Date: 13/09/23

* Find the domain of $z = \sqrt{2-x^2-y^2}$

Sol^n:

z is real when

$$2-x^2-y^2 \geq 0$$

$$\Rightarrow 2 \geq x^2+y^2$$

$$D_f = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 2\}$$

* Domain and Range of $u = \dfrac{1}{x^2+y^2+z^2}$

u is defined when,

$$x^2+y^2+z^2 \neq 0$$

$$\Rightarrow x \neq 0, \ y \neq 0, \ z \neq 0$$

$$(x,y,z) \neq (0,0,0)$$

$$D_f = \mathbb{R}^3 - \{(0,0,0)\}$$

$$\Rightarrow \lim_{x \to a} f(x) = L$$

$|x-a| < \delta$ then

$|f(x) - L| < \varepsilon$

$\varepsilon, \delta > 0$, ε, δ very small.

Date: 18/09/23

(1) Simultaneous limit / Double limit at (a,b)

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad \text{or} \quad \lim_{\substack{x\to a \\ y\to b}} f(x,y)$$

l সংজ্ঞায়িত হতে হবে ; না হলে doesn't exist.

(2) Iterative limit at (a,b)

$$\lim_{x\to a} \lim_{y\to b} f(x,y) = \lim_{y\to b} \lim_{x\to a} f(x,y) = L$$

if simultaneous exist, then iterative also exist at that point.

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* Find iterative limit of $f(x,y) = \frac{x^3-y^3}{x^3+y^3}$ at $(0,0)$.

Sol^n:

$$\text{L.H.S.} = \lim_{x\to a} \lim_{y\to b} f(x,y)$$

$$= \lim_{x\to 0} \lim_{y\to 0} \frac{x^3 - y^3}{x^3+y^3}$$

$$= \lim_{x\to 0} \frac{x^3 - 0}{x^3 + 0}$$

$$= \lim_{x\to 0} 1$$

$$= 1$$

$$\text{R.H.S.} = \lim_{y\to b} \lim_{x\to a} f(x,y)$$

$$= \lim_{y\to 0} \lim_{x\to 0} \frac{x^3 - y^3}{x^3+y^3}$$

$$= \lim_{y\to 0} \frac{0 - y^3}{0 + y^3}$$

$$= \lim_{y\to 0} (-1)$$

$$= -1$$

Since, L.H.S $\neq$ R.H.S.

Iterative limit doesn't exist at $(0,0)$.

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* Find simultaneous limit of $f(x,y) = \frac{x^3 - y^3}{x^3 + y^3}$ at (0,0).

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$$

$$= \frac{0-0}{0+0} = \frac{0}{0}$$

simultaneous limit doesn't exist at (0,0).

* Show that, Iterative limit of $f(x,y) = \frac{xy}{x^2+y^2}$ exists at (0,0) but simultaneous limit doesn't exist.

$$L.H.S. = \lim_{x\to a} \lim_{y\to b} f(x,y)$$

$$= \lim_{x\to 0} \lim_{y\to 0} \left(\frac{xy}{x^2+y^2}\right)$$

$$= \lim_{x\to 0} \frac{x \times 0}{x^2 + \cancel{y^2}\, 0^2}$$

$$= \lim_{x\to 0} \frac{0}{x^2}$$

$$= \lim_{x\to 0} 0$$

$$= 0$$

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$$\text{R.H.S.} = \lim_{y \to b} \lim_{x \to a} f(x,y)$$

$$= \lim_{y \to 0} \lim_{x \to 0} \left(\frac{xy}{x^2+y^2} \right)$$

$$= \lim_{y \to 0} \frac{\cancel{x}\, 0 \times y}{0^2 + y^2}$$

$$= \lim_{y \to 0} \frac{0}{y^2}$$

$$= \lim_{y \to 0} 0$$

$$= 0$$

Since L.H.S. = R.H.S.

$\therefore$ Iterative limit exists at $(0,0)$.

Again,

$$\lim_{(x,y) \to (a,b)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}$$

$$= \frac{0 \times 0}{0^2 + 0^2} = \frac{0}{0}$$

Simultaneous limit doesn't exist at $(0,0)$.

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Simultaneous limit $\Rightarrow$

* show that,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{x^2+y^2} \text{ does not exist.}$$

Let,

$x = my$

Now,
$x^2+y^2 = 0$
$\Rightarrow x^2 = -y^2$
$\Rightarrow x = iy$
$\therefore x = my$

Then,

$$\frac{2xy}{x^2+y^2} = \frac{2my.y}{m^2y^2+y^2}$$

$$= \frac{2my^2}{y^2(m^2+1)}$$

$$= \frac{2m}{m^2+1}$$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{x^2+y^2} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{2m}{1+m^2}$$

$$= \frac{2m}{1+m^2}$$; which is not unique

since m gives different value along diffenent curve. thus the limiting value is not unique. Hence, $\lim_{\substack{y \to 0 \\ x \to 0}} \frac{2xy}{x^2+y^2}$ doesn't exist.

Continuity

Single variable ⇒

(i) $\lim_{x\to a} f(x)$ exists

(ii) $f(a)$ exists

(iii) $\lim_{x\to a} f(x) = f(a)$

Multi variable ⇒

(i) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists [Output ⇒ unique, finite]

(ii) $f(a,b)$ exists

(iii) $\lim_{(x,y)\to(a,b)} = f(a,b)$

$$* \quad f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

Show that $f(x,y)$ is discontinious at $(0,0)$.

Sol^n:

(i) $\lim_{(x,y)\to(0,0)} f(x,y)$

$= \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

$x^2+y^2=0$

$\Rightarrow x^2 = -y^2$

$\therefore x = my$

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$$= \lim_{(x,y)\to(0,0)} \frac{m^2y^2 - y^2}{m^2y^2 + y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{y^2}(m^2-1)}{\cancel{y^2}(m^2+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m^2-1}{m^2+1}$$

$= \dfrac{m^2-1}{m^2+1}$, which is not unique since m gives different value among different curves.

