Kinetic Theory of Jet Dynamics in the Stochastic Barotropic and 2D Navier-Stokes Equations

Freddy Bouchet · Cesare Nardini · Tomás Tangarife

Abstract We discuss the dynamics of zonal (or unidirectional) jets for barotropic flows forced by Gaussian stochastic fields with white in time correlation functions. This problem contains the stochastic dynamics of 2D Navier-Stokes equation as a special case. We consider the limit of weak forces and dissipation, when there is a time scale separation between the inertial time scale (fast) and the spin-up or spin-down time (large) needed to reach an average energy balance. In this limit, we show that an adiabatic reduction (or stochastic averaging) of the dynamics can be performed. We then obtain a kinetic equation that describes the slow evolution of zonal jets over a very long time scale, where the effect of non-zonal turbulence has been integrated out. The main theoretical difficulty, achieved in this work, is to analyze the stationary distribution of a Lyapunov equation that describes quasi-Gaussian fluctuations around each zonal jet, in the inertial limit. This is necessary to prove that there is no ultraviolet divergence at leading order, in such a way that the asymptotic expansion is self-consistent. We obtain at leading order a Fokker–Planck equation, associated to a stochastic kinetic equation, that describes the slow jet dynamics. Its deterministic part is related to well known phenomenological theories (for instance Stochastic Structural Stability Theory) and to quasi-linear approximations, whereas the stochastic part allows to go beyond the computation of the most probable zonal jet. We argue that the effect of the stochastic part may be of huge importance when, as for instance in the proximity of phase transitions, more than one attractor of the dynamics is present.

Keywords Turbulence · Geostrophic turbulence · Non-equilibrium steady states · Stochastic partial differential equations · Stochastic averaging · Stochastic 2D Navier Stokes equations · Stochastic barotropic equations · Atmosphere dynamics

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1 Introduction

Turbulence in planetary atmospheres leads very often to self organization and to jet formation (please see for example the special issue of Journal of Atmospherical Science, named "Jets and Annular Structures in Geophysical Fluids" that contains the paper [24]). Those jet behaviors are at the basis of midlatitude atmosphere dynamics [75] and quantifying their statistics is fundamental for understanding climate dynamics. A similar self-organization into jets has also been observed in two-dimensional turbulence [8, 79].

In this paper, we study the jet formation problem in the simplest possible theoretical framework: the two-dimensional equations for a barotropic flow with a beta effect β. These equations, also called the barotropic quasi-geostrophic equations, are relevant for the understanding of large scale planetary flows [64]. When $\beta = 0$, they reduce to the two-dimensional Euler or Navier-Stokes equations. All the formal theoretical framework developed in this work could be easily extended to the equivalent barotropic quasi-geostrophic model (also called the Charney–Hasegawa–Mima equation), to the multi-layer quasi-geostrophic models or to quasi-geostrophic models for continuously stratified fluids [64].

The aim of this work is to consider an approach based on statistical mechanics. The equilibrium statistical mechanics of two-dimensional and quasi-two-dimensional turbulence is now well understood [9, 48, 54, 67, 68]: it explains self-organization and why zonal jets (east-west jets) are natural attractors of the dynamics. However, a drawback of the equilibrium approach is that the set of equilibrium states is huge, as it is parametrized by energy, enstrophy and all other inertial invariant of the dynamics. Moreover, whereas many observed jets are actually close to equilibrium states, some other jets, for instance Jupiter's ones, seem to be far from any equilibrium state. It is thus essential to consider a non-equilibrium statistical mechanics approach, taking into account forces and dissipation, in order to understand how real jets are actually selected by a non-equilibrium dynamics. In this work, we consider the case when the equations are forced by a stochastic force. We then use classical tools of statistical mechanics and field theory (stochastic averaging, projection techniques) in order to develop the kinetic theory from the original dynamics.

Any known relevant kinetic approach is associated with an asymptotic expansion where a small parameter is clearly identified. Our small parameter α [8, 9] is the ratio of an inertial time scale (associated to the jet velocity and domain size) divided by the forcing time scale or equivalently the dissipation time scale (the spin-up or spin-down time scale, needed to reach a statistically stationary energy balance). As discussed below, when the force is a white in time stochastic force, the average energy input rate is known a-priori, and the value of α can actually be expressed in terms of the control parameters. We call the limit $\alpha \ll 1$ the small force and dissipation limit.

For small α, the phenomenology is the following. At any times the velocity field is close to a zonal jet $\mathbf{v} \simeq U(y,t)\mathbf{e}_x$. $U(y,t)\mathbf{e}_x$ is a steady solution of the inertial dynamics. This zonal jet then evolves extremely slowly under the effect of the stochastic forces, of the turbulence (roughly speaking Reynolds stresses), and of the small dissipation. The non-zonal degrees of freedom have a turbulent motion which is strongly affected by the dominant zonal flow $U(y)$. These turbulent fluctuations are effectively damped by the non-normal linear dynamics close to the zonal jets. This inviscid damping of the fluctuations is accompanied by a direct transfer of energy from the fluctuations to the zonal flow. The understanding and the quantification of these processes is the aim of this work. The final result will be a kinetic equation that describes the slow dynamics of the zonal jets, where the effects of the non-zonal turbulence has been integrated out.

From a technical point of view, we start from the Fokker–Planck equation describing exactly the evolution of the Probability Density Function (to be more precise, Functional)

(PDF) of the potential vorticity field (or vorticity field for the two-dimensional Navier-Stokes equations). We make an asymptotic expansion of this Fokker–Planck equation, by carefully identifying the slow and fast processes and the order of magnitude of all fields. At a formal level we follow an extremely classical route, described for instance in the theory of adiabatic averaging of stochastic processes [33], also called stochastic reduction (see also [11, 30, 39] for counterpart in mathematics). This leads to a new Fokker–Planck equation that describes the slow evolution of the zonal jet U; such formal computations are tedious but involve no difficulties.

The main theoretical challenge is to check that the asymptotic expansion is self-consistent. We have to prove that all quantities appearing in the kinetic theory remain finite, keeping in such a way the order of magnitude initially assumed. In field problems, like this one, this is not granted as ultraviolet divergences often occur. For example, the slow Fokker–Planck equation involves a non-linear force that is computed from the Ornstein–Uhlenbeck process, corresponding to the linearized dynamics close to U stochastically forced. This process is characterized by the two-points correlation function dynamics, which is described by a Lyapunov equation. We then need to prove that this Lyapunov equation has a stationary solution for $\alpha = 0$, in order for the theory to be self-consistent.

A large part of our work deals with the analysis of this Ornstein–Uhlenbeck process, and the Lyapunov equation, in the limit $\alpha \to 0$. The fact that the Lyapunov equation has a stationary solution in the limit $\alpha \to 0$ is striking, as this is the inertial limit in which the equation for the non-zonal degrees of freedom contains a forcing term but no dissipation acts to damp them. The issue is quite subtle, as we prove that the vorticity-vorticity correlation function diverges point-wise, as expected based on enstrophy conservation. However we also prove that any integrated quantity, for instance the velocity-velocity autocorrelation function, reaches a stationary solution due to the effect of the zonal shear and a combination of phase mixing and global effects related to the non-normality of the linearized operator. As discussed thoroughly in the text, this allows to prove the convergence in the inertial limit of key physical quantities such as the Reynolds stress. Most of this analysis strongly relies on the asymptotic behavior of the linearized deterministic Euler equation, that we have studied for that purpose in a previous paper [7].

The linearized equation close to a base flow $U(y)$ has a family of trivial zero modes that correspond to any function of y only. The main hypothesis of our work is that the base flows U are linearly stable (they have no exponentially growing mode), but also that they have no neutral modes besides the trivial zonal ones. A linear dynamics with no mode at all may seem strange at first sight, but this is possible for an infinite dimensional non-normal linear operator. Actually, the tendency of turbulent flows to produce jets that expel modes has been recognized long ago [38]. Moreover, as discussed in [7], the absence of non-trivial modes is a generic situation for zonal flows, even if it is certainly not always the case.

We now describe the physics corresponding to the Fokker–Planck equation for the slow jets dynamics. It is equivalent to a stochastic dynamics for the velocity field that we call the kinetic equation. The kinetic equation has a deterministic drift $F(U)$ and a stochastic force which is a Gaussian process with white in time correlation function. $F(U)$ is the Reynolds stress that corresponds to the linearized dynamics close to the base flow U. If we consider the deterministic drift only, the equation is then related to Stochastic Structural Stability Theory (SSST) first proposed on a phenomenological basis by Farrell, Ioannou [1, 26, 27], for quasi-geostrophic turbulence. It is also related to a quasi-linear approximation plus an hypothesis where zonal average and ensemble average are assumed to be the same, as discussed in details in Sect. 2.4. More recently, an interpretation in terms of a second order closure (CE2) has also been given [50, 51, 74]. All these different forms of quasi-linear approximations have thoroughly been studied numerically, sometimes with stochastic forces

and sometimes with deterministic ones [22]. Very interesting empirical studies (based on numerical simulations) have been performed recently in order to study the validity of this type of approximation [51, 52, 62, 74], for the barotropic equations or for more complex dynamics. The SSST equations have also been used to compare theoretical prediction of the transition from a turbulence without a coherent structure to a turbulence with zonal jets [1, 63, 72]. Our first conclusion is that kinetic theory provides a strong support to quasi-linear types of approximations, in the limit of weak forces and dissipation $\alpha \ll 1$, in order to compute the attractors for the slow jet dynamics.

Beside the deterministic drift $F(U)$, the kinetic theory also predicts the existence of a small noise. Moreover it predicts the Fokker–Planck equation describing the full Probability Density Functional of the zonal jet U. This was not described, even phenomenologically, in any previous approach. This is an essential correction in many respects. First, it allows to describe the Gaussian fluctuations of the zonal jet. We also note that the probability of arbitrarily large deviations from the deterministic attractors can be computed from this Fokker–Planck equation. For instance, we may implement large deviations theory in the small noise limit. This is typically the kind of result that cannot be obtained from a standard cumulant expansion or closure based on moments. The possibility to compute the full PDF is extremely important especially in cases where the deterministic part of the dynamics has more than one attractor. This case actually happens, as noticed by Ioannou and Farrell (see Sect. 6). Then, our approach may predict the actual probability of each attractor and the transition probabilities from one attractor to the other. We remark anyway that, from a practical point of view, a further step forward should be done to obtain explicit results in this direction.

Our work has been clearly inspired by the kinetic theory of systems with long range interactions [3, 45, 61], described at first order by Vlasov equation and at next order by Lenard–Balescu equation. We have proven in previous works that this kinetic theory leads to algebraic relaxations and anomalous diffusion [6, 10, 13, 78]. The Euler equation is also an example of system with long range interaction, and there is a strong analogy between the 2D Euler and Vlasov equations. Quasilinear approximation for the relaxation towards equilibria of either the 2D Euler equation [15] or the point vortex dynamics [16, 17, 25] have actually been proposed and studied in the past.

All the above results started from deterministic dynamics, with no external forces. In order to prepare this work, and to extend these kinetic theories to the case with non-equilibrium stochastic forces, we have first considered the Vlasov equation with stochastic forces [56, 57]. A kinetic equation was then obtained, similar to the one in this paper, but with much less technical difficulties. The reason is that the Landau damping, which is then responsible for the inviscid relaxation, can be studied analytically through explicit formulas at the linear level [45, 61]. Non-linear Landau damping has also been recently established [55]. The kinetic equation for the stochastic dynamics [56] has the very interesting property to exhibit phase transitions and multistability, leading to a dynamics with random transitions from one attractor to the other [57]. We stress again that beside the formal structure, the main theoretical difficulty is the analysis of the Lyapunov equation which is a central object of these kinetic theories. In order to extend this approach to the barotropic equations, the current work discusses the first study of the Lyapunov equations for either the 2D Euler, the 2D Navier–Stokes or the barotropic flow equations in the inertial limit.

The barotropic flow equations include the 2D Stochastic Navier-Stokes equations as a special case. During last decade, a very interesting set of mathematical works have proved results related to the existence and uniqueness of invariant measures, their inertial limit, their ergodicity, the validity of the law of large number [12, 20, 28, 29, 35, 36, 40–42,

53, 70, 77] and of large deviations principles [34, 37, 73]. Some of the results are summarized in a recent book [44]. In order to be applied for real physical situations, these works should be extended in order to deal with a large scale dissipation mechanism, for instance large scale linear friction. We also note an interesting work considering the 2D Navier-Stokes equations forced by random vorticity patches [49]. In the case when there exists a scale separation between the forcing scales and the largest scale, our theory is probably very close to approach through Rapid Distortion Theory, or WKB Rapid Distortion Theory (see for instance [59, 60] for three dimensional flows and [58] for two dimensional near wall turbulence). The mathematical literature also contains a lot of interesting studies about stochastic averaging in partial differential equations [43], but we do not know any example dealing with the 2D Navier-Stokes equations or the barotropic flow equations.

In Sect. 2, we discuss the model, the energy and enstrophy balances, non-dimensional parameters, the quasi-linear approximation and the Fokker–Planck equations for the potential vorticity PDF. We develop the formal aspects of the kinetic theory, or stochastic averaging, in Sect. 3. This section ends with the derivation of the kinetic equation: the Fokker–Planck equations for the slow evolution of the jet (33), and its corresponding stochastic dynamics, Eq. (34) or (35). Section 4 comes back on energy and enstrophy balances. The analysis of the Lyapunov equation is performed in Sect. 5. We establish that it has a stationary solution in the inertial limit, discuss the divergence of the vorticity-vorticity correlation function, the nature of its singularity and how they are regularized in a universal way by a small linear friction or by a small viscosity. Section 6 discusses the importance of the stochastic part of the kinetic equation, explains how it predicts zonal jet PDF with jets arbitrarily far from Gaussian fluctuations, and stress the existence of cases with multiple attractors, phase transitions and bistability. We discuss open issues and perspectives in Sect. 7. The paper contains also three appendices reporting the most technical details of Sects. 3 and 5.

2 Quasi-geostrophic Dynamics and Fokker–Planck Equation

2.1 2D Barotropic Flow upon a Topography

We are interested in the non-equilibrium dynamics associated to the 2D motion of a barotropic flow, with topography h, on a periodic domain $\mathcal{D} = [0, 2\pi l_x L] \times [0, 2\pi L]$ with aspect ratio $1/l_x$:

$$\begin{cases} \dfrac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = -\lambda \omega - \nu_{n,d}(-\Delta)^n \omega + \sqrt{\sigma} \eta, \\[2mm] \mathbf{v} = \mathbf{e}_z \times \nabla \psi, \ \omega = \Delta \psi, \ q = \omega + h(y), \end{cases} \tag{1}$$

where ω, q, $\mathbf{v}$ and ψ are respectively the vorticity, potential vorticity, the non-divergent velocity, and the stream function. In the equations above, $\mathbf{r} = (x, y)$ are space vectors, x is the zonal coordinate and y the meridional one, λ is a Rayleigh (or Ekman) friction coefficient and $\nu_{n,d}$ is the hyper-viscosity coefficient (or viscosity for $n = 1$). All these fields are periodic $f(x + 2\pi l_x L, y) = f(x, y)$ and $f(x, y + 2\pi L) = f(x, y)$. We have introduced a forcing term η, assumed to be a white in time Gaussian noise with autocorrelation function $\mathbb{E}[\eta(\mathbf{r}_1, t_1)\eta(\mathbf{r}_2, t_2)] = C(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2)$, where C is an even positive definite function, periodic with respect to x and y. As discussed below, σ is the average energy input rate. When $h = 0$, the 2D barotropic flow equations are the 2D-Navier-Stokes equations. Here

we assume that the noise autocorrelation function C is translationally invariant in both direction. The fact that C is zonally invariant is important for some of the computations in this paper. However the hypothesis that C is meridionally invariant is not important and generalization to non-meridionally invariant forcing would be straightforward.

Dynamical Invariants of Perfect Barotropic Flows   Equations (1) with $\lambda = \sigma = v_{n,d} = 0$ describe a perfect barotropic flow. The equations are then Hamiltonian and they conserve the energy

$$\mathcal{E}[q] = \frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \mathbf{v}^2 = -\frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, (q - h)\psi, \tag{2}$$

and the Casimir functionals

$$\mathcal{C}_s[q] = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, s(q), \tag{3}$$

for any sufficiently regular function s. The potential enstrophy

$$\mathcal{C}_2[q] = \frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, q^2$$

is one of the invariants. When $h = 0$, the perfect barotropic flow equations clearly reduce to the 2D Euler equations.

Averaged Energy Input Rate and Non-dimensional Equations   Because the force is a white in time Gaussian process, we can compute a-priori the average, with respect to noise realizations, of the input rate for quadratic invariants. Without loss of generality, we assume that

$$-2\pi^2 l_x L^2 (\Delta^{-1} C)(\mathbf{0}) = 1,$$

where Δ^{-1} denotes the inverse Laplacian operator; indeed, multiplying C by an arbitrary positive constant amounts at renormalizing σ. Then, with the above choice, the average energy input rate is σ and the average energy input rate by unit of mass is $\varepsilon = \sigma/4\pi^2 l_x L^2$. Moreover, the average potential enstrophy input rate is given by

$$2\pi^2 l_x L^2 C(\mathbf{0})\sigma.$$

We consider the energy balance for Eq. (1), with $E = \mathbb{E}[\mathcal{E}[q]]$:

$$\frac{dE}{dt} = -2\lambda E - v_{n,d} H_n + \sigma, \tag{4}$$

where $H_n = -\mathbb{E}[\int_{\mathcal{D}} \psi(-\Delta)^n \omega]$. For most of the turbulent flows we are interested in, the ratio $2\lambda E/v_{n,d} H_n$ will be extremely large (viscosity is negligible for energy dissipation). Then, in a statistically stationary regime, the approximate average energy is $E \simeq \sigma/2\lambda$. We perform a transformation to non-dimensional variables such that in the new units the domain is $\mathcal{D} = [0, 2\pi l_x) \times [0, 2\pi)$ and the approximate average energy is 1. This is done introducing a non-dimensional time $t' = t/\tau$ and a non-dimensional spatial variable $\mathbf{r}' = \mathbf{r}/L$ with $\tau = L^2 \sqrt{2\lambda/\sigma}$. The non-dimensional physical variables are $q' = \tau q$, $\mathbf{v}' = \tau \mathbf{v}/L$, $h' = \tau h$, and the non-dimensional parameters are defined by

$$\alpha = \lambda \tau = L^2 \sqrt{\frac{2\lambda^3}{\sigma}} = \frac{L}{2\pi} \sqrt{\frac{2\lambda^3}{\varepsilon l_x}},$$

$\nu_n = \nu_{n,d}\tau/L^{2n} = \nu_{n,d}\sqrt{2\lambda/\sigma}/L^{2n-2}$. We consider a rescaled stochastic Gaussian field η' with $\mathbb{E}[\eta'(\mathbf{r}'_1, t'_1)\eta'(\mathbf{r}'_2, t'_2)] = C'(\mathbf{r}'_1 - \mathbf{r}'_2)\delta(t'_1 - t'_2)$ with $C'(\mathbf{r}') = L^4 C(\mathbf{r})$. Performing the non-dimensionalization procedure explained above, the barotropic equations read

$$\begin{cases} \dfrac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = -\alpha\omega - \nu_n(-\Delta)^n\omega + \sqrt{2\alpha}\eta, \\[2mm] \mathbf{v} = \mathbf{e}_z \times \nabla\psi, \ \omega = \Delta\psi, \ q = \omega + h(y), \end{cases} \tag{5}$$

where, for easiness in the notations, we drop here and in the following the primes. We note that in non-dimensional units, α represents an inverse Reynolds number based on the large scale dissipation of energy and ν_n is an inverse Reynolds number based on the viscosity or hyper-viscosity term that acts predominantly at small scales. The non dimensional energy balance is

$$\frac{dE}{dt} = -2\alpha E - \nu_n H_n + 2\alpha. \tag{6}$$

2.2 2D Barotropic Flow with a Beta-Effect

For a 2D flow on a rotating sphere, the Coriolis parameter depends on the latitude. For mid-latitudes, such a dependence can be approached by a linear function, the so-called beta effect: this corresponds to consider a linear topography function $h(y) = \beta_d y$. In a periodic geometry, such a function is meaningless, and so is the associated potential vorticity $q = \omega + h$. However, consistent equations can be written for the vorticity ω and the velocity:

$$\begin{cases} \dfrac{\partial\omega}{\partial t} + \mathbf{v}\cdot\nabla\omega + \beta v_y = -\alpha\omega - \nu_n(-\Delta)^n\omega + \sqrt{2\alpha}\eta, \\[2mm] \mathbf{v} = \mathbf{e}_z \times \nabla\psi, \ \omega = \Delta\psi \end{cases} \tag{7}$$

Observe that, in this case, we have directly written the equations in a non-dimensional form; the parameter β as a function of dimensional quantities is given by

$$\beta = L^3\sqrt{\frac{2\lambda}{\sigma}}\beta_d = \left(\frac{L}{L_\beta}\right)^2,$$

where β_d is the dimensional beta effect, and we have defined a Rhines scale $L_\beta = \sqrt{L/\tau\beta_d}$ based on the large scale velocity $U = L/\tau$.

