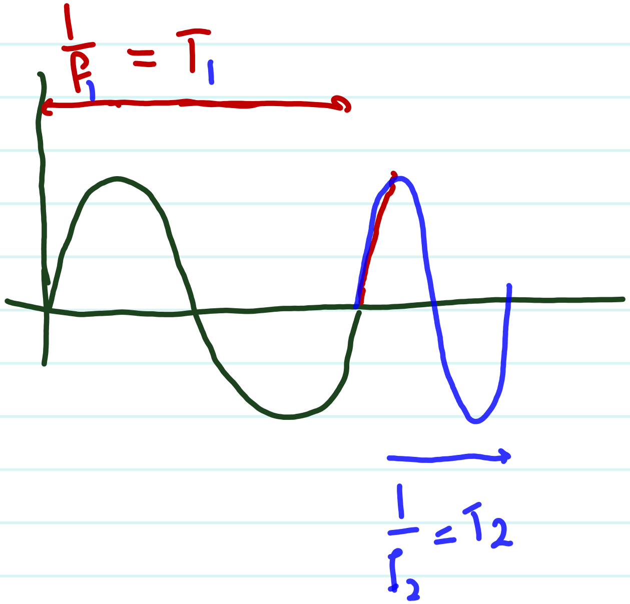
March 22nd 2023



$T_2 < T_1$

$f_2 > f_1$

1 Hz = $\boxed{1 \text{ sec}}$

10 Hz = $\frac{1}{10}$ sec = $\underline{100 \text{ ms}}$

AR(na)

$$\underline{y_t + a_1 y_{t-1} + \dots + a_{na} y_{t-na} = e_t} \qquad e_t \sim WN(0, \sigma_e^2)$$

$E[y(t-\tau)$.

Autocovariance eq.

$$\underbrace{E[y_t * y_{t-\tau}]} + a_1 E[\underline{y_{t-1} \cdot y_{t-\tau}}] + \dots + a_{na} E[y_{t-na} \cdot y_{t-\tau}] = E[\underline{e_t \cdot y_{t-\tau}}]$$

$$\boxed{R_y(\tau) + a_1 R_y(\tau-1) + \dots + a_{na} R_y(\tau - na) = R_{ye}(\tau)}$$

$$\underline{R_{ye}(\tau)} = \begin{cases} g(-\tau) \cdot \sigma_e^2 : & \tau \le 0 \\ 0 : & \tau > 0 \end{cases}$$

$$ACF = \frac{\text{Autocov}}{\text{Variance}}$$

$$R_y(\tau) = E[y_t \cdot y_{t-\tau}] \rightarrow R_y(0) = E[y_t \cdot y_t] = E\{y_t^2\} = var(y)$$

How to derive impulse Response g_t !

$\delta_t \leftarrow e_t$

$g_t \leftarrow y_t$

$$\boxed{g_t + a_1 g_{t-1} + \cdots + a_{na} g_{t-na} = \delta_t}$$

\# 1000 , seed : 6313

Ex : AR(1) $\boxed{y_t + 0.5 y_{t-1} = e_t}$; $e_t \sim WN(0, 1)$

— num = [1 0]

den = [1 0.5]

$\longrightarrow R_y(\tau) + 0.5 R_y(\tau-1) = R_{ye}(\tau)$

Var = $R_y(0)$, $\tau = 0$

$$\boxed{\tau=0} \longrightarrow R_y(0) + 0.5 \underbrace{R_y(-1)}_{R_y(1)} = \underbrace{R_{ye}(0)}_{g(0) \cdot \sigma_e^2}$$

1

#1 $\boxed{R_y(0) + 0.5 R_y(1) = 1}$

$\underline{\tau=1}$: $R_y(1)+0.5R_y(0) = \overbrace{R_{ye}(1)}^{0}$

#2 $\boxed{R_y(1)+0.5R_y(0)=0}$

$\underline{\text{Find } g(0)}$:

