# Midterm Exam 2

**1.) (10) Show that if 5 divides $n$ with remainder 3, 10 divides $2n^2$ with remainder 8**

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|---|---|
| $2n^2 = 2(5n+3)^2$ | Substitution |
| $= 50n^2 + 60n + 18$ | Simplification |
| $= (50n^2 + 60n + 10) + 8$ | Expand |
| $= 5(10n^2 + 12n + 2) + 8$ | Distribution |
| $2n^2 = 5y + 8$ | Divisible by 5 with a remainder of 8 |

**2. (15) Find GCD(184, 18), GCD(981, 243) and GCD(5655, 819).**

**a.)** $\frac{184}{18} = 10\ with\ 4\ remainder$

$\frac{18}{4} = 4\ with\ 2\ remainder$

$\frac{4}{2} = 2\ evenly$

$GCD = 2$

**b.)** $\frac{981}{243} = 4\ with\ 9\ remainder$

$\frac{243}{9} = 27\ evenly$

$GCD = 9$

**c.)** $\frac{5655}{819} = 6\ with\ 741\ remainder$

$\frac{819}{741} = 1\ with\ 78$

$\frac{741}{78} = 9\ with\ 39\ remainder$

$\frac{78}{39} = 2\ evenly$

$GCD = 39$

**3. (15) Find the remainders of $\frac{3^{16}}{8}$ , $\frac{4^{31}}{5}$ and $\frac{2^{40}}{11}$.**

| | |
|---|---|
| **a.)** $\frac{(3^{16})}{8} = \frac{(3^4)^4}{8}$ | Exponentiation |
| $\frac{(81)^4}{8} = \frac{1^4}{8}$ | Arithmetic and $81 = \frac{1}{8}, let\ f(x) = x^4 and \left[f(81) = \frac{f(1)}{8}\right] \rightarrow$ $[81^4 = \frac{1^4}{8}]$ |

$\frac{1}{8} = 1$ Arithmetic and Modulo

**b.)** $\frac{4^{31}}{5} = \frac{4}{5} * \frac{(4^2)^{15}}{5}$ Exponentiation

$\frac{4}{5} * \frac{(16)^{15}}{5} = \frac{4}{5} * \frac{(1)^{15}}{5}$ Arithmetic and $16 = \frac{1}{5}, let\ f(x) = x^{15} and \left[f(16) = \frac{f(1)}{8}\right] \rightarrow [16^{15} = \frac{1^{15}}{8}]$

$\frac{4}{5} * \frac{1}{5}$ Arithmetic

$4 * 1 = 4$ Modulo

**c.)** $\frac{2^{40}}{11} = \frac{(2^4)^{10}}{11}$ Exponentiation

$\frac{(16)^{10}}{11} = \frac{5^{10}}{11}$ Arithmetic and $16 = \frac{5}{11}, let\ f(x) = x^{10} and \left[f(16) = \frac{f(5)}{11}\right] \rightarrow [16^{10} = \frac{5^{10}}{11}]$

$\frac{(5^2)^5}{11} = \frac{25^5}{11}$ Exponentiation and Arithmetic

$\frac{25^5}{11} = \frac{3^5}{11}$ $25 = \frac{3}{11}, let\ f(x) = x^5 and \left[f(25) = \frac{f(3)}{11}\right] \rightarrow [25^5 = \frac{3^5}{11}]$

$\frac{243}{11}$ Arithmetic

$1$ Modulo

**4. (20) Prove by induction that $\sum_{i=1}^{n}(2i^3 + 7i^2 + 3i + 4) = \frac{n}{6}(3n^3 + 20n^2 + 33n + 40)$**

Induction on n

$base\ case\ n = 1$

$\sum_{i=1}^{1}(2 + 7 + 3 + 4) = \frac{1}{6}(3 + 20 + 33 + 40)$ Substitution

$16 = 16$ Arithmetic

Induction Hypothesis: $\sum_{i=1}^{k}(2i^3 + 7i^2 + 3i + 4) = \frac{k}{6}(3k^3 + 20k^2 + 33k + 40)$

Induction Step: $\sum_{i=1}^{k+1}(2i^3 + 7i^2 + 3i + 4) = \frac{k+1}{6}(3(k + 1)^3 + 20(k + 1)^2 + 33k + 73)$

$\sum_{i=1}^{k+1}(2i^3 + 7i^2 + 3i + 4) = \frac{k+1}{6}(3k^3 + 29k^2 + 82k + 96)$

$\sum_{i=1}^{k}(2i^3 + 7i^2 + 3i + 4) + (2(k + 1)^3 + 7(k + 1)^2 + 3k + 7)$ Definition of Summation

$\frac{k}{6}(3k^3 + 20k^2 + 33k + 40) + (2(k + 1)^3 + 7(k + 1)^2 + 3k + 7)$ Induction Hypothesis