In the case of the barotropic model, we note that an alternative choice for the non-dimensional parameters is often used (see for instance [21, 31]), based on a Rhines scales built with an estimate of the RMS velocity $U_{\text{rms}} = \sqrt{\varepsilon/\lambda}$ leading to $L_{\text{Rhines}} = (\varepsilon/\beta^2\lambda)^{1/4}$. Such a choice, may be more relevant in regimes where the jet width is actually of the order of magnitude of this Rhine scale. This choice formally leads to same non-dimensional equations, but the expression of the non-dimensional quantities as a function of the dimensional ones are different. As far as the following theory is concerned, the best choice of time unit is the relaxation time of non-zonal perturbations. This time scale may be regime dependent.

The perfect barotropic equations ($\alpha = \nu_n = 0$), in doubly periodic geometry, with beta effect ($\beta \neq 0$), conserve only two quantities: the kinetic energy

$$\mathcal{E}[\omega] = -\frac{1}{2}\int_{\mathcal{D}} d\mathbf{r}\,\omega\psi, \tag{8}$$

and the enstrophy

$$\mathcal{Z}[\omega] = \frac{1}{2} \int_{\mathcal{D}} d\mathbf{r}\, \omega^2. \tag{9}$$

For $\beta = 0$, the beta-plane equations reduce to the 2D-Navier Stokes equations discussed in previous section.

For sake of simplicity, in the following, we consider the case of a viscous dissipation $n = 1$, and we denote the viscosity $\nu = \nu_1$. One of the consequences of our theoretical work is that, in the limit $\alpha \ll 1$ and $\nu_n \ll \alpha$, the results will be independent on ν_n for generic cases, and at leading order in the small parameters.

2.3 Decomposition into Zonal and Non-zonal Flow

The physical phenomenon we are interested in is the formation of large scales structures (jets and vortices). Such large scale features are slowly dissipated, mainly due to the friction α. This dissipation is balanced by Reynolds stresses due to the transfer of energy from the forcing scale until the scale of these structures. This phenomenology is commonly observed in planetary atmospheres (Earth, Jupiter, Saturn) and in numerical simulations. For the barotropic equations (5) and (7), the regime corresponding to this phenomenology is observed when $\alpha \ll 1$ (time scale for forcing and dissipation $1/\lambda$ much larger than time scale for the inertial dynamics τ) and when $2\lambda E/\nu_{n,d}H_n \gg 1$ (turbulent regime). For this reason, we study in the following the limit $\nu_n \to 0$, $\alpha \to 0$ ($\alpha \ll 1$ and $\nu_n \ll \alpha$).

For sake of simplicity, we consider below the case when the zonal symmetry (invariance by translation along x) is not broken. Then the large scale structure will be a zonal jet characterized by either a zonal velocity field $\mathbf{v}_z = U(y)\mathbf{e}_x$ or its corresponding zonal potential vorticity $q_z(y) = -U'(y) + h(y)$. For reasons that will become clear in the following discussion (we will explain that this is a natural hypothesis and prove that it is self-consistent in the limit $\alpha \ll 1$), the perturbation to this zonal velocity field is of order $\sqrt{\alpha}$.

Defining the zonal projection $\langle . \rangle$ as

$$\langle f \rangle(y) = \frac{1}{2\pi l_x} \int_0^{2\pi l_x} dx\, f(\mathbf{r}),$$

the zonal part of the potential vorticity will be denoted by $q_z \equiv \langle q \rangle$; the rescaled non-zonal part of the flow $q_m = \omega_m$ is then defined through the decomposition

$$q(\mathbf{r}) = q_z(y) + \sqrt{\alpha}\omega_m(\mathbf{r}).$$

The zonal and non-zonal velocities are then defined through $U'(y) = -q_z(y) + h(y)$, the periodicity condition, and

$$\mathbf{v}(\mathbf{r}) = U(y)\mathbf{e}_x + \sqrt{\alpha}\mathbf{v}_m(\mathbf{r}).$$

We now project the barotropic equation (5) into zonal

$$\frac{\partial q_z}{\partial t} = -\alpha \frac{\partial}{\partial y}\langle v_m^{(y)}\omega_m \rangle - \alpha\omega_z + \nu \frac{\partial^2 \omega_z}{\partial y^2} + \sqrt{2\alpha}\eta_z \tag{10}$$

and non-zonal part

$$\frac{\partial \omega_m}{\partial t} + L_U[\omega_m] = \sqrt{2}\eta_m - \sqrt{\alpha}\mathbf{v}_m.\nabla\omega_m + \sqrt{\alpha}\langle \mathbf{v}_m.\nabla\omega_m \rangle, \tag{11}$$

where $\eta_z = \langle \eta \rangle$ is a Gaussian field with correlation function $\mathbb{E}[\eta_z(y_1, t_1)\eta_z(y_2, t_2)] = C_z(y_1 - y_2)\delta(t_1 - t_2)$ with $C_z = \langle C \rangle$, $\eta_m = \eta - \langle \eta \rangle$ is a Gaussian field with correlation function $\mathbb{E}[\eta_m(\mathbf{r}_1, t_1)\eta_m(\mathbf{r}_2, t_2)] = C_m(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2)$ with $C_m = C - \langle C \rangle$, and where

$$L_U[\omega_m] = U(y)\frac{\partial \omega_m}{\partial x} + q_z'(y)\frac{\partial \psi_m}{\partial x} + \alpha \omega_m - \nu \Delta \omega_m, \quad \omega_m = \Delta \psi_m. \tag{12}$$

Observe that the cross correlation between η_z and η_m is exactly zero, due to the translational invariance along the zonal direction of C. Moreover, in the previous equations, $\frac{\partial \omega_m}{\partial t} + L_U[\omega_m] = 0$ is the deterministic dynamics linearized close to the zonal base flow U, whose corresponding potential vorticity is q_z. In the following we will also consider the operator

$$L_U^0[\omega_m] = U(y)\frac{\partial \omega_m}{\partial x} + q_z'(y)\frac{\partial \psi_m}{\partial x}, \tag{13}$$

which is the operator for the linearized inertial dynamics (with no dissipation) close to the base flow U. In all the following, we assume that the base flow $U(y)$ is linearly stable (the operator L_U^0 has no unstable mode). We remark that the action of L_U^0 on zonal functions $f(y)$ is trivial: any zonal function is a neutral mode of L_U^0. We will thus consider the operator L_U^0 acting on non-zonal functions only (functions which have a zero zonal average). We assume that L_U^0 restricted to non-zonal functions has no normal mode at all (which is possible as L_U^0 is a non-normal operator). This hypothesis will be important for the results presented in Sect. 5. This hypothesis may seem restrictive, but as explained in [7] this is a generic case, probably the most common one.

The equation for the zonal potential vorticity evolution (10) can readily be integrated in order to get an equation for the zonal flow evolution

$$\frac{\partial U}{\partial t} = \alpha \langle v_m^{(y)} \omega_m \rangle - \alpha U + \nu \frac{\partial^2 U}{\partial y^2} + \sqrt{2\alpha}\eta_U,$$

where η_U is a Gaussian field with correlation function $\mathbb{E}[\eta_U(y_1, t_1)\eta_U(y_2, t_2)] = C_U(y_1 - y_2)\delta(t_1 - t_2)$ with $\frac{d^2 C_U}{dy_1^2}(y_1 - y_2) = -C_z(y_1 - y_2)$.

We see that the zonal potential vorticity is coupled to the non-zonal one through the zonal average of the advection term. If our rescaling of the equations is correct, we clearly see that the natural time scale for the evolution of the zonal flow is $1/\alpha$. By contrast the natural time scale for the evolution of the non-zonal perturbation is one. These remarks show that in the limit $\alpha \ll 1$, we have a time scale separation between the slow zonal evolution and a rapid non-zonal evolution. Our aim is to use this remark in order to (i) describe precisely the stochastic behavior of the Reynolds stress in this limit by integrating out the non-zonal turbulence, (ii) prove that our rescaling of the equations and the time scale separation hypothesis are self-consistent.

The term $\sqrt{\alpha}\mathbf{v}_m.\nabla\omega_m - \sqrt{\alpha}\langle \mathbf{v}_m.\nabla\omega_m \rangle$ describes the interactions between non-zonal degrees of freedom (sometimes called eddy-eddy interactions). If our rescaling is correct, these terms should be negligible at leading order. Neglecting these terms leads to the so called quasi-linear dynamics, which is described in next section.

2.4 The Quasi-linear Dynamics

By neglecting the interactions between non-zonal degrees of freedom in (11), one obtains the quasi-linear approximation of the barotropic equations:

$$\begin{cases} \dfrac{\partial q_z}{\partial t} = -\alpha \dfrac{\partial}{\partial y} \langle v_m^{(y)} \omega_m \rangle - \alpha \omega_z + \nu \dfrac{\partial^2 \omega_z}{\partial y^2} + \sqrt{2\alpha}\,\eta_z, \\[2ex] \dfrac{\partial \omega_m}{\partial t} + L_U[\omega_m] = \sqrt{2}\,\eta_m \end{cases} \tag{14}$$

In two-dimensional and geostrophic turbulence, the fluctuations around a given mean flow are often weak, so this approximation is natural in this context. If our rescaling is relevant, this corresponds to the limit $\alpha \ll 1$.

The approximation leading to the quasi-linear dynamics amounts at suppressing some of the triads interactions. As a consequence, the inertial quasi-linear dynamics has the same quadratic invariants as the initial barotropic equations: the energy and potential enstrophy.

As discussed in the previous paragraphs, and as can be seen in (14), in the regime $\alpha \ll 1$, it seems natural to assume that there is a separation between the time scale for the fluctuation dynamics (second equation of (14)), which is of order 1, and the time scale for the zonal flow (first equation of (14)), which is of order $1/\alpha$. At leading order, the evolution of ω_m can be considered with the zonal velocity profile held fixed. The second equation of (14), with a fixed velocity profile, is then linear. Thus, denoting by $\mathbb{E}$ the average over the realization of the noise η_m for fixed U, the distribution of ω_m is completely characterized by the two-points correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)]$. The evolution of g is given by the so-called Lyapunov equation, which is obtained by applying the Itō formula (with U fixed):

$$\frac{\partial g}{\partial t} + L_U^{(1)} g + L_U^{(2)} g = 2C_m,$$

where $L_U^{(1)}$ (resp. $L_U^{(2)}$) is the linearized operator L_U (12) acting on the variable $\mathbf{r}_1$ (resp. $\mathbf{r}_2$).

If we assume that this Gaussian process has a stationary distribution, then considering again the time scale separation $\alpha \ll 1$, one is led to the natural hypothesis that at leading order

$$\frac{\partial q_z}{\partial t} = -\alpha \frac{\partial}{\partial y} \mathbb{E}_U \langle v_m^{(y)} \omega_m \rangle - \alpha \omega_z + \nu \frac{\partial^2 \omega_z}{\partial y^2} + \sqrt{2\alpha}\,\eta_z, \tag{15}$$

where $\mathbb{E}_U \langle v_m^{(y)} \omega_m \rangle$ is the average over the stationary Gaussian process corresponding to the dynamics given by the second equation of (14) at fixed U. In our problem, invariant by translation along the zonal direction, we note that $-\mathbb{E}_U \langle v_m^{(y)} \omega_m \rangle$ is the divergence of a Reynolds stress [65]. We call this equation the kinetic equation, because it is the analogous of the kinetic equation obtained in a deterministic context in the kinetic theory of plasma physics [45, 61], systems with long-range interactions [5, 6, 10] and the 2D Euler equations [15, 16]. The time scale separation hypothesis, leading to consider the asymptotic solution of the Lyapunov equation is referred to as the Bogolyubov hypothesis in classical kinetic theory [45, 61].

Another model related to the quasi-linear dynamics is

$$
\begin{cases}
\dfrac{\partial q_z}{\partial t} = -\alpha \dfrac{\partial}{\partial y} \mathbb{E}_m \big[\langle v_m^{(y)} \omega_m \rangle \big] - \alpha \omega_z + \nu \dfrac{\partial^2 \omega_z}{\partial y^2} + \sqrt{2\alpha}\, \eta_z \\[2ex]
\dfrac{\partial g}{\partial t} + \big(L_U^{(1)} + L_U^{(2)} \big) g = 2 C_m,
\end{cases}
\tag{16}
$$

where $\mathbb{E}_m$ is the average corresponding to the Gaussian process with two-point correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}_m[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)]$. By contrast to the kinetic equation (15), in Eq. (16) g and U (or q_z) evolve simultaneously. The Reynolds stress divergence $-\mathbb{E}_m[\langle v_m^{(y)} \omega_m \rangle]$, or the gradient of Reynolds stress divergence $-\frac{\partial}{\partial y} \mathbb{E}_m[\langle v_m^{(y)} \omega_m \rangle]$, can be computed as a linear transform of g. This model, first proposed by Farrell and Ioannou [1, 26, 27], is referred to as Structural Stochastic Stability Theory (SSST). Farrell and Ioannou have also used this model in a deterministic context [22], where the correlation C_m is then added phenomenologically in order to model the effect of small scale turbulence (the effect of the neglected non linear terms in the quasi-linear dynamics). This model, or an analogous one in a deterministic context, is also referred to as CE2 (Cumulant expansion of order 2) [50–52].

As previously observed, the second equation in (16) can be deduced from the second equation of (14) as an average over the realizations of the non-zonal part of the noise η_m, with U held fixed. It can be considered as an approximation of the quasilinear dynamics, where the instantaneous value of the non-linear term $\langle v_m^{(y)} \omega_m \rangle$ is replaced by its ensemble average, with U held fixed. We see no way how this could be relevant without a good time scale separation $\alpha \ll 1$. With a good time scale separation, both the quasilinear dynamics (14) and SSST-CE2 (16) are likely to be described at leading order by the kinetic equation (15). We see no reason how SSST dynamics could be relevant to describe the quasilinear dynamics beyond the limit where it is an approximation of the kinetic equation. However, from a practical point of view, SSST-CE2 is extremely interesting as it provides a closed dynamical system which may be extremely easily computed numerically (by comparison to both direct numerical simulations, the quasilinear dynamics or the kinetic equation) in order to obtain the average zonal velocity and its Gaussian corrections. It is probably the best tool to understand qualitatively the dynamics of barotropic flows in the limit of small forces and dissipations.

2.5 Fokker–Planck Equation

We will use the remark that we have a time scale separation between zonal and non-zonal degrees of freedom in order to average out the effect of the non-zonal turbulence. This corresponds to treat the zonal degrees of freedom adiabatically. This kind of problems are described in the theoretical physics literature as adiabatic elimination of fast variables [33] or stochastic averaging in the mathematical literature. Our aim is to perform the stochastic averaging of the barotropic flow equation and to find the equation that describes the slow evolution of zonal flows.

There are several ways to perform this task, for instance using path integral formalism, working directly with the stochastic equations, using Mori–Zwanzig or projection formalisms, and so on. A typical way in turbulence is to write the hierarchy of equations for the moments of the hydrodynamic variables. As discussed in the introduction and in Sect. 6 this approach may lead to interesting results sometimes, but it can sometimes be misleading, and when it works it can describe quasi-Gaussian fluctuations at best. The formalism we find the simplest, the more convenient, and the more amenable to a clear mathematical study is the one based on the (functional) Fokker–Planck equations.

At a formal level, we will perform adiabatic reduction of the Fokker–Planck equation using the classical approach, as described for instance in Gardiner textbook [33]. Whereas Gardiner treats examples with a finite number of degrees of freedom, we are concerned in this paper with a field problem. At a formal level, this does not make much difference and the formalism can be directly generalized to this case. At a deeper level this however hides some mathematical difficulties, some of them being related to ultraviolet divergences. We will show that such ultraviolet divergences do not occur, at least for the quantities arising at leading order.

The starting point of our analysis is to write the Fokker–Planck equation associated with Eqs. (10) and (11). We consider the time dependent probability distribution function $P_t[q_z, \omega_m]$ for the potential vorticity field and, for easiness in the notations, we omit time t in the following. The distribution $P[q_z, \omega_m]$ is a functional of the two fields q_z and ω_m and is a formal generalization of the probability distribution function for variables in finite dimensional spaces. From the stochastic dynamics (10)–(11), using standard techniques (functional Itō calculus), one can prove that P solves the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \mathcal{L}_0 P + \sqrt{\alpha} \mathcal{L}_n P + \alpha \mathcal{L}_z P, \tag{17}$$

where the leading order term is

$$\mathcal{L}_0 P \equiv \int d\mathbf{r}_1 \frac{\delta}{\delta \omega_m(\mathbf{r}_1)} \left[L_U[\omega_m](\mathbf{r}_1) P + \int d\mathbf{r}_2 \, C_m(\mathbf{r}_1 - \mathbf{r}_2) \frac{\delta P}{\delta \omega_m(\mathbf{r}_2)} \right], \tag{18}$$

the zonal part of the perturbation is

$$\mathcal{L}_z P \equiv \int dy_1 \frac{\delta}{\delta q_z(y_1)} \left[\left(\frac{\partial}{\partial y} \langle v_m^{(y)} \omega_m \rangle(y_1) + \omega_z(y_1) - \frac{\nu}{\alpha} \Delta \omega_z(y_1) \right) P \right. $$
$$\left. + \int dy_2 \, C_z(y_1 - y_2) \frac{\delta P}{\delta q_z(y_2)} \right], \tag{19}$$

and the nonlinear part of the perturbation is

$$\mathcal{L}_n P \equiv \int d\mathbf{r}_1 \frac{\delta}{\delta \omega_m(\mathbf{r}_1)} \left[\left(\mathbf{v}_m . \nabla \omega_m(\mathbf{r}_1) - \langle \mathbf{v}_m . \nabla \omega_m(\mathbf{r}_1) \rangle \right) P \right]. \tag{20}$$

In previous equations, $\frac{\delta}{\delta q(y)}$ are functional derivatives with respect to the field q at point y. We observe that $\mathcal{L}_n$ is a linear operator which describes the effect of the non-linear terms in the fluid mechanics equations.

The operator $\mathcal{L}_0$ is the Fokker–Planck operator that corresponds to the linearized dynamics close to the zonal flow U, forced by a Gaussian noise, white in time and with spatial correlations C_m. This Fokker–Planck operator acts on the non-zonal variables only and depends parametrically on U. This is in accordance with the fact that on time scales of order 1, the zonal flow does not evolve and only the non-zonal degrees of freedom evolve significantly. It should also be remarked that this term contains dissipation terms of order α and ν. These dissipation terms could have equivalently been included in $\mathcal{L}_z$, but we keep them on $\mathcal{L}_0$ for later convenience. However it will be an important part of our work to prove that in the limit $\nu \ll \alpha \ll 1$, at leading order, the operator $\mathcal{L}_0$ with or without these dissipation terms will have the same stationary distributions. This is a crucial point that will be made clear in Sect. 5.

At order $\sqrt{\alpha}$, the nonlinear part of the perturbation $\mathcal{L}_n$ contains the non-linear interactions between non-zonal degrees of freedom. We will prove that at leading order $\mathcal{L}_n$ has no effect, justifying the quasilinear approach. This is non-trivial as $\mathcal{L}_n$ formally arise at an order in $\sqrt{\alpha}$ lower than $\mathcal{L}_z$. At order α, the zonal part of the perturbation $\mathcal{L}_z$ contains the terms that describe the coupling between the zonal and non-zonal flow, the dynamics due to friction acting on zonal scales and the zonal part of the stochastic forces.

