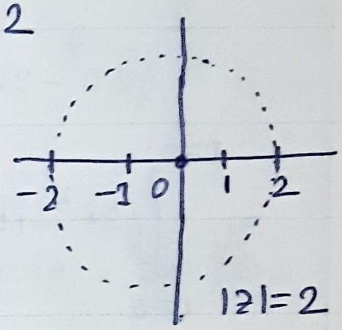
Tutorial sheet-1 (Unit-5)

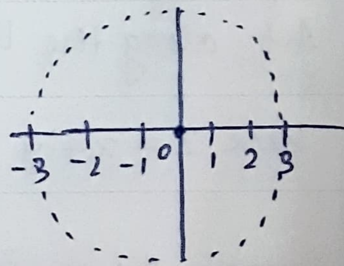
1. Evaluate $\oint_C \frac{e^{-z}}{z+1}\,dz$ where C is a circle $|z|=2$



Ans here let $f(z)=e^{-z}$ Now, $z=-1$ is interior point of the circle $|z|=2$ so, by 1st form of Cauchy's integral formula

$$\oint_C \frac{e^{-z}}{z+1}\,dz = 2\pi i \times f(-1) = 2\pi i \times e^{-(-1)} = 2\pi i e \quad \text{(Ans)}$$

2. Evaluate $\oint_C \frac{e^{2z}}{(z+1)(z-2)}\,dz$ where C is a circle $|z|=3$



Ans here $z=-1$, $z=2$ both are interior point of the circle

$$\frac{1}{(z+1)(z-2)} = \frac{-1/3}{z+1} + \frac{1/3}{z-2}$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = -\frac{1}{3}\oint_C \frac{e^{2z}}{z+1}\,dz + \frac{1}{3}\oint_C \frac{e^{2z}}{z-2}\,dz$$

$\therefore$ By Cauchy's integral formula (1st form) we have

$$= -\frac{1}{3}\times 2\pi i \times f(-1) + \frac{1}{3}\times 2\pi i \times f(2)$$

$$= -\frac{1}{3}\times 2\pi i \times e^{-2} + \frac{1}{3}\times 2\pi i \times e^{4} = \frac{2\pi i}{3}\left(e^{4}-e^{-2}\right) \quad \text{(Ans)}$$

3. Evaluate $\oint_C \bar{z}\,dz$ from $A(0,0)$ to $B(4,2)$ along the curve C and

$z = t^2 + it$

Ans $z = t^2 + it \quad \therefore dz = (2t + i)\,dt$

$A(0,0)$ to $B(4,2)$ $\therefore t^2 : 0$ to 4 ie. $t : 0$ to 2

$\bar{z} = t^2 - it \quad \therefore \oint_C \bar{z}\,dz = \int_{t=0}^{2} (t^2 - it)(2t + i)\,dt$

$$= \int_{t=0}^{2} (2t^3 - 2it^2 + it^2 - i^2 t)\,dt$$

$$= \int_{t=0}^{2} (2t^3 - it^2 + t)\,dt = \left[\frac{\cancel{2}t^4}{\cancel{4}_2} - i\frac{t^3}{3} + \frac{t^2}{2}\right]_0^2$$

$$= \frac{2^4}{2} - i\frac{2^3}{3} + \frac{2^2}{2} = 8 + 2 - i\frac{8}{3} = 10 - i\frac{8}{3} \text{ (Ans).}$$

4. Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along the line $y = \frac{x}{2}$