$\delta_t \leftarrow e_t$

$g_t \leftarrow y_t$

$g_t + 0.5 g_{t-1} = \delta_t$

$g(0) + 0.5\, \underset{\substack{\downarrow \\ 0}}{g(-1)} = \delta(0)$

$\boxed{g(0)=1}$

$$\begin{cases} R_y(0) + 0.5R_y(1) = 1 \\ 0.5R_y(0) + R_y(1) = 0 \end{cases}$$

Clamle's rule

$$\begin{cases} a_1 x + a_2 y = a_3 \\ b_1 x + b_2 y = b_3 \end{cases}$$

a_1, a_2, a_3
b_1, b_2, b_3 are known

x, y are unknown

$$x = \frac{\begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$$

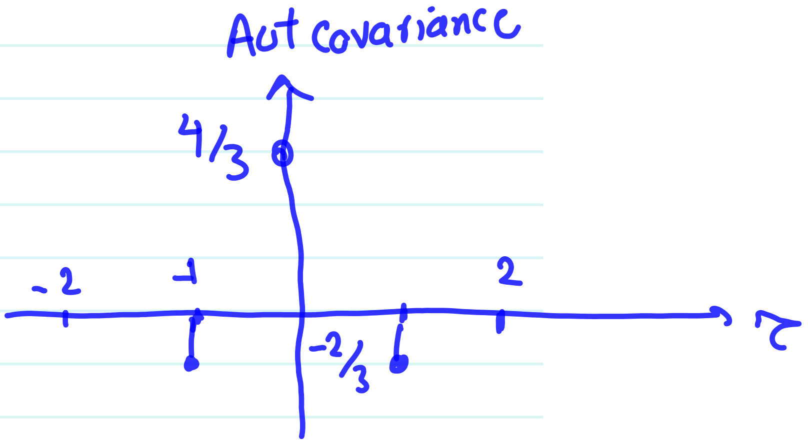
L

$$R_y(0) = \frac{\begin{vmatrix} 1 & 0.5 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0.5 \\ 0.5 & 1 \end{vmatrix}} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$R_y(1) = \frac{\begin{vmatrix} 1 & 1 \\ .5 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & .5 \\ .5 & 1 \end{vmatrix}} = \frac{-0.5}{3/4} = \frac{-1/2}{3/4} = \frac{-2}{3}$$

$$R_y(0) = \text{variance} = \frac{4}{3}$$

$$R_y(1) = \frac{-2}{3}$$



$\boxed{\tau=2}$ $$R_y(2) + 0.5 R_y(1) = \overbrace{R_{ye}(2)}^{0}$$

$$R_y(2) = -0.5 R_y(1)$$

$\boxed{\tau=3}$ $$R_y(3) = (-.5)\ R_y(2)$$

$$R_y(\tau) = \begin{cases} \frac{4}{3} & \tau = 0 \\ \frac{-2}{3} & \tau = 1 \\ \left(\frac{-1}{2}\right) R_y(\tau-1) & \tau > 1 \end{cases}$$

Ex MA(1)

$$y_t = e_t + 0.25\, e_{t-1} \qquad ; \quad \epsilon_t \sim WN(0,1)$$

$$R_y(\tau) = R_{ye}(\tau) + 0.25\, R_{ye}(\tau-1)$$

num = [1 .25]

den = [1 0]

$$\tau = 0 \longrightarrow R_y(0) = \underset{\underset{\underset{1}{\downarrow}\ \ \underset{1}{\downarrow}}{g(0)\cdot\sigma_e^2}}{\underset{\downarrow}{R_{ye}(0)}} + 0.25\, \underset{\underset{\underset{.25}{\downarrow}\ \ \underset{1}{\downarrow}}{g(1)\cdot\sigma_e^2}}{\underset{\downarrow}{R_{ye}(-1)}}$$

$$g_t = \delta_t + 0.25\, \delta_{t-1}$$

$$g(0) = \delta(0) + 0.25\, \underset{\underset{0}{\downarrow}}{\delta(-1)}$$

$$\boxed{g(0) = 1}$$

$$g(1) = \overset{\overset{0}{\uparrow}}{\delta(1)} + 0.25\, \delta(0)$$

$$\boxed{g(1) = 0.25}$$

$$g(2) = \delta(2) + .25\, \delta(1) = 0$$

$$g_t = \begin{cases} 1 & : \ t=0 \\ \boxed{0.25 \quad : \ t=1} & n_b \\ 0 & : \ Else \end{cases}$$

$R_y(0) = 1 + \frac{1}{16} = \frac{17}{16}$ = variance

$\boxed{\tau=1}$ $R_y(1) = \overset{0}{R_{ye}(1)} + .25\,\overset{g(0)\cdot\sigma_e^2 \,=\, 1\cdot 1}{R_{ye}(0)}$

$R_y(1) = .25$

$\boxed{\tau=2}$ $R_y(2) = \overset{0}{R_{ye}(2)} + .25\,\overset{0}{R_{ye}(1)}$

$$R_{ye}(\tau) = \begin{cases} \frac{17}{16} & : \tau = 0 \\ \boxed{\frac{1}{4} \quad : \quad \tau = 1} \\ 0 & : \tau > 1 \end{cases}$$

Ex: Variance of the following:

num=[1 0.25]

den=[1 0.5]

ARMA(1,1)

