CSE 221: Algorithms

Quicksort

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References

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Why do we want to study Quicksort?

One of the most widely used, and extensively studied, sorting algorithms.

Divide and conquer

Quicksort an *n*-element array:

1. *Divide* Partition the array into subarrays around a *pivot* x such that the elements in lower subarray $\leq x \leq$ elements in the upper subarray.

2. *Conquer* Recursively sort the two subarrays.
3. *Combine* Trivial – just concatenate the lower subarray, *pivot*, and the upper subarray.

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Key

Linear-time partitioning algorithm.

Quicksort in action

1	2	3	4	5	6	7	8
6	10	13	5	8	3	2	11

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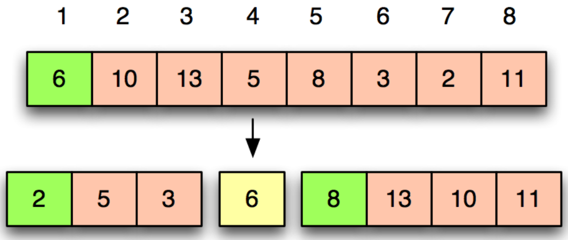

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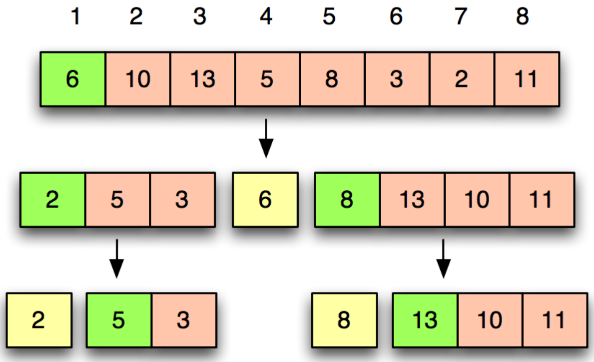

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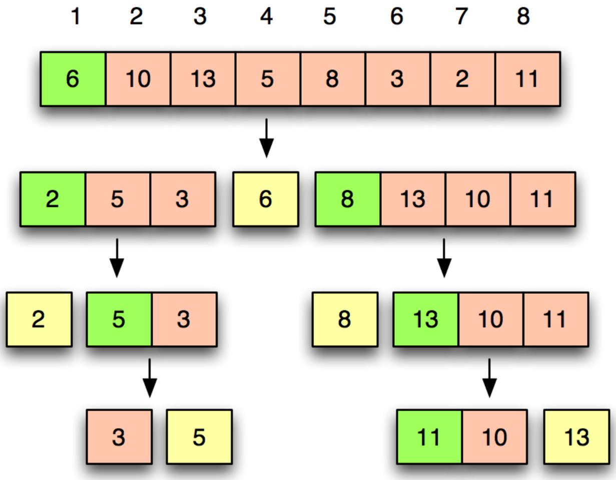

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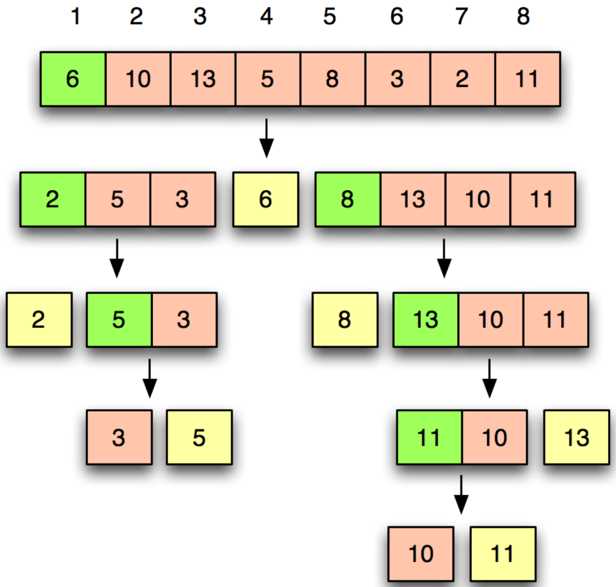