Ans along the line $y = \frac{x}{2}$ or $x = 2y$ $\qquad z = x + iy$

$\qquad\qquad\qquad\qquad\qquad\qquad dz = dx + i\,dy$

here $z = x + iy$ so, $\bar{z} = x - iy$ $\quad (\bar{z})^2 = (x - iy)^2$

$$= x^2 - 2ixy + i^2 y^2$$

$$= (x^2 - y^2) - i\,2xy$$

Now, $z : 0$ to $2+i$

$\therefore x : 0$ to 2

$y : 0$ to 1 $\quad$ either convert in x or in y

$$\therefore \int_0^{2+i} (\bar{z})^2\,dz = \int_{x=0}^{2} \left(\left(x^2 - \frac{x^2}{4}\right) - i\,2x \cdot \frac{x}{2}\right)\left(dx + i\frac{dx}{2}\right)$$

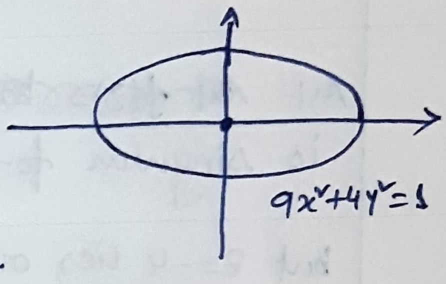
$$= \left(1 + \frac{i}{2}\right)\int_{x=0}^{2} \left(\frac{3}{4}x^2 - ix^2\right)dx = \left(1 + \frac{i}{2}\right)\left[\frac{\cancel{3}}{4} \times \frac{x^3}{\cancel{3}} - i\frac{x^3}{3}\right]_0^2$$

$$= (1 + i/2)\,(2 - 8i/3)$$

$= 2 + i - \frac{8i}{3} + \frac{4}{3}$

$= \frac{10}{3} - i\frac{5}{3} = \frac{5}{3}(2-i)$ (Ans).

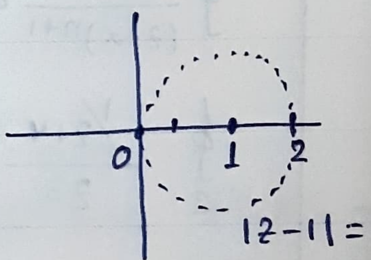
5. Evaluate $\oint_C \frac{\cos z}{z} dz$ where C is an ellipse $9x^2+4y^2=1$



here, let $f(z) = \cos z$. Now, the integral is in the form $\oint_C \frac{f(z)}{z-\alpha} dz$. here $\alpha = 0$ is interior of the ellipse so, by Cauchy's integral formula

$\oint_C \frac{f(z)}{(z-\alpha)} dz = 2\pi i \times f(\alpha)$ so, $\oint_C \frac{\cos z}{z} dz = 2\pi i \times \cos(0) = 2\pi i$ (Ans).

6. Evaluate $\oint_C \frac{3z^2+z}{z^2-1} dz$ where $C: |z-1| = 1$



$\oint_C \frac{3z^2+z}{z^2-1} dz = \oint_C \frac{3z^2+z}{(z+1)(z-1)} dz$

Now, $\frac{1}{(z+1)(z-1)} = \frac{-1/2}{z+1} + \frac{1/2}{z-1}$

$\therefore \oint_C \frac{3z^2+z}{(z+1)(z-1)} dz = -\frac{1}{2}\oint_C \frac{3z^2+z}{z+1} dz + \frac{1}{2}\oint_C \frac{3z^2+z}{z-1} dz$

Let $f(z) = 3z^2+z$

Now, $z = 1, -1$ are the points where the integrals are singular

but $z = -1$ lies outside the circle $|z-1| = 1$ so,

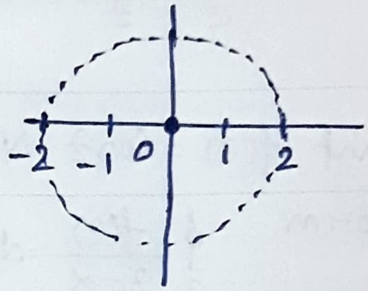
$\oint_C \frac{3z^2+z}{z+1} dz = 0$ by Cauchy's theorem