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

So, $f(x,y)$ is discontinious at $(0,0)$.

Date:........................

PARTIAL DERIVATIVES

(1) $f_x(x,y) = \frac{\delta f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$

(2) $f_y(x,y) = \frac{\delta f}{\delta y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$

(3) $f_{xy}(x,y) = \frac{\delta}{\partial y}\left(\frac{\partial f}{\delta x}\right)$

$= \frac{\delta}{\delta y}(f_x)$

$= \lim_{k \to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$

(4) $f_{yx}(x,y) = \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta y}\right)$

$= \frac{\delta}{\delta x}(f_y)$

$= \lim_{h \to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$

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* $f(x,y) = x^2 + xy + y^2$, then using analytical definition find $f_x(-1,1)$, $f_y(2,5)$, and $f_{xy}(-1,1)$.

Solution:

$$\frac{\delta f}{\delta x} = f_x(x,y)$$

$$= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + (x+h)y + y^2 - x^2 - xy - y^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{xy} + hy + \cancel{y^2} - \cancel{x^2} - \cancel{xy} - \cancel{y^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + y)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h + y$$

$$= 2x + 0 + y$$

$$= 2x + y$$

$$f_x(-1,1) = 2(-1) + 1 = -1$$

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$$f_y(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$\Rightarrow f(2,5) = \lim_{k \to 0} \frac{f(2,5+k) - f(2,5)}{k}$$

$$= \lim_{k \to 0} \frac{2^2 + 2\cdot(5+k) + (5+k)^2 - 2^2 - 10 - 25}{k}$$

$$= \lim_{k \to 0} \frac{4 + 10 + 2k + 25 + 10k + k^2 - 4 - 10 - 25}{k}$$

$$= \lim_{k \to 0} \frac{\cancel{k}(2 + 10 + k)}{\cancel{k}}$$

$$= 2 + 10 + 0$$

$$= 12$$

$$f_{xy} = \frac{\delta}{\delta y}(f_x) = \lim_{k \to 0} \frac{f_x(x, y+k) + f_x(x,y)}{k}$$

$$f_{xy}(-1,1) = \lim_{k \to 0} \frac{f_x(-1, 1+k) - f_x(-1,1)}{k}$$

$$= \lim_{k \to 0} \frac{\{-2 + (1+k)\} - \{-2+1\}}{k}$$

$$= \lim_{k \to 0} \frac{-2 + 1 + k + 1}{k}$$

$$= \lim_{k \to 0} \frac{k}{k} = 1$$

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Differentiability

The ~~differen~~ function $f(x,y)$ is said to be differentiable at the point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-hf_x(a,b)-kf_y(a,b)}{\sqrt{h^2+k^2}}=0$$

$$* \quad f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2} & ;\text{ when } (x,y)\neq(0,0)\\ 0 & ;\text{ when } (x,y)=(0,0)\end{cases}$$

Check differentiability at $(0,0)$.

SOLUTION:

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$\Rightarrow f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{h^3}{h^2}-0}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{h\to 0} 1$$

$$=1$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-\frac{\cancel{k^3}k}{\cancel{k^2}} - 0}{k}$$

$$= \lim_{k\to 0} -\frac{k}{k}$$

$$= \lim_{k\to 0} -1$$

$$= -1$$

$$f_x(0,0) = 1 \quad , \quad f_y(0,0) = -1$$

We know,

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \left[\frac{h^3 \cdot 1}{} \right.$$

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We know,

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-hf_x(a,b)-kf_y(a,b)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h^3-k^3}{h^2+k^2}-0-h-(-1)k\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h^3-k^3-h(h^2+k^2)+k(h^2+k^2)}{h^2+k^2}\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{\cancel{h^3}-\cancel{k^3}-\cancel{h^3}-hk^2+h^2k+\cancel{k^3}}{h^2+k^2}\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\cdot\frac{hk(h-k)}{h^2+k^2}$$

$$= \lim_{(h,k)\to(0,0)} \frac{hk(h-k)}{(h^2+k^2)^{3/2}}$$

$$h^2+k^2=0$$
$$\Rightarrow h=ik$$
$$\therefore h=mk$$

$$= \lim_{(h,k)\to(0,0)} \frac{mk^2(mk-k)}{(m^2k^2+k^2)^{3/2}}$$

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$$= \lim_{(h,k)\to(0,0)} \frac{mk^3(m-1)}{\{k^2(m^2+1)\}^{3/2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{mk^3(m-1)}{k^3(m^2+1)^{3/2}}$$

$$= \frac{m(m-1)}{(m^2+1)^{3/2}} \neq 0$$

So, the function is not differentiable at $(0,0)$.

CHAIN RULE ⇒

Suppose that, $z = f(u(x,y), v(x,y))$

Then ⇒

$$\frac{\delta z}{\delta x} = \frac{\partial f}{\delta u}\cdot\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\cdot\frac{\delta v}{\delta x}$$

$$\frac{\delta z}{\delta y} = \frac{\delta f}{\delta u}\cdot\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\cdot\frac{\delta v}{\delta y}$$

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* Suppose that, $z = f(u,v)$ where $u = e^x \cos y$ and $v = e^x \sin y$.

Show that, (i) $\frac{\delta z}{\delta x} = u.\frac{\delta f}{\delta u} + v.\frac{\delta f}{\delta v}$

(ii) $\frac{\delta z}{\delta y} = -v\frac{\delta f}{\delta u} + u.\frac{\delta f}{\delta v}$

(i) $\frac{\delta z}{\delta x} = \frac{\delta f}{\delta u}.\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}.\frac{\delta v}{\delta x}$

$= \frac{\delta f}{\delta u}.\frac{\delta}{\delta x}(e^x \cos y) + \frac{\delta f}{\delta v}.\frac{\delta}{\delta x}(e^x \sin y)$

$= \frac{\delta f}{\delta u} e^x \cos y + \frac{\delta f}{\delta v}.e^x \sin y$

$= u.\frac{\delta f}{\delta u} + v.\frac{\delta f}{\delta v}$

$=$ R.H.S.

