Introduction to Conditional Probability and Bayes' Theorem

1. Introduction to Probability

Definition

Probability measures the likelihood of an event occurring, expressed as a value between 0 (impossible) and 1 (certain).

Formula

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Key Terms

- **Event**: A possible outcome or combination of outcomes.
- **Sample Space (S)**: The set of all possible outcomes.

2. Conditional Probability

Definition

The probability of an event A occurring given that another event B has already occurred.

- **Notation**: $P(A \mid B)$.

Formula

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

- $P(A \cap B)$: The probability of both events A and B occurring.
- $P(B)$: The probability of event B.

Key Properties of Conditional Probability

- $P(A \mid B) \neq P(B \mid A)$: Conditional probabilities are not symmetric.
- **Chain Rule**:

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

Example

Scenario: A deck of 52 cards. What is the probability of drawing a king (A) given the card is a face card (B)?

$$P(B) \quad \text{¿} \frac{12}{52} \text{(face cards: kings, queens, jacks)},$$

$$P(A \cap B) \quad \text{¿} \frac{4}{52} \text{(all kings)},$$

$$P(A \mid B) \quad \text{¿} \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}.$$

Bayes' Theorem

Definition

A formula to find the probability of event A given B, using prior knowledge about A and B.

Formula

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}, P(B) \neq 0$$

- $P(A)$: Prior probability of A.
- $P(B)$: Total probability of B.
- $P(B \mid A)$: Likelihood of B given A.

Steps to Apply Bayes' Theorem

1. Determine $P(A)$, $P(B \mid A)$, and $P(B \mid A^c)$ (where A^c is the complement of A).
2. Calculate $P(B)$ using the **Total Probability Rule**:

$$P(B) = P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c).$$

3. Substitute values into Bayes' formula.

Example

Scenario: A medical test detects a disease with:

- $P(\text{Disease}) = 0.01$, $P(\text{No Disease}) = 0.99$.
- $P(\text{Positive Test} \mid \text{Disease}) = 0.95$ (True positive rate).

- $P(\text{Positive Test} \mid \text{No Disease}) = 0.05$ (False positive rate).

Goal: Find $P(\text{Disease} \mid \text{Positive Test})$.

1. Compute $P(\text{Positive Test})$:

$$P(\text{Positive Test}) = P(\text{Positive Test} \mid \text{Disease}) \cdot P(\text{Disease}) + P(\text{Positive Test} \mid \text{No Disease}) \cdot P(\text{No Disease}).$$

$$P(\text{Positive Test}) = (0.95 \cdot 0.01) + (0.05 \cdot 0.99) = 0.0095 + 0.0495 = 0.059.$$

2. Apply Bayes' Theorem:

$$P(\text{Disease} \mid \text{Positive Test}) = \frac{P(\text{Positive Test} \mid \text{Disease}) \cdot P(\text{Disease})}{P(\text{Positive Test})}.$$

$$P(\text{Disease} \mid \text{Positive Test}) = \frac{0.95 \cdot 0.01}{0.059} = 0.161.$$

Result: $P(\text{Disease} \mid \text{Positive Test}) \approx 16.1\%$.

Applications

- **Medical Diagnosis**: Identifying disease probabilities.
- **Spam Filtering**: Determining if an email is spam based on keywords.
- **Machine Learning**: Classification problems.

Summary of Key Concepts

Concept	Definition	Formula
Conditional Probability	Likelihood of A given B.	$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
Bayes' Theorem	Relates conditional probabilities to prior and likelihood probabilities.	$P(A \mid B) = \frac{P(B \mid A) \cdot P(A}{P(B)}$

Conclusion

- **Conditional probability** forms the foundation of understanding dependent events.
- **Bayes' Theorem** provides a powerful framework for updating probabilities based on new evidence, making it a cornerstone of modern applications like diagnostics and predictive modeling.