Task12

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This task is about minimizing the expression $E[L] = \sum_i p_i \; L_i$ under the constraint $\sum_i 2^{-L_i}$ and then prove that $L_i = log_2 \; (\frac{1}{p_i})$

Proof: This is a minimizing problem that can be solved with Lagrange method:

$$min \;\; H(x) = \sum_{i \geq 1}^{m} p_i \; .L_i + \lambda.(1 - \sum_{i \geq 1}^{m} 2^{-L_i})$$

$$s.t \qquad \sum_{i \geq 1}^{m} 2^{-L_i}$$

$$L_i \geq 0$$

With Lagrangian optimization, it obtains:

$$\mathcal{L} = \sum_{i \geq 1}^{m} p_i \; .L_i + \lambda.(1 - \sum_{i \geq 1}^{m} 2^{-L_i})$$

Solving the derivatives with respect to i and set them to zero, obtains:

$$\frac{\partial \mathcal{L}}{\partial L_i} = p_i + \lambda. \ln(2).2^{-L_i} = 0$$

$$\Leftrightarrow 2^{-L_i} = -\frac{p_i}{\lambda. \ln(2)} \qquad (1)$$

Now,lets derivative with respect to λ:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \sum_{i \geq 1}^{m} 2^{-L_i}$$

$$1 - \sum_{i=1}^{m} 2^{-L_i} = 0$$

$$\sum_{i \geq 1}^{m} 2^{-L_i} = 1 \qquad (2)$$

Combining (1) into (2) in order to calculate the value of λ yields:

$$\sum_{i=1}^{m} -\frac{p_i}{\lambda. \ln(2)} = 1 \Leftrightarrow -\frac{1}{\lambda. \ln(2)}. \sum_{i=1}^{m} p_i = 1$$

$$\Leftrightarrow \lambda = -\frac{1}{\ln(2)} \qquad (3), since \sum_{i \geq 1}^{m} p_i \to 1$$

The last step is calculating L_i, and it can be don by combining (3) into (1):

$$p_i - \frac{\ln(1)}{\ln(2)}.2^{-L_i} = 0 \Leftrightarrow p_i = 2^{-L_i}$$

$$log_2\ p_i = -L_i \Leftrightarrow L_i = log_2 \frac{1}{p_i}$$

Compute the second derivative on $L_i = log_2 \frac{1}{p_i}$ yields $\frac{1}{2^{L_i}}.\ln(2) > 0$ which means $E(L_i)$ reaches a global minimum and $L_i = log_2 \frac{1}{p_i}$ is its minimizer $\square$