Now, $\oint_C \frac{3z^2+z}{z-1} dz = 2\pi i \times f(1) = 2\pi i \times (3+1) = 8\pi i$

$$\therefore \oint_C \frac{3z^2+z}{z^2-1}\,dz$$

$$= -\frac{1}{2}\times 0 + \frac{1}{2}\times \cancel{8}^{\,4}\pi i = 4\pi i \text{ (Ans)}$$

7. Evaluate $\oint_C \frac{dz}{z^3(z+4)}$ where $C: |z|=2$



~~At~~ Here the integral is singular for $z=0$ & $z=-4$

but $z=-4$ lies outside the circle so, $f(z) = 1/z+4$ is analytic within and on the circle

so, by Cauchy integral formula (2nd form)

$$\oint \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = \frac{2\pi i}{n!}\times f^{n}(\alpha) \text{ we have,}$$

$$\oint_C \frac{1/z+4}{z^3}\,dz = \frac{2\pi i}{2!}\times f''(0) \quad [\because \alpha=0 \text{ here which is an interior pt. of the circle}]$$

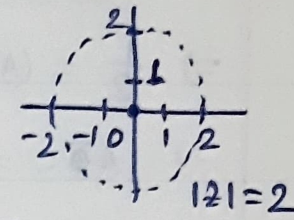
now, $f(z) = \frac{1}{z+4}$ $\therefore f'(z) = -\frac{1}{(z+4)^2}$

$$\therefore f''(z) = \frac{2}{(z+4)^3}$$

$$\therefore f''(0) = \frac{2}{4^3} = \frac{1}{32}$$

$$\therefore \oint_C \frac{1/z+4}{z^3}\,dz = \frac{2\pi i}{2!}\times\frac{1}{32} = \frac{\pi i}{32} \text{ (Ans).}$$

8. Evaluate $\oint_C \frac{z^3-2z+1}{(z-i)^2}\,dz$ When $C: |z|=2$



As here the integral be comes singular at $z=i$ Which lies inside the circle $|z|=2$. Let $f(z)=z^3-2z+1$.

$\therefore$ By cauchy's integral formula (2nd form)

$$\oint \frac{f(z)}{(z-i)^2}\,dz = \frac{2\pi i}{1!}\times f'(i)$$

$f(z)=z^3-2z+1 \Rightarrow f'(z)=3z^2-2 \quad f'(i)=3i^2-2=-5$

$$\therefore \oint_C \frac{z^3-2z+1}{(z-i)^2}\,dz = -10\pi i$$

9. Evaluate $\oint_C \frac{e^z}{(z^2-\pi^2)^2}$ Where $C: |z|=4$

As here the singularities of integrals are $z^2-\pi^2=0 \Rightarrow z=\pm\pi$

Let $f(z)=e^z$ Which is analytic Within & on $|z|=4$

and $z=\pi$ & $z=-\pi$ both lies inside the circle $[\because \pi=\frac{22}{7}]$

Now, $$\frac{1}{(z^2-\pi^2)^2} = \frac{1}{(z+\pi)^2(z-\pi)^2} = \frac{A}{(z+\pi)} + \frac{B}{(z+\pi)^2} + \frac{C}{(z-\pi)} + \frac{D}{(z-\pi)^2}$$

$$\therefore 1 = A(z+\pi)(z-\pi)^2 + B(z-\pi)^2 + C(z+\pi)^2(z-\pi) + D(z+\pi)^2$$

put $z=\pi$ $\quad 1 = 4\pi^2 D \Rightarrow D = \boxed{\frac{1}{4\pi^2}}$

put $z=-\pi$ $\quad 1 = 4\pi^2 B \Rightarrow B = \boxed{\frac{1}{4\pi^2}}$

Equating the co-efficient of z^3 : $0 = A+C$

Equating the co-efficient of Const: $1 = \pi^3 A + B\pi^2 - C\pi^3 + D\pi^2$

$$1 = \pi^3 A + \frac{1}{4} - C\pi^3 + \frac{1}{4}$$

$$\Rightarrow \frac{1}{2} = (A - C)\pi^3$$

$$\Rightarrow A - C = \frac{1}{2\pi^3}$$

$$A + C = 0$$
$$A - C = \frac{1}{2\pi^3}$$
$$+ \quad \overline{}$$
$$2A = \frac{1}{2\pi^3}$$
$$A = \frac{1}{4\pi^3}$$
$$C = -\frac{1}{4\pi^3}$$