$y_t + 0.5y_{t-1} = e_t + 0.25e_{t-1}$ $e_t \sim WN(0,1)$

$\boxed{g_t + 0.5g_{t-1} = \delta_t + 0.25g_{t-1}}$ ⟶ $\boxed{g(0) = 1}$

$g(1) + .5g(0) = .25$

$\boxed{g(1) = -.25}$

$R_y(\tau) + 0.5R_y(\tau-1) = R_{ye}(\tau) + 0.25R_{ye}(\tau-1)$

$\nearrow R_y(1)$

$$\boxed{\tau=0}\quad R_y(0)+0.5\overbrace{R_y(-1)}=\underbrace{R_{ye}(0)}_{\underset{1\qquad 1}{g(0)\cdot\sigma_e^2}}+\underbrace{0.25\,R_y(-1)}_{\underset{-.25\qquad 1}{g(1)\cdot\sigma_e^2}}$$

$$\boxed{\#1\quad R_y(0)+0.5\,R_y(1)=1-\frac{1}{16}=\frac{15}{16}}$$

$$\boxed{\tau=1}\quad R_y(1)+0.5\,R_y(0)=\overbrace{R_{ye}(1)}^{\to 0}+0.25\overbrace{R_{ye}(0)}^{\overset{\nearrow 1}{g(0)\cdot\sigma_e^2}}$$

$$\boxed{\#2\quad R_y(1)+.5\,R_y(0)=\frac{1}{4}}$$

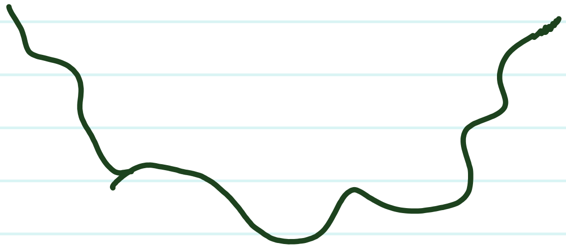
$$\begin{cases} R_y(0)+0.5R_y(1)=\frac{15}{16}\\ 0.5R_y(0)+R_y(1)=\frac{1}{4}\end{cases}\rightarrow\boxed{R_y(0)=\frac{13}{12}}\qquad R_y(1)=\frac{-7}{24}$$

Variance

$$y_t + a_1 y_{t-1} + \dots + a_{na} y_{t-na} = e_t + b_1 e_{t-1} + \dots + b_{nb} e_{t-na}$$

ARMA

$$(1 - a_1 q^{-1} - a_2 q^{-2} - \dots - a_{na} q^{-na})\, y_t = (1 - b_1 q^{-1} - \dots - b_{nb} q^{-nb})\, e_t$$



ARIMA(na, d , nb)

$$(1 - a_1 q^{-1} - \dots - \boxed{a_{na}} q^{-na})\underline{(1 - q^{-1})^{d}\, y_t} = (1 - b_1 q^{-1} - \dots \; b_{\boxed{nb}} q^{-nb})\, e_t$$

ARIMA(1, 1, 1)

$$(1 - a_1 q^{-1})(1 - q^{-1}) y_t = (1 - b_1 q^{-1}) e_t$$

$$(1 - q^{-1} - a_1 q^{-1} + a_1 q^{-2}) y_t = e_t - b_1 e_{t-1}$$

$$\overbrace{y_t - y_{t-1}}^{y'_t} - a_1 (\overbrace{y_{t-1} - y_{t-2}}^{y'_{t-1}}) = e_t - b_1 e_{t-1}$$

$$y'_t - a_1 y'_{t-1} = e_t - b e_{t-1}$$

ARIMA(1, 1, 1)

$$(1 - 0.5 q^{-1})(\underbrace{1 - \overset{\textcircled{1}}{q^{-1}}}) y_t = (1 - .25 q^{-1}) e_t \qquad e_t \sim wn(0, 1)$$

$$(1 - q^{-1} - 0.5 q^{-1} + 0.5 q^{-2}) y_t = e_t - 0.25 e_{t-1}$$

→ $y_t - 1.5y_{t-1} + 0.5y_{t-2} = e_t - 0.25e_{t-1}$

num = [1 -.25 0]

den = [1 -1.5 0.5]

Ex: $(1-0.5q^{-1})(1-q^{-1})^{\boxed{2}}\,y_t = (1-0.25q^{-1})\,e_t$

ARIMA(1, 2, 1)

$\boxed{e_t \sim WN(0,1)}$

$N = 1000$

$$\boxed{y_t - 2.5y_{t-1} + 2y_{t-2} - 0.5y_{t-3} = e_t - 0.25e_{t-1}}$$

Start date = Jan 1st 2018

Daily

num = [1 -.25 0 0]

den = [1 -2.5 2 -0.5]

$e_t \sim WN(\mu_e, \sigma_e^2)$


30
20
10
0
−10
0
Jan
2018
Jul
Jan
2019
Jul
Jan
2020
Jul