Proved

(ii) $$\frac{\delta z}{\delta y} = \frac{\delta f}{\delta u} \cdot \frac{\delta u}{\delta y} + \frac{\delta f}{\delta v} \cdot \frac{\delta v}{\delta y}$$

$$= \frac{\delta f}{\delta u} \cdot \frac{\delta}{\delta y}(e^x \cos y) + \frac{\delta f}{\delta v} \cdot \frac{\delta}{\delta y}(e^x \sin y)$$

$$= \frac{\delta f}{\delta u}(-e^x \sin y) + \frac{\delta f}{\delta v}(e^x \cos y)$$

$$= -v \frac{\delta f}{\delta u} + u \frac{\delta f}{\delta v}$$

= R.H.S.

Proved

* Suppose that,

$$u = f(x-y, y-z, z-x)$$

Show that, $u_x + u_y + u_z = 0$

Solution ⇒

Let, $p = x-y$, $q = y-z$, $r = z-x$

$\therefore u = f(p(x,y), q(y,z), r(z,x))$

$$\therefore u_x = \frac{\delta u}{\delta x} = \frac{\delta f}{\delta p} \cdot \frac{\delta p}{\delta x} + \frac{\delta f}{\delta q} \cdot \frac{\delta q}{\delta x} + \frac{\delta f}{\delta r} \cdot \frac{\delta r}{\delta x}$$

$$= \frac{\delta f}{\delta p} \cdot \frac{\delta}{\delta x}(x-y) + \frac{\delta f}{\delta q} \cdot \frac{\delta}{\delta x}(y-z) + \frac{\delta f}{\delta r} \cdot \frac{\delta}{\delta x}(z-x)$$

$$= \frac{\delta f}{\delta p} - \frac{\delta f}{\delta r}$$

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$$\therefore u_y = \frac{\delta f}{\delta p}\cdot\frac{\delta p}{\delta y} + \frac{\delta f}{\delta q}\cdot\frac{\delta q}{\delta y} + \frac{\delta f}{\delta r}\cdot\frac{\delta r}{\delta y}$$

$$= \frac{\delta f}{\delta p}\cdot\frac{\delta}{\delta y}(x-y) + \frac{\delta f}{\delta q}\frac{\delta}{\delta y}(y-z) + \frac{\delta f}{\delta r}\cdot\frac{\delta}{\delta y}(z-x)$$

$$= -\frac{\delta f}{\delta p} + \frac{\delta f}{\delta q}$$

$$\therefore u_z = \frac{\delta f}{\delta p}\cdot\frac{\delta p}{\delta z} + \frac{\delta f}{\delta q}\cdot\frac{\delta q}{\delta z} + \frac{\delta f}{\delta r}\cdot\frac{\delta r}{\delta z}$$

$$= \frac{\delta f}{\delta p}\frac{\delta}{\delta z}(x-y) + \frac{\delta f}{\delta q}\cdot\frac{\delta}{\delta z}(y-z) + \frac{\delta f}{\delta r}\cdot\frac{\delta}{\delta z}(z-x)$$

$$= -\frac{\delta f}{\delta q} + \frac{\delta f}{\delta r}$$

$$L.H.S. = U_x + U_y + U_z$$

$$= \frac{\delta f}{\delta p} - \frac{\delta f}{\delta r} - \frac{\delta f}{\delta p} + \frac{\delta f}{\delta q} - \frac{\delta f}{\delta q} + \frac{\delta f}{\delta r}$$

$$= 0$$

$$= R.H.S.$$

Proved

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Homogenous function ⇒ সব পদের Power same.

* $f(x, y) = 3x^5 + 4x^4y + 9x^3y^2 + y^5$ ✓

* $f(x, y) = 3x^4 + 9x^2y + y^4 + 9$ × Homo.

Euler's Theorem ⇒

* If $u = u(x, y)$ is a homogenous function of degree n.

$$x \cdot \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = nu$$

$$\Rightarrow x u_x + y u_y = nu$$

* If $u(x, y, z)$

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} + z \frac{\delta u}{\delta z} = nu$$

$$\therefore x u_x + y u_y + z u_z = nu.$$

*Verify Euler's theorem for $u = \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$

Sol^n:

$$u = x^2y^{-1} + y^2z^{-1} + z^2x^{-1}$$

Here, u is a homogenous function of x, y, z of degree $n=1$.

We have to show that,

$$xu_x + yu_y + zu_z = nu = 1 \times u = u$$

L.H.S. =

$$xu_x + yu_y + zu_z$$

$$= x\left(\frac{2x}{y} - \frac{z^2}{x^2}\right) + y\left(\frac{2y}{z} - \frac{x^2}{y^2}\right) + z\left(-\frac{y^2}{z^2} + \frac{2z}{x}\right)$$

$$= \frac{2x^2}{y} - \frac{xz^2}{x^2} + \frac{2y^2}{z} - \frac{x^2y}{y^2} - \frac{y^2z}{z^2} + \frac{2z^2}{x}$$

$$= \frac{2x^2}{y} - \frac{z^2}{x} + \frac{2y^2}{z} - \frac{x^2}{y} - \frac{y^2}{z} + \frac{2z^2}{x}$$

$$= \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$$

$= u$

$=$ R.H.S. Proved.

