Q] Prove that the function $u = x^2 - y^2 + xy$ is a harmonic function. Determine it's harmonic conjugate.

Ans) Given $f^n$ = $u = x^2 - y^2 + xy$

To check $f^n$ is Harmonic or not:

$\therefore$ by laplace $eq^n$.

$$u_{xx} + u_{yy} = 0$$

$\therefore$ $\boxed{u_x = 2x - 0 + y}$, $u_{xx} = 2 + 0$

Simillary, $\boxed{u_y = 0 - 2y + x}$, $u_{yy} = -2 + 0$

$\therefore$ $u_{xx} = 2$, $u_{yy} = -2$

$\therefore$ $2 - 2 = 0$

now we will check then harmonic Conjugate.

$\therefore$ we use here C-R $eq^n$.

$$u_x = v_y \quad \& \quad u_y = -v_x$$

i.e = $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ — (1) & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ — (2)

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$\therefore$ To find harmonic Conjuge i.e. $v(x,y)$

$\therefore$ $$dv = \left(\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial v}{\partial y}\right) dy$$

$\therefore$ we have, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \to -u_y$ from eqn ②

Simillarly $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \to u_x$ from eqn ①

$u_x = 2x - 0 + y = 2x + y$

$u_y = 0 - 2y + x = -2y + x$

$\therefore$ $dv = -(-2y + x)dx + (2x + y)\,dy$

we want $v$ $\therefore$ integrate this.

$\therefore$ $\int dv = v = \int -(-2y + x)\,dx + (2x + y)\,dy$

$= \int ((2y - x)dx + (2x + y)dy)$

$= \int (2y - x)\,dx + \int (2x + y)\,dy$

$= 2xy - \frac{x^2}{2} + 2xy + \frac{y^2}{2} + c$

$= 4xy - \frac{x^2}{2} + \frac{y^2}{2} + c$

$v = -\frac{x^2}{2} + \frac{y^2}{2} + 4xy + c$

$\therefore$ Harmonic Conjugate $v(x, y)$ is

$$-\frac{x^2}{2} + \frac{y^2}{2} + 4xy + c$$