3 Stochastic Averaging and the Fokker–Planck Equation for the Slow Evolution of Zonal Velocity Profiles

In this section, we formally perform the perturbative expansion in power of $\sqrt{\alpha}$, for the Fokker–Planck equation (17). The aim of the computation is to obtain a new equation describing the slow zonal part only. We will compute explicitly only the leading order relevant terms. This formal computation will make sense only if the Gaussian process corresponding to a stochastically forced linearized equation has a stationary solution. That last point is thus the real physical issue and the most important mathematical one. We consider this question in Sect. 5.

3.1 Stationary Distribution of the Fast Non-zonal Variables

At leading order when α is small, we have

$$\frac{\partial P}{\partial t} = \mathcal{L}_0 P. \tag{21}$$

Let us first consider the special case when the zonal flow is a deterministic one: $P[q, \omega_m] = \delta(q - q_z)Q(\omega_m)$. Then, (21) is a Fokker–Planck equation corresponding to the dynamics of the non-zonal degrees of freedom only. It is the Fokker–Planck equation associated to the barotropic equations linearized around a fixed base flow with zonal velocity U and forced by a Gaussian noise delta correlated in time and with spatial correlation function C_m:

$$\frac{\partial \omega_m}{\partial t} + L_U[\omega_m] = \sqrt{2}\eta_m, \qquad \mathbb{E}\big[\eta_m(\mathbf{r}_1, t_1)\eta_m(\mathbf{r}_2, t_2)\big] = C_m(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2) \tag{22}$$

When U is held fixed, this is a linear stochastic Gaussian process, or Ornstein-Uhlenbeck process. Thus, it is completely characterized by the two-points correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)]$ (the average $\mathbb{E}[\omega_m]$ is equal to zero, here the average $\mathbb{E}$ refers to an average over the realization of the noise η_m, for fixed U). The evolution of g is given by the so-called Lyapunov equation, which is obtained by applying the Itō formula to (22):

$$\frac{\partial g}{\partial t} + L_U^{(1)}g + L_U^{(2)}g = 2C_m, \tag{23}$$

where $L_U^{(1)}$ (resp. $L_U^{(2)}$) is the linearized operator L_U (12) acting on the variable $\mathbf{r}_1$ (resp. $\mathbf{r}_2$).

For fixed α and ν, the linear operator L_U (12) is dissipative. If the linearized dynamics is stable, then the Ornstein-Uhlenbeck will have a stationary distribution. However, as we are computing an asymptotic expansion for small α and ν, we need to consider the limit $\nu \to 0$

and $\alpha \to 0$ of this stationary distribution. In order to do that we will consider the Lyapunov equation

$$\frac{\partial g}{\partial t} + L_U^{0(1)} g + L_U^{0(2)} g = 2C_m, \tag{24}$$

where L_U^0 is the inertial linear dynamics (13). In this section, we assume that the corresponding Gaussian process has a stationary distribution. This assumption may seem paradoxical as there is no dissipation in L_U^0; however, it will be proved in Sect. 5 that this is correct. We denote by $g^\infty[q_z]$ the stationary two-points correlation function, $g^\infty[q_z](\mathbf{r}_1, \mathbf{r}_2) = \lim_{t \to \infty} g(\mathbf{r}_1, \mathbf{r}_2, t)$, and by $(g^\infty)^{-1}[q_z](\mathbf{r}_1, \mathbf{r}_2)$ its inverse.[1] Then, the stationary distribution of the linear equation (22) close to the base flow with potential vorticity q_z is

$$G[q_z, \omega_m] = \frac{1}{\sqrt{\det[q_z]}} \exp\left(-\frac{1}{2} \int d\mathbf{r}_1 \, d\mathbf{r}_2 \, \omega_m(\mathbf{r}_1) \big(g^\infty\big)^{-1}[q_z](\mathbf{r}_1, \mathbf{r}_2) \omega_m(\mathbf{r}_2)\right), \tag{25}$$

where the normalization constant depends on $\det q_z$, the determinant of $2\pi g^\infty[q_z]$. The stationary solution to (21) with any initial condition $P[q, \omega_m] = \delta(q - q_z) Q(\omega_m)$ is thus given by $\delta(q - q_z) G[q_z, \omega_m]$.

For any q_z, the average of any observable $A[\omega_m]$ over the stationary Gaussian distribution will be denoted

$$\mathbb{E}_U[A] \equiv \int \mathcal{D}[\omega_m] A[\omega_m] G[q_z, \omega_m]$$

and, for any two observables $A[\omega_m]$ and $B[\omega_m]$, the correlation of A at time t with B at time zero will be denoted

$$\mathbb{E}_U\big[A(t)B(0)\big] \equiv \int \mathcal{D}[\omega_m] A[\omega_m] e^{t\mathcal{L}_0} \big[B[\omega_m] G[q_z, \omega_m]\big]. \tag{26}$$

The covariance of A at time t with B at time zero is denoted by

$$\mathbb{E}_U\big[[A(t)B(0)]\big] \equiv \mathbb{E}_U\big[(A(t) - \mathbb{E}_U[A])(B(0) - \mathbb{E}_U[B])\big]. \tag{27}$$

As already pointed out, we assume that for all q_z of interest the Gaussian process corresponding to the inertial linear dynamics (13) has a stationary distribution. Then, the operator $\exp(t\mathcal{L}_0)$ has a limit for time going to infinity. We can then define $\mathcal{P}$ the projector on the asymptotic solutions of this equation

$$\mathcal{P}[P] = \lim_{t \to \infty} \exp(t\mathcal{L}_0) P$$

for any P. We observe that, because $\mathcal{L}_0$ does not act on zonal degrees of freedom, $\mathcal{P}[\delta(q_z - q)Q(\omega_m)] = \delta(q_z - q)G[q_z, \omega_m]$ for any Q. Moreover, using the linearity of $\mathcal{L}_0$, we conclude that for any P

$$\mathcal{P}[P] = G[q_z, \omega_m] \int \mathcal{D}[\omega_m] P[q_z, \omega_m]. \tag{28}$$

The above equation expresses that for any q_z, the fastly evolving non-zonal degrees of freedom relax to the stationary distribution G. With this last relation, it is easily checked that $\mathcal{P}$

---

[1] Here, inverse is understood in the linear operator sense: $\int d\mathbf{r}' \, g^\infty[q_z](\mathbf{r}_1, \mathbf{r}')(g^\infty)^{-1}[q_z](\mathbf{r}', \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)$.

is a projector: $\mathcal{P}^2 = \mathcal{P}$. Moreover, because $\mathcal{P}$ commutes with $\mathcal{L}_0$ and projects on its stationary distribution, we have

$$\mathcal{P}\mathcal{L}_0 = \mathcal{L}_0\mathcal{P} = 0.$$

3.2 Adiabatic Elimination of the Fast Variables

We decompose any PDF P into its slow and fast components:

$$P = P_s + P_f$$

with $P_s \equiv \mathcal{P}P$ and $P_f \equiv (1 - \mathcal{P})P$.

Using (28), we obtain

$$P_s[q_z, \omega_m] = R[q_z]G[q_z, \omega_m],$$

with $R[q_z] = \int \mathcal{D}[\omega_m]P[q_z, \omega_m]$. R is the marginal distribution of P on the zonal modes and describes the statistics of the zonal flow; P_s describes the statistics of the zonal flow assuming that for any q_z, the fast non-zonal degrees of freedom instantaneously relax to their stationary Gaussian distribution (25). We thus expect R and P_s to evolve slowly, and P_f contains the small corrections to P_s. The goal of the stochastic reduction presented here is to get a closed equation for the evolution of R, valid for small α.

Applying the projector operator $\mathcal{P}$ to Eq. (17) and using $\mathcal{P}\mathcal{L}_0 = 0$, we get the evolution equation for P_s

$$\frac{\partial P_s}{\partial t} = \mathcal{P}(\alpha\mathcal{L}_z + \sqrt{\alpha}\mathcal{L}_n)(P_s + P_f).$$

As expected, this equation evolves on a time scale much larger than 1, and this is due to the relation $\mathcal{P}\mathcal{L}_0 = 0$, which is essentially the definition of $\mathcal{P}$.

Using (28) and (20) we remark that $\mathcal{P}\mathcal{L}_n$ involves the integral over $\mathcal{D}[\omega_m]$ of a divergence with respect to ω_m. As a consequence,

$$\mathcal{P}\mathcal{L}_n = 0. \tag{29}$$

An important consequence of this relation is that the first-order non-linear correction to the quasi-linear dynamics is exactly zero:

$$\frac{\partial P_s}{\partial t} = \alpha\mathcal{P}\mathcal{L}_z(P_s + P_f). \tag{30}$$

Applying now the operator $(1 - \mathcal{P})$ to Eq. (17), using $(1 - \mathcal{P})\mathcal{L}_0 = \mathcal{L}_0(1 - \mathcal{P})$ and (29), we obtain

$$\frac{\partial P_f}{\partial t} = \mathcal{L}_0 P_f + \left(\sqrt{\alpha}\mathcal{L}_n + \alpha(1 - \mathcal{P})\mathcal{L}_z\right)(P_s + P_f). \tag{31}$$

Our goal is to solve (31) and to inject the solution into (30), which will become a closed equation for the slowly evolving PDF P_s. We solve Eqs. (30, 31) using Laplace transform. The Laplace transform of a function of time f is defined by

$$\tilde{f}(s) = \int_0^\infty dt\, e^{-st} f(t),$$

where the real part of s is sufficiently large for the integral to converge. Using $(\widetilde{\partial_t f})(s) = s\tilde{f}(s) - f(0)$ and the fact that the operators don't depend explicitly on time, equations (30) and (31) become

$$\begin{cases} s\tilde{P}_s(s) = \alpha\mathcal{P}\mathcal{L}_z(\tilde{P}_s + \tilde{P}_f) + P_s(0) \\ s\tilde{P}_f(s) = \mathcal{L}_0\tilde{P}_f + \big(\alpha(1-\mathcal{P})\mathcal{L}_z + \sqrt{\alpha}\mathcal{L}_n\big)(\tilde{P}_s + \tilde{P}_f) + P_f(0). \end{cases}$$

For simplicity, we assume that the initial condition satisfies $P_f(0) = 0$. In most classical cases, this assumption is not a restriction, as relaxation towards such a distribution is exponential and fast. For our case, this may be trickier because as discussed in Sect. 5, the relaxation will be algebraic for time scales much smaller than $1/\alpha$. However, we do not discuss this point in detail before the conclusion (Sect. 7).

Denoting by $\frac{1}{\mathcal{L}}$ the inverse of a generic operator $\mathcal{L}$, the second equation can be solved as

$$\tilde{P}_f(s) = \sqrt{\alpha}\frac{1}{s - \mathcal{L}_0 - \sqrt{\alpha}\mathcal{L}_n - \alpha(1-\mathcal{P})\mathcal{L}_z}\big[\sqrt{\alpha}(1-\mathcal{P})\mathcal{L}_z + \mathcal{L}_n\big]\tilde{P}_s$$

and this solution can be injected into the first equation:

$$s\tilde{P}_s(s) = \left[\alpha\mathcal{P}\mathcal{L}_z + \alpha^{3/2}\mathcal{P}\mathcal{L}_z\frac{1}{s - \mathcal{L}_0 - \sqrt{\alpha}\mathcal{L}_n - \alpha(1-\mathcal{P})\mathcal{L}_z}\big[\sqrt{\alpha}(1-\mathcal{P})\mathcal{L}_z + \mathcal{L}_n\big]\right]\tilde{P}_s$$
$$+ P_s(0).$$

At this stage we made no approximation and this last result is exact.

We look at an expansion in powers of $\sqrt{\alpha}$ for the above equation. At order 4 in $\sqrt{\alpha}$, we get

$$s\tilde{P}_s(s) = \left\{\alpha\mathcal{P}\mathcal{L}_z + \alpha^{3/2}\mathcal{P}\mathcal{L}_z\frac{1}{s - \mathcal{L}_0}\mathcal{L}_n + \alpha^2\mathcal{P}\mathcal{L}_z\left[\frac{1}{s - \mathcal{L}_0}(1-\mathcal{P})\mathcal{L}_z + \left(\frac{1}{s - \mathcal{L}_0}\mathcal{L}_n\right)^2\right]\right\}\tilde{P}_s$$
$$+ P_s(0) + \mathcal{O}\big(\alpha^{5/2}\big).$$

We use that $\frac{1}{s - \mathcal{L}_0}$ is the Laplace transform of $\exp(t\mathcal{L}_0)$ and that the inverse Laplace transform of a product is a convolution to conclude that

$$\frac{\partial P_s}{\partial t} = \alpha\mathcal{P}\mathcal{L}_z P_s + \alpha^{3/2}\mathcal{P}\mathcal{L}_z\int_0^\infty dt'\big[e^{t'\mathcal{L}_0}\mathcal{L}_n P_s(t - t')\big]$$

$$+ \alpha^2\mathcal{P}\mathcal{L}_z\int_0^\infty dt'\, e^{t'\mathcal{L}_0}(1-\mathcal{P})\mathcal{L}_z P_s(t - t')$$

$$+ \alpha^2\mathcal{P}\mathcal{L}_z\int_0^\infty dt'\int_0^\infty dt''\, e^{t'\mathcal{L}_0}\mathcal{L}_n e^{t''\mathcal{L}_0}\mathcal{L}_n P_s(t - t' - t'') + \mathcal{O}\big(\alpha^{5/2}\big).$$

We observe now that the evolution equation for P_s contains memory terms. However, in the limit $\alpha \ll 1$, P_s evolves very slowly. Thus, if we make a Markovianization of the time integrals replacing $P_s(t - t')$ by $P_s(t)$, we make an error of order $\frac{\partial P_s}{\partial t}(t)$ which is of order α. For example, the first integral in the last equation is

$$\int_0^\infty dt'\big[e^{t'\mathcal{L}_0}\mathcal{L}_n P_s(t - t')\big] = \int_0^\infty dt'\, e^{t'\mathcal{L}_0}\mathcal{L}_n P_s(t) + \mathcal{O}(\alpha).$$

The evolution equation for P_s is then

$$
\frac{\partial P_s}{\partial t} = \left\{ \alpha \mathcal{P} \mathcal{L}_z + \alpha^{3/2} \mathcal{P} \mathcal{L}_z \int_0^\infty \mathrm{d}t'\, \mathrm{e}^{t'\mathcal{L}_0} \mathcal{L}_n \right.
$$
$$
\left. + \alpha^2 \mathcal{P} \mathcal{L}_z \int_0^\infty \mathrm{d}t'\, \mathrm{e}^{t'\mathcal{L}_0} \left[(1-\mathcal{P})\mathcal{L}_z + \int_0^\infty \mathrm{d}t''\, \mathcal{L}_n \mathrm{e}^{t''\mathcal{L}_0} \mathcal{L}_n \right] \right\} P_s + \mathcal{O}(\alpha^{5/2}). \quad (32)
$$

We now have a closed differential equation for the slow probability distribution function P_s.

3.3 The Fokker–Planck Equation for the Slow Evolution of the Zonal Flow

The explicit computation of each term involved in (32), which is reported in Appendix A, leads to the final Fokker–Planck equation for the zonal jets:

$$
\frac{\partial R}{\partial \tau} = \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left\{ \left[\frac{\partial F_1[U]}{\partial y_1} + \omega_z(y_1) - \frac{\nu}{\alpha} \frac{\partial^2 \omega_z}{\partial y_1^2} \right] R[q_z] \right.
$$
$$
\left. + \int \mathrm{d}y_2 \frac{\delta}{\delta q_z(y_2)} \big(C_R(y_1, y_2)[q_z] R[q_z] \big) \right\}, \quad (33)
$$

which evolves over the time scale $\tau = \alpha t$, with the drift term

$$
F_1[U] = F[U] + \alpha \int_0^\infty \mathrm{d}t' \int_0^\infty \mathrm{d}t''\, \mathbb{E}_U \big[\langle v_m^{(y)} \omega_m \rangle(y_1) M[q_z, \omega_m](t', t'') \big],
$$

with M given in Appendix A, with

$$
F[U] = \mathbb{E}_U \big[\langle v_m^{(y)} \omega_m \rangle \big],
$$

and with the diffusion coefficient

$$
C_R(y_1, y_2)[q_z] = C_z(y_1 - y_2) + \alpha \int_0^\infty \mathrm{d}t'\, \frac{\partial^2}{\partial y_2 \partial y_1} \mathbb{E}_U \big[\big[\langle v_m^{(y)} \omega_m \rangle(y_1, t') \langle v_m^{(y)} \omega_m \rangle(y_2, 0) \big] \big].
$$

From its definition, we see that $F[U]$ is the opposite of the Reynolds stress divergence computed from the quasi-linear approximation.

The Fokker–Planck equation (33) is equivalent to a non-linear stochastic partial differential equation for the potential vorticity q_z,

$$
\frac{\partial q_z}{\partial \tau} = -\frac{\partial F_1}{\partial y}[U] - \omega_z + \frac{\nu}{\alpha} \frac{\partial^2 \omega_z}{\partial y^2} + \zeta, \quad (34)
$$

where ζ is a white in time Gaussian noise with spatial correlation C_R. As C_R depends itself on the velocity field U, this is a non-linear noise.

At first order in α, we recover the deterministic kinetic equation (15) discussed in Sect. 2.4. The main result here is that the first contribution of the non-linear operator, the order $O(\alpha^{3/2})$ in (32), is exactly zero. We could have applied the same stochastic reduction techniques to the quasi-linear dynamics (14) and we would have then obtained the same deterministic kinetic equation at leading order, as expected. We thus have shown that at order 3 in $\sqrt{\alpha}$, the quasi-linear approximation and the full non-linear dynamics give the same results for the zonal flow statistics.

At next order, we see a correction to the drift $F[U]$ due to the non-linear interactions. At this order, the quasilinear dynamics and non-linear dynamics differ. We also see the appearance of the noise term, which has a qualitatively different effect than the drift term. We note that at this order, we would have obtained the same non-linear noise C_R from the quasilinear dynamics. Pushing our computations to next orders, we would obtain higher order corrections to the drift and noise terms, for instance due to eddy-eddy (non-zonal) non-linear interactions, but also corrections through fourth order operators.

We note that we can also write directly an equation equivalent to (34) for the velocity profile

$$\frac{\partial U}{\partial \tau} = F_1[U] - U + \frac{\nu}{\alpha} \frac{\partial^2 U}{\partial y^2} + \xi, \tag{35}$$

where ξ is a white in time Gaussian noise with spatial correlation

$$\mathbb{E}_\xi \big[\xi(y_1, t_1) \xi(y_2, t_2) \big] = \Bigg(C_U(y_1 - y_2)$$
$$+ \alpha \int_0^\infty \mathrm{d}t' \, \mathbb{E}_U \big[\big[\langle v_m^{(y)} \omega_m \rangle (y_1, t') \langle v_m^{(y)} \omega_m \rangle (y_2, 0) \big] \big] \Bigg) \delta(t_1 - t_2). \tag{36}$$

Whereas at a formal level, correction to F and the noise term appear at the same order, their qualitative effect is quite different. For instance if one is interested in large deviations from the most probable states, correction of order α to F will still be vanishingly small, whereas the effect of the noise will be essential. At leading order, the large fluctuations will be given by

$$\frac{\partial U}{\partial \tau} = F[U] - U + \frac{\nu}{\alpha} \frac{\partial^2 U}{\partial y^2} + \xi, \tag{37}$$

Equation (37) then appears to be the minimal model in order to describe the evolution of zonal jet in the limit of weak forcing and dissipation. We will comment further on this issue in Sect. 6, dealing with bistability of zonal jets and phase transitions.

4 Energy and Enstrophy Balances

We discuss here the energy balance in the inertial limit $\alpha \ll 1$ and the consistency of the stochastic reduction at the level of the energy. It is thus essential to distinguish the different ways to define the averages, for the original stochastic equation (5) or for the zonal Fokker–Planck equation (33). We recall that $\mathbb{E}[\cdot] = \int \mathcal{D}[q_z] \mathcal{D}[\omega_m] \cdot P[q_z, \omega_m]$ denotes the average with respect to the full PDF P_t whose evolution is given by the Fokker–Planck equation (17), or equivalently, the average over realizations of the noise η in Eq. (10)–(11); $\mathbb{E}_U[\cdot] = \int \mathcal{D}[\omega_m] \cdot G[q_z, \omega_m]$ denotes the average with respect to the stationary Gaussian distribution of the non-zonal fluctuations G, defined in (25) and $\mathbb{E}_R[\cdot] = \int \mathcal{D}[q_z] \cdot R[q_z]$ denotes the average with respect to the slowly evolving zonal PDF R whose evolution is given by the zonal Fokker–Planck equation (33). $\mathbb{E}_R$ is equivalently an average over the realizations of the noise ζ in Eq. (34).