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Maximum – Minimum

Maximum and minimum of a function :-

Suppose that, $f(x,y)$ be the given function

(1) find $\frac{\delta f}{\delta x}$, $\frac{\delta f}{\delta y}$, $r = \frac{\delta^2 f}{\delta x^2}$, $s = \frac{\delta^2 f}{\delta x \delta y}$, $t = \frac{\delta^2 f}{\delta y^2}$

(2) For critical point,

Consider $\frac{\delta f}{\delta x} = 0$ —— (1) and

$\frac{\delta f}{\delta y} = 0$ —— (2)

After solving (1) and (2), you will find some values of (x,y).

Suppose that, the point is $(x,y) = (a,b)$.

(3) Now, put $(x,y) = (a,b)$ at r, s & t.

(i) if $r < 0$, $t < 0$ and $rt - s^2 > 0$, then $f(x,y)$ has $\boxed{\text{max}}$ value at $(x,y) = (a,b)$ and the value is $f(a,b)$

(ii) if $r > 0$, $t > 0$ and $rt - s^2 > 0$, then $f(x,y)$ has $\boxed{\text{min}}$ value at $(x,y) = (a,b)$ and the min value is $f(a,b)$

(iii) If the above conditions violate, then $f(x,y)$ has neither max nor min value at (a,b) and (a,b) is called $\boxed{\text{saddle point}}$ then.

Victoria

Date:........................

* Find max and min value of the function ; $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$\Rightarrow$ We have,

$$\frac{\delta f}{\delta x} = 3x^2 - 3 \quad ; \quad \frac{\delta f}{\delta y} = 3y^2 - 12$$

$$r = \frac{\delta^2 f}{\delta x^2} = 6x \quad ; \quad t = \frac{\delta^2 f}{\delta y^2} = 6y; \quad s = \frac{\delta^2 f}{\delta x \delta y} = 0$$

For critical point,

$\frac{\delta f}{\delta x} = 0$	$\frac{\delta f}{\delta y} = 0$
$\Rightarrow 3x^2 - 3 = 0$	$\Rightarrow 3y^2 - 12 = 0$
$\therefore x = \pm 1$	$\therefore y = \pm 2$

Thus we have critical points $(1,2), (1,-2),$
$(-1,2), (-1,-2)$

At $(1,2) \rightarrow$

$r = 6x = 6 \times 1 = 6 > 0$; $t = 6y = 12 > 0$;

$rt - s^2 = 72 > 0$

$\therefore f(x,y)$ has minimum value at $(1,2)$

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$\therefore$ Minimum value, $f(1,2) = 1^3 + 2^3 + 3.1 - 12.2 + 20$
$= 2$

At $(1,-2) \rightarrow$

$r = 6 > 0$; $t = -12 < 0$

$\therefore f(x,y)$ has no max or min value at $(1,-2)$

Hence $(1,-2)$ is a saddle point.

At $(-1,2) \rightarrow$

$r = -6 < 0$; $t = 12 > 0$

$\therefore f(x,y)$ has no max or min value at $(-1,2)$

Hence $(-1,2)$ is a saddle point.

At $(-1,-2) \rightarrow$

$r = -6 < 0$; $t = -12 < 0$; $rt - s^2 = (-6)(-12)$
$= 72 > 0$

$f(x,y)$ has a maximum value at $(-1,-2)$

$\therefore$ Maximum value, $f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$
$= 38$

(Ans.)

Date:........................

Lagrange Multipliers:

$$\vec{\nabla} = \hat{i}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f(x,y)$$

$$= \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}$$

$$\nabla f = \lambda \nabla g \xrightarrow{} \text{Lagrange Multiplier.}$$

* Use lagrange multipliers to find the three real numbers whose sum is 12 and the sum of whose squares is minimum.

$\Rightarrow$ Suppose that, the three real numbers are x, y and z.

Given that, $x+y+z = 12$ --- (i)

$$\Rightarrow x+y+z-12 = 0$$

$$\therefore g(x,y,z) = x+y+z-12 \text{ --- (ii)}$$

We have to minimize,

$$f(x,y,z) = x^2+y^2+z^2$$

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For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \left(\hat{i} \frac{\delta f}{\delta x} + \hat{j} \frac{\delta f}{\delta y} + \hat{k} \frac{\delta f}{\delta z} \right) = \lambda \cdot \left(\hat{i} \frac{\delta g}{\delta x} + \hat{j} \frac{\delta g}{\delta y} + \hat{k} \frac{\delta g}{\delta z} \right)$$

$$\Rightarrow 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k}$$

$\Rightarrow 2x\hat{i} = \lambda\hat{i}$ | $2y = \lambda$ | $2z = \lambda$

$\therefore 2x = \lambda$

$\therefore 2x = 2y = 2z$

$\Rightarrow x = y = z$

From (i) $\rightarrow$

$$x + x + x = 12$$

$$\Rightarrow 3x = 12$$

$\therefore x = 4, \; y = 4, \; z = 4.$

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Date:.......................

* For a rectangle whose perimeter is 20 m, use the lagrange multiplier method to find the dimensions that will maximize area.

Suppose that, the dimensions are x and y.

Given that,

$$2(x+y) = 20$$

$$\Rightarrow x+y = 10 \quad - - - - \text{(i)}$$

$$\Rightarrow x+y-10 = 0$$

$$g(x,y) = x+y-10 \quad - - - - \text{(ii)}$$

We have to maximize,

$$f(x,y) = xy$$

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For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \hat{i}\frac{\delta f}{\delta x} + \hat{j}\frac{\delta f}{\delta y} = \lambda\left(\hat{i}\frac{\delta g}{\delta x} + \hat{j}\frac{\delta g}{\delta y}\right)$$

$$\Rightarrow \hat{i}\, y + \hat{j}\, x = \lambda \hat{i} + \lambda \hat{j}$$

$\hat{i} y = \lambda \hat{i}$ | $\hat{j} x = \lambda \hat{j}$

$\Rightarrow \lambda = y$ | $\Rightarrow \lambda = x$

$\therefore x = y$

From (i),

$$x + x = 10$$

$$\Rightarrow 2x = 10$$

$$\therefore x = 5,\ y = 5.$$

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* Find the points on the circle $x^2+y^2=80$ which are closest and farthest from the point $(1,2)$.