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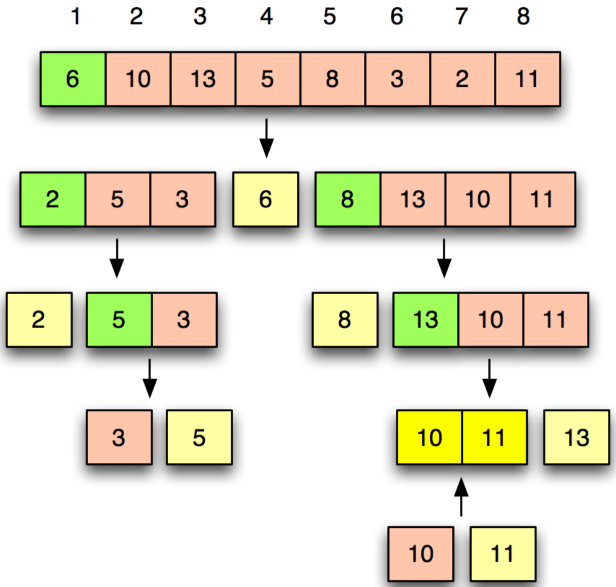

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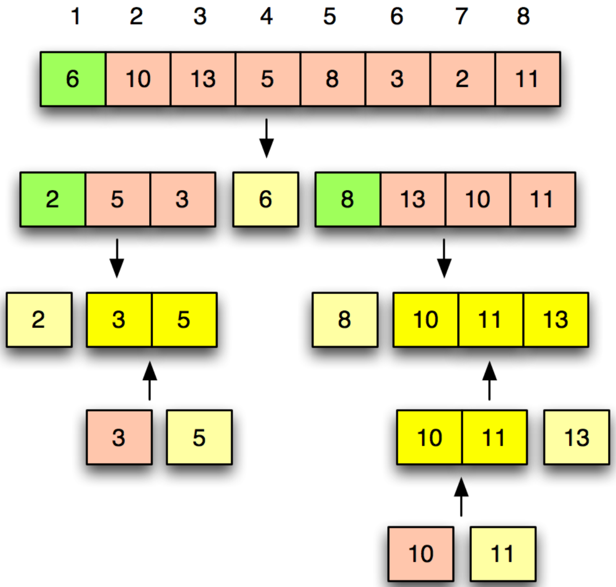

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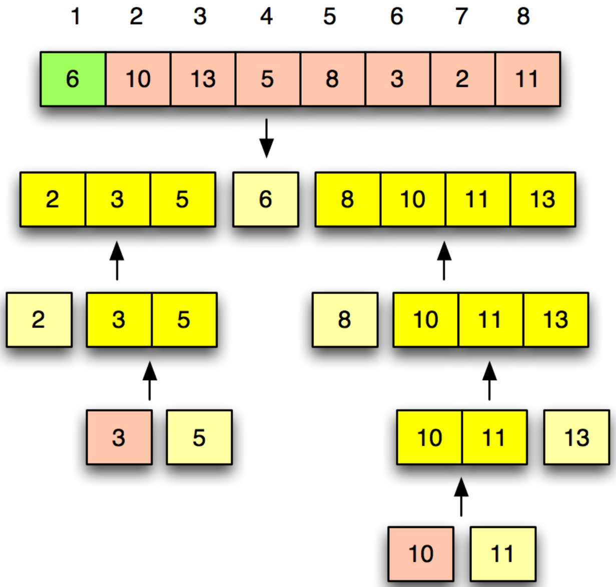