As discussed in the presentation of the barotropic equations, we are interested in the regime where the dissipation of energy is dominated by the one due to the large-scale linear friction: $2\lambda E / \nu_{n,d} H_n \gg 1$ in Eq. (4). As a consequence, we will consider in this section only the case of zero viscosity, $\nu = 0$.

4.1 Energy Balance for the Barotropic Equations

The total energy balance of the non-dimensional barotropic equations (5) given by Eq. (6) is reported here for convenience:

$$\frac{dE}{dt} = -2\alpha E + 2\alpha \tag{38}$$

where we recall that $E = \mathbb{E}[\mathcal{E}]$. With the orthogonal decomposition into zonal and non-zonal degrees of freedom, we have a natural decomposition of the energy contained in zonal and non-zonal degrees of freedom: $E = E_z + \alpha E_m$, with $E_z = \mathbb{E}[\frac{1}{2}\int_{\mathcal{D}} U^2]$ and $E_m = \mathbb{E}[\frac{1}{2}\int_{\mathcal{D}} \mathbf{v}_m^2]$.

Zonal Energy Balance   From the definition of E_z, either by direct computation from the Fokker–Planck equation (17) or from Eq. (10) applying the Itō formula, we have

$$\frac{dE_z}{dt} = \alpha 2\pi l_x \int dy\, \mathbb{E}\big[\langle v_m^{(y)} \omega_m\rangle(y) U(y)\big] - 2\alpha E_z + 2\alpha\sigma_z \tag{39}$$

with $\sigma_z = -2\pi^2 l_x(\Delta^{-1}C_z)(0)$ the rate of energy injected by the forcing directly into the zonal degrees of freedom. In addition with the expected energy dissipation and direct energy injection by σ_z, the first term on the right hand side describes the energy production due to the non-zonal fluctuations.

Non-zonal Energy Balance   Using $E = E_z + \alpha E_m$ and Eqs. (38), (39), we obtain

$$\frac{dE_m}{dt} = -2\pi l_x \int dy\, \mathbb{E}\big[\langle v_m^{(y)} \omega_m\rangle(y) U(y)\big] - 2\alpha E_m + 2\sigma_m, \tag{40}$$

where $\sigma_m = 1 - \sigma_z$ is the rate of energy injection by the forcing on non-zonal degrees of freedom. Clearly, Eqs. (39) and (40) are exact and fully equivalent to the energy balance (38).

4.2 Energy and Enstrophy Balance for the Kinetic Equation

We now show that the energy balance of the kinetic equation (34) is consistent with the exact energy balances written above, at leading order in α. We denote by $\tilde{E}_z = \mathbb{E}_R[\frac{1}{2}\int_{\mathcal{D}} U^2]$ the average zonal energy for the kinetic equation and by $\alpha\tilde{E}_m = \alpha\mathbb{E}_U[\frac{1}{2}\int_{\mathcal{D}} \mathbf{v}_m^2]$ the energy contained in non-zonal degrees of freedom. We start from the energy balance for zonal degrees of freedom. Again, working at the level of the zonal Fokker–Planck (33) equation or of the kinetic equation (34) give the same result:

$$\frac{d\tilde{E}_z}{d\tau} = 2\pi l_x \mathbb{E}_R\bigg[\int dy\, F_1[U](y) U(y)\bigg] - 2\tilde{E}_z - 2\pi l_x \mathbb{E}_R\bigg[\int dy\big(\Delta^{-1}C_R\big)(\mathbf{0})\bigg].$$

To approximate the above equation at leading order in α, we observe that $F_1[U] = F[U] + O(\alpha) = \mathbb{E}_U[\langle v_m^{(y)}\omega_m\rangle](y_1) + O(\alpha)$, and $C_R = C_z + O(\alpha)$. Then, the stationary energy balance for zonal degrees of freedom reduces to

$$\frac{d\tilde{E}_z}{d\tau} = 2\pi l_x \mathbb{E}_R\bigg[\int dy\, \mathbb{E}_U\big[\langle v_m^{(y)}\omega_m\rangle\big] U\bigg] - 2\tilde{E}_z + 2\sigma_z + O(\alpha). \tag{41}$$

In the limit $\alpha \to 0$, the full PDF $P[q_z, \omega_m]$ is given by the slowly evolving part $P_s[q_z, \omega_m] = G[q_z, \omega_m]R[q_z]$. Then, the rate of energy transferred from the fluctuations to the zonal flow becomes

$$\mathbb{E}_R\left[\int dy\, \mathbb{E}_U\left[\langle v_m^{(y)}\omega_m\rangle\right]U\right] + O(\alpha) = \int dy\, \mathbb{E}\left[\langle v_m^{(y)}\omega_m\rangle U\right];$$

and Eq. (41) reduces to the energy balance for zonal degrees of freedom (39) at leading order in α. This proves the consistency of the zonal energy balance for the kinetic equation and the barotropic equations: $E_z = \tilde{E}_z + O(\alpha)$.

4.2.1 Non-zonal Energy Balance and Total Transfer of Energy to the Zonal Flow

Let us now consider the energy balance for $\alpha \tilde{E}_m$. Applying Itō formula to (22), we get, in the stationary state and for any U,

$$0 = -2\pi l_x \int dy\, \mathbb{E}_U\left[\langle v_m^{(y)}\omega_m\rangle(y)\right]U(y) - 2\alpha\tilde{E}_m + 2\sigma_m, \qquad (42)$$

with $\sigma_m = -2\pi^2 l_x(\Delta^{-1}C_m)(\mathbf{0})$. Equation (42) does not contain time evolution which is consistent with the definition of $\tilde{E}_m$ in the kinetic equation through a stationary average. In the limit $\alpha \ll 1$, in agreement with the scaling of the variables we expect the energy of the non-zonal degrees of freedom to be of order α, or $\tilde{E}_m = O(1)$. This is essential for the consistency of the asymptotic expansion and will be proved in Sect. 5.

From $\tilde{E}_m = O(1)$ and Eq. (42), the energy dissipated in the non-zonal fluctuations per unit time is negligible, so the stationary energy balance for the non-zonal fluctuations gives

$$\pi l_x \int dy\, \mathbb{E}\left[\langle v_m^{(y)}\omega_m\rangle U\right] = \pi l_x \int dy\, \mathbb{E}_U\left[\langle v_m^{(y)}\omega_m\rangle(y)\right]U(y) + O(\alpha) = \sigma_m + O(\alpha).$$

Injecting this relation in the stationary energy balance for the zonal degrees of freedom (41) gives

$$E_z = \pi l_x \int dy\, \mathbb{E}_R\left[\langle v_m^{(y)}\omega_m\rangle U\right] + \sigma_z = \sigma_z + \sigma_m + O(\alpha) = 1 + O(\alpha).$$

The barotropic equations (5) are in units so that $E = 1$ in a stationary state: the above relation expresses the fact that, in the limit $\alpha \ll 1$, all the energy is concentrated in the zonal degrees of freedom: $E_z = E + O(\alpha)$.

4.2.2 Enstrophy Balance for the Kinetic Equation

We conclude considering the enstrophy balance for the kinetic equation. As we will see below, the concentration property found for the energy does not hold for the enstrophy. The zonal and non-zonal enstrophy balances can be obtained with a very similar reasoning as the one done for the energy and, at leading order in α, one can use the full Fokker–Planck (17) or the approximated one (33), (22) and obtain consistent results. Denoting $Z_z = \mathbb{E}[\frac{1}{2}\int_D q_z^2]$ the enstrophy of zonal degrees of freedom and $\alpha Z_m = \alpha\mathbb{E}[\frac{1}{2}\int_D \omega_m^2]$ the non-zonal degrees of freedom one, we have

$$0 = -2\pi l_x \mathbb{E}\left[\int dy\, U''(y)\langle v_m^{(y)}\omega_m\rangle(y)\right] - 2Z_z + 4\pi^2 l_x C_z(0), \qquad (43)$$

$$0 = 2\pi l_x \mathbb{E}\left[\int dy\, U''(y)\langle v_m^{(y)}\omega_m\rangle(y)\right] - 2\alpha Z_m + 4\pi^2 l_x C_m(\mathbf{0}), \qquad (44)$$

where the first equation is the stationary enstrophy balance for Z_z and the second one for Z_m. The above equations refer explicitly to the case of the 2D Euler equations, but the generalization to the barotropic equations is straightforward.

As will be discussed in next section, the enstrophy in the non-zonal degrees of freedom doesn't converge in the inertial limit: more precisely, $\alpha Z_m = O(1)$. Then, the enstrophy in the zonal degrees of freedom is

$$Z_z = -\alpha Z_m + 2\pi^2 l_x \big(C_z(0) + C_m(\mathbf{0})\big) = I_Z - \alpha Z_m,$$

with the total enstrophy injected by the forcing I_Z. Then, by contrast with the zonal energy, the zonal enstrophy is not equal to the enstrophy injected plus correction of order α. There is no concentration of the enstrophy in the zonal degrees of freedom, and a non-vanishing part of the enstrophy injected is dissipated in the non-zonal fluctuations.

Clearly there is no self-similar cascade in the problem considered here, energy goes directly from the fluctuations at any scales to the zonal flow through the effect of the advection by the zonal flow. These results are however in agreement with the phenomenology of a transfer of the energy to the largest scales, while the excess enstrophy is transferred to the smallest scales, however with dynamical processes that are non-local in Fourier space. This non-equilibrium transfer of energy to the largest scales is also consistent with predictions from equilibrium statistical mechanics which, roughly speaking, predicts that the most probable flow concentrates its energy at the largest possible scales.

5 The Lyapunov Equation in the Inertial Limit

As discussed in Sect. 3, it is essential to make sure that the Gaussian process corresponding to the inertial linearized evolution of non-zonal degrees of freedom close a base flow U, see Eqs. (13) and (24), has a stationary distribution. We discuss this issue in this section. We consider the linear dynamics with stochastic forces (Eq. (22)), that we recall here for convenience

$$\frac{\partial \omega_m}{\partial t} + L_U^0[\omega_m] = \sqrt{2}\eta_m, \qquad \mathbb{E}\big[\eta_m(\mathbf{r}_1, t)\eta_m(\mathbf{r}_2, t')\big] = C_m(\mathbf{r}_1 - \mathbf{r}_2)\delta\big(t - t'\big), \qquad (45)$$

where

$$L_U^0[\omega_m] = U(y)\frac{\partial \omega_m}{\partial x} + \big(-U''(y) + h'(y)\big)\frac{\partial \psi_m}{\partial x} \qquad (46)$$

is the linearized evolution operator close to the zonal flow U.

Equation (45) describes a linear stochastic Gaussian process, or Ornstein-Uhlenbeck process. Thus, it is completely characterized by the two-points correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)]$. The evolution of g is given by the so-called Lyapunov equation, which is obtained by applying the Itō formula to (45):

$$\frac{\partial g}{\partial t} + L_U^{0(1)}g + L_U^{0(2)}g = 2C_m. \qquad (47)$$

We will prove that Eq. (47) has a asymptotic limit g^∞ for large time. This may seem paradoxical as we deal with a linearized dynamics with a stochastic force and no dissipation mechanism. We explain in this section that the Orr mechanism (the effect of the

shear through a non-normal linearized dynamics) acts as an effective dissipation. However this effects is not uniform on all observables. We will prove that g has a limit g^∞ in the sense of distribution, from which we will be able to prove that velocity-like observables have a limit. By contrast, observables involving the vorticity gradient will diverge. Moreover, if the kinetic energy contained in the non-zonal degrees of freedom converges, the enstrophy diverges. This non-uniformity for the convergence of the two-points correlations functions is also related to the fact that the convergences will be algebraic in time, rather than exponential. The statement that "the Gaussian process corresponding to the inertial linearized evolution close to a base flow U has a stationary distribution" must thus be understood with care: not all observable converge and the convergences are algebraic.

An observable like the Reynolds stress divergence involves both the velocity and the vorticity gradient. It is thus not obvious that it has an asymptotic limit. We will also prove that the long time limits of the Reynolds stress divergence $-\mathbb{E}_U[\langle v_m^{(y)}\omega_m\rangle]$ and of its gradient $-\mathbb{E}_U[\frac{\partial}{\partial y}\langle v_m^{(y)}\omega_m\rangle]$ are actually well defined. The results in this section ensure that the asymptotic expansion performed in Sect. 3 is well posed, at leading order.

Because some quantities (such as the enstrophy) diverge in the inertial limit, it is of interest to understand how the Gaussian process is regularized by a small viscosity or linear friction. This corresponds to replace the operator L_U^0 in Eq. (47) with the operator L_U defined in Eq. (12). Moreover, to be able to separate the effect of the viscosity and of the Rayleigh friction, we will introduce in the following the operators

$$L_U^\nu[\omega_m] = L_U^0[\omega_m] - \nu\Delta\omega_m, \tag{48}$$

and

$$L_U^\alpha[\omega_m] = L_U^0[\omega_m] + \alpha\omega_m, \tag{49}$$

in which the superscript indicates which of the terms have been retained from the ordinal operator L_U.

For $\alpha = \nu = 0$, the two-point correlation function g converges as a distribution. $g(\mathbf{r}, \mathbf{r}', t)$ diverges point-wise only for value of y and y' such that $U(y) = U(y')$, for instance $y = y'$. We will prove that for small α this divergence is regularized in a universal way (independent on U) close to points $(\mathbf{r}, \mathbf{r}')$ such that $U(y) = U(y')$, over a scale $\lambda_\alpha = \frac{2\alpha}{kU'(y)}$, where k is the wavenumber of the meridional perturbation. We stress that only the local behavior close to the divergence point is universal. All quantities converge exponentially over a time scale $1/\alpha$.

We will argue that a small viscosity $\nu \neq 0$ also leads to a universal regularization of the divergence, over a scale $\lambda_\nu = (\frac{2\nu}{kU'(y)})^{1/3}$. When both α and ν are not equal to zero, the way the singularity is regularized depends on the ratio of the length scales λ_α and λ_ν. More detailed results are discussed in this section.

Our analysis is based on the evaluation of the long time asymptotics for the deterministic inertial linearized dynamics [7]. As far as we know, such theoretical results are currently available only for the linearized Euler equation [7], $h = \beta = 0$. We will discuss the Lyapunov equation in the case $\beta \neq 0$ in a forthcoming paper.

5.1 General Discussion

5.1.1 Solution of the Lyapunov Equation from the Solution of the Deterministic Linear Equation

In this first section, we show how to obtain a formal solution of the Lyapunov equation (47) using the solution of the deterministic dynamics $\frac{\partial}{\partial t} + L_U^0$ with appropriate initial conditions. This discussion can be trivially extended to the cases in which the operator appearing in Eq. (45) is L_U^v, L_U^α or L_U, by simply replacing the operator L_U^0 with the appropriate one wherever it appears.

We expand the force correlation function C_m in Fourier series, we have

$$C_m(x, y) = \sum_{k>0,l} c_{kl} \cos(kx + ly).$$

We note that because C_m is a correlation, it is a positive definite function. This explains why sin contribution are zero in this expansion. Moreover for all $k \in \mathbb{N}^*$ and $l \in \mathbb{Z}$, we have $c_{kl} \geq 0$. The expression $c_{kl} \cos(kx + ly) + c_{k,-l} \cos(kx - ly)$ is the most general positive definite function involving the Fourier components e^{ikx} and e^{ikl} or their complex conjugates. Here we have assumed that the correlation function is homogeneous (it depends only on $x_1 - x_2$ and on $y_1 - y_2$). Its generalization to the case of an inhomogeneous force, for instance for the case of a channel would be straightforward.

Because the Lyapunov equation is linear, the contribution of the effect of all forcing terms just add up

$$g = \sum_{k>0,l} c_{kl} g_{kl}, \tag{50}$$

where g_{kl} is the solution of the Lyapunov equation (47) with right hand side $2\cos(kx + ly)$. By direct computation it is easy to check that

$$g_{kl}(\mathbf{r}_1, \mathbf{r}_2, t) = \int_0^t e^{-t_1 L_U^0} [e_{kl}](x_1, y_1) e^{-t_1 L_U^0} [e_{kl}^*](x_2, y_2) \, dt_1 + \text{C.C.} \tag{51}$$

with $e_{kl}(x, y) = e^{i(kx+ly)}$, and where C.C. stands for the complex conjugate.[2] We note that $e^{-t_1 L_U^0}[e_{kl}]$ is the solution at time t_1 of the deterministic linear dynamics $\partial_t + L_U^0$ with initial condition e_{kl}.

Let us observe that from the solution of the Lyapunov equation, we can also easily obtain the evolution of $\mathbb{E}[\omega_m(\mathbf{r}_1, t)(S\omega_m)(\mathbf{r}_2, t)]$ where S is a linear operator. We have

$$\mathbb{E}[\omega_m(\mathbf{r}_1, t)(S\omega_m)(\mathbf{r}_2, t)] = S^{(2)}[g]$$

where $S^{(2)}$ is the linear operator S acting on the second variable. The typical operators S that we will use in the following are those necessary to obtain the stream function and the velocity from the vorticity.

---

[2]A similar formula can easily been deduced for any stochastic force of the form $C(\mathbf{r}_1, \mathbf{r}_2) = f(\mathbf{r}_1)f(\mathbf{r}_2)$ from the explicit solution of the Gaussian process from the stochastic integral $\int_0^t e^{-t_1 L_U^0}[f] dW_{t_1}$. Here the key point is that the Fourier basis diagonalizes any translationally invariant correlation function.

5.1.2 Two Explicitly Solvable Examples

Before discussing the long-time behavior of the Gaussian process in Eq. (45) for a general zonal flow U, we discuss here two simple examples for which the deterministic linear dynamics $\partial_t + L_U^0$ can be solved analytically. The two examples are the perfect and the viscous advection by a linear shear. These two simple examples will put in evidence the mechanisms that will ensure the convergence of the long-time limit of two-points correlations in the inertial limit.

For a linear shear this mechanism is the Orr mechanism: the transport of the vorticity along each streamline leads to a phase mixing in the computation of all integrated quantities, for instance the velocity. Then the velocity or the stream function decay algebraically for large times, the exponent for this decay being related to the singularity of the Laplacian green function. For more general profile U for which $U''(y) \neq 0$, this shear mechanism still exists, but is also accompanied by global effects due to the fact that vorticity affects the velocity field globally. In many cases, those global effects are the dominant one [7]. They will be taken into account in Sect. 5.2.

Perfect Advection by a Linear Shear   To treat an example that can be worked out explicitly, we consider the perturbation by a stochastic force of a linear shear $U(y) = sy$, which corresponds to set in Eq. (45) $L_U^0 = sy\partial_x$. For sake of simplicity, we only treat here the case $s > 0$, the corresponding generalization to $s < 0$ being trivial. The following discussion applies to flows that are periodic in the longitudinal direction x, with period $2\pi l_x$, either to the case of the domain $\mathcal{D} = [0, 2\pi l_x] \times (-\infty, \infty)$, or to flows in a zonal channel with walls at $y = \pm L$.

The Lyapunov equation we have to consider for this problem is

$$\frac{\partial g}{\partial t} + L_U^{0(1)} g + L_U^{0(2)} g = 2C_m,$$

for the vorticity-vorticity correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)]$. For sake of simplicity, we consider the case where the forcing acts on a single wave vector

$$C_m = \frac{\varepsilon(k^2 + l^2)}{4}\left[e^{ik(x_1 - x_2) + il(y_1 - y_2)} + \text{C.C.}\right] = \frac{\varepsilon(k^2 + l^2)}{2}\cos\left[k(x_1 - x_2) + l(y_1 - y_1)\right], \tag{52}$$

where C.C. stands for the complex conjugation of the first term. As explained in the previous section, contributions to the Lyapunov equation from other forcing modes just add up (see Eq. (50)). ε is the average energy input rate per unit of mass (unit $m^2 s^{-3}$) (in this section and the following ones dealing with the case of a linear shear, ω_m is the non-zonal vorticity and has dimensions s^{-1}, whereas in section dealing with the kinetic theory $\sqrt{\alpha}\omega_m$ is the non-zonal vorticity).