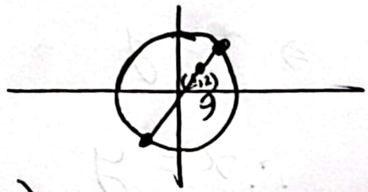
Soln:

Suppose that,

(x,y) be the points on the circle.

$x^2+y^2=80$

$\Rightarrow x^2+y^2-80=0$ ---- (i)



$\therefore g(x,y)=x^2+y^2-80$ ---- (ii)

Now, the distance between (x,y) and $(1,2)$ is, $d=\sqrt{(x-1)^2+(y-2)^2}$

$\Rightarrow d^2=(x-1)^2+(y-2)^2$

We have to maximize or minimize,

$f(x,y)=(x-1)^2+(y-2)^2$ ---- (iii)

For extremum,

$\nabla f=\lambda \nabla g$

$\Rightarrow \left(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}\right)\{(x-1)^2+(y-2)^2\}=\lambda\left\{\left(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}\right)(x^2+y^2-80)\right\}$

$\Rightarrow 2\hat{i}(x-1) + 2\hat{j}(y-2) = 2x\hat{i}\lambda + 2y\lambda\hat{j}$

$\Rightarrow 2(x-1) = 2x\lambda$, ~~y(y~~ $2(y-2) = 2y\lambda$

$\Rightarrow \lambda = \frac{x-1}{x}$, $\Rightarrow \lambda = \frac{y-2}{y}$

$\frac{x-1}{x} = \frac{y-2}{y}$

$\Rightarrow xy - y = xy - 2x$

$\therefore y = 2x$

(i) implies that,

$x^2 + y^2 = 80$

$\Rightarrow x^2 + 4x^2 = 80$

$\Rightarrow 5x^2 = 80$

$\therefore x = \pm 4$

when, $x = 4, y = 8 \Rightarrow (x,y) = (4,8)$

$x = -4, y = -8 \Rightarrow (x,y) = (-4,-8)$

From (iii),

at $(x,y) = (4,8)$

$f(x,y) = 3^2 + 6^2 = 9+36 = 45$

$f(x,y) = (-5)^2 + (-10)^2 = 125.$

MULTIPLE INTEGRAL:

$$\int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

* Prove that, $\int_0^{\ln 2}\int_0^1 xy\,e^{xy^2}\,dy\,dx = \frac{1}{2}[1-\ln 2]$

$$L.H.S. = \int_0^{\ln 2}\int_0^1 xye^{xy^2}\,dy\,dx$$

$$= \int_0^{\ln 2} x\left[\int_0^1 \cdot y e^{xy^2}\,dy\right]dx$$

$$= \int_0^{\ln 2} x\left[\frac{1}{2}\int_0^1 e^{xy^2}\,2y\,dy\right]dx$$

$$= \int_0^{\ln 2} x\left[\frac{1}{2}\int_0^1 e^{xy^2}\,dy^2\right]dx$$

$$= \frac{1}{2}\int_0^{\ln 2} x\left[\frac{e^{xy^2}}{x}\right]_0^1 dx$$

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$$= \frac{1}{2}\int_0^{\ln 2} x\left[\frac{e^x}{x} - \frac{1}{x}\right] dx$$

$$= \frac{1}{2}\int_0^{\ln 2} x \cdot \frac{1}{x}\left(e^x - 1\right) dx$$

$$= \frac{1}{2}\int_0^{\ln 2} \left(e^x - 1\right) dx$$

$$= \frac{1}{2}\left[e^x - x\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left\{\left(e^{\ln 2} - \ln 2\right) - \left(e^0 - 0\right)\right\}$$

$$= \frac{1}{2}\left[2 - \ln 2 - 1\right]$$

$$= \frac{1}{2}\left(1 - \ln 2\right)$$

$= R.H.S.$

Proved

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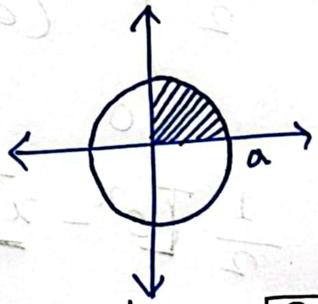
* Evaluate $\iint_R xy\,dy\,dx$, where ~~R is~~ R is the quadrant of the circle $x^2+y^2=a^2$, where $x \geq 0$ and $y \geq 0$.

SOLUTION:

From figure, in the region R, x varies from $x=0$ to $x=a$ and y varies from $y=0$ to $y=\sqrt{a^2-x^2}$.



Here, we can write,

$$\iint_R xy\,dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x \cdot \frac{1}{2}\,(a^2-x^2-0)\,dx$$

$$= \int_0^a \frac{x}{2}\,(a^2-x^2)\,dx$$

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$$= \frac{1}{2}\int_0^a (a^2 \cdot x - x^3)\, dx$$

$$= \frac{1}{2}\left[a^2 \cdot \frac{x^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4} \right)$$

$$= \frac{1}{2} \times \frac{a^4}{4}$$

$$= \frac{a^4}{8}$$ (Ans.)

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Date:..................

