COT3100 Final Exam 2021

1.)

q	p	r	$(p \vee r)$	$(q \wedge p)$	$(p \vee r) \vee (q \wedge p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	F	T
T	F	F	F	F	F
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	F	F	F

2.)$\big((p \wedge \neg q \wedge r) \vee (q \wedge p \wedge) \vee (\neg r \wedge s \wedge r)\big) = p \wedge r$

$\big((\neg q \wedge p \wedge r) \vee (q \wedge p \wedge r) \vee (\neg r \wedge r \wedge s)\big) = p \wedge r$ Commutativity

$\big((\neg q \wedge p \wedge r) \vee (q \wedge p \wedge r) \vee (F \wedge s)\big) = p \wedge r$ Negation

$\big((\neg q \wedge p \wedge r) \vee (q \wedge p \wedge r) \vee F\big) = p \wedge r$ Domination

$\big((\neg q \wedge p \wedge r) \vee (q \wedge p \wedge r)\big) = p \wedge r$ Identity

$\big((\neg q \wedge p) \wedge r \vee (q \wedge p) \wedge r\big) = p \wedge r$ 2x Associativity

$\big((\neg q \wedge p) \wedge (q \wedge p) \vee r \wedge r\big) = p \wedge r$ Commutativity

$\big((\neg q \wedge p) \wedge (q \wedge p) \vee r\big) = p \wedge r$ Idempotence

$\big((\neg q \wedge p) \wedge (r \wedge p) \vee (r \wedge q)\big) = p \wedge r$ Distributivity

$\big((\neg q \wedge p) \wedge (q \wedge r) \vee p \wedge r\big) = p \wedge r$ 3x Commutativity

$(F \vee p \wedge r) = p \wedge r$ Negation (because $(\neg q \wedge p) \wedge (q \wedge r)$ will always contain F)

$(\boldsymbol{p} \wedge \boldsymbol{r}) = \boldsymbol{p} \wedge \boldsymbol{r}$ **Identity**

3.)$\Big((A - D) \cup \big((B \cup C) - D\big)\Big) = \big((A \cup B \cup C) - D\big)$

Case 1: $x \in \Big((A - D) \cup \big((B \cup C) - D\big)\Big)$ Consider an element x

$x \in A \; and \; x \notin D \; or \; x \in B \; or \; x \in C \; and \; x \notin D$ Definition of Union and Relative Complacent

Case 1.1 $x \in A \; and \; x \notin D \; (which \; is \; A - D)$

$x \in A \cup B \cup C$ Def. Union, S = any set $(B \cup C)$

$x \in (A \cup B \cup C) - D$ Relative Complacent and add parenthesis

Case 1.2 $x \in B \; and \; x \notin D$

$x \in B \cup A \cup C$ Def. Union, S = any set $(A \cup C)$

$x \in (A \cup B \cup C) - D$ Relative Complacent, Reorder, Add parenthesis

Case 1.3 $x \in C \; and \; x \notin D$

$x \in C \cup A \cup B$ Def. Union, S = any set $(A \cup B)$

$x \in (A \cup B \cup C) - D$ Relative Complacent, Reorder, Add parenthesis

$\therefore \left((\boldsymbol{A} - \boldsymbol{D}) \cup \left((\boldsymbol{B} \cup \boldsymbol{C}) - \boldsymbol{D}\right)\right) = \left((\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}) - \boldsymbol{D}\right)$ **By Universal Generalization**

Case 2 $x \in \left((A \cup B \cup C) - D\right)$ Consider an element x

$x \in A \; or \; x \in B \; or \; x \in C \; and \; x \notin D$ Definition of Union and Relative Complacent

Case 2.1 $x \in A \; and \; x \notin D$

$x \in A - D$ Relative Complacent

$x \in A - D \cup \left((B \cup C) - D\right)$ Def. Union, S = any set $\left((B \cup C) - D\right)$

Case 2.2 $x \in B \; and \; x \notin D$

$x \in B \cup C - D$ Def. Union, S = any set (C),Relative Complacent

$x \in (B \cup C) - D \cup A - D$ Def. Union, S = any set $(A - D)$

$x \in A - D \cup \left((B \cup C) - D\right)$ Reorder, Add parenthesis

Case 2.3 $x \in C \; and \; x \notin D$

$x \in C \cup B - D$ Def. Union, S = any set (B),Relative Complacent

$x \in (C \cup B) - D \cup A - D$ Def. Union, S = any set $(A - D)$

$x \in A - D \cup \left((B \cup C) - D\right)$ Reorder, Add parenthesis

$\therefore \left((\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}) - \boldsymbol{D}\right) = \left((\boldsymbol{A} - \boldsymbol{D}) \cup \left((\boldsymbol{B} \cup \boldsymbol{C}) - \boldsymbol{D}\right)\right)$ **By Universal Generalization**

4.) $\left((A \cup B \cup C) - D\right) \subseteq (A \cup ((B \cup C) - D))$

Case 1: $x \in \left((A \cup B \cup C) - D\right)$ Consider an element x

$x \in A \; or \; x \in B \; or \; x \in C \; and \; x \notin D$ Definition of Union and Relative Complacent