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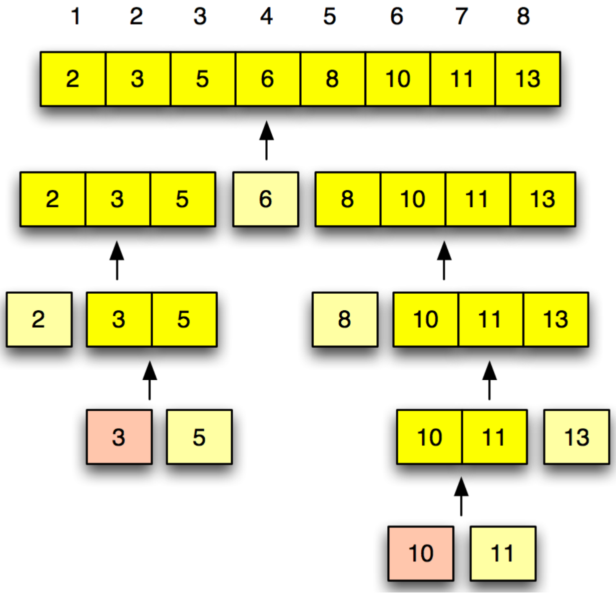

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Partitioning algorithm

Algorithm

PARTITION(A, p, q) $\triangleright$ $A[p \,..\, q]$

$x \leftarrow A[p]$ $\triangleright$ pivot $= A[p]$

$i \leftarrow p$

for $j \leftarrow p + 1$ **to** q

  do if $A[j] \leq x$

    then $i \leftarrow i + 1$

      exchange $A[i] \leftrightarrow A[j]$

exchange $A[p] \leftrightarrow A[i]$

return i

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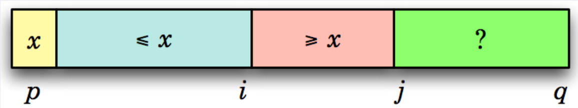
Invariant



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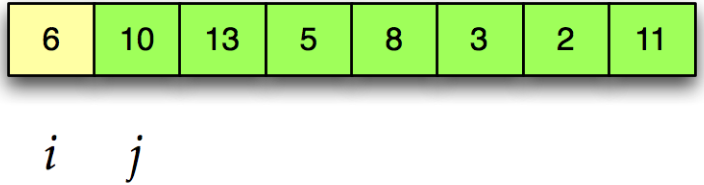

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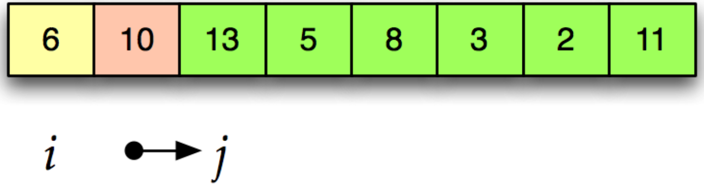

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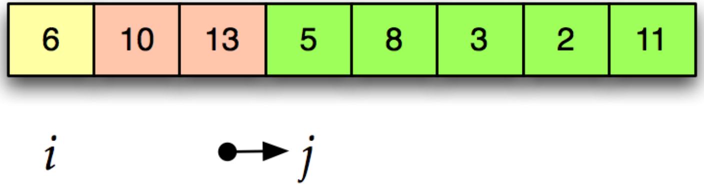

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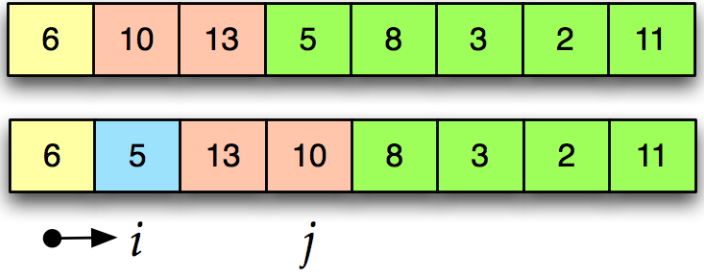

