Tutorial sheet - 1 (unit-4)

1. Test whether $f(z) = z^3$ is analytic

$\Rightarrow f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3$

$= x^3 + i\,3x^2y - 3xy^2 - iy^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$= u + iv$

Now, $u = x^3 - 3xy^2 \quad \therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \frac{\partial u}{\partial y} = -6xy$

$v = 3x^2y - y^3 \quad \therefore \frac{\partial v}{\partial x} = 6xy \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ so, $f(z)$ satisfies C-R eqn also

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

hence $f(z)$ is analytic everywhere.

2. If $f(z)$ and $\overline{f(z)}$ are analytic fun of z then prove that $f(z)$ is const.

Ans let $f(z) = u+iv \quad \therefore \overline{f(z)} = \overline{u+iv} = u - iv$

$\because f(z)$ is analytic so, u & v satisfies C-R Eqn

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ —— (1)

$\because \overline{f(z)} = u - iv$ is analytic so, the real part and imaginary part of $\overline{f(z)}$ satisfies C-R eq^n.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial y}(-v)$ & $\frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}(-v)$ $\left[\because \begin{array}{l} \text{Real } \overline{f(z)} = u \\ \text{Im } \overline{f(z)} = -v \end{array}\right]$

$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ —(2)

By (1) & (2) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$ $\therefore 2\frac{\partial v}{\partial y} = 0 \Rightarrow v = f(x, c)$ —(3)

$\hookrightarrow \therefore v$ is independent of y.

Now, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x}$

$\therefore 2\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0 \Rightarrow v = g(y, c)$ —(4)

$\hookrightarrow \therefore v$ is independent of x.

$\therefore$ By (3) & (4) v is independent of x & y hence $\boxed{v = \text{Const.}}$

similarly we can show $\boxed{u = \text{Const.}}$

Show that the fun^n $f(z) = e^x(\cos y + i \sin y)$ is analytic and find its derivative.

Ans $f(z) = e^x \cos y + i e^x \sin y = u(x,y) + iv(x,y)$

Now, $\frac{\partial u}{\partial x} = e^x \cos y$ $\quad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$ $\quad \frac{\partial v}{\partial y} = e^x \cos y$

hence, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ so, u, v are satisfying C-R Eq^{ns}

also, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous. hence $f(z)$ is analytic.

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$

$= e^x(\cos y + i \sin y)$

$= e^x e^{iy} = e^{x+iy} = e^z$ (Ans).

4. P.T. if v is harmonic conjugate of u and u is harmonic conjugate of v then f(z) is constant.

Ans If v is harmonic conjugate of u then,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——— (1)}$$

similarly if u is harmonic conjugate of v then,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \text{——— (2)}$$

By (1) & (2) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ $\therefore 2\frac{\partial u}{\partial x} = 0$ so, $\frac{\partial u}{\partial x} = 0$ so, u is independent of x.

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\therefore 2\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} = 0$

So, u is independent of y.

$\therefore u = \text{Const.}$ similarly $v = \text{Const.}$

5. Show that the funⁿ $u = 2\log(x^2+y^2)$ is harmonic.

$$\Rightarrow \frac{\partial u}{\partial x} = \cancel{2}\frac{1}{x^2+y^2}\cdot\cancel{2}x = \frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2).0 - y.2x}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

here, u has continuous 1st & 2nd order partial derivatives.

More over, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$

$\therefore$ u satisfies Laplace Eqⁿ hence u is harmonic.

6. Show that an analytic funⁿ with i) Constant real part is constant
ii) Constant modulus is constant.

Ans (i) Let $f(z) = u + iv$ $\quad u$ = Real part of $f(z)$
v = Imaginary part of $f(z)$.

If $u = c$ $\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$

$\because f(z)$ is analytic then, u & v satisfy C-R eqⁿs then

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\quad \therefore \frac{\partial v}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$

$\Rightarrow v = \text{Const.}$

$\therefore f(z) = \text{Const.}$

(ii) $|f(z)| = \sqrt{u^2+v^2}$ if $|f(z)| = \text{Const.} = c_1$

$\therefore \sqrt{u^2+v^2} = c_1 \Rightarrow u^2+v^2 = c_1^2$ $\quad \therefore 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$

Diff w.r.to x $\Rightarrow \boxed{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0}$ —(1)

Diff w.r.to y $\Rightarrow \boxed{u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0}$ —(2)

$\because f(z)$ is analytic so, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ by (1) $\quad u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$ —(3) $\times u$

by (2) $\quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$ —(4) $\times v$

$$\therefore \; u^2\frac{\partial u}{\partial x} - \cancel{uv\frac{\partial u}{\partial y}} = 0$$

$$+ \; \cancel{uv\frac{\partial u}{\partial y}} + v^2\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x}(u^2+v^2) = 0 \Rightarrow c_1^2\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = 0}$$

