Algorithms Worksheet 1

For each part of a question write the answer and include workings. Q1 is worth four marks, the other two are worth two marks each. There are also two marks for attendance.

1. This question is about estimating the algorithmic complexity of evaluating a polynomial. Here, consider fixed sized variables, so multiplication and addition take roughly one step, irrespective of how many digits the number has. Once again, powers are calculated by multiplication and we are working out how many steps it takes to work out $p(x)$ when we are given some x value, say $x = x_0$.

   a) What is the big-oh complexity of evaluating, that is finding the value of $p(x_0)$, of an order n polynomial

   $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

   using straight-forward substitution?

   b) Horner's method is a quicker method for evaluating a polynomial. If x_o is the value that the polynomial needs to be evaluated on, let $b_n = a_n$ and then

   $$b_{n-1} = a_{n-1} + x_o b_n$$

   and

   $$b_{n-2} = a_{n-2} + x_0 b_{n-1}$$

   right down to

   $$b_0 = a_0 + x_0 b_1$$

   and you could check that $b_0 = p(x_0)$ is the answer. You don't have to check that, the question here is what is the big-oh complexity of this algorithm?

   Solution: So calculating x^i is $i-1$ multiplications, there are faster ways to work out powers, but you are told that they are calculated by multiplication; multiplying by a_i is one more, so that is i calculations, thus evaluating the polynomial is $1+2+3+\ldots n$ multiplications, along with $n-1$ additions; thus this is $\Theta(n^2)$. However, using Horner's method each b_i is a few calculations and there are n b_is, so that means it is $\Theta(n)$.

2. This question is about the asymptotic behavior of different functions, in each case give big-Theta for $T(n)$; if $T(n)$ was the worst case run-time this would give big-Oh. There is no need to give any working for this problem.

   a) $T(n) = n^5 + \frac{1}{n} + n(n-1)(n+2)^4$

   b) $T(n) = 2^n + n!$

   c) $T(n) = \sum_{i=0}^{n} i$

   d) $T(n) = \sqrt{n}n + n$

   e) $T(n) = (n^5 + 345n^4 + 36n)/(n^2 + 2n + 1)$

   f) $T(n) = 1/(n^2 + 2n + 1)$

   g) $T(n) = [(n+1)(n+2)(n+3)]/[(n+4)(n+5)]$

   h) $T(n) = n!/(n-1)!$

Solution: So just take the leading term in n each time

a) $\Theta(n^6)$
b) $\Theta(n!)$
c) $\Theta(n^2)$
d) $\Theta(\sqrt{n}n)$
e) $\Theta(n^3)$
f) $\Theta(1)$
g) $\Theta(n)$
h) $\Theta(n)$

3. Show that

$$\lim_{x\to\infty}\frac{\ln\ln x}{\ln x}=0$$

where $\ln\ln x$ is a common way to write $\ln(\ln(x))$. It might be useful to recall the chain rule

$$\frac{df(v(x))}{dx}=\frac{df}{dv}\frac{dv}{dx}$$

for example, in this case $v=\log x$ and $f(v)=\ln v$ so $f(v(x))=\ln\ln x$. You should also recall that $d\ln x/dx=1/x$. The point of this question is that it shows

$$O(\log\log n)\subset O(\log n)$$

So

$$\frac{d\ln\ln x}{dx}=\frac{d\ln\ln x}{d\ln x}\frac{d\ln x}{dx}=\frac{1}{\ln x}\frac{1}{x}$$

hence, by L'Hôpidal's rule

$$\lim_{x\to\infty}\frac{\ln\ln x}{\ln x}=\lim_{x\to\infty}\frac{1}{\ln x}=0$$