+z^2+\cdots] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

ii) $1<|z|<2$

$1<|z| \Rightarrow \left|\frac{1}{z}\right|<1$ & $|z|<2 \Rightarrow |z/2|<1$

$$\therefore f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = -\frac{1}{z}\frac{1}{(1-1/z)} - \frac{1}{2}\frac{1}{(1-z/2)}$$

$$= -\frac{1}{z}(1-1/z)^{-1} - \frac{1}{2}(1-z/2)^{-1} = -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$- 1/2\left[1+z/2+(z/2)^2+\cdots\right]$$

iii) $|z|>2 \Rightarrow \frac{2}{|z|}<1$

$\therefore f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$

$= -\frac{1}{z}(1-1/z)^{-1} + \frac{1}{z}(1-2/z)^{-1}$

$= -1/z\left[1+1/z+\frac{1}{z^2}+\cdots\right] + \frac{1}{z}\left[1+2/z+(2/z)^2+\cdots\right]$

iv) $0<|z-1|<1$ let $u=z-1$

$\Rightarrow z=u+1$

$\therefore f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$

$= -\frac{1}{u+1-1} + \frac{1}{u+1-2}$

$= -\frac{1}{u} + \frac{1}{u-1}$

Now, $0<|z-1|<1 \Rightarrow 0<|u|<1$

$= -\frac{1}{u} - \frac{1}{1}(1-u)^{-1}$

$= -\frac{1}{u} - [1+u+u^2+\cdots]$

$= -\frac{1}{z-1} - [1+(z-1)+(z-1)^2+\cdots]$

8. Find Residues of $f(z) = \frac{z^3}{(z-1)^2(z-2)(z-3)}$ and its poles

Let here $z=1$ is a pole of order 2

$z=2$ " " " " 1

$z=3$ " " " " 1

$\therefore$ Res $(z=1) = \frac{1}{(2-1)!} \lim_{z\to 1}\left[\frac{d}{dz}\left(\cancel{(z-1)^2}\frac{z^3}{\cancel{(z-1)^2}(z-2)(z-3)}\right)\right]$

$$\text{Res}(z=1) = \lim_{z\to 1}\left[\frac{d}{dz}\left(\frac{z^3}{(z-2)(z-3)}\right)\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left\{\frac{z^3}{z^2-5z+6}\right\}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z^2-5z+6).3z^2 - z^3.(2z-5)}{(z^2-5z+6)^2}\right]$$

$$= \frac{(1-5+6)3 - 1(2-5)}{(1-5+6)^2} = \frac{6+3}{4} = \boxed{\frac{9}{4}}$$

$$\text{Res}(z=2) = \lim_{z\to 2}\cancel{(z-2)}\frac{z^3}{(z-1)^2\cancel{(z-2)}(z-3)}$$

$$= \lim_{z\to 2}\frac{z^3}{(z-1)^2(z-3)} = \frac{8}{1.(-1)} = \boxed{-8}$$

$$\text{Res}(z=3) = \lim_{z\to 3}\cancel{(z-3)}\frac{z^3}{(z-1)^2(z-2)\cancel{(z-3)}}$$

$$= \lim_{z\to 3}\frac{z^3}{(z-1)^2(z-2)} = \frac{27}{4\times 1} = \boxed{\frac{27}{4}}$$

9. Find residue of $f(z) = \frac{e^z}{\cos \pi z}$ & hence evaluate $\oint_C f(z)dz$ where $C: |z|=1$

Ans here the singularities of $f(z)$ are $\cos\pi z = 0$

$$\Rightarrow \cos\pi z = \cos(2n+1)\pi/2$$

$$\Rightarrow z = (2n+1)\frac{1}{2}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow z = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$$

Now, for the circle $|z|=1$ the points $z=1/2$ & $z=-1/2$ are the interior points and the others lie outside $|z|=1$.

$\therefore z = \frac{1}{2}, -\frac{1}{2}$ are poles of order 1.

$$\therefore \text{Res}(z = \tfrac{1}{2}) = \lim_{z \to \frac{1}{2}} (z - \tfrac{1}{2}) \frac{e^z}{\cos \pi z} \quad [\text{form } \tfrac{0}{0}]$$

$\therefore$ By L'Hospital Rule

$$= \lim_{z \to \frac{1}{2}} \frac{e^z(z - \frac{1}{2}) + e^z}{-\pi \sin \pi z}$$

$$= \frac{e^{\frac{1}{2}}(\frac{1}{2} - \frac{1}{2}) + e^{\frac{1}{2}}}{-\pi \sin \frac{\pi}{2}} = \frac{e^{\frac{1}{2}}}{-\pi} =$$

$$\text{Res}(z = -\tfrac{1}{2}) = \lim_{z \to -\frac{1}{2}} \frac{(z + \frac{1}{2}) e^z}{\cos \pi z} \quad [\text{form } \tfrac{0}{0}]$$

$\therefore$ By L'Hospital Rule

$$= \lim_{z \to -\frac{1}{2}} \frac{e^z(z + \frac{1}{2}) + e^z}{-\pi \sin \pi z}$$

$$= \frac{e^{-\frac{1}{2}} \times 0 + e^{-\frac{1}{2}}}{-\pi \sin(-\frac{\pi}{2})} = \frac{e^{-\frac{1}{2}}}{\pi}$$

now, $$\oint_C f(z)\, dz = 2\pi i \left[\frac{e^{\frac{1}{2}}}{-\pi} + \frac{e^{-\frac{1}{2}}}{\pi} \right]$$

$$= 2i \left[-e^{\frac{1}{2}} + e^{-\frac{1}{2}} \right] \text{ (Ans).}$$

10. Find the nature & location of the singularities of the fun?

i) $\frac{z - \sin z}{z^2} \Rightarrow z = 0$ is a singularity now,

$$\Rightarrow \frac{z - [z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots]}{z^2}$$

$$\Rightarrow \frac{\cancel{z} - \cancel{z} + \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots}{z^2} = \frac{z}{3!} - \frac{z^3}{5!} + \cdots$$ No negative power of z so,

$z=0$ is removable singularity.

ii) $(z+1)\sin\dfrac{1}{z-2}$

here $z=2$ is the singularity now,

$$(z+1)\left[\left(\frac{1}{z-2}\right)-\frac{(1/z-2)^3}{3!}+\frac{(1/z-2)^5}{5!}-\cdots\right]$$

the Laurent's series about $z=2$ contains infinite no. of negative term of power of $(z-2)$ so, ~~z~~ $z=2$ is essential singularity.

iii) $\dfrac{1}{\cos z-\sin z}$

Singularities are $\cos z-\sin z=0 \Rightarrow \cos z=\sin z \Rightarrow \tan z=1$

$\Rightarrow z=\pi/4$

$\therefore$ $z=\pi/4$ is a simple pole as,

$$\lim_{z\to\pi/4}\frac{1}{\cos z-\sin z}=\infty \text{ but } \lim_{z\to\pi/4}(z-\pi/4)\frac{1}{\cos z-\sin z}\quad\left[\frac{0}{0}\right]$$

$$=\lim_{z\to\pi/4}\frac{1}{-\sin z-\cos z}$$

$$=\frac{1}{-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}=\frac{1}{-\frac{2}{\sqrt{2}}}=-\frac{1}{\sqrt{2}}$$ finite