$\frac{k}{6}(3k^3 + 20k^2 + 33k + 40) + (2k^3 + 13k^2 + 23k + 16)$

$\left(\frac{3k^4}{6} + \frac{20k^3}{6} + \frac{33k^2}{6} + \frac{40k}{6}\right) + (2k^3 + 13k^2 + 23k + 16)$ Distribute

$\left(\frac{k^4}{2} + \frac{10k^3}{3} + \frac{11k^2}{2} + \frac{20k}{3}\right) + (2k^3 + 13k^2 + 23k + 16)$ Simplify

$\frac{k^4}{2} + \frac{16k^3}{3} + \frac{349k^2}{2} + \frac{89k}{3} + 16$ Add like terms

$\frac{1}{6}(3k^4 + 32k^3 + 1047k^2 + 178k + 96)$ Distribution

$\frac{1}{6}(k(3k^3 + 32k^2 + 1047k + 178) + 96)$ Distribution

$\frac{k}{6}(3k^3 + 32k^2 + 1047k + 178 + 16)$ Distribution

$\frac{k+1}{6}(3k^3 + 29k^2 + 82k + 96) = \frac{k+1}{6}(3k^3 + 29k^2 + 82k + 96)$ Distribution (left = right)

$\therefore \sum_{i=1}^{n}(2i^3 + 7i^2 + 3i + 4) = \frac{\mathrm{n}}{6}(3\mathrm{n}^3 + 20\mathrm{n}^2 + 33\mathrm{n} + 40)$ By induction

**5. (20) Prove by induction that for positive integers $\boldsymbol{n}$, $\mathbf{3}|(\mathbf{7}^{\boldsymbol{n}} + \mathbf{5}^{\mathbf{2}\boldsymbol{n}+\mathbf{1}})$**

Induction on n

$base\ case\ n = 1$

$3|\left(7^1 + 5^{2(1)+1}\right) = \frac{132}{3} = 44$ Substitution and Arithmetic

Induction Hypothesis: $3|\left(7^k + 5^{2(k)+1}\right)$

Induction Step: $3|\left(7^{k+1} + 5^{2(k+1)+1}\right)$

$\left(7^{k+1} + 5^{2k+3}\right)$ Arithmetic

$7(7^k) + 5^{2k+3}$ Exponentiation

$7\left(3x - 5^{2k+1}\right) + 5^{2k+3}$ Induction Hypothesis

$7\left(3x - 5^{2k+1}\right) + 25 * 5^{2k+1}$ Exponentiation

$7 * 3x + 3 + 25 * 5^{2k+1}$ Distribution

$3(7x + 5^{2k+1})$ is divisible by 3 Distribution

$\therefore 3|(7^n + 5^{2n+1})$ By induction

**6. (20) Prove by induction that $\sum_{i=1}^{n}(3^{i+2}) = \frac{27}{2}(3^n - 1)$.**

Induction on n

$base\ case\ n = 1$

| | |
|---|---|
| $\sum_{i=1}^{1}(3^{1+2}) = \frac{27}{2}(3^1 - 1)$ | Substitution |
| $27 = 27$ | Arithmetic |

Induction Hypothesis: $\sum_{i=1}^{k}(3^{i+2}) = \frac{27}{2}(3^k - 1)$

Induction Step: $\sum_{i=1}^{k+1}(3^{i+2}) = \frac{27}{2}(3^{k+1} - 1)$

| | |
|---|---|
| $\sum_{i=1}^{k}(3^{i+2}) + (3^{k+3}) = \frac{27}{2}(3^{k+1} - 1)$ | Definition of Summation |
| $\frac{27}{2}(3^k - 1) + (3^{k+3}) = \frac{27}{2}(3^{k+1} - 1)$ | Induction Hypothesis |
| $\frac{27}{2}\left(\frac{1}{27}3^{k+3} - 1\right) + (3^{k+3}) = \frac{27}{2}(3^{k+1} - 1)$ | Exponentiation |
| $\frac{27}{2}\left(\frac{1}{27}3^{k+3} - 1\right) + \frac{2}{2}(3^{k+3}) = \frac{27}{2}(3^{k+1} - 1)$ | Arithmetic |
| $\left(\frac{1}{2}3^{k+3} - \frac{27}{2}\right) + \frac{2}{2}(3^{k+3})) = \frac{27}{2}(3^{k+1} - 1)$ | Distribution |
| $(\frac{3}{2}3^{k+3} - \frac{27}{2}) = \frac{27}{2}(3^{k+1} - 1)$ | Arithmetic |
| $\frac{3}{2}\left(3^{k+3} - 9\right) = \frac{27}{2}(3^{k+1} - 1)$ | Distribution |
| $\frac{3}{2}\left(9 * 3^{k+1} - 9\right) = \frac{27}{2}(3^{k+1} - 1)$ | Exponentiation |
| $\frac{27}{2}\left(3^{k+1} - 1\right) = \frac{27}{2}(3^{k+1} - 1)$ | Distribution |
| $\therefore \sum_{i=1}^{n}(3^{i+2}) = \frac{27}{2}(3^n - 1)$ | By induction |