Jacobian's Rules:

Suppose that,

$$u = u(x, y, z)$$

$$v = v(x, y, z)$$

$$w = w(x, y, z)$$

Then,

$$J = \frac{\partial (u, v, w)}{\partial (x, y, z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$du\,dv\,dw = |J|\,dx\,dy\,dz$$

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□ By transforming to polar co-ordinates show that

$$\int_0^\infty\int_0^\infty dx\,dy = \int_0^{\infty}\int_0^{\pi/2} r\,d\theta\,dr$$

Solution:

We know that, In polar-coordinate system

$x(r,\theta) = x = r\cos\theta$

$y(r,\theta) = y = r\sin\theta$

We can write,

$$dx\,dy = |J|\,dr\,d\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

$= r \times 1$

$= r$

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Then (i) becomes,

$$dx\,dy = |r|\,dr\,d\theta = r\,dr\,d\theta$$

when $x=0$, $0 = r\cos\theta$

$\Rightarrow r=0$ or $\cos\theta = 0$

$\therefore r=0$ or $\theta = \frac{\pi}{2}$

when $x=\infty$, $r\cos\theta = \infty$

$\therefore r=\infty$ $[\because \cos\theta \neq \infty]$

when $y=0$, $r\sin\theta = 0$

$\Rightarrow r=0$ or $\sin\theta = 0$

$\therefore r=0$ or $\theta = 0$

when $y=\infty$, $r\sin\theta = \infty$

$\therefore r=\infty$ $[\because \sin\theta \neq \infty]$

$$\therefore \int_0^\infty\int_0^\infty dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^\infty r\,dr\,d\theta$$

$$= \int_0^\infty\int_0^{\frac{\pi}{2}} r\,d\theta\,dr = R.H.S$$

Proved

Date:........................

Q. Transform $\iiint dx\,dy\,dz$ to spherical polar co-ordinates.

Sol^n:

$$x = r\cos\varphi\sin\theta = x(r,\theta,\varphi)$$

$$y = r\sin\varphi\sin\theta = y(r,\theta,\varphi)$$

$$z = r\cos\theta = z(r,\theta,\varphi)$$

$$J = \frac{\delta(x,y,z)}{\delta(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\varphi\sin\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\varphi\sin\theta\,(r^2\sin^2\theta\cos\varphi) - r\cos\varphi\cos\theta\,(-r\cos\varphi\sin\theta\cos\theta)$$
$$- r\sin\varphi\sin\theta\,(-r\sin\varphi\sin^2\theta - r\sin\varphi\cos^2\theta)$$

$$= r^2\sin^3\theta\cos^2\varphi + r^2\cos^2\varphi\sin\theta\cos^2\theta + r^2\sin^2\varphi\sin^3\theta +$$
$$r^2\sin^2\varphi\sin\theta\cos^2\theta$$

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$= r^2 \sin^3\theta\,(\cos^2\phi + \sin^2\phi) +$

$r^2 \cos^2\theta \sin\theta\,(\cos^2\phi + \sin^2\phi)$

$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta$

$= r^2\sin\theta\,(\sin^2\theta + \cos^2\theta)$

$= r^2\sin\theta$

since, $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$

~~Since~~, $\therefore\ dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$\therefore \iiint dx\,dy\,dz = \iiint r^2\sin\theta\,dr\,d\theta\,d\phi$

(Ans.)

$\cancel{f_x = \dfrac{\partial f}{\partial x} = \lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{}}$

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H.W.

If $f(x,y) = x^2 + 3xy + 2y^3$, then $f_{xx}(1,2)$ and $f_{xy}(1,2)$.

Solution:

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\{(x+h)^2 + 3(x+h)y + 2y^3\} - x^2 - 3xy + 2y^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3xy} + 3hy + \cancel{2y^3} - \cancel{x^2} - \cancel{3xy} - \cancel{2y^3}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 3hy}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h+3y)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h + 3y$$

$$= 2x + 0 + 3y$$

$$= 2x + 3y$$

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$$f_{xx}(1,2) = \frac{\partial}{\partial x}(f_x)$$

$$= \lim_{h\to 0} \frac{f_x(1+h,2) - f_x(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{2(1+h)+3\times 2 - 2\times 1 - 3\times 2}{h}$$

$$= \lim_{h\to 0} \frac{2+2h+6-2-6}{h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} 2$$

$$= 2$$

$$f_{xy}(1,2) = \frac{\partial}{\partial y}(f_x)$$

$$= \lim_{k\to 0} \frac{f_x(1,2+k) - f_x(1,2)}{k}$$

$$= \lim_{k\to 0} \frac{2\times 1 + 3(2+k) - 2\times 1 - 3\times 2}{k}$$

$$= \lim_{k\to 0} \frac{2+6+3k-2-6}{k} = \lim_{k\to 0} \frac{3\cancel{k}}{\cancel{k}} = \lim_{k\to 0} 3 = 3$$

(Ans.)

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Differentiability:

If the function, $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{\sqrt{x^2+y^2}} ; & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

Then, examine the differentiability of $f(x,y)$ at $(0,0)$.

Sol^n:

By analytical definition of partial derivative

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{h \cdot 0 (h^2 - 0^2)}{\sqrt{h^2+0^2}} - 0 \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \times 0$$

$$= \lim_{h \to 0} 0 = 0$$

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~~$f_y(x,y) =$~~

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(\cancel{x}\,0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left(\frac{0\cdot k\cdot(0^2-k^2)}{\sqrt{0^2+k^2}} - 0\right)$$

$$= \lim_{k\to 0} \frac{1}{k} \times 0$$

$$= \lim_{k\to 0} 0$$

$$= 0$$

By analytical definition of differentiability,

$$L.H.L. = \lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

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Date:

$$= \lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h\cdot k\cdot(h^2-k^2)}{\sqrt{h^2+k^2}}-0-h\times 0-k\times 0\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \times \frac{h\cdot k(h^2-k^2)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{hk(h^2-k^2)}{h^2+k^2} \quad ---- (i)$$

Then,

Let, $k = mh$

$$\lim_{(h,k)\to(0,0)} \frac{hk(h^2-k^2)}{h^2+k^2}$$

$$= \lim_{(h,k)\to(0,0)} \frac{mh^2(h^2-m^2h^2)}{h^2+m^2h^2}$$

$$= \lim_{(h,k)\to(0,0)} \frac{mh^4(1-m^2)}{h^2(1+m^2)}$$

$h^2+k^2 = 0$

$\Rightarrow k^2 = -h^2$

$\Rightarrow k = \sqrt{-h^2}$

$\Rightarrow k = ih$

$\therefore k = mh$

$$= \lim_{(h,k)\to(0,0)} \frac{mh^2 \cdot \cancel{h}(1-m^2)}{1+m^2}$$

$$= \frac{m\times 0(1-m^2)}{1+m^2}$$

$$= 0$$

$= R.H.S.$

The function is differentiable at $(0,0)$.