$$\therefore \oint_C \frac{e^z}{(z^2 - \pi^2)^2}\, dz$$

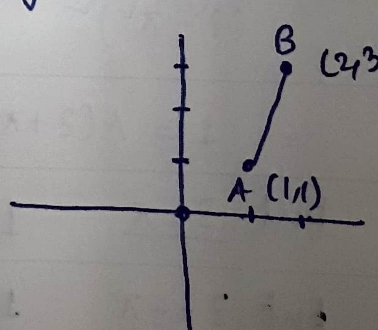
$$= \frac{1}{4\pi^3}\oint_C \frac{e^z}{z+\pi}\, dz + \frac{1}{4\pi^2}\oint_C \frac{e^z}{(z+\pi)^2}\, dz - \frac{1}{4\pi^3}\oint_C \frac{e^z}{(z-\pi)}\, dz$$
$$+ \frac{1}{4\pi^2}\oint_C \frac{e^z}{(z-\pi)^2}\, dz$$

$$= \frac{1}{\underset{2}{\cancel{4}}\pi^{\cancel{3}_2}} \times \cancel{2}\pi i \times f(-\pi) + \frac{1}{\underset{2}{\cancel{4}}\pi^{\cancel{2}_1}} \times \frac{\cancel{2}\pi i}{1!} \times f'(-\pi) - \frac{1}{\underset{2}{\cancel{4}}\pi^3} \times \cancel{2}\pi i \times f(\pi)$$
$$+ \frac{1}{\underset{2}{\cancel{4}}\pi^2} \times \frac{\cancel{2}\pi i}{1!} \times f'(\pi)$$

$$= \frac{i}{2\pi^2}e^{-\pi} + \frac{i}{2\pi}e^{-\pi} - \frac{i}{2\pi^2}e^{\pi} + \frac{i}{2\pi}e^{\pi}$$

$$= \frac{i}{2\pi}\left(\frac{1}{\pi} + 1\right)e^{-\pi} - \frac{i}{2\pi}e^{\pi}\left(\frac{1}{\pi} - 1\right)$$

10. Evaluate $\displaystyle\oint_{1+i}^{2+3i} (z^2 + z)\, dz$ along the line joining the points $(1,1)$ and $(2,3)$.



Eqn of line joining $A(1,1)$ to $B(2,3)$ is

$$\frac{y-1}{x-1} = \frac{3-1}{2-1}$$

$$\Rightarrow \frac{(y-1)}{(x-1)} = 2$$

$$\Rightarrow y - 1 = 2x - 2$$

$$\Rightarrow \boxed{y = 2x - 1}$$

Now, $z = x + iy$

$dz = dx + i\,dy$ on the line $y = 2x - 1$ $\therefore$ $dy = 2\,dx$

$\therefore dz = dx + i\,2\,dx$

$\Rightarrow dz = (1+2i)\,dx$

$\therefore z : 1+i$ to $2+3i$

$\Rightarrow x : 1$ to 2

$y : 1$ to 3

$$\int_{z=1+i}^{2+3i} (z^2 + z)\,dz$$

$$= \int_{x=1}^{2} \{(x+iy)^2 + (x+iy)\}\,(dx + i\,dy)$$

$$= \int_{x=1}^{2} \{(x + i(2x-1))^2 + (x + i(2x-1))\}\,(1+2i)\,dx$$

= Now, simple integration