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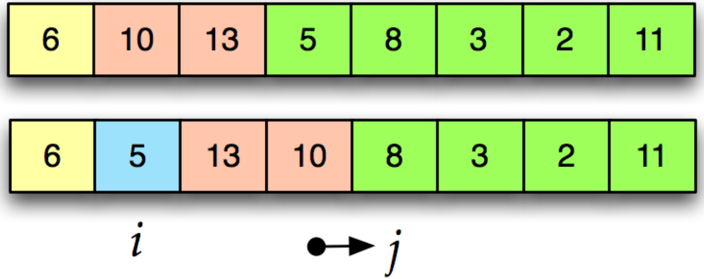

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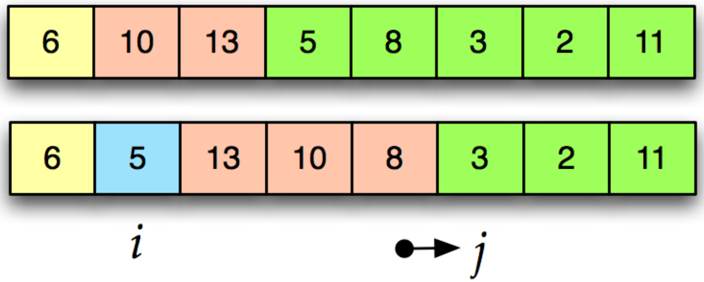

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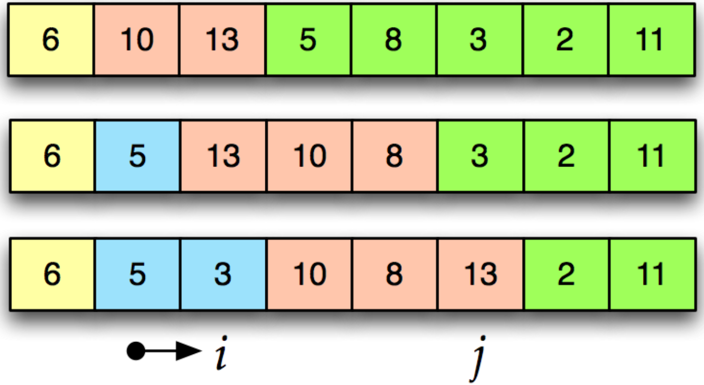

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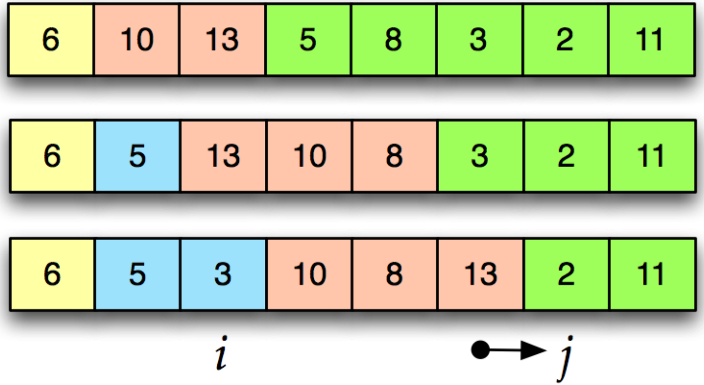

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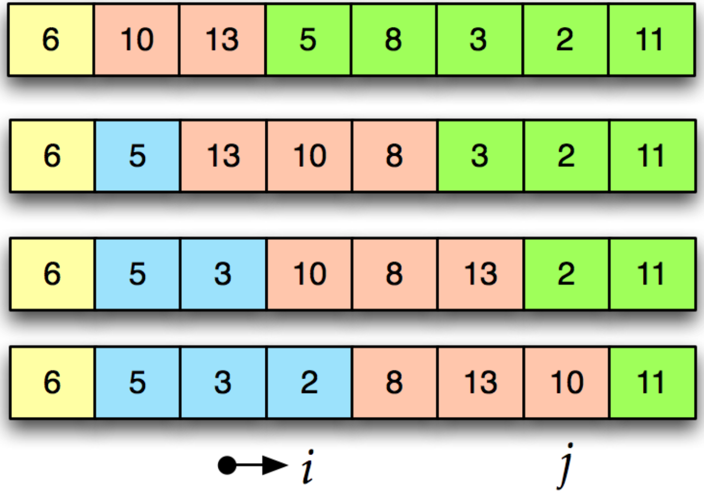