(ii) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

show that both $f_x(0,0)$ and $f_y(0,0)$ both exist but $f(x,y)$ is discontinious at $(0,0)$.

Soln:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{h\times 0}{h^2+0^2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \times 0 = 0$$

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$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left(\frac{0\times k}{0^2 + k^2}\right)$$

$$= \lim_{k\to 0} \frac{1}{k} \times 0$$

$$= \lim_{k\to 0} 0$$

$$= 0$$

Here, $f_x(0,0)$ and $f_y(0,0)$ both exist.

Then, $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x\cdot mx}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$; which is not unique.

$x^2+y^2=0$
$\Rightarrow y^2 = -x^2$
$\therefore y = mx$

So, $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ doesn't exist and $f(x,y)$ is discontinuous at $(0,0)$.

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(iii) Let, $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

show that, $f_y(x,0) = x$; $f_x(0,y) = -y$ and

~~$f(x,y)(0,0) =$~~ $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Solution:

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$L.H.S. = f_y(x,0) = \lim_{k\to 0} \frac{f(x,0+k) - f(x,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(x,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k} \times \left[\frac{x \times k(x^2-k^2)}{x^2+k^2} - 0 \right]$$

$$= \lim_{k\to 0} \frac{1}{\cancel{k}} \frac{x\cancel{k}(x^2-k^2)}{x^2+k^2}$$

$$= \lim_{k\to 0} \frac{x(x^2-k^2)}{x^2+k^2}$$

$$= \frac{x(x^2-0^2)}{x^2+0^2}$$

$$= \frac{x^3}{x^2} = x = R.H.S.$$

[showed]

Date:...................

L.H.S.

$= f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y) - f(0,y)}{h}$

$\cancel{= \lim_{h\to 0} \frac{f(0,h,y) - f(0,0)}{h}}$

$= \lim_{h\to 0} \frac{f(h,y) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{hy(h^2-y^2)}{h^2+y^2} - 0\right]$

$= \lim_{h\to 0} \frac{1}{\cancel{h}} \times \frac{\cancel{h}y(h^2-y^2)}{h^2+y^2}$

$= \lim_{h\to 0} \frac{y(h^2-y^2)}{h^2+y^2}$

$= \frac{y(0^2-y^2)}{h^2+y^2}$

$= -\frac{y^3}{y^2}$

$= -y$

$=$ R.H.S.

[Proved]

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Date:.......................

~~L.H.S~~ $f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x)$

$\Rightarrow f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$

$= \lim_{k\to 0} \frac{-0-k-0}{k}$

$= \lim_{k\to 0} \frac{-k}{k}$

$= \lim_{k\to 0} (-1)$

$= -1$

$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y)$

$\Rightarrow f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$

$= \lim_{h\to 0} \frac{0+h-0}{h}$

$= \lim_{h\to 0} \frac{h}{h}$

$= \lim_{h\to 0} 1$

$= 1$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$ Proved

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Chain Rule

(i) If $\omega = f\left(\frac{y-x}{xy}, \frac{z-y}{yz}\right)$ then show that

$$x^2\frac{\partial \omega}{\partial x} + y^2\frac{\partial \omega}{\partial y} + z^2\frac{\partial \omega}{\partial z} = 0$$

Solution:

Let $p = \frac{y-x}{xy}$, $q = \frac{z-y}{yz}$

$$\therefore \omega = f\left(p\left(\frac{y-x}{xy}\right), q\left(\frac{z-y}{yz}\right)\right)$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x}$$

$$= \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial x}\left(\frac{y-x}{xy}\right)$$

$$= \frac{\partial f}{\partial x}\cdot\frac{\partial}{\partial x}\left(\frac{y}{xy} - \frac{x}{xy}\right)$$

$$= \frac{\partial f}{\partial x}\partial\left(\frac{1}{\cancel{y}x} - \frac{1}{y}\right)$$

$$= \frac{\partial f}{\partial x}\cdot\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y}$$

$$= \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial y}\left(\frac{y-x}{xy}\right) + \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial y}\left(\frac{z-y}{yz}\right)$$

$$= \frac{\partial f}{\partial p}\cdot\frac{1}{y^2} - \frac{\partial f}{\partial q}\cdot\frac{1}{y^2}$$

$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial z}$$

$$= \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial z}\left(\frac{z-y}{yz}\right)$$

$$= \frac{\partial f}{\partial q}\cdot\frac{1}{z^2}$$

$$\therefore \text{L.H.S.} = x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z}$$

$$\cancel{= -x^2\cdot\frac{1}{x^2}\frac{\partial f}{\partial p} +}$$

$$= -\cancel{x^2}\cdot\frac{1}{\cancel{x^2}}\cdot\frac{\partial f}{\partial p} + y^2\left(\frac{\partial f}{\partial p}\cdot\frac{1}{y^2} - \frac{\partial f}{\partial q}\cdot\frac{1}{y^2}\right) +$$

$$\cancel{z^2}\times\frac{1}{\cancel{z^2}}\times\frac{\partial f}{\partial q}$$

$$= -\frac{\partial f}{\partial p} + \frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} + \frac{\partial f}{\partial q}$$

$= 0$

$=$ R.H.S. Proved

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(ii) If $u = f(x^2+2yz, y^2+2zx)$, then show that $(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0.$

Solution:

Let, $p = x^2+2yz$, $q = y^2+2zx$.