The deterministic evolution of the linearized dynamics L_U^0 with initial condition $e^{i(kx + ly)}$ is

$$e^{-tL_U^0}\left[e^{i(kx+ly)}\right] = e^{ikx + ily - iksyt}, \tag{53}$$

as can be easily checked. From (51), we have

$$g(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{\varepsilon(k^2 + l^2)}{4s}\frac{e^{ik(x_1 - x_2) + il(y_1 - y_2)}}{-ik(y_1 - y_2)}\left[e^{-iks(y_1 - y_2)t} - 1\right] + \text{C.C.}, \tag{54}$$

for $y_1 \neq y_2$ and $g(x_1, y, x_2, y, t) = \frac{\varepsilon(k^2 + l^2)}{2}\cos(k(x_1 - x_2))t$.

This result readily shows that the square of the perturbation vorticity $g(\mathbf{r}, \mathbf{r}, t)$ diverges proportionally to time t. This is expected as the average enstrophy input rate per unit of area is $\varepsilon(k^2 + l^2)$, and there is no dissipation mechanism. However, we also remark that for $y_1 \neq y_2$ the autocorrelation function is a fast oscillating function. As a consequence g will have a well defined limit in the sense of distributions:

$$
\begin{aligned}
g^\infty(\mathbf{r}_1, \mathbf{r}_2) &= \lim_{t \to \infty} g(\mathbf{r}_1, \mathbf{r}_2, t) \\
&= \frac{\varepsilon(k^2 + l^2)}{4s} e^{ik(x_1 - x_2) + il(y_1 - y_2)} \frac{1}{k} \left[\pi \delta(y_1 - y_2) - iPV\left(\frac{1}{y_1 - y_2}\right) \right] + \text{C.C.},
\end{aligned}
\tag{55}
$$

where PV stands for the Cauchy Principal Value (we have used Plemelj formula, Eq. (93)). Equivalently, we have

$$
\begin{aligned}
g^\infty(\mathbf{r}_1, \mathbf{r}_2) = \frac{\varepsilon(k^2 + l^2)}{2s} &\left[\frac{\cos k(x_1 - x_2)}{k} \pi \delta(y_1 - y_2) \right. \\
&\left. + \frac{\sin[k(x_1 - x_2) + l(y_1 - y_2)]}{k} PV\left(\frac{1}{y_1 - y_2}\right) \right].
\end{aligned}
\tag{56}
$$

The fact that the stationary vorticity-vorticity correlation function has a limit in the sense of distributions is a very important result. It means that every observable that can be obtained by integration of a smooth function over g^∞ will have a well defined stationary limit. Actually the formula above will be valid when integrated over any continuous function. For instance, we can use it to compute the velocity-vorticity or velocity-velocity correlation functions. Then all these quantities will have a definite stationary value. This is a remarkable fact, as we force continuously the perfect flow and no dissipation is present. Looking at the prefactor $\frac{\varepsilon(k^2 + l^2)}{2s}$, we remark that it is an injection rate $\varepsilon(k^2 + l^2)$ (the injection rate of enstrophy per unit of area) divided by twice the shear. By analogy with the equivalent formula in classical Ornstein-Uhlenbeck process, we see that the shear s acts as an effective damping mechanism. The effect of the shear, called Orr mechanism, leads to phase mixing which acts as an effective dissipation for the physical quantities dominated by the large scales. All integrated quantities, for instance the kinetic energy, are proportional to $\frac{\varepsilon(k^2 + l^2)}{2s}$ and are independent on the linear friction or viscosity at leading order. We also note that in the statistically stationary state the enstrophy is infinite.

As an example, we compute the vorticity-stream function stationary correlation function

$$
h^\infty(\mathbf{r}_1, \mathbf{r}_2) = \lim_{t \to \infty} \mathbb{E}\left[\omega(\mathbf{r}_1, t) \psi(\mathbf{r}_2, t) \right]
\tag{57}
$$

From Eq. (56) we obtain

$$
\begin{aligned}
h^\infty(\mathbf{r}_1, \mathbf{r}_2) = \frac{\varepsilon(k^2 + l^2)}{2s} &\left[\frac{\cos k(x_1 - x_2)}{k} \pi H_k(y_1 - y_2) \right. \\
&\left. + PV \int \frac{\sin[k(x_1 - x_2) + l(y_1 - y_3)]}{k} \frac{H_k(y_2 - y_3)}{y_1 - y_3} dy_3 \right],
\end{aligned}
\tag{58}
$$

where H_k is the Green function for the Laplacian in the kth sector, with the appropriate boundary conditions. From this last expression, and using $v^{(x)} = -\partial \psi / \partial y$ and $v^{(y)} = \partial \psi / \partial x$ it is easy to obtain the correlation function between the vorticity and the velocity field. We

note that the Green function $H_k(y_1 - y_2)$ is not a smooth function: its derivative has a jump in $y_1 = y_2$. This implies that the correlation between the vorticity and the x component of the velocity is defined point-wise only for $y_1 \neq y_2$ or defined globally as a distribution. With analogous computations one can deduce other stationary two-points correlations: for example, the Reynolds stress divergence converges to a well defined function.

In this section, we have discussed how the vorticity-vorticity correlation function has a stationary limit in the sense of distribution. When looking at this solution point-wise, it is singular for $y_1 = y_2$. In the following section, we explain how this singularity is regularized either by a linear friction or by viscosity. The discussion in this section was relying on the analytic solution of the linear dynamics close to a linear shear (Eq. (53)). Such an analytical solution is not known for generic base flows, so that we need more refined techniques, as will be explained in Sect. 5.2. We will obtain the same conclusion: the vorticity-vorticity correlation function converges as a distribution, and is regularized in a universal way.

Advection by a Linear Shear: Regularization by a Linear Friction   We consider the same problem as in the last paragraph, but adding a linear friction. We will see how the singularity close to $y_1 = y_2$ is regularized by a linear friction. We solve

$$\frac{\partial g_\alpha}{\partial t} + L_U^{\alpha(1)} g_\alpha + L_U^{\alpha(2)} g_\alpha = 2C_m,$$

with $L_U^\alpha = sy\partial_x - \alpha$ and C_m given by Eq. (52). It is straightforward to observe that the evolution under the linearized dynamics L_U^α is given by

$$\mathrm{e}^{-tL_U^\alpha}\left[\mathrm{e}^{i(kx+ly)}\right] = \mathrm{e}^{ikx+ily-iksyt}\mathrm{e}^{-\alpha t}. \tag{59}$$

With very similar computations to those of the last section, we can obtain the stationary value of the vorticity-vorticity correlation function

$$g_\alpha^\infty(\mathbf{r}_1, \mathbf{r}_2) = \lim_{t\to\infty} g_\alpha(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{\varepsilon(k^2+l^2)}{4s}\mathrm{e}^{ik(x_1-x_2)+il(y_1-y_2)}\frac{1}{k}F_{\frac{2\alpha}{ks}}(y_1 - y_2) + \text{C.C.}, \tag{60}$$

where

$$F_\lambda(y) = \frac{-i}{y - i\lambda}. \tag{61}$$

We can thus observe that the function $F_\lambda(y)$ is a regularization of the Plemelj formula (Eq. (93)) on the length scale λ. Indeed we have

$$F_\lambda(y) = \frac{1}{\lambda}F_1\left(\frac{y}{\lambda}\right) = \frac{\lambda}{y^2+\lambda^2} - i\frac{y}{y^2+\lambda^2}$$

where the real part of F_λ is even while the imaginary part is odd. Moreover the real part of F_λ is a regularization of $\pi\delta(y)$,

$$\lim_{\lambda\to 0^+} \Re\left[F_\lambda(y)\right] = \lim_{\lambda\to 0^+} \frac{\lambda}{y^2+\lambda^2} = \pi\delta(y),$$

and the imaginary part of F_λ is a regularization of $-PV(1/y)$,

$$\lim_{\lambda\to 0^+} \Im\left[F_\lambda(y)\right] = -\lim_{\alpha\to 0} \frac{y}{y^2+\lambda^2} = -PV\left(\frac{1}{y}\right)$$

where $\Re$ and $\Im$ stand, respectively, for the real and the imaginary parts. We note that for $y \gg \lambda$, $\lim_{\lambda \to 0^+} \Re[F_\lambda(y)] = 0$, and $\lim_{\lambda \to 0^+} \Im[F_\lambda(y)] = 1/y$.

We thus finally obtain

$$g_\alpha^\infty(\mathbf{r}_1, \mathbf{r}_2) \underset{\frac{\alpha}{ks} \ll 1}{\sim} \frac{\varepsilon(k^2 + l^2)}{2s} \left\{ \cos k(x_1 - x_2) \frac{1}{k} \Re\left[F_{\frac{2\alpha}{ks}}(y_1 - y_2) \right] \right.$$

$$\left. - \sin\left[k(x_1 - x_2) + l(y_1 - y_2) \right] \frac{1}{k} \Im\left[F_{\frac{2\alpha}{ks}}(y_1 - y_2) \right] \right\}, \tag{62}$$

where one should observe that, because $\Re[F_\alpha(ky)]$ decays sufficiently fast to zero for $y \gg \alpha$, the factor $\cos[k(x_1 - x_2) + l(y_1 - y_2)]$ has been replaced by $\cos[k(x_1 - x_2)]$. Equation (62) is a regularization of the vorticity-vorticity correlation function found in (56), by the effect of a small linear friction $\alpha > 0$.

Quantities which were found to be divergent in the last section are now regularized by the presence of a small Rayleigh friction. For example the point-wise rms. non zonal enstrophy density

$$\mathbb{E}_U\left[\omega_m^2\right] = g_\alpha^\infty(0, 0) = \frac{\varepsilon(k^2 + l^2)}{4\alpha}. \tag{63}$$

Advection by a Linear Shear in a Viscous Fluid   In this paragraph, we study the regularization of the solution to the Lyapunov equation by a small viscosity considering the perturbation by a stochastic force of a linear shear $U(y) = sy$ in a viscous flow. We consider the domain $\mathcal{D} = [0, 2\pi l_x) \times (-\infty, \infty)$ and periodic boundary conditions in the x direction.

We solve the Lyapunov equation

$$\frac{\partial g}{\partial t} + L_U^{\nu(1)} g + L_U^{\nu(2)} g = 2C_m,$$

for the vorticity-vorticity correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)]$, where $L_U^\nu = sy\partial_x - \nu\Delta$, C_m is given by Eq. (52) and ε the injection rate of energy per unit of mass (ε has the dimensions $\text{m}^2 \text{s}^{-3}$).

The deterministic evolution of the linear dynamics $\frac{\partial}{\partial t} + L_U^\nu = 0$ with the initial condition $e^{i(kx+ly)}$ is given by

$$e^{-tL_U^\nu}\left[e^{i(kx+ly)}\right] = e^{ikx} e^{i(l-skt)y} e^{-\nu(k^2+l^2)t + \nu sklt^2 - \frac{1}{3}\nu s^2 k^2 t^3}. \tag{64}$$

This solution, first derived by Lord Kelvin, can be obtained by the method of characteristics. Alternatively one can directly check a-posteriori that (64) is a solution to the deterministic equation. We can see that the second exponential in (64) just gives the inertial evolution of the perturbation and the third one describes the effect of the viscosity. We thus see that the vorticity is damped to zero on the time-scale $(1/s^2k^2\nu)^{1/3}$. This time scale is the typical time for an initial perturbation with longitudinal scale $1/k$ to be stretched by the shear until it reaches a scale where it is damped by viscosity.

We can now calculate the asymptotic solution to the Lyapunov equation. It is given by

$$g_\nu^\infty(\mathbf{r}_1, \mathbf{r}_2) = \frac{\varepsilon(k^2 + l^2)}{4} \int_0^\infty e^{-tL_U^{\nu(1)}}\left[e^{ikx_1} e^{ily_1}\right] e^{-tL_U^{\nu(2)}}\left[e^{-ikx_2} e^{-ily_2}\right] dt + \text{C.C.}$$

$$= \frac{\varepsilon(k^2 + l^2)}{4s} e^{ik(x_1-x_2)+il(y_1-y_2)} \frac{1}{k} H_{(\frac{2\nu}{ks})^{1/3}}(y_1 - y_2) + \text{C.C.} \tag{65}$$

where we have used Eq. (64) and the change of variable $t_1 = (2\nu k^2 s^2)^{1/3} t$ to write the second equality, and where

$$H_\lambda(y) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{iy}{\lambda} t_1 - \lambda^2 (k^2 + l^2) t_1 + \lambda l t_1^2 - \frac{1}{3} t_1^3} \, dt_1. \tag{66}$$

In Appendix B we prove that $H_\lambda(y) \underset{\lambda \to 0}{=} G_\lambda(y) + \mathcal{O}(\lambda)$ with

$$G_\lambda(y) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{iy}{\lambda} t_1 - \frac{1}{3} t_1^3} \, dt_1. \tag{67}$$

We note that $\mathrm{Hi}(Y) = G_\lambda(i\lambda Y)/\pi$ is one of the two Scorer's functions, that solve the differential equation $d^2 \mathrm{Hi}/dY^2 - Y\mathrm{Hi} = 1/\pi$, and is related to the family of Airy functions. We stress that the asymptotic behavior for large y of the real parts of G_λ and H_λ are different $\mathfrak{R}[H_\lambda(y)] \underset{y \gg \sqrt{\frac{2}{k^2+l^2}}}{\sim} (k^2 + l^2) \frac{\lambda^3}{y^2}$ whereas $\mathfrak{R}[G_\lambda(y)] \underset{y \gg \lambda}{\sim} 2\frac{\lambda^3}{y^4}$ but in any cases those are subdominant for small λ (please see Appendix B for more details).

As explained in Appendix B, G_λ is a regularization of the distributions in Plemelj formula at the length scale λ and has the dimension of the inverse of a length. The integral over y of the real part of G_λ is π, consistently with the fact that it is a regularization of $\pi \delta(y)$; the imaginary part is a regularization of Cauchy Principal value of $1/y$, with

$$\mathfrak{I}[G_\lambda(y)] \underset{y \gg \lambda}{\sim} -\frac{1}{y}. \tag{68}$$

We thus conclude that

$$g_\nu^\infty(\mathbf{r}_1, \mathbf{r}_2) \underset{(\frac{2\nu}{sk})^{1/3} l \ll 1}{\sim} \frac{\varepsilon(k^2 + l^2)}{2s} \left\{ \frac{\cos[k(x_1 - x_2)]}{k} \mathfrak{R}[G_{(\frac{2\nu}{ks})^{1/3}}(y_1 - y_2)] \right.$$
$$\left. - \frac{\sin[k(x_1 - x_2) + l(y_1 - y_2)]}{k} \mathfrak{I}[G_{(\frac{2\nu}{ks})^{1/3}}(y_1 - y_2)] \right\}. \tag{69}$$

Observe that, because $\mathfrak{R}[G_{(\frac{2\nu}{ks})^{1/3}}(y)]$ decays sufficiently fast to zero for $y \gg (\frac{2\nu}{ks})^{1/3}$, the factor $\cos[k(x_1 - x_2) + l(y_1 - y_2)]$ has been replaced by its value for $y_1 = y_2$. The first term of Eq. (69) is a local contribution, for values of $y_1 - y_2$ of order $(\frac{2\nu}{ks})^{1/3}$, whereas the second term is a global contribution. Whereas the local contribution is independent on l and depends on $y_1 - y_2$ through the shape function G, the global contribution has a phase dependance through $l(y_1 - y_2)$.

By contrast, the point wise value of the two-point vorticity correlation function for $(y_1 - y_2) \sim (\frac{2\nu}{ks})^{1/3}$ diverges for large Reynolds number. For instance

$$\mathbb{E}_U[\omega_m^2] = g_\nu^\infty(0, 0) \underset{(\frac{2\nu}{sk})^{1/3} l \ll 1}{\sim} \frac{\varepsilon(k^2 + l^2)}{2s} \left(\frac{s}{2k^2 \nu}\right)^{1/3} \int_0^\infty e^{-\frac{1}{3} t_1^3} \, dt_1,$$

and one can note that $\frac{s}{2\nu k^2}$ is a Reynolds number based on the local shear and the scale of the non-zonal perturbation. We observe that the enstrophy density $\mathbb{E}_U[\omega_m^2]$ is regularized by viscosity but diverges for large Reynolds number to the power $1/3$.

We conclude by stating that physical quantities involving higher order derivatives will also be regularized by the viscosity and diverge with the Reynolds number. For instance the

palinstrophy density will diverge as

$$\mathbb{E}_U\left[(\nabla\omega_m)^2\right] \underset{(\frac{2\nu}{sk})^{1/3}l\ll 1}{\sim} C_P \frac{\varepsilon(k^2+l^2)}{2s}\left(\frac{ks}{2\nu}\right)^{2/3}\left(\frac{s}{2\nu k^2}\right)^{1/3},$$

where C_P is a non-dimensional constant.

In this section, we have discussed the case of a force at a longitudinal scale $1/k$ and transverse scale $1/l$. The conclusion for any other forces can easily be obtained by super-position of the contributions from all scales. The general conclusion is that in the limit of large Reynolds number $\frac{s}{2\nu k^2} \gg 1$, the two-point correlation function converges as a distribution, and converges point-wise for values of $y_1 - y_2$ much larger than $(\frac{2\nu}{k_m s})^{1/3}$ where $1/k_m$ is the maximal scale for the forcing. As a consequence, the velocity-velocity and velocity-vorticity correlation functions have a limit independent on the Reynolds number, and the kinetic energy density ($\mathrm{m}^2\,\mathrm{s}^{-2}$) is roughly proportional to $\frac{\varepsilon}{2s}$ where ε is the average energy input rate.

We have made explicit computations only in the case of a linear shear $U(y) = sy$, with $s > 0$. Explicit computations are not easily done in more complex situations in the presence of viscosity. However we expect that for generic U, the singularities of the stationary solutions to the Lyapunov equations are regularized in a universal way.

Advection by a Linear Shear in a Viscous Fluid with Linear Friction   When both linear friction and viscosity are present, the analysis above can be easily generalized. The way the two points correlation function is regularized depends on the relative value of the two length scales $\lambda_\alpha = \frac{2\alpha}{ks}$ and $\lambda_\nu = (\frac{2\nu}{ks})^{1/3}$. When $\lambda_\alpha \gg \lambda_\nu$, the regularization is of a friction type and formula (62) will be correct. When $\lambda_\alpha \ll \lambda_\nu$, the regularization is of a viscous type and formula (69) will be correct.

We stress that, whatever the values of the length scales λ_α and λ_ν, the real part of the regularizing function always decays proportionally to $1/y^4$ for small enough values of y and proportionally to $1/y^2$ for large enough values of y. The location of the crossover between these two behaviors depends on the values of the length scales λ_α and λ_ν. A careful discussion of this issue is addressed in Appendix B.

Here, we only point out that three cases are possible: (i) when $\lambda_\alpha \gg \lambda_\nu$ the regularization is of friction type and the crossover happens for $y \sim \lambda_\alpha$; (ii) when $\lambda_\alpha \ll \lambda_\nu^3(k^2+l^2)$ the regularization is of viscous type and the crossover happens for $y \sim \sqrt{2/(k^2+l^2)}$; ($iii$) when $\lambda_\alpha \ll \lambda_\nu$ but $\lambda_\alpha \gg \lambda_\nu^3(k^2+l^2)$ the singularity is also of viscous type, but in this case the crossover happens for $y \sim \sqrt{2\lambda_\nu^3/\lambda_\alpha}$. The emergence of this last length scale $\sqrt{2\lambda_\nu^3/\lambda_\alpha}$ is due to the different multiplicative factors of the $1/y^2$ decay due to friction and in the $1/y^4$ decay due to viscosity (please see Appendix B).