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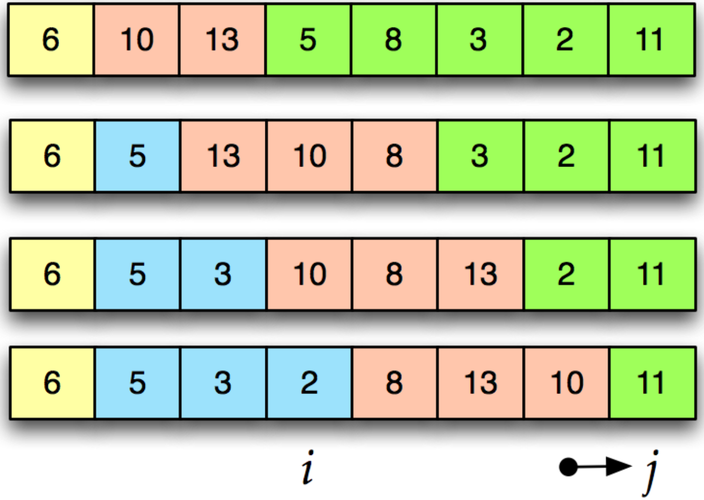

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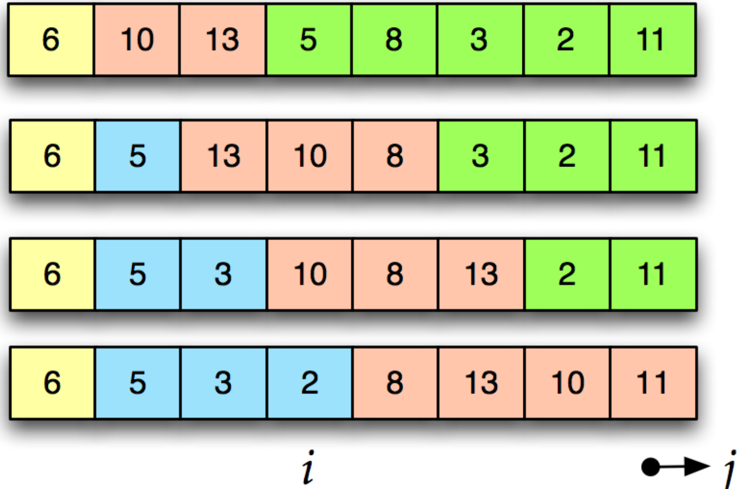

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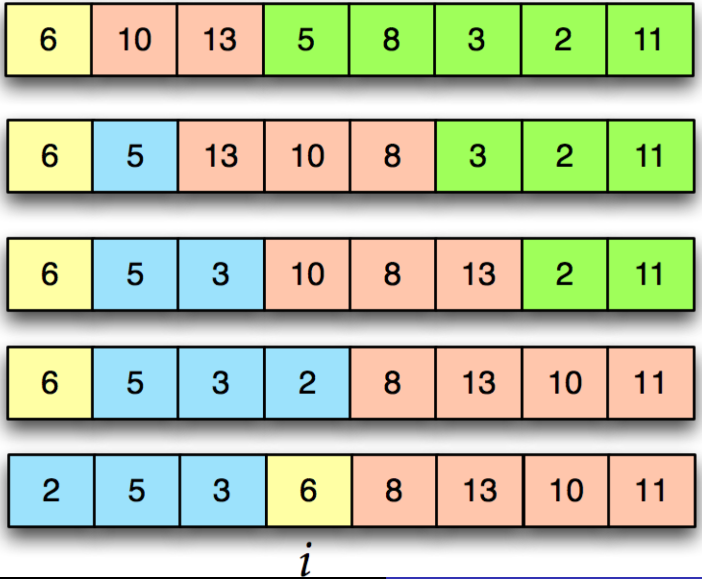