$\therefore u = f(p(x^2+2yz), q(y^2+2zx))$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial x}(x^2+2yz) + \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial x}(y^2+2zx)$$

$$= 2x\cdot\frac{\partial f}{\partial p} + 2z\cdot\frac{\partial f}{\partial q}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial y}(x^2+2yz) + \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial y}(y^2+2zx)$$

$$= \frac{\partial f}{\partial p}\,2z + \frac{\partial f}{\partial q}\cdot 2y$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial z}(x^2+2yz) + \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial z}(y^2+2zx)$$

$$= \frac{\partial f}{\partial p}\cdot 2y + \frac{\partial f}{\partial q}\cdot 2x$$

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$\cancel{L.H.S = f(x^2+2yz\,,\, y^2+2zx)}$

$$L.H.S. = (y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$$

$$= (y^2 - zx)\left(2x\cdot\frac{\partial f}{\partial p} + 2z\frac{\partial f}{\partial q}\right) + (x^2 - yz)\left(2z\frac{\partial f}{\partial p} + 2y\frac{\partial f}{\partial q}\right) + (z^2 - xy)\left(\frac{\partial f}{\partial p}\,2y + \frac{\partial f}{\partial q}\,2x\right)$$

$$= \cancel{2y}\; \cancel{2xy^2\frac{\partial f}{\partial p}} - \cancel{2zx^2\frac{\partial f}{\partial p}} + \cancel{2y^2z\frac{\partial f}{\partial q}} - \cancel{2z^2x\frac{\partial f}{\partial q}} + \cancel{2zx^2\frac{\partial f}{\partial p}} - \cancel{2yz^2\frac{\partial f}{\partial p}} + \cancel{2x^2y\frac{\partial f}{\partial q}} - \cancel{2y^2z\frac{\partial f}{\partial q}} + \cancel{2yz^2\frac{\partial f}{\partial p}} - \cancel{2xy^2\frac{\partial f}{\partial p}} + \cancel{2xz^2\frac{\partial f}{\partial q}} - \cancel{2x^2y\frac{\partial f}{\partial q}}$$

$$= 0$$

$$= R.H.S.$$

Prooved

(iii) If $z = z(x,y)$ and $x = e^{u} + e^{-v}$;

$y = e^{-u} - e^{v}$. then show that,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}.$$

$$= \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial u}(e^{u} + e^{-v}) + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial u}(e^{-u} - e^{v})$$

$$= \frac{\partial z}{\partial x}(e^{u}) + \frac{\partial z}{\partial y}\cdot(-e^{-u})$$

$$= e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial v}(e^{u} + e^{-v}) + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial v}(e^{-u} - e^{v})$$

$$= -\frac{\partial z}{\partial x}\cdot e^{-v} + \frac{\partial z}{\partial y}e^{v}.$$

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$$L.H.S. = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial x} e^{u} - e^{-u}\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} e^{-v} - \frac{\partial z}{\partial y} e^{v}$$

$$= \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$= R.H.S.$

Proved

Euler's Theorem:

(i) Verify Euler's theorem for $u(x,y,z) = x^3+y^3+z^3$.

Solution:

Here, u is a homogenous function of degree $= 3$. We have to show that,

$$xU_x + yU_y + zU_z = 3u$$

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$\therefore$ L.H.S. $= xU_x + yU_y + zU_z$

$= x.\cancel{\ } 3x^2 + y.3y^2 + z.3z^2$

$= 3x^3 + 3y^3 + 3z^3$

$= 3(x^3+y^3+z^3)$

$= 3u$

$=$ R.H.S.

Proved

(ii) Verify Euler's theorem for $u(x,y,z) = x^2+y^2+z^2$.

Solution:

Here, u is a homogenous function of x, y, z of degree $= 2$.

We have to show that,

$$xu_x + yu_y + zu_z = 2u$$

L.H.S. $= xU_x + yU_y + zU_z$

$= x.2x + y.2y + z.2z$

$= 2x^2+2y^2+2z^2$

$= 2(x^2+y^2+z^2) = 2u =$ R.H.S.

Proved

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(iii) Verify Euler's theorem for

$$u(x,y,z) = \frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}.$$

Solution $\Rightarrow$

$$u = xy^2z^{-1} + yz^2x^{-1} + zx^2y^{-1}$$

Here, u is a homogenous function of x, y, z of degree $= 2$.

$$x.U_x + yU_y + zU_z = 2u.$$

L.H.S. $= xU_x + yU_y + zU_z$

$$= x\left(\frac{y^2}{z} - \frac{yz^2}{x^2} + \frac{2zx}{y}\right) + y\left(\frac{2xy}{z} + \frac{z^2}{x} - \frac{zx^2}{y^2}\right) + z\left(-\frac{xy^2}{z^2} + \frac{2yz}{x} + \frac{x^2}{y}\right)$$

$$= \frac{xy^2}{z} - \frac{xyz^2}{x^2} + \frac{2zx^2}{y} + \frac{2xy^2}{z} + \frac{yz^2}{x} - \frac{x^2yz}{y^2} - \frac{xy^2z}{z^2} + \frac{2yz^2}{x} + \frac{x^2z}{y}$$

$$= \cancel{\frac{xy^2}{z}} - \cancel{\frac{yz^2}{x}} + \frac{2zx^2}{y} + \frac{2xy^2}{z} + \cancel{\frac{yz^2}{x}} - \cancel{\frac{x^2z}{y}} - \cancel{\frac{xy^2}{z}} + \frac{2yz^2}{x} + \cancel{\frac{x^2z}{y}}$$

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$$= 2 \cdot \frac{xy^2}{z} + 2\frac{yz^2}{x} + 2\frac{zx^2}{y}$$

$$= 2\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right)$$

$$= 2u$$

$$= R.H.S$$

Proved.