Matrix, Eigenvalues, Eigenvectors, and U-V Decomposition

1. Introduction to Matrices

Definition

A matrix is a rectangular array of numbers arranged in rows and columns.

- **Notation**: A matrix A with m rows and n columns is denoted as $A_{m \times n}$.

Key Concepts

- **Elements**: Individual entries in the matrix, denoted as a_{ij}, where i is the row index, and j is the column index.
- **Types of Matrices**:
  - **Square Matrix**: $m = n$.
  - **Identity Matrix**: Diagonal elements are 1, and others are 0.
  - **Diagonal Matrix**: Non-diagonal elements are 0.
  - **Zero Matrix**: All elements are 0.

Eigenvalues and Eigenvectors

1. Eigenvalues and Eigenvectors: Definition

- **Eigenvector**: A non-zero vector v that only changes by a scalar factor when a matrix A is applied to it:

$$A v = \lambda v$$

- **Eigenvalue**: The scalar λ associated with the eigenvector v.

2. How to Compute

- Solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

  - A: The matrix.
  - λ: Eigenvalue.
  - I: Identity matrix.
- Once λ is found, substitute it back into $(A - \lambda I) v = 0$ to find v.

3. Properties of Eigenvalues and Eigenvectors

- A $n \times n$ matrix has at most n eigenvalues.
- Eigenvalues can be real or complex.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.

4. Applications

- Principal Component Analysis (PCA): Dimensionality reduction in data science.
- Stability analysis in differential equations.
- Quantum mechanics (e.g., energy states of particles).

U-V Decomposition (Singular Value Decomposition, SVD)

1. Definition

- **SVD**: A factorization of a matrix A into three matrices:

$$A = U \Sigma V^T$$

  - U: Orthogonal matrix representing left singular vectors.
  - Σ: Diagonal matrix containing singular values (square roots of eigenvalues of $A^T A$).
  - V^T: Transpose of the orthogonal matrix V, representing right singular vectors.

2. Key Steps

1. Compute $A^T A$ and find its eigenvalues.
2. Compute singular values as the square roots of these eigenvalues.
3. Compute the eigenvectors of $A^T A$ (right singular vectors) and $A A^T$ (left singular vectors).

3. Applications

- **Dimensionality Reduction**: Extracting key features in datasets (e.g., image compression).
- **Data Compression**: Approximate matrix A with reduced dimensions.
- **Recommendation Systems**: Latent semantic analysis in collaborative filtering.

4. Relation to Eigenvalues and Eigenvectors

- Singular values (σ) are the square roots of the eigenvalues of $A^T A$.

- Eigenvectors of $A^T A$ and $A A^T$ form V and U, respectively.

Summary of Key Concepts

Concept	Definition	Application
Matrix	Rectangular array of numbers, with rows and columns.	Representing linear systems, transformations.
Eigenvalue	Scalar factor by which an eigenvector is scaled during transformation by a matrix.	PCA, stability analysis, quantum mechanics.
Eigenvector	Vector unchanged in direction under a matrix transformation.	Describes principal directions of data or systems.
SVD (U-V Decomposition)	Factorizes a matrix into orthogonal and diagonal components ($U \Sigma V^T$).	Data compression, dimensionality reduction, recommendation systems.

Conclusion

- **Matrices** provide the foundation for representing linear transformations.
- **Eigenvalues** and **eigenvectors** are crucial for understanding matrix properties and their impact in various domains.
- **SVD** (U-V decomposition) is a versatile tool for analyzing and simplifying data, with applications in machine learning and signal processing.