COT4210 Discrete Structures – Exam 2

Fall 2023

1. (30) Let $G = (V, \sum, R, S)$ be a grammar with $V = \{Q, R, T\}$; $\sum = \{\mathbf{q}, \mathbf{r}, \mathbf{t}\}$; and the set of rules:

   S → Q
   Q → **q** | R**q**T
   R → **r** | **r**T | QQ**r** | ε
   T → **t** | S| **t**T

   a. (25) Convert G to Chomsky Normal Form.

      $S \to Q$
      $Q \to q|Rqt$
      $R \to r|rT|QQr|\varepsilon$
      $T \to t|S|tT$

      Add S and remove any ε:
      $S_0 \to Q|\varepsilon$
      $Q \to q|Rqt$
      $R \to r|rT|QQr$
      $T \to t|Q|tT$

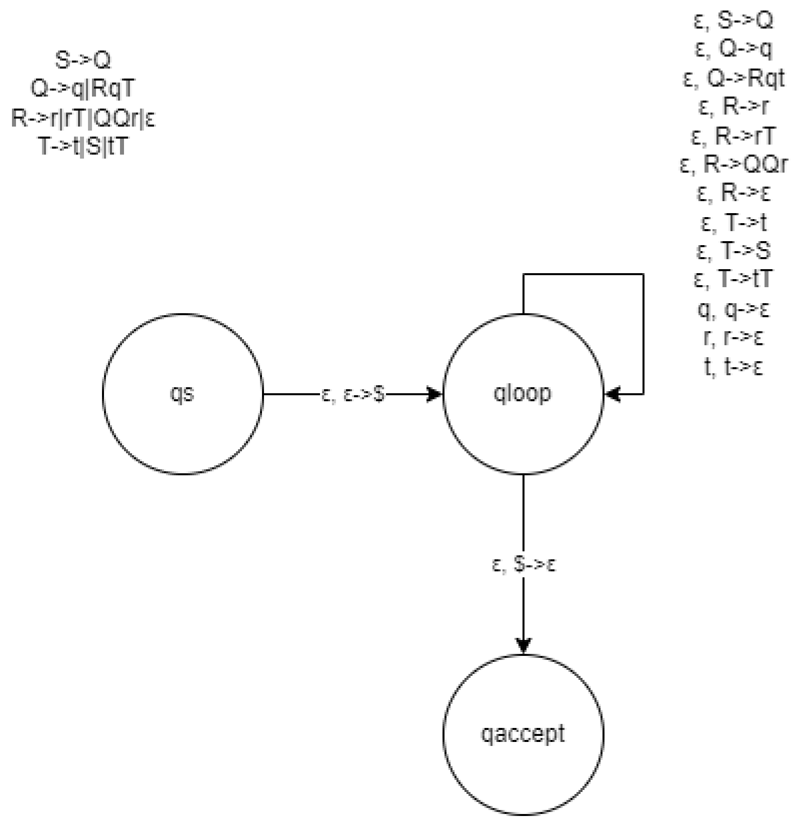
      Remove single rewrites:
      $S_0 \to Q|\varepsilon$
      $Q \to q|Rqt$
      $R \to r|rT|QQr$
      $T \to t|q|Rqt|tT$

      Remove mixed/multiple terminals:
      $S_0 \to Q|\varepsilon$
      $Q \to q|RQ_0T_0$
      $R \to r|R_0T|QQR_0$
      $T \to t|q|RQ_0T_0|T_0T$
      $Q_0 \to q$
      $R_0 \to r$
      $T_0 \to t$

      Remove long rewrites to get CNF:
      $S_0 \to Q|\varepsilon$
      $Q \to q|RQ_1$
      $R \to r|R_0T|Q_2R_0$
      $T \to t|q|RQ_1|T_0T$
      $Q_0 \to q$
      $Q_1 \to Q_0T_0$
      $Q_2 \to QQ$

$R_0 \to r$
$T_0 \to t$

b. (5) Convert the language of G to a PDA.



2. (20) Show that the class of context-free languages is closed under homomorphism.