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Quicksort algorithm

Algorithm

$\textsc{Quicksort}(A, p, r) \triangleright A[p \,..\, r]$

1 **if** $p < r$

2 **then** $q \leftarrow \textsc{Partition}(A, p, r)$

3 $\textsc{Quicksort}(A, p, q - 1)$

4 $\textsc{Quicksort}(A, q + 1, r)$

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if $p < r$
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    $\textsc{Quicksort}(A, p, q-1)$
    $\textsc{Quicksort}(A, q+1, r)$

Initial call

$\textsc{Quicksort}(A, 1, n)$

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- Worst-case happens when pivot is always the minimum or maximum element.

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Worst-case analysis

(Note: the worst-case running time for partitioning is $\Theta(n)$.)

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \boxed{\Theta(n^2)} \end{aligned}$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

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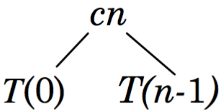

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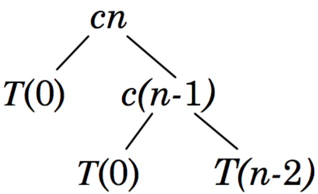

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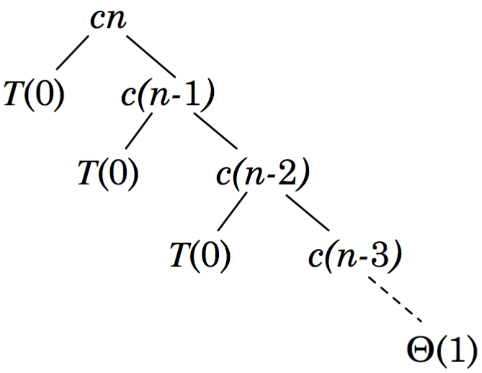

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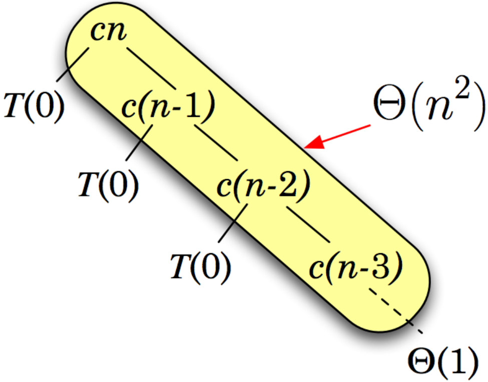

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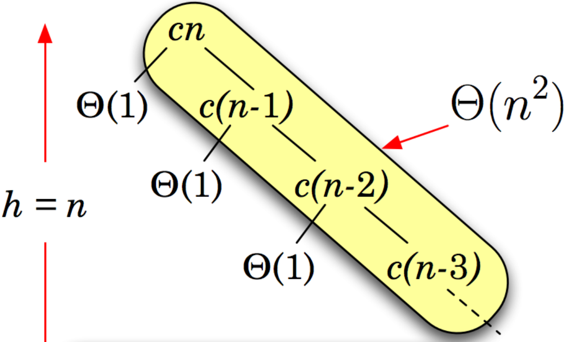

$$\begin{aligned} T(n) &= \Theta(n) + \Theta(n^2) \\ &= \Theta(n^2) \end{aligned}$$

Best- and almost-worst case performances

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- What if the split is always $\frac{1}{10} : \frac{9}{10}$?