Case 1.1 $x \in A \; and \; x \notin D$

$x \in A - D$ Relative Complacent

$x \in (A \cup ((B \cup C) - D)$ Def. of Union, $x \in A \cup S$, S= any set

Case 1.2 $x \in B \ and \ x \notin D$

$x \in B - D$ Relative Complacent

$x \in (A \cup ((B \cup C) - D)$ Def of Union, $x \in B \cup S$, any set S

Case 1.3 $x \in C \ and \ x \notin D$

$x \in C - D$ Relative Complacent

$x \in (A \cup ((B \cup C) - D)$ Def. of Union, $x \in C \cup S$, any set S

$\therefore \left((\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}) - \boldsymbol{D}\right) \subseteq (\boldsymbol{A} \cup ((\boldsymbol{B} \cup \boldsymbol{C}) - \boldsymbol{D}))$ **Universal Generalization**

5.) Disprove $(A \cup \left((B \cup C) - D\right)) \subseteq ((A \cup B \cup C) - D)$

A	B	C	D	$A \cup \left((B \cup C) - D\right)$	$(A \cup B \cup C) - D)$	$(A \cup \left((B \cup C) - D\right)) \subseteq ((A \cup B \cup C) - D)$
Y	Y	Y	Y	Y	N	**N**
Y	Y	Y	N	Y	Y	Y
Y	Y	N	Y	Y	N	**N**
Y	Y	N	N	Y	Y	Y
Y	N	Y	Y	Y	N	**N**
Y	N	Y	N	Y	Y	Y
Y	N	N	Y	Y	N	**N**
Y	N	N	N	Y	Y	Y
N	Y	Y	Y	N	N	Y
N	Y	Y	N	Y	Y	Y
N	Y	N	Y	N	N	Y
N	Y	N	N	Y	Y	Y
N	N	Y	Y	N	N	Y
N	N	Y	N	Y	Y	Y
N	N	N	Y	N	N	Y
N	N	N	N	N	N	Y

6.) $\sum_{i=1}^{n}(2i^2 + 2i + 2) = \frac{1}{3}(2n^3 + 6n^2 + 10n)$

Induction on n

Base case $n = 1$

$\sum_{i=1}^{1}(2i^2 + 2i + 2) = 2 + 2 + 2 = 6$

$\frac{1}{3}\left(2(1)^3 + 6(1)^2 + 10(1)\right) = \frac{18}{3} = 6$

Induction Hypothesis: $\sum_{i=1}^{k}(2i^2 + 2i + 2) = \frac{1}{3}(2k^3 + 6k^2 + 10k)$

Induction: $\sum_{i=1}^{k+1}(2i^2+2i+2)=\frac{1}{3}(2(k+1)^3+6(k+1)^2+10(k+1))$

$\sum_{i=1}^{k+1}(2i^2+2i+2)=\frac{2k^3+12k^2+28k+18}{3}$ Arithmetic

$\sum_{i=1}^{k+1}(2i^2+2i+2)=\sum_{i=1}^{k}(2i^2+2i+2)+2(k+1)^2+2(k+1)+2$ Summation

$\sum_{i=1}^{k+1}(2i^2+2i+2)=\frac{(2k^3+6k^2+10k)}{3}+2k^2+6k+6$ I.H and Arithmetic

$\sum_{i=1}^{k+1}(2i^2+2i+2)=\frac{2k^3+6k^2+10k}{3}+\frac{6k^2}{3}+\frac{18k}{3}+\frac{18}{3}$ Arithmetic

$\sum_{i=1}^{k+1}(2i^2+2i+2)=\frac{2k^3+12k^2+28k+18}{3}$ Arithmetic

$\therefore \boldsymbol{\sum_{i=1}^{n}\left(2i^2+2i+2\right)=\frac{1}{3}(2n^3+6n^2+10n)}$ **Induction Principle**

7.) $41|(2^{n+3}+5^{3n-1})$

Induction on n

Base case $n=1$

$41|\left(2^{1+3}+5^{3(1)-1}\right)=\frac{41}{41}$ Arithmetic

Induction Hypothesis: $41|\left(2^{k+3}+5^{3k-1}\right)$

Induction Step: $41|\left(2^{(k+1)+3}+5^{3(k+1)-1}\right)$

$2^{k+4}+5^{3k+2}$ Arithmetic

$2(2^{k+3})+5^{3k+2}$ Exponents

$2(41x-5^{3k-1})+5^{3k+2}$ Induction Hypothesis

$2\left(41x-5^{3k-1}\right)+125*5^{3k-1}$ Powers

$2*41x+123*5^{3k-1}$ Distribution

$41\left(2x+3*5^{3k-1}\right)$ is divisible by 41 Distribution

$\therefore \boldsymbol{41|(2^{n+3}+5^{3n-1})}$ **By Induction**

8.) Accepting the following premises, conclude $\neg p$:

1. $q \rightarrow (r \wedge s)$	2.$\neg(r \vee t)$
3. $\neg s \rightarrow q$	4.$\neg(p \wedge s)$

$a)\ \neg r \wedge \neg t$	Premise 2, De Morgan's
$b)\ \therefore \neg r$	Conjunctive Simplification
$c)\ q \rightarrow r$	Premise 1 and Conjunctive Simplification
$d)\ q \rightarrow r \wedge \neg r$	$b)\neg r$
$e)\ \neg q$	Modus Tollens
$f)\ \neg p \vee \neg s$	Premise 4, De Morgan's
$g)\ \neg p \vee q$	Premise 3
$h)p \rightarrow q \wedge \neg q$	De Morgan's, $e)\neg q$
$\boldsymbol{i})\ \therefore \neg \boldsymbol{p}$	**Modus Tollens**

9.) HUDSONICUS

$\left(\left(\frac{9!}{2!*2!}+\frac{9!}{2!*2!}+\frac{9!}{2!}+\frac{9!}{2!*2!}+\frac{9!}{2!*2!}\right)+\left(\frac{9!}{2!*2!}+\frac{9!}{2!*2!}+\frac{9!}{2!}\right)\right)=907200$ combinations that begin with a vowel or end with a consonant

$\left(\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}+\frac{8!}{2!}+\frac{8!}{2!*2!}+\frac{8!}{2!*2!}\right)=$ 181440 duplicate combinations due to the inclusion-exclusion principle

$907200-181440=\mathbf{725760}$ **total actual ways to rearrange the word HUDSONICUS**

10.)

$\binom{5+22-1}{22}=\binom{26}{22}=14950$ capturing at least five paperback

$\binom{5+20-1}{20}=\binom{24}{20}=10626$ capturing at most 1 loose-leaf

$\binom{5+16-1}{16}=\binom{20}{17}=1140$ capturing at most 3 spiral types

Therefore $14950-(10626-1140)=\mathbf{5464}$ **unique combinations**

11.)

$\binom{n}{k}p^k(1-P)^{n-k}$

$\binom{10}{7}(0.55)^7(0.45)^3 = \mathbf{16.65\%}$ **chance of exactly 7 being St. Augustine-subtypes**

12.)

$\binom{n}{k}p^k(1-P)^{n-k}$

$\binom{10}{1}(0.15)^1(.85)^9 = \mathbf{34.74\%}$ **chance of at least one being a Zoysia subtype**

13.)

$\binom{n}{k}p^k(1-P)^{n-k}$

70% chance of either St. Augustine-subtype or Zoysia-subtype, 30% chance of neither.

$\binom{10}{10}(0.30)^{10}(.85)^0 = \mathbf{0.00059\%}$ **chance there are no St. Augustine or Zoysia-subtypes captured**

14.) $R \subseteq \mathbb{R} \times \mathbb{R}$ with $\{(x,y)|x^3 = -y^3\}$

Not Reflexive because x will result in a negative version of itself, for example $x^3 = 1$ will result in $y^3 = -1$.

Irreflexive because x will never be the same as y, whatever x is y will be the negative version of that number.

Symmetric because y will always be the opposite of x so the equation will contain both (x,y) & (y,x) for each value that is within the subset. For example $(1,-1)$ & $(-1,1)$ and $(4,-4)$ & $(-4,4)$ exist.

Not Anti-Symmetric because it is symmetric

Not Transitive since $x^3 = -y^3$ and $-y^3 = z^3$ but $x^3 \,! \neq z^3$. For example, $(10,-10) \in R, (-10,10) \in R, (10,10) \notin R$

Complete since xRy or yRx, all distinct pairs will relate at least one way or the other.

Not an Ordering Relation as it is not anti-symmetric or transitive

Not an Equivalence Relation because it is not reflexive and transitive, only symmetric

15.) $R \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ with $\{(x, y) | |x| = |y|\}$

Reflexive because x will always be the same as y, for example $|5| = |5|$

Not Irreflexive because it is reflexive

Symmetric because x will always be the same as y, for example $(1,1), (50,50)$ ect.

Anti-Symmetric because it is composed only of self-related entries and don't break the anti-symmetric property because the elements aren't distinct

Transitive because $|x| = |y|,\ |y| = |z|\ therefore\ |x| = |z|$. For example, $(2,2), (2,2), (2,2);\ x, y, z = 2$.

Complete since it contains xRy and yRx, all pairs relate to each other one way or the other

Not an Ordering Relation as it isn't irreflexive

Is an Equivalence relation because it is reflexive, symmetric and transitive.

16.) $f: \mathbb{N}^+ \to \mathbb{N}^+$ and let $f(x)$ return the sum of the digits of x.

Not Injective because if $a = 13\ f(a) = 4$ and if $b = 22\ f(b) = 4$ as well and therefore breaks the rule of $a \neq b \to f(a) \neq f(b)$

Surjective because for every $f(x)$ there will be a natural positive x $(\mathbb{N}^+ = x)$

Not Bijective because it is not Injective