In similar way $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = 0$

Now, $f'(z) = u_x + i v_x = 0$

$\therefore \boxed{f(z) = \text{Const.}}$

7. If $f(z) = u + iv$ is an analytic fun of z show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Ans $|f(z)|^2 = u^2 + v^2$ $\qquad \frac{\partial}{\partial x}(|f(z)|^2) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$

$$\frac{\partial^2}{\partial x^2}(|f(z)|^2) = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} \quad \text{—(1)}$$

Similarly $\frac{\partial^2}{\partial y^2}(|f(z)|^2) = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2}$ —(2)

Now, adding (1) & (2) we have,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)$$
$$+ 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{—(3)}$$

$\because f(z)$ is analytic so, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ also, u & v both are harmonic as $f(z)$ is analytic.

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ by (3) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)$

$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ $\left[\because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$

8. If $f(z)=u+iv$ is an analytic fun^{n} of z then show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|=0$$

Ans $|f(z)|=\sqrt{u^2+v^2}$ $\quad \log|f(z)|=\log(u^2+v^2)^{1/2}=\frac{1}{2}\log(u^2+v^2)$

$$\frac{\partial}{\partial x}(\log|f(z)|)=\frac{1}{\cancel{2}}\,\frac{1}{u^2+v^2}\,\cancel{2}u\cdot\frac{\partial u}{\partial x}+\frac{1}{\cancel{2}}\,\frac{1}{u^2+v^2}\,\cancel{2}v\,\frac{\partial v}{\partial x}$$

$$=\frac{uu_x+vv_x}{u^2+v^2} \quad \text{——— (1)}$$

Similarly $\frac{\partial}{\partial y}(\log|f(z)|)=\frac{uu_y+vv_y}{u^2+v^2}$ ——— (2)

$$\frac{\partial^2}{\partial x^2}(\log|f(z)|)=\frac{\partial}{\partial x}\left\{\frac{uu_x+vv_x}{u^2+v^2}\right\}$$

$$=\frac{(u^2+v^2)(u_x^2+uu_{xx}+v_x^2+v\,v_{xx})-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$=\frac{1}{u^2+v^2}(uu_{xx}+vv_{xx}+u_x^2+v_x^2)-\frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2 \quad \text{——— (3)}$$

Similarly

$$\frac{\partial^2}{\partial y^2}(\log|f(z)|)=\frac{1}{u^2+v^2}(uu_{yy}+vv_{yy}+u_y^2+v_y^2)$$
$$-\frac{2}{u^2+v^2}(uu_y+vv_y)^2 \quad \text{——— (4)}$$

$\therefore\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log(|f(z)|)$ (adding (3) & (4))

$$=\frac{1}{u^2+v^2}\left\{u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+u_x^2+u_y^2+v_x^2+v_y^2\right\}$$
$$-\frac{2}{(u^2+v^2)^2}\left\{(uu_x+vv_x)^2+(uu_y+vv_y)^2\right\}$$

$\because$ $f(z)$ is analytic so, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ [CR-Eq^{n}]

also, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ & $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$ [Laplace Eq^{n}]

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)|$

$$= \frac{1}{u^2+v^2}\left(2\times(u_x^2+v_x^2)\right) - \frac{2}{(u^2+v^2)^2}\left\{(uu_x+vv_x)^2+(-uv_x+vu_x)^2\right.$$

$$= \frac{2(u_x^2+v_x^2)}{(u^2+v^2)} - \frac{2}{(u^2+v^2)^2}\times\left\{u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)\right\}$$

$$= \frac{2(u_x^2+v_x^2)}{(u^2+v^2)} - \frac{2(u_x^2+v_x^2)}{(u^2+v^2)} = 0$$

9. Show that the fun^n $u = e^x \cos y$ is harmonic and find the harmonic conjugate of u.

Ans $u = e^x \cos y$ $\quad u_x = e^x \cos y \quad u_y = -e^x \sin y$

$u_{xx} = e^x \cos y \quad u_{yy} = -e^x \cos y . \quad u_{xy} = -e^x \sin y.$

also, $u_{xx} + u_{yy} = 0$

$\therefore u$ is harmonic.

let v be harmonic conjugate of u then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore \frac{\partial v}{\partial y} = e^x \cos y$ & $\frac{\partial v}{\partial x} = e^x \sin y$ —— (1)

$\therefore v = e^x \sin y + f(x)$

$v_x = e^x \sin y + f'(x)$ —— (2)

comparing (1) & (2). $f'(x) = 0 \Rightarrow f(x) = c$

$\therefore \boxed{v = e^x \sin y + c}$