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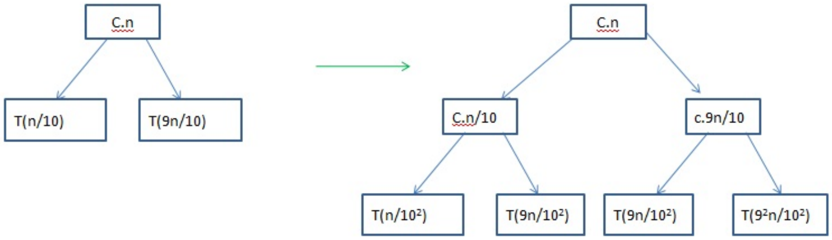
- Best-case happens when pivot is the median element, creating equal size partitions.

$$T(n) = 2T(n/2) + \Theta(n) = \Theta(n \lg n)$$

- What if the split is always $\frac{1}{10} : \frac{9}{10}$?
- $T(n) = T(n/10) + T(9n/10) + Cn$
- Lets draw the recursion tree for this recurrence formula.

Recursion Tree for split proportion 1/10: 9/10



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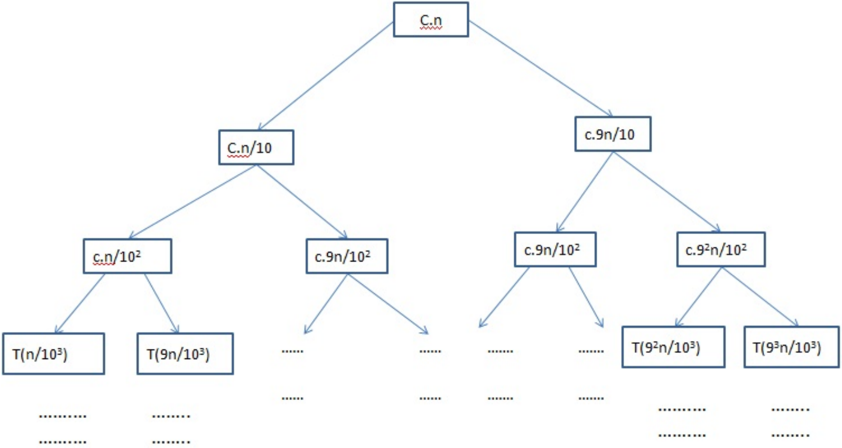

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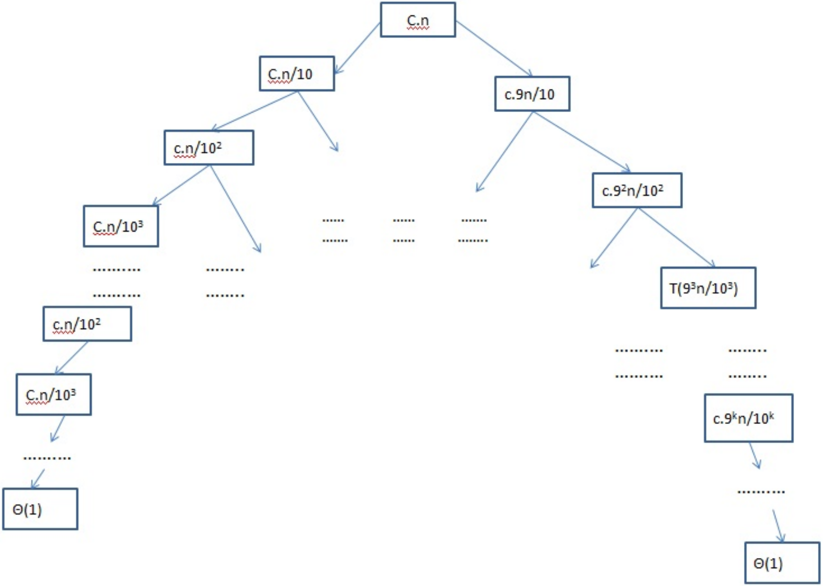