5.2 Stationary Solutions to the Lyapunov Equation in the Inertial Limit

In the previous section, we have considered a base flow with constant shear, for which analytical solution to the Lyapunov equation can be computed, and provides a qualitative understanding of its solution. We have concluded that it has a stationary solution, in the sense of distributions, even without dissipation. This solution diverges locally for $y_1 = y_2$, related to an infinite enstrophy. We have explained how those local divergences are regularized by a small linear friction or viscosity. The aim of this section is to prove that the same conclusions remain valid for any generic shear flow $U(y)$, which is assumed to be linearly stable. We also

prove that the Reynolds stress divergence $-\mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle]$ and its gradient $-\mathbb{E}_U[\frac{\partial}{\partial y} \langle v_m^{(y)} \omega_m \rangle]$ have finite values. Convergence results for other two-point correlations are also discussed.

At a rough qualitative level, the reason why this is valid is the same as the one discussed in the previous section: the effect of the shear (Orr-mechanism). However this explanation can not be considered as satisfactory. The constant shear flow case has a zero gradient of vorticity, that's why the equation are so simple and can be solved analytically. Whenever the vorticity gradient is non-zero, the hydrodynamic problem becomes drastically different, coupling all fluctuations globally due to the long range interactions involved through the computation of the velocity. Any explanation based on local shear only is then doubtful. Moreover, most of jets in geophysical situations have points with zero shear $U'(y) = 0$. This is also a necessity for jets in doubly periodic geometries. Then the local shear effect can not be advocated.

As already suggested, explicit analytical results are hopeless in this case. In order to prove the result, we rely on two main ingredients. First we use the fact that the stationary solution of the Lyapunov equation can be computed from solutions of the deterministic linearized dynamics, as expressed by formulas (50) and (51). Second we prove that these formulas have limits for large times based on results on the asymptotic behavior of the linearized 2D Euler equations, discussed in the work [7]. At a qualitative level, the results of this paper show that the flow can be divided into areas dominated by the shear for which the Orr mechanism is responsible for a phase mixing leading to an effective dissipation. In other flow regions, for instance close to jet extrema, where no shear is present, a global mechanism called vorticity depletion at the stationary streamlines wipes out any fluctuations, extremely rapidly.

We discuss in this section only the case with no beta effect $\beta = 0$, the case with beta effect will be discussed in a forthcoming publication.

5.2.1 The Orr Mechanism and Vorticity Depletion at the Stationary Streamlines

In this section we summarize existing results on the large time asymptotics of the linearized Euler equations [7]. We consider the linear deterministic advection with no dissipation. Because the corresponding linear operator L_U^0 is not normal, a set of eigenfunctions spanning the whole Hilbert space on which L_U^0 acts does not necessarily exist. Because U is stable, L_U^0 has no eigenmodes corresponding to exponential growth. Moreover it is a very common situation that the Euler operator L_U^0 has no modes at all (neither neutral nor stable nor unstable). A simple example for which this can be easily checked is the case of the linear shear treated in Sect. 5.1.2. We assume in the following that L_U^0 has no mode and $\beta = 0$.

While the vorticity shows filaments at finer and finer scales when time increases, non-local averages of the vorticity (such as the stream-function or the velocity) converge to zero in the long-time limit. This relaxation mechanism with no dissipation is very general for advection equations and it has an analog in plasma physics in the context of the Vlasov equation, where it is called Landau damping [55]. This phenomenon was first studied in the context of the Euler equations in the case of a linear profile $U(y) = sy$ in [14]. The work [7] was the first to study the resolvent and to establish results about the asymptotic behavior of the linearized equations in the case in which the profile $U(y)$ has stationary points y_c such that $U'(y_c) = 0$, which is actually the generic case.

We consider the deterministic linear dynamics $\partial_t \tilde{\omega} + L_U^0 \tilde{\omega} = 0$ with initial condition $e^{ikx} f(y)$. The solution is of the form $\tilde{\omega}(x, y, t) = e^{ikx} \tilde{\omega}_k(y, t)$. From [7], we know that[3]

$$\tilde{\omega}_k(y, t) \underset{t \to \infty}{\sim} \tilde{\omega}_k^\infty(y) e^{-ikU(y)t}. \tag{70}$$

We thus see that the vorticity oscillates on a finer and finer scale as the time goes on. By contrast to the behavior of the vorticity, any integral of the vorticity with a differentiable kernel decays to zero. For instance, the results for the x and y components of the velocity and for the stream function are:

$$\tilde{v}_k^{(x)}(y, t) \underset{t \to \infty}{\sim} \frac{\tilde{\omega}_k^\infty(y)}{ikU'(y)} \frac{e^{-ikU(y)t}}{t}, \tag{71}$$

$$\tilde{v}_k^{(y)}(y, t) \underset{t \to \infty}{\sim} \frac{\tilde{\omega}_k^\infty(y)}{ik(U'(y))^2} \frac{e^{-ikU(y)t}}{t^2}, \tag{72}$$

and

$$\tilde{\psi}_k(y, t) \underset{t \to \infty}{\sim} \frac{\tilde{\omega}_k^\infty(y)}{(ikU'(y))^2} \frac{e^{-ikU(y)t}}{t^2}. \tag{73}$$

In all the above formulas, higher order corrections are present and decay with higher powers in $1/t$. One should also observe that $\tilde{\omega}_k^\infty(y)$ depends on the initial condition $f(y)$. The asymptotic profile $\tilde{\omega}_k^\infty(y)$ could be computed numerically, for instance from the resolvent of the operator L_U^0. An essential point is that $\tilde{\omega}_k^\infty(y)$ has in general no local approximation, it is not a simple function of the local shear but depends on the whole profile U.

These results have been proven for every shear flow U, also in the presence of stationary points y_c such that $U'(y_c) = 0$. Moreover, it has been proved in [7] that at the stationary points $\tilde{\omega}_k^\infty(y_c) = 0$. This phenomenon has been called vorticity depletion at the stationary streamlines. It has been observed numerically that the extend of the area for which $\tilde{\omega}_k^\infty(y_c) \simeq 0$ can be very large, up to half of the total domain, meaning that in a large part of the domain, the shear is not the explanation for the asymptotic decay. The formula for the vorticity Eq. (70) is valid for any y. The formulas for the velocity and stream functions are valid for any $y \neq y_c$. Exactly at the specific point $y = y_c$, the damping is still algebraic with preliminary explanation given in [7], but a complete theoretical prediction is not yet available.

Using these results, we will study the stationary solutions of the Lyapunov equation with no dissipation.

5.2.2 The Vorticity Auto-correlation Function Converges to a Distribution in the Perfect Flow Limit

We prove now that the Lyapunov equation has a stationary solution, in the sense of distributions. The hypothesis are the same as in previous section: U is linearly stable, has no mode, and $\beta = 0$.

In order to understand the long time behavior of the vorticity-vorticity correlation function, we consider Eq. (51). We consider Eq. (70), where $\tilde{\omega}_k(y, t) e^{ikx}$ is the solution of the

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[3]We prefer to use in this section the notation with a tilde to denote the solutions of the deterministic linear dynamics $\partial_t + L_U^0$.

deterministic equation with initial condition $e^{ikx+ily}$ (we note that $\tilde{\omega}_k$ then depends on l through the initial condition e^{ily}). One can check that the integrand of the r.h.s. of this equation behaves for long time as

$$\tilde{\omega}_k(y_1, t_1)\tilde{\omega}_k^*(y_2, t_1) \underset{t_1 \to \infty}{\sim} \tilde{\omega}_k^\infty(y_1)\tilde{\omega}_k^{\infty*}(y_2)\, e^{-ik[U(y_1)-U(y_2)]t_1}. \tag{74}$$

For y_1 and y_2 such that $U(y_1) \neq U(y_2)$ a computation analogous to the one in Sect. 5.1.2, shows that g_{kl}, and hence g^∞ diverge proportionally to time t. This is related to an infinite value of enstrophy.

In the following, we write formulas for the case when U is a monotonic profile. Then each frequency correspond to a single streamline. In the opposite case, two streamlines may have the same frequency, and resonances between streamlines should be considered. The formula would then be more intricate, but the result can be easily obtained from (70) and the conclusion that the limit exists is still true.[4] From (74), we get

$$g_{kl}(\mathbf{r}_1, \mathbf{r}_2, t) = \int_0^t \tilde{\omega}_k(y_1, t_1)e^{ikx_1}\tilde{\omega}_k^*(y_2, t_1)e^{-ikx_2}\, dt_1 + \text{C.C.}$$

$$= \left\{ \frac{\tilde{\omega}_k^\infty(y_1)\tilde{\omega}_k^{\infty*}(y_2)}{-ik[U(y_1) - U(y_2)]}\left[e^{-ik[U(y_1)-U(y_2)]t} - 1\right] + g^r(y_1, y_2, t) \right\}e^{ik(x_1-x_2)}$$

$$+ \text{C.C.}, \tag{75}$$

where g^r is a function which remains continuous in the $t \to \infty$ limit. We thus obtain, for the stationary vorticity-vorticity correlation function

$$g_{kl}^\infty(\mathbf{r}_1, \mathbf{r}_2) = \left\{ \frac{\pi|\tilde{\omega}_k^\infty(y_1)|^2}{|kU'(y_1)|}\delta(y_1 - y_2) - iPV\left(\frac{\tilde{\omega}_k^\infty(y_1)\tilde{\omega}_k^{\infty,*}(y_2)}{k(U(y_1) - U(y_2))}\right) \right.$$

$$\left. + g^r(y_1, y_2, \infty) \right\}e^{ik(x_1-x_2)} + \text{C.C.}, \tag{76}$$

and conclude that the Lyapunov equation has a stationary solution understood as a distribution.

Another quantity of interest is the velocity auto-correlation function and the kinetic energy density. We note that the velocity auto-correlation function is a quadratic quantity, that can be obtained directly as a linear transform from the vorticity-vorticity correlation function g, or from the deterministic solution to the linearized operator. From (50), we see that the contributions from each forcing modes add up. Then

$$\mathbb{E}_U\left[\mathbf{v}_m(\mathbf{r}_1, t) \cdot \mathbf{v}_m(\mathbf{r}_2, t)\right] = \sum_{k>0,l} c_{kl}E_{kl}(\mathbf{r}_1, \mathbf{r}_2),$$

with

$$E_{kl}(\mathbf{r}_1, \mathbf{r}_2) = e^{ik(x_1-x_2)}\int_0^\infty \tilde{\mathbf{v}}_k(y_1, u).\tilde{\mathbf{v}}_k^*(y_2, u)\, du + \text{C.C.}, \tag{77}$$

where $\tilde{\mathbf{v}}_k e^{ikx}$ is the velocity of the deterministic solution to the linearized equations $\partial_t\tilde{\omega} + L_U^0\tilde{\omega} = 0$, with initial condition $e^{ikx+ily}$, as in the previous section. Alternatively, $E_{kl} =$

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[4]We note that the difficult theoretical result related to those resonances is to establish (70).

$\mathbf{V}^{(1)}\mathbf{V}^{(2)*}g_{kl}^{\infty}$, where $\mathbf{V} = -\nabla[\Delta^{-1}(.)] \times \mathbf{e}_z$ is the linear operator giving the velocity from the vorticity and $\mathbf{V}^{(1)}$, resp. $\mathbf{V}^{(2)}$ are the operator $\mathbf{V}$ acting on the first, or second variable respectively. From (71, 72), it is clear that E_{kl} and thus the velocity autocorrelation functions have finite values in the inertial limit, even if no dissipation is present in this limit.

We note that $\alpha \mathbb{E}_U[\mathbf{v}_m(\mathbf{r}_1, t) \cdot \mathbf{v}_m(\mathbf{r}_1, t)]$ is the non-zonal kinetic energy density (the kinetic energy contained in the non-zonal degrees of freedom). We thus conclude that in the limit of very small α, the non-zonal kinetic energy is proportional to α, and that its value can be estimated from the Lyapunov equation with $\alpha = 0$. Those results are extremely important, as they prove that the scaling we have adopted all along this work in order to make an asymptotic expansion is self-consistent.

5.2.3 Convergence of the Reynolds Stress Divergence and of Its Gradient

The stationary value of the Reynolds stress divergence $-\mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle(y)]$ and its gradient $-\mathbb{E}_U[\frac{\partial}{\partial y}\langle v_m^{(y)} \omega_m \rangle(y)]$ are central objects in the kinetic theory. Indeed, they enter in the final kinetic equation when written, respectively, for the evolution of the zonal velocity (35) or for the evolution of the zonal vorticity (34). We have thus to prove that they are finite.

We note that the Reynolds stress is a quadratic quantity that can be obtained directly as a linear transform from the vorticity-vorticity correlation function g. From (50), we see that the contributions from each forcing modes add up. Then

$$F(y) \equiv \mathbb{E}_U\big[\langle v_m^{(y)} \omega_m \rangle(y)\big] = \sum_{k>0,l} c_{kl} F_{kl}(y),$$

with

$$F_{kl}(y) = \lim_{t\to\infty} \int_0^t \tilde{\omega}_k(y, u)\tilde{v}_k^{(y)*}(y, u)\,\mathrm{d}u + \text{C.C.}, \tag{78}$$

where $\tilde{\omega}_k \mathrm{e}^{ikx}$ and $\tilde{v}_k^{(y)} \mathrm{e}^{ikx}$ are the deterministic solutions to the linearized equations $\partial_t\tilde{\omega} + L_U^0\tilde{\omega} = 0$ with initial condition $\mathrm{e}^{ikx+ily}$, as in previous section.

Proving that the long-time limit of the integral in Eq. (78) converges is straightforward. Using Eqs. (70) and (72), one easily realizes that the integrands behaves as $1/u^2$ for large u and thus the integral converges. One conclude that the Reynolds stress divergence is finite.

The computation needed to prove that also the gradient of the Reynolds stress divergence $\sum_{k>0,l} c_{kl}\partial F_{kl}/\partial y$ exists, has to be done with more care. Using (78), (70) and (72), one may be led to the conclusion that a contribution proportional to $1/u$ exists, that would lead to a logarithmic divergence once integrated over time. This is actually not the case because the computation shows that the leading order contributions to the Reynolds stress cancels out exactly, as can be checked easily. As a consequence, we conclude that also the divergence of the Reynolds stress is finite.

In order to reach this conclusion, we have used (70) and (72), which have been established for those values of y that are not stationary for the profile $U(y)$. For a stationary point $y = y_c$ no theoretical results are available for the asymptotic behavior of the velocity field. However, based on numerical evidences reported in [7], we discuss in Appendix C that the Reynolds stress divergence is finite also for $y = y_c$. This will be also checked by direct numerical computations of the Lyapunov equation discussed below.

5.2.4 The Effect of the Small Rayleigh Friction and Viscosity on Stationary Solutions of the Lyapunov Equation

We briefly comment on the effect of the small linear friction, or of a small viscosity. From the discussions in Sects. 5.2.1 and 5.2.3, we have proved that the kinetic energy density and Reynolds stress have limit values independent on α or the viscosity. By contrast, the vorticity-vorticity correlation function $g^\infty(\mathbf{r}_1, \mathbf{r}_2)$ diverges point-wise for $y_1 = y_2$ (more generally for any two points for which the streamlines have the same frequency), and that it has a well defined limit as a distribution. This also implies that the enstrophy contained in non-zonal degrees of freedom is virtually infinite for $\alpha = 0$. We address how this is regularized for small values of α, or due to a small viscosity.

We have already discussed this issue for the particular case of a linear shear $U(y) = sy$, in Sect. 5.1.2, both with and without viscosity. For the general case of any stable base flow U and for a linear friction, we will conclude below that the same conclusion as for the linear shear holds. More precisely, we can prove that in the vicinity of $y_1 = y_2$, the singular behavior is regularized in a universal way (with shape functions independent on U) over a scale $\alpha/kU'(y)$. Moreover we can prove that $\mathbb{E}_U[\omega_m^2(\mathbf{r})]$ diverges proportionally to $1/\alpha$, such that the non-zonal enstrophy density $\alpha\mathbb{E}_U[\omega_m^2(\mathbf{r})]$ has a finite limit (this could have been expected in order to balance the finite enstrophy input rate provided by the stochastic force).

When viscous dissipation is present, we think that it is still true that the divergence of g has a universal regularization, and we think that the scaling for the divergence and enstrophy found for the linear shear are also valid for a generic velocity profile U. We do not prove this result, but discuss why this is plausible at the end of this section.

In order to prove those results, we need to consider the operator $L_U^\alpha = L_U^0 + \alpha$ instead of the inertial L_U^0 for the analysis of the Lyapunov equation. The effect of replacing L_U^0 by L_U^α in Sect. 5.1.1 corresponds simply to add the exponential damping $e^{-2\alpha t_1}$ to the integrands of expressions like the one in Eq. (75). This observation gives us the possibility of understanding how diverging quantities are actually regularized by the presence of a small friction for any zonal flow U. Straightforward computation then leads to a regularization of the inertial result (76) as

$$g_{kl}^\infty(\mathbf{r}_1, \mathbf{r}_2) = \left\{ \frac{\tilde{\omega}_k^\infty(y_1)\tilde{\omega}_k^{\infty*}(y_2)}{ik(U(y_1) - U(y_2)) + 2\alpha} + g_\alpha^r(y_1, y_2, \infty) \right\} e^{ik(x_1 - x_2)} + \text{C.C.} \qquad (79)$$

where, as before, g_α^r has a finite limit when $\alpha \to 0$. In the small α limit and for $y_1 \sim y_2$, we have

$$g_{kl}^\infty(\mathbf{r}_1, \mathbf{r}_2) \underset{\frac{\alpha}{|kU'(y_1)|} \ll 1, y_1 \sim y_2}{\sim} 2\left\{ \cos k(x_1 - x_2) \frac{|\tilde{\omega}_k^\infty(y_1)|^2}{|kU'(y_1)|} \Re\left[F_{\frac{2\alpha}{|kU'(y_1)|}}(y_1 - y_2) \right] \right.$$

$$\left. + \Im\left[\tilde{\omega}_k^\infty(y_1)\tilde{\omega}_k^{\infty*}(y_2)e^{ik(x_1 - x_2)} \right] \frac{1}{kU'(y_1)} \Im\left[F_{\frac{2\alpha}{|kU'(y_1)|}}(y_1 - y_2) \right] \right\}$$

$$+ g_\alpha^{r,2}(\mathbf{r}_1, \mathbf{r}_2), \qquad (80)$$

where $g_\alpha^{r,2}$ has a finite limit when $\alpha \to 0$. We also recall that if $U'(y) = 0$, then $\tilde{\omega}_k^\infty(y) = 0$. Comparing (62) with (80), we conclude that the vorticity-vorticity correlation function is regularized in a universal way close to $y_1 = y_2$.

The fact that the $\mathbb{E}_U[\omega_m^2(\mathbf{r})]$ diverges proportionally to $1/\alpha$, is obtained by observing that Eq. (74) should be replaced by

$$\tilde{\omega}_k(y_1, t_1)\tilde{\omega}_k^*(y_2, t_1) \sim_{t_1 \to \infty} \tilde{\omega}_k^\infty(y_1)\tilde{\omega}_k^{\infty*}(y_2)\, e^{-ik[U(y_1)-U(y_2)]t_1 - 2\alpha t_1}. \qquad (81)$$

We thus have

$$\mathbb{E}_U\left[\omega_m^2(\mathbf{r})\right] = \sum_{k>0,l} c_{kl} g_{kl}^\infty(\mathbf{r}, \mathbf{r}) = \frac{|\tilde{\omega}_k^\infty(y)|^2}{2\alpha} + g_\alpha^r(y, y, \infty) \qquad (82)$$

We have not worked out the case with a small viscosity. This would require first to describe precisely the large time asymptotics of the linearized 2D Navier–Stokes equations. Such results should involve the classical literature on the study of viscous critical layers [23], through matched asymptotics expansions between what happens close to the critical layers and away from it. Those classical results exhibit naturally the scale $(\frac{2\nu}{ks})^{1/3}$ and the family of Airy functions. It should be possible to extent those classical deterministic results to the Lyapunov equation, using the relation between the deterministic solutions and the solutions to the Lyapunov equation discussed at the beginning of this section. For this reason, we guess that the regularization obtained for the linear shear is universal and should explain the regularization for generic velocity profile U. One important technical drawback is that we guess that the classical results for the deterministic equation do not yet exist yet for the case where U has stationary streamlines ($U'(y_c) = 0$) and when several streamlines have the same frequency.

5.3 Numerical Solutions of the Lyapunov Equation

In the previous section we have proved that the Lyapunov equation for the linearized Euler dynamics has a stationary solution. In this section we compute numerically this stationary solution.