Suppose we have a context free language $G = (V, \Sigma, R, S)$ and any homomorphism h on terminal symbols Σ, we can construct a grammar $h(G) = (V_h, \Sigma_h, R_h, S_h)$ by replacing each terminal symbol with the result of that symbol in function h, so where

$\Sigma_h = \left(h(\Sigma_1), h(\Sigma_2), \ldots h(\Sigma_n)\right)$
$V_h = V$
$R_h = R$
$S_h = S$
$L(G) =$

For example, language $G = (\{S\}, \{a, b\}, R, S)$ with
$S \to aSb |\ SS \mid \varepsilon$

Then language $h(G) = (\{S\}, \{h(a), h(b)\}, R, S)$
$S \to h(a)Sh(b) |\ SS \mid \varepsilon$

So if $h(a) = 00, h(b) = 11$ then
$S \to 00S11 \mid SS \mid \varepsilon$

3. (15) Let $REPEAT_{!"\#} = \{\langle M\rangle | M$ is a DFA and for every $s \in L(M)$, $s = uv$ where $u = v\}$. Show that $REPEAT_{!"\#}$ is decidable.

   Proof: Construction, assume M_{Repeat} which decides $REPEAT_{!"\#}$

   M_{Repeat} on input <M>:

   - WLOG, we receive DFA M
   - Where for every $s \in L(M)$
     - where s can be split into even length substrings u and v
     - And u = v
     - Then we accept
   - Otherwise reject.

   This machine clearly accepts a DFA if it's language only contains strings where $s = uv$ and $u = v$ and rejects otherwise. Even if |s| is infinite it will still be decidable. For example, if s = a* this would reject since a* cannot be evenly split into u and v.

4. (15) Let $SUBSET_{!"\#} = \{\langle M_{\$}, M_{\%}\rangle \mid M_{\$}$ and $M_{\%}$ are DFAs and $L(M_{\$}) \subseteq L(M_{\%})\}$. Show that $SUBSET_{!"\#}$ is decidable.

   Proof: Construction, assume M_{Subset} which decides $SUBSET_{!"\#}$

   M_{Repeat} on input <$M_{\$}, M_{\%}$>:

   - WLOG, we receive DFA $M_{\$}$ and $M_{\%}$
   - Where:
     - Reject if only $M_{\$}$ is infinite
     - Accept if every $s \in L(M_{\$})$ is also $s \in L(M_{\%})$ (so if $L(M_{\$}) \subseteq L(M_{\%})$)
     - Otherwise reject.

   This machine clearly accepts a DFA if it's $L(M_{\$}) \subseteq L(M_{\%})$ rejects otherwise.

5. (20) A Turing machine *M* is **verbose** on the input string *s* if, when finished computing on *s*, *M* does not shrink the input – that is, leaves at least as many non-blank characters on the left of the tape as *s* has total characters.

   Let $VERBOSE_{TM}$ = { <*M*, *w*> | *M* is a TM and is verbose on *s* }, and show that $VERBOSE_{TM}$ is undecidable by reduction from A_{TM}. Do not use Rice's theorem.

   Proof: If $M_{VERBOSE}$ decides $VERBOSE_{TM}$. Assume BWOC that $M_{VERBOSE}$ is decidable:

   By definition of decidability let $M_{VERBOSE}$ be its decider where:

- We receive string s and Turing machine M
- We accept if M is a TM and is verbose on s
- We reject otherwise

Then we build a M_{accept} where:

- We mimic A_{TM} and we receive Turing machine M and string s
- We construct a new Turing machine $M_{Internal}$ that,
  - Receives string x and TM M
  - Enumerator w where w is the number of things on the tape
  - we reject if w is greater than or equal to the length of string x
    - This causes M to decide
  - Otherwise simulate the contrariwise of TM M on s
    - This mimics the opposite of M whether is accepts, or rejects. Also means that it mimics if it never halts

Interrogate $M_{Internal}$ using $M_{Verbose}$ which causes M_{Accept} to decide A_{TM} and creates a contradiction. Therefore *VERBOSE*$_{TM}$ must be undecidable.