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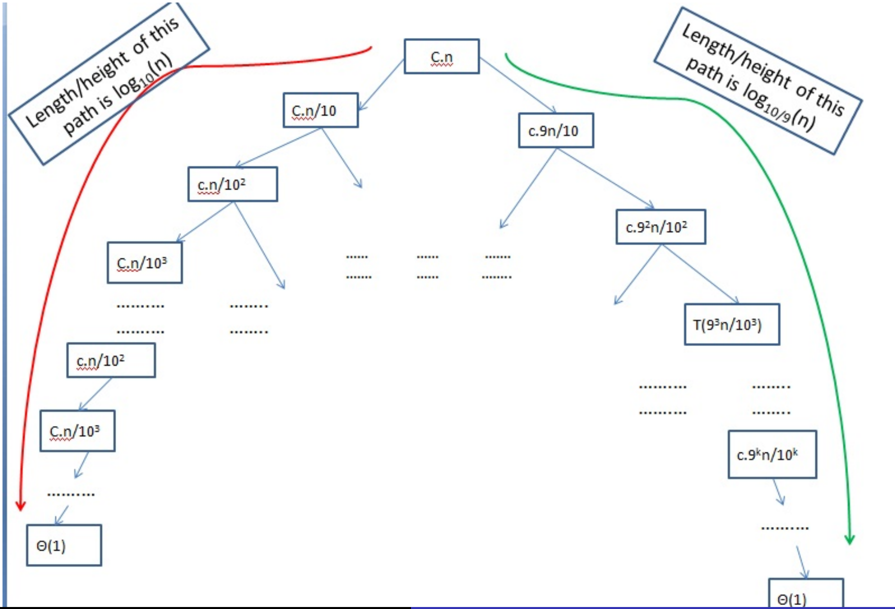

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- Then total cost will be $Cn \log_{10/9} n + Cn$ (total cost at each level is Cn, recall the calculation of total cost for quicksort having split proportion $1/2 : 1/2$)

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- Simplify it as $T(n) \leq Cn \log_2 n + Cn$
- $T(n) \leq Cn \lg n + Cn$

- $\log_{10} n \leq \log_{10/9} n$
- Left most branch ends before all other branches, it is the path having smallest height
- Right most branch ends after all other branches, so it is the path having maximum height
- So length/height of other branches in between these two branches will be in between $\log_{10} n$ and $\log_{10} n$
- Lets consider, all branches have the maximum height $\log_{10/9} n$
- Then total cost will be $Cn \log_{10/9} n + Cn$ (total cost at each level is Cn, recall the calculation of total cost for quicksort having split proportion $1/2 : 1/2$)
- But actually total cost would be less then this cost as all branches are not of maximum height
- So, $T(n) \leq Cn \log_{10/9} n + Cn$
- Simplify it as $T(n) \leq Cn \log_2 n + Cn$
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Very close to worst-case produces $O(n \lg n)$, not $O(n^2)$.

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How to ensure that we don't *usually* hit the worst-case?

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$\textsc{Randomized-Partition}(A, p, r) \triangleright A[p \mathinner{..} r]$

1 $i \leftarrow \textsc{Random}(p, r)$ $\triangleright i = [p \mathinner{..} r]$
2 exchange $A[p] \leftrightarrow A[i]$
3 **return** $\textsc{Partition}(A, p, r)$

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$\textsc{Randomized-Partition}(A, p, r) \triangleright A[p \,..\, r]$

```
i ← RANDOM(p, r)                    ▷ i = [p .. r]
exchange A[p] ↔ A[i]
return PARTITION(A, p, r)
```

$\textsc{Randomized-Quicksort}(A, p, r)$

```
if p < r
   then q ← RANDOMIZED-PARTITION(A, p, r)
        RANDOMIZED-QUICKSORT(A, p, q − 1)
        RANDOMIZED-QUICKSORT(A, q + 1, r)
```

Conclusion

- One of the most widely used sorting algorithm.
- While it runs in $O(n^2)$ time in the worst-case, it runs in $O(n \lg n)$ time on the average.
- Runs almost twice as fast as merge-sort.
- Can be *tuned* substantially.
- Many programming language runtime libraries provide some variant of Quicksort (java.util.Arrays.sort() in Java, qsort() in C, etc).

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Questions to ask (and remember)

- What are the worst, best and average case performances?
- Is it in-place?
- Is it stable?