For the Lyapunov equation with viscosity, or hyperviscosity, assuming a stable base flow, we expect the stationary solution to be well behaved. Then a natural way to solve the Lyapunov equation would be to discretize the linear operator $L_U^{(1)} + L_U^{(2)}$ and to directly solve the approximate dynamics. This is the traditional way and such a technique has for instance been used in most of previous works using SSST-CE2 [1, 26, 27, 52, 63, 72].

However, we are here specifically interested in the inertial limit. Then one could solve the Lyapunov equation for finite values of α and ν and then study the asymptotic behavior of the results when these parameters go to zero. While feasible, this route seems extremely difficult, as the numerical discretization would have to be increased as ν goes to zero. We have then chosen to try another route in the following. We will make a direct numerical computation of F, the divergence of the Reynolds stress which appears in the kinetic equation (35), which we have proved to be well behaved. In order to do that, we first establish an integral equation verified by h_{kl}, the vorticity-stream function stationary correlation function, which will be precisely defined below. As we will show, this integral equation does not suffer from the instability in the small viscosity limit that would hinder a partial differential equation approach, and can be solved even if $\nu = 0$.

Using this technique, we illustrate the theoretical results of last section: F has a finite limit for the zero friction limit $\alpha \to 0$, the energy in the non-zonal degrees of freedom has a finite limit, and the energy dissipation rate for the non-zonal degrees of freedom goes to zero when α goes to zero. We also check the analyticity breaking, and its order, for the correlation function $h_{kl}(y_1, y_2)$ at the points $y_1 = y_2$.

5.3.1 Computing Stationary Solution of the Lyapunov Equation from an Integral Equation

By definition, the stationary vorticity autocorrelation function g_{kl}^∞ is the solution of the stationary Lyapunov equation (with linear friction and no viscosity)

$$L_U^{\alpha(1)} g_{kl}^\infty + L_U^{\alpha(2)} g_{kl}^\infty = 2\cos\bigl(k(x_1 - x_2) + l(y_1 - y_2)\bigr).$$

Equivalently, we have $g_{kl}^\infty(x_1, x_2, y_1, y_2) = e^{ik(x_1 - x_2)}\tilde{g}_{kl}(y_1, y_2) + \text{C.C.}$ with

$$L_{U,k}^{\alpha(1)}\tilde{g}_{kl} + L_{U,-k}^{\alpha(2)}\tilde{g}_{kl} = e^{il(y_1 - y_2)} \tag{83}$$

where

$$L_{U,k}^\alpha = ikU - ik\bigl(U'' - \beta\bigr)\Delta_k^{-1} + \alpha.$$

We define h_{kl} such that $\tilde{g}_{kl} = \Delta_k^{(2)} h_{kl}$, where $\Delta_k^{(2)}$ is the Laplacian operator with respect to the second variable. The drift term in Eq. (33) can be directly computed from h_{kl}, with

$$F(y) \equiv \mathbb{E}_U\bigl[\langle v_m^{(y)}\omega_m\rangle(y)\bigr] = \sum_{k>0,l} c_{kl} F_{kl}(y),$$

where

$$F_{kl}(y) = 2k\Im\bigl[h_{kl}(y, y)\bigr].$$

In terms of h_{kl}, the stationary Lyapunov equation (83) becomes

$$\Delta_k^{(2)} h_{kl}(y_1, y_2) = \frac{ik(U''(y_1) - \beta)h_{kl}^*(y_2, y_1) - ik(U''(y_2) - \beta)h_{kl}(y_1, y_2) + e^{il(y_1 - y_2)}}{ik(U(y_1) - U(y_2)) + 2\alpha}. \tag{84}$$

We note that this equation is not a differential equation for $h_{kl}(y_1, y_2)$ for each y_1 fixed, as it involves $h_{kl}^*(y_2, y_1)$.

In the previous section, we have shown that g_{kl}^∞, and thus $\tilde{g}_{kl} = \Delta_k^{(2)} h_{kl}$ diverge pointwise for $y_1 \to y_2$ in the limit $\alpha \to 0$. This is related to the vanishing of the denominator for $\alpha = 0$ and $y_1 = y_2$ in Eq. (84). On the other hand, it can be proved, with a very similar reasoning to the one used in Sect. 5.2.3 for F_{kl}, that h_{kl} is well-defined as a function, even in the limit $\alpha \to 0$. Thus, we chose to turn Eq. (84) to an integral equation. Inverting the Laplacian operator using the Green function H_k (see Eq. (58)), we obtain

$$\begin{aligned}
h_{kl}(y_1, y_2) = {} & -\frac{i}{k}\int \frac{H_k(y_2, y_2')e^{il(y_1 - y_2')}}{U(y_1) - U(y_2') - \frac{2i\alpha}{k}}\,dy_2' \\
& + \int \frac{H_k(y_2, y_2')}{U(y_1) - U(y_2') - \frac{2i\alpha}{k}}\Bigl[(U''(y_1) - \beta)h_{kl}^*(y_2', y_1) \\
& - (U''(y_2') - \beta)h_{kl}(y_1, y_2')\Bigr]\,dy_2'.
\end{aligned} \tag{85}$$

The generalization to the 2D barotropic equation upon a topography is straightforward, replacing β by $h'(y)$ in (85). The main advantage of this equation is that it involves only well-behaved functions, even in the limit of no dissipation $\alpha \to 0$. Indeed, in this limit, the integrals converge to their Cauchy principal values. Moreover, the fact that it doesn't involve any space derivative will make it easy to solve numerically, as discussed in next section.

5.3.2 Numerical Implementation

In order to numerically compute solutions of Eq. (85), we chose an iterative scheme. We compute the sequence $\{h_N\}_{N \geq 0}$ with

$$h_{N+1} = S + T[h_N],$$

where S is the first term in the right hand side of Eq. (85) and T is the integral operator of the second term. If this sequence converges, then $h_{kl} = \lim_{N \to \infty} h_N$.

We note that we have not been able to establish conditions for which T is contracting. As a consequence, the convergence of the algorithm is not guaranteed, and we will establish the convergence empirically on a case by case basis. More precisely, the convergence of the iterative algorithm is checked by plotting $\log \|h_N - h_{N-1}\|$ as a function of the number of iterations N, where $\|.\|$ is the L^2 norm.

The computation of integrals of the form $\int \frac{f_k(y,y')}{U(y)-U(y')-2i\alpha/k} \, \mathrm{d}y'$ requires a particular attention. Indeed, the singularity of the denominator at the points such that $U(y') = U(y)$ can be the source of important numerical errors, and we find that the result strongly depends on the resolution if it is not precise enough. The resolution required to get robust results is easily understood: in order to avoid numerical errors, the denominator must satisfy $|U(y) - U(y \pm \Delta y)| \ll \frac{2\alpha}{k}$ for a discretization step Δy. More precisely, for sufficiently small Δy, we have $U(y) - U(y \pm \Delta y) \simeq \pm U'(y) \Delta y$, so that the condition becomes $\Delta y \ll \frac{2\alpha}{k|U'|}$. The numerical results confirm this scaling, for base flows with no stationary points. This is an important remark, because the iterative algorithm may converge and yet give a wrong result if the condition $\Delta y \ll \frac{2\alpha}{k|U'|}$ is not respected. For base flows with stationary points, the iterative algorithm often gives problems of convergence in the small α limit.

We have recently devised a different algorithm that systematically converges and allows to compute stationary solution of the Lyapunov equation for zero viscosity, with and without stationary points. This much more robust and versatile algorithm will be presented in a following paper.
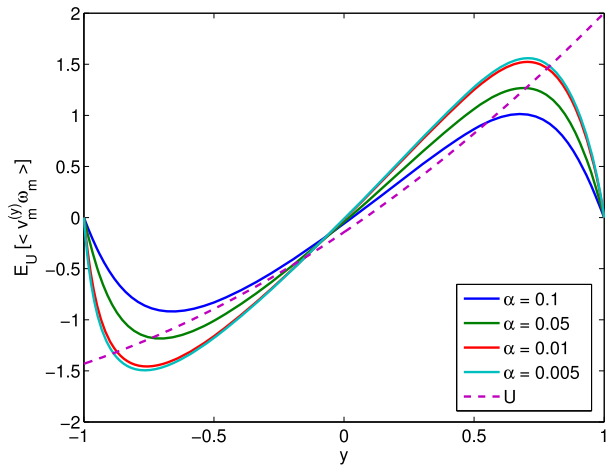
5.3.3 Reynolds Stress and Stationary Correlation Functions

The properties of F and g_{kl}^∞ discussed at a theoretical level in Sects. 5.1 and 5.2 are now illustrated in two examples.

Parabolic Profile We consider the case of the 2D Euler equations ($\beta = h = 0$) in the channel $(x, y) \in [0, 2\pi) \times [-1, 1]$ with rigid boundary conditions $\psi_m(x, \pm 1) = 0$, $\mathbf{v}_m(x + 2\pi, y) = \mathbf{v}_m(x, y)$, and with a parabolic base flow $U(y) = A(y + 2)^2 - U_0$, where the constants A and U_0 are chosen so that the total energy is 1 and the total momentum is 0. This flow has no inflection point, whence its linear stability by direct application of Rayleigh's inflection point theorem [23]. Moreover, this flow has no stationary points (y_0 such that $U'(y_0) = 0$). We will consider next the case of a cosine profile, which has stationary points. We moreover chose here, for sake of simplicity, to force only one mode $k = l = 1$. This corresponds to the forcing correlation function $C(\mathbf{r}) = c_{11} \cos(x + y)$, with $c_{11} = 4.72$.

We have numerically computed the solution of (85) with the iterative scheme previously explained. As mentioned above, the necessary resolution to use depends on the value of α; in the present case, it ranges from $\Delta y = 1/60$ for the largest value $\alpha = 0.1$ to $\Delta y = 1/300$ for the smallest one, $\alpha = 0.005$. Figure 1 shows the Reynolds stress divergence $F(y) = 2kc_{kl} \Im[h_{kl}(y, y)]$ and it clearly illustrates the convergence of F for $\alpha \to 0$, as theoretically

Fig. 1 The Reynolds stress divergence $F(y) = \mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle] = 2kc_{kl} \Im \times [h_{kl}(y, y)]$ in the case of a parabolic base profile in a channel geometry, with $k = l = 1$ and different values of the friction coefficient α. We check the convergence of F to a smooth function in the inertial limit, as theoretically predicted in Sect. 5

described in Sect. 5.2. We also note that the Reynolds stress divergence and the base flow profile U have the same sign, except in a small region near $y = 0$. Looking at the evolution equation for U (37)

$$\partial_t U = \alpha F - 2\alpha U + \alpha \xi, \tag{86}$$

this observation implies that the Reynolds stress is actually forcing the zonal flow.

Others correlation functions can be very easily computed from h_{kl} and the analyticity properties that we have predicted can be directly checked. For instance, the vorticity auto-correlation function $g_{kl}^\infty(x_1, x_2, y_1, y_2) = e^{ik(x_1-x_2)}\Delta_k^{(2)}h_{kl}(y_1, y_2) + \text{C.C.}$ is represented in Fig. 2. We clearly see the divergence at the point $y_1 = y_2 = 0$ when $\alpha \to 0$. Moreover, we recover the universal shape of $\Delta_k^{(2)}h_{kl}(y_1, y_2)$ near this divergence, as expected from Eq. (79): for $\frac{2\alpha}{ks_0} \ll y \ll 1$, where $s_0 = U'(0)$ is the local shear at $y = 0$, we have

$$\begin{aligned}
\Re\left[\Delta_k^{(2)}h_{kl}(y, 0)\right] &\sim \frac{2\alpha}{k^2(U(y) - U(0))^2 + 4\alpha^2}, \\
\Im\left[\Delta_k^{(2)}h_{kl}(y, 0)\right] &\sim \frac{-k(U(y) - U(0))}{k^2(U(y) - U(0))^2 + 4\alpha^2}.
\end{aligned} \tag{87}$$

The comparison between this theoretical prediction and the numerical results is shown in Fig. 3.

Cosine Profile The second example we consider is the zonal base flow $U(y) = \cos y$ in the domain $(x, y) = [0, 2\pi l_x) \times [-\pi, \pi)$ with periodic boundary conditions, which is usually referred to as the Kolmogorov flow [7]. This flow is stable and the linearized operator associated to this flow has no normal modes for aspect ratio $l_x < 1$ [7]. We choose the parameters $l_x = 0.5$, $k = 2$, $l = 0$, corresponding to the forcing correlation function $C(\mathbf{r}) = c_{20}\cos(2x)$, with $c_{20} = 1.29$. The Reynolds stress divergence F is plotted in Fig. 4. It converges to a smooth function in the inertial limit. For all points such that $U'(y) \neq 0$, this was expected from the theoretical results of Sect. 5. We note that we have also a convergence of $F(y)$ to a finite limit at the stationary points $y = 0, \pi$, as discussed at the end of Sect. 5.2.3. We observe that the Reynolds stress is forcing the flow except in some regions around the zeros of U, like in the case of the parabolic zonal flow.
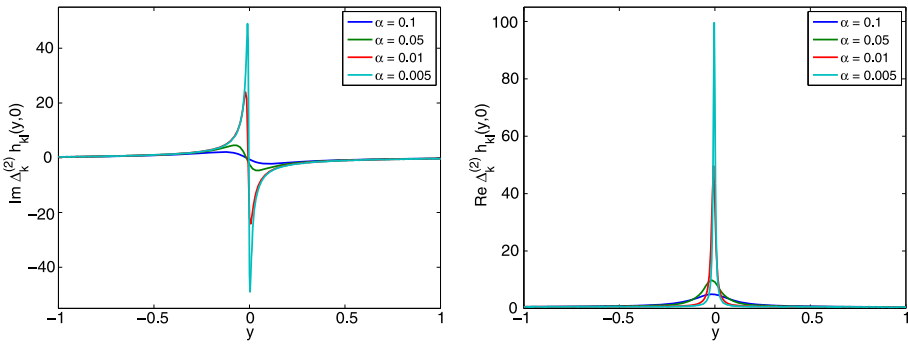
Fig. 2 Real and imaginary parts of the k-th Fourier component of the stationary vorticity auto-correlation function $\Delta_k^{(2)} h_{kl}(y, y_2)|_{y_2=0}$ in the case of a parabolic base profile in a channel geometry, with $k = l = 1$ and different values of the friction coefficient α. The plots clearly show the expected divergence at $y = 0$
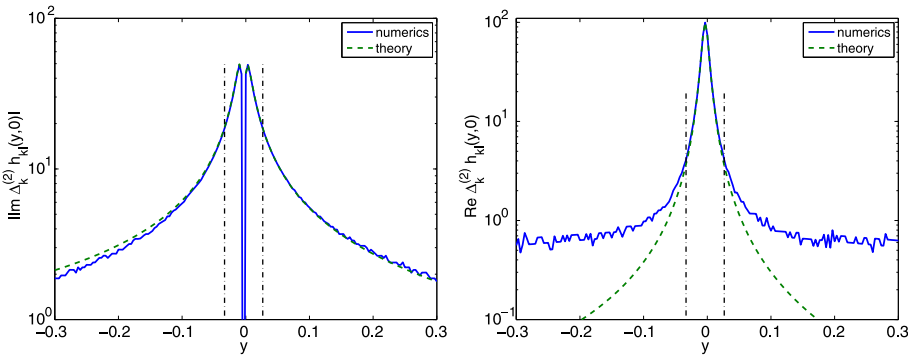

Fig. 3 Divergence of the stationary vorticity auto-correlation function $\Delta_k^{(2)} h_{kl}(y, y_2)|_{y_2=0}$ near $y = 0$, in the case of a parabolic base profile in a channel geometry, with $k = l = 1$ and $\alpha = 0.005$. As expected, the comparison between the numerical result and the universal shape (87) is very good in the range $1 \gg y \gg \frac{2\alpha}{ks_0} \simeq 0.006$ (the *area between the vertical lines*)

5.3.4 Conservation Laws

In the channel geometry considered, the linear momentum is always zero, $P_x = \int dy \, v_x = 0$. This implies the constraint $\int dy \, U(y) = 0$. At the level of the kinetic equation (86), this implies that the integral of the divergence of the Reynolds stress is zero: $\int dy \, \mathfrak{I}[h_{kl}(y, y)] = 0$, which is also a trivial consequence of the definition of h_{kl}. This constraint is fulfilled by the numerical results in Fig. 1 within an error of order 10^{-8}.

We now discuss the energy and enstrophy balance derived in Sects. 4 and 5. The theoretical results predict a vanishing of the non-zonal energy in the inertial limit, and a divergence of the non-zonal enstrophy as $1/\alpha$. Figure 5 shows the three rates of energy involved in (42) as a function of $1/\alpha$: the power injected by the forcing (by definition equal to 1)

$$\sigma_m = -2\pi^2 l_x \left(\Delta^{-1} C_m \right)(\mathbf{0}) = 1,$$

Fig. 4 The Reynolds stress divergence $F(y) = \mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle] = 2kc_{kl}\,\Im \times [h_{kl}(y,y)]$ in the case of a cosine base profile $U(y) = \cos y$ in a periodic geometry, with $k = 2$, $l = 0$ and different values of the friction coefficient α. Again, we observe the convergence of F towards a smooth function when $\alpha \to 0$, even at the stationary points $y = 0$ and $y = \pi$ where we do not have full theoretical predictions
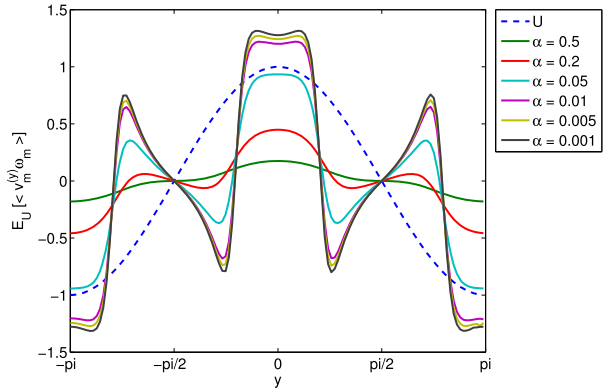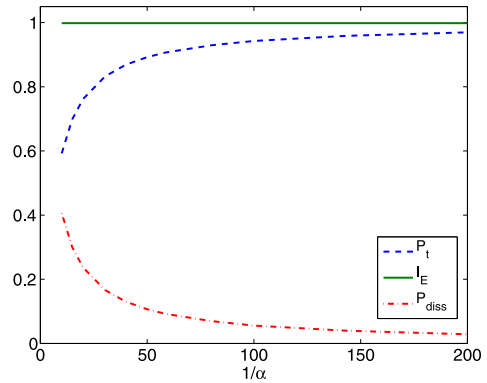

Fig. 5 Stationary energy balance for the fluctuations, in the case of a parabolic base profile in a channel geometry, with $k = l = 1$ and different values of the friction coefficient α. We see that in the inertial limit $\alpha \ll 1$, all the energy injected in the fluctuations is transferred to the zonal flow, while the energy dissipated in the fluctuations vanishes

the power transferred to the zonal flow

$$P_t = \pi l_x \int dy\, \mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle(y)] U(y) = \pi l_x \int dy\, 2kc_{kl}\,\Im[h_{kl}(y,y)]U(y),$$

and the rate of energy dissipated in the non-zonal degrees of freedom

$$P_{diss} = \alpha E_m = -2\alpha \int dy\, \Re[h_{kl}(y,y)].$$

It illustrates the fundamental property that in the inertial limit, all the energy injected in the fluctuations is transferred to the zonal flow, as expected from the discussion in Sect. 4. We note that the energy balance (42) is verified in our numerical solutions up to errors of order 10^{-4}.

Enstrophy has a very different behavior. From the general results of Sect. 5, we know that the dissipated enstrophy is not expected to go to zero as $\alpha \to 0$, so the transferred enstrophy is not expected to be close to the injection rate. In the case of a parabolic profile $U(y) = A(y+2)^2 - U_0$, this property is even more dramatic as the transferred enstrophy vanishes, it is indeed

$$Z_t = -\pi l_x \mathbb{E}\left[\int dy\, U''(y)\langle v_m^{(y)} \omega_m \rangle(y)\right] = -\pi l_x \int dy\, 2A \cdot 2kc_{kl}\,\Im[h_{kl}(y,y)] = 0$$

which is zero because it is proportional to the conservation of the momentum. Whatever the value of α, no enstrophy is transferred to the zonal flow and all the enstrophy injected by the stochastic forcing is dissipated in the fluctuations. When it comes to the numerical results, checking that the enstrophy balance is fulfilled amounts at checking the conservation of the momentum $\int dy\, \mathfrak{I}[h_{kl}(y, y)] = 0$, which has already been discussed.

6 Attractors of the Slow Dynamics and Multistability of Atmosphere Jets

In Sect. 3, we have obtained that the jet has a slow evolution whose dynamics is described by Eq. (37), reported here for convenience:

$$\frac{\partial U}{\partial \tau} = F[U] - U + \frac{\nu}{\alpha} \frac{\partial^2 U}{\partial y^2} + \xi, \tag{88}$$

where ξ is Gaussian process with autocorrelations given by (36). In most known situations, there is no noise acting on the largest scales. Then the correlation function of ξ, see Eq. (36), is of an order α smaller than $F[U]$. In the limit we consider, we thus expect the noise to be small. Then one may wonder how important is its effect for practical purposes. We discuss this issue here, with emphasis on the case where the deterministic equation $\frac{\partial U}{\partial \tau} = F[U] - U + \nu/\alpha \partial_y^2 U$ has more than one attractor.

First it is useful to note that the deterministic equation $\frac{\partial U}{\partial \tau} = F[U] - U + \nu/\alpha \partial_y^2 U$ is unable to describe the statistics of the small fluctuations close to a an attractor U_0. A first very interesting result that can be derived from (88) is the statistics for the Gaussian fluctuations of the jet close to its most probable value.

If we now assume that the deterministic equation $\frac{\partial U}{\partial \tau} = F[U] - U + \nu/\alpha \partial_y^2 U$ has more than one attractor, for instance U_0 and U_1, the small noise ξ becomes essential in order to describe the relative probability of the two attractors and the transition probabilities between them. These can be computed numerically from (88) by means of large deviation theory or, equivalently using a path integral representations of the stochastic process (88) and instanton theory. One may wonder if such bistability or multistability situations actually exist.

Multistability and phase transitions are actually very common for turbulent flows (for instance, paths of the Kuroshio current [69], atmospheric flows [76], Earth's magnetic field reversal and MHD experiments [2], two-dimensional turbulence simulations and experiments [8, 46, 47, 71], and three-dimensional flows [66] show this kind of behavior). Figure 6 shows random transitions in the 2D Navier-Stokes equations [8]. These transitions may in principle be described in the kinetic approach developed in the present paper; however, this would require an extension of the work to cases where non zonal attractors are also considered. This is in principle possible, but the Lyapunov equation is then more tricky. We postpone such an issue for future works.

Remaining in the zonal framework assumed in this paper, we refer to works by N. Bakas, N. Constantinou, B. Farrell, and P. Ioannou [19, 26, 27] for existence of bistability. In those three works the possibility for multiple attractors for SSST dynamics (the deterministic part of Eq. (88)) is discussed. Figure 7, shows such a case, where the evolution of SSST dynamics has been computed for some fixed parameters and fixed force spectrum. The two spatiotemporal diagrams of the zonally averaged velocity profile U obtained from the barotropic equations, and the two asymptotic profiles U obtained both from SSST dynamics and the barotropic equations show that for this set of parameters the deterministic part of the dynamics has at least two attractors, and that these attractors are observed in the barotropic
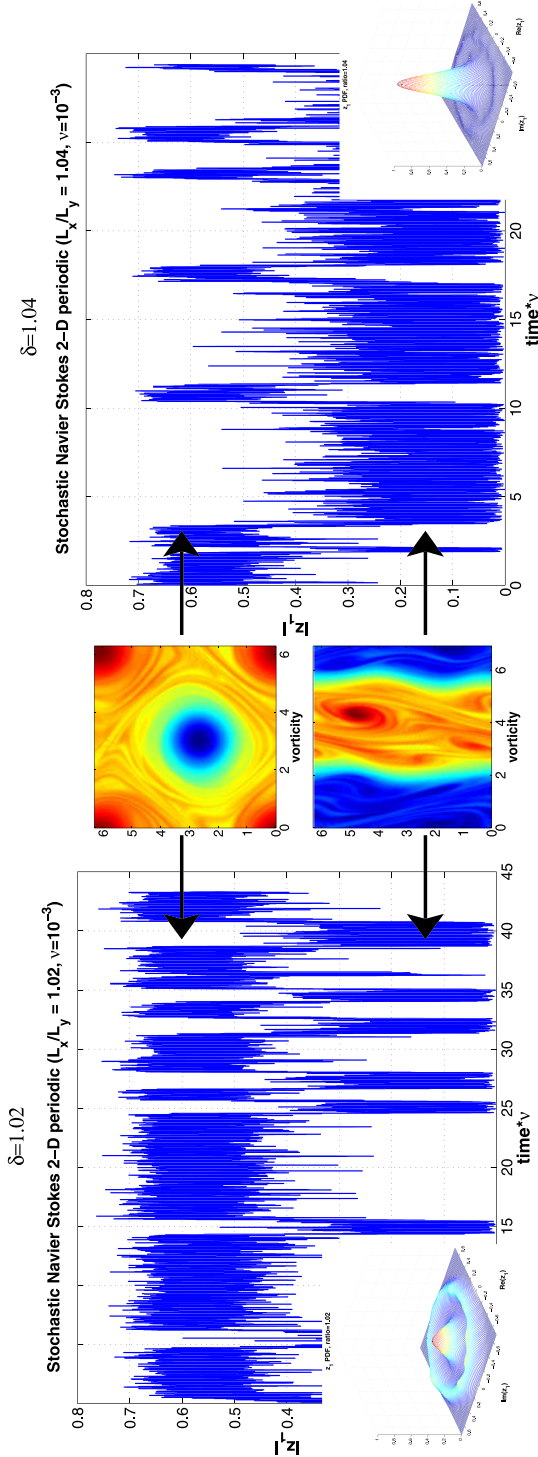
Fig. 6 Figure taken from [8] showing rare transitions (illustrated by the Fourier component of the largest y mode) between two large scale attractors of the periodic 2D Navier-Stokes equations. The system spends the majority of its time close to the vortex dipole and zonal flow configurations
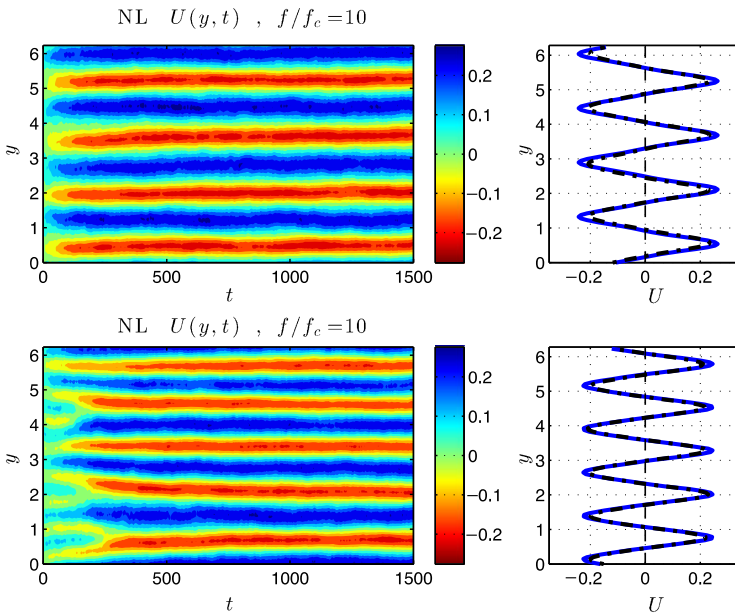
Fig. 7 Evolution of the zonally averaged velocity profile as obtained from non-linear simulations (*left*) and comparison of the stationary velocity profile obtained from non-linear simulations and from the deterministic part of the kinetic equation (88), or equivalently SSST [19, 26, 27]. The *upper* and *lower pictures* are obtained for the same values of the physical parameters but with different initial conditions. The figure shows that for a given set of parameters it can converge towards two attractors, U_0 and U_1. Courtesy N. Constantinou

equations. In order to compute the relative probability of the attractors and the transition probabilities from one to the other, one can rely on the Fokker–Planck equation (33) or on the stochastic dynamics (88). These transitions involve states that are arbitrarily far from the attractors U_0 and U_1.

7 Conclusion

In this paper, we have considered the kinetic theory of the equations for a barotropic flow with beta effect or topography and with stochastic forces, in the limit of weak forces and dissipation $\alpha \to 0$. We have formally computed the leading order dynamics, using the framework of stochastic averaging for this adiabatic process. At leading order, we have obtained a Fokker–Planck equation, or equivalently a stochastic dynamics, that describes the slow jet dynamics (37). We have discussed that, at leading order, the deterministic part of this dynamics are the equations known as Stochastic Structural Stability Theory or, equivalently what can be obtained from a second order closure theory. This provides a strong support for those equations if one wants to compute the attractors for the jet dynamics, when α is sufficiently small. This result also shows that those models do not describe completely Gaussian fluctuations close to these attractors. Moreover, in phenomena for which large deviations beyond Gaussian become essential, a kinetic approach is still relevant, and the full kinetic equation (37) should be considered.

We have also proven that at leading order, quasilinear dynamics will lead to the same kinetic equation (37) in the limit of weak forces and dissipation. We stress however that this will be wrong at next order.

Our main hypothesis is that the zonal jets with profile $U(y)$ are stable and have no non-zonal neutral modes. Whereas the stability is a crucial hypothesis, the non-neutral mode hypothesis is in principle not essential. However, if this hypothesis would not be fulfilled, a different kinetic equation would be obtained.

For the kinetic theory to be meaningful, the base flows have to be attractors of the inertial dynamics. This is actually the case for some of zonal jets. However non-zonal attractors also exist, for instance dipoles for the 2D Euler equations [8, 18]. The kinetic theory could be extended in principle to non-zonal attractors. It would be easy to write phenomenologically the kinetic equation, just as was done for zonal jets. We stress however that proving the self-consistency of the theory, which is the key point from a theoretical point of view, would be much more tricky. This requires the analysis of the asymptotic solutions of the Lyapunov equation, which would be technically more difficult for non-zonal attractors than for zonal jets.

We stress that we have proven the consistency of the kinetic theory only for the case of the two-dimensional Navier-Stokes equations. With a beta effect or a topography, the formal structure of the problem will be very similar, so that the formal generalization of the kinetic theory is straightforward. However no theoretical results for the asymptotic decay of the linearized equation are yet available in the literature. As a consequence we cannot draw, for the moment, any conclusion about the convergence of the velocity-vorticity correlation functions and of the Reynolds stress without dissipation in these cases.

A number of interesting issues have not been addressed in this work and should be considered in the future. Whereas the limit $\alpha \to 0$ is the appropriate one in order to set up a clear theoretical framework, an essentially question arises from a practical point of view: how small α should be? How does the answer depend on the force spectrum? These issues could certainly be addressed both numerically and theoretically.

We believe that our theoretical approach is the correct way to describe the statistics of the largest scale of the flow. However, for small but finite α, especially if the forcing has a narrow band spectrum, it is likely that non-linear effects would become important. In some regimes, we expect to see together with a quasi-linear dynamics at the largest scales, either enstrophy cascade, or inverse energy cascade, or both, at scales much smaller than a typical jet scale. Those are described by Kraichnan type cascades for the 2D-Navier-Stokes equations [4], or by zonostrophic turbulence if $\beta \neq 0$ [31, 32]. Those nonlinear effects, basically small scale turbulent mixing, would then be the correct mechanism that would regularize the Lyapunov equations. They would also be responsible for the loss of information about the initial condition, discussed in Sect. 3.2. In such a case, how would our large scale theory relate with smaller scale cascade pictures for finite α? What would be the crossover scale and how should it depend on α? While answer using dimensional analysis can be given easily, a more detailed theoretical study would be very interesting. Another related issue is the effects of the small scale part of the force spectrum. All over the paper, we have assumed that the spatial correlation C is smooth and that its spectrum converges sufficiently fast for large wave numbers $|\mathbf{k}|$, in order to insure the convergence of the sum of the contributions of all sectors k to the Lyapunov equation. Could the required hypothesis be made more explicit?

We have made explicit computations up to leading order only. It would however be extremely interesting to compute next order corrections. This may be important in order to have very precise predictions for small but finite α and also to give a clear indication of the order of magnitude of α required for the theory to be trusted. Undoubtedly the answer

will depend much on the force spectrum. We stress that the next order corrections will be different from an approach using a cumulant expansion and that it should not be hindered with the usual inconsistency observed in cumulant expansions.

Another important issue is the validity of our approach for more complex dynamics, for instance for the class of quasi-geostrophic dynamics. At a formal level, the same approach can be applied. However, the dynamics is expected to be quite different. For instance, whereas the fact that there is no non-zonal neutral modes is probably generic for barotropic flows, it is probably exceptional for equivalent barotropic quasi-geostrophic dynamics as soon as the Rossby deformation radius is smaller than the domain size. Also if one considers a deterministic force in addition to the stochastic one, which is essential for instance for Earth jet dynamics, then the same formalism can be applied, but the phenomenology is expected to be quite different. This calls for new specific studies.

Appendix A: Explicit Computation of the Operators in the Zonal Fokker–Planck Equation

We compute here each term of the equation for the slowly varying part of the PDF (32), which we report:

$$\frac{\partial P_s}{\partial t} = \left\{ \alpha \mathcal{P}\mathcal{L}_z + \alpha^{3/2} \mathcal{P}\mathcal{L}_z \int_0^\infty dt' \, e^{t'\mathcal{L}_0} \mathcal{L}_n \right.$$
$$\left. + \alpha^2 \mathcal{P}\mathcal{L}_z \int_0^\infty dt' \, e^{t'\mathcal{L}_0} \left[(1-\mathcal{P})\mathcal{L}_z + \int_0^\infty dt'' \, \mathcal{L}_n e^{t''\mathcal{L}_0} \mathcal{L}_n \right] \right\} P_s + \mathcal{O}(\alpha^{5/2}). \quad (89)$$

This will lead to the final reduced Fokker–Planck equation for the zonal flow PDF R, (33).

− The first term in Eq. (89) gives

$$\mathcal{P}\mathcal{L}_z P_s = G[q_z, \omega_m] \int dy_1 \frac{\delta}{\delta q_z(y_1)} \left[\left(\frac{\partial}{\partial y} \mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle](y_1) + \omega_z(y_1) - \frac{\nu}{\alpha} \Delta \omega_z(y_1) \right) R \right.$$
$$\left. + \int dy_2 \, C_z(y_1, y_2) \frac{\delta R}{\delta q_z(y_2)} \right]. \quad (90)$$

− The next term is exactly zero: we have

$$\mathcal{L}_n P_s = \int d\mathbf{r}_1 \frac{\delta}{\delta \omega_m(\mathbf{r}_1)} \left[\left(\mathbf{v}_m . \nabla \omega_m(\mathbf{r}_1) - \langle \mathbf{v}_m . \nabla \omega_m(\mathbf{r}_1) \rangle \right) G[q_z, \omega_m] R[q_z] \right].$$

We remark that the action of the functional derivative $\frac{\delta}{\delta \omega_m(\mathbf{r}_1)}$ will produce factors which are either linear or cubic in the variable ω_m. When applying $\mathcal{L}_z$, those terms will be

multiplied by constant or quadratic factors. The projection $\mathcal{P}$ will then lead to the computation of odd moments of the centered Gaussian distribution G, which are all null. As a consequence, we conclude that

$$\mathcal{P}\mathcal{L}_z e^{t'\mathcal{L}_0} \mathcal{L}_n P_s = 0. \tag{91}$$

This is a very important result: at first-order, the corrections to the quasi-linear dynamics due to the full nonlinearity exactly vanish. This is a partial explanation to the success of the quasi-linear dynamics: the non-linear interactions between the non-zonal degrees of freedom (eddy-eddy interaction) don't have any influence on the statistics of the zonal flow until the order α^2.

– Terms of order $O(\alpha^2)$ in Eq. (89) give non-trivial contributions. We write

$$e^{t'\mathcal{L}_0} \mathcal{L}_n e^{t''\mathcal{L}_0} \mathcal{L}_n P_s = M[q_z, \omega_m](t', t'') G[q_z, \omega_m] R[q_z].$$

Then,

$$\mathcal{P}\mathcal{L}_z e^{t'\mathcal{L}_0} \mathcal{L}_n e^{t''\mathcal{L}_0} \mathcal{L}_n P_s$$

$$= G[q_z, \omega_m] \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left[\frac{\partial}{\partial y_1} \mathbb{E}_U \big[\langle v_m^{(y)} \omega_m \rangle(y_1) M[q_z, \omega_m](t', t'') \big] R[q_z] \right]$$

$$+ G[q_z, \omega_m] \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left[\left(\omega_z(y_1) - \frac{\nu}{\alpha} \Delta \omega_z(y_1) \right) \mathbb{E}_U \big[M[q_z, \omega_m](t', t'') \big] R[q_z] \right]$$

$$+ G[q_z, \omega_m] \int \mathrm{d}y_1 \mathrm{d}y_2 \, C_z(y_1, y_2) \frac{\delta^2}{\delta q_z(y_1)\delta q_z(y_2)} \big[\mathbb{E}_U \big[M[q_z, \omega_m](t', t'') \big] R[q_z] \big].$$

We first consider the average value of M. As $\mathbb{E}_U$ is an average over the stationary measure G, we have

$$\mathbb{E}_U \big[M[q_z, \omega_m](t', t'') \big] = \mathbb{E}_U \big[M_1[q_z, \omega_m](t' - t'') \big]$$

with M_1 defined by

$$\mathcal{L}_n e^{t'-t''\mathcal{L}_0} \mathcal{L}_n G[q_z, \omega_m] = M_1[q_z, \omega_m](t' - t'') G[q_z, \omega_m].$$

Using the expression of $\mathcal{L}_n$ as a functional derivative with respect to ω_m (Eq. (20)), we then easily show that $\mathbb{E}_U[M[q_z, \omega_m](t', t'')] = 0$, so that the interesting quantity simplifies to

$$\mathcal{P}\mathcal{L}_z e^{t'\mathcal{L}_0} \mathcal{L}_n e^{t''\mathcal{L}_0} \mathcal{L}_n P_s$$

$$= G[q_z, \omega_m] \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left(\frac{\partial}{\partial y_1} \mathbb{E}_U \big[\langle v_m^{(y)} \omega_m \rangle(y_1) M[q_z, \omega_m](t', t'') \big] R[q_z] \right).$$

– In the last term, we have

$$(1 - \mathcal{P})\mathcal{L}_z P_s = G[q_z, \omega_m] \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left[\frac{\partial}{\partial y} \big(\langle v_m^{(y)} \omega_m \rangle - \mathbb{E}_U \big[\langle v_m^{(y)} \omega_m \rangle \big] \big)(y_1) R \right],$$

thus

$$\mathcal{L}_z e^{t'\mathcal{L}_0}(1-\mathcal{P})\mathcal{L}_z P_s$$

$$= \int \mathrm{d}y_2 \frac{\delta}{\delta q_z(y_2)} \left\{ \left(\frac{\partial}{\partial y_2} \langle v_m^{(y)} \omega_m \rangle (y_2) + \omega_z(y_2) - \frac{\nu}{\alpha} \Delta \omega_z(y_2) \right. \right.$$

$$+ \int \mathrm{d}y_3 \, C_z(y_2, y_3) \frac{\delta}{\delta q_z(y_3)} \bigg) \cdots$$

$$\times e^{t'\mathcal{L}_0} \left[G[q_z, \omega_m] \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left[\frac{\partial}{\partial y_1} \left(\langle v_m^{(y)} \omega_m \rangle - \mathbb{E}_U[\langle v_m^{(y)} \omega_m \rangle] \right) (y_1) R \right] \right] \right\}.$$

Using the definition of the correlation and covariance (26) and (27) we conclude that

$$\mathcal{P}\mathcal{L}_z e^{t'\mathcal{L}_0}(1-\mathcal{P})\mathcal{L}_z P_s = G[q_z, \omega_m] \int \mathrm{d}y_2 \mathrm{d}y_1 \frac{\delta^2}{\delta q_z(y_1)\delta q_z(y_2)} \frac{\partial^2}{\partial y_2 \partial y_1}$$

$$\times \left\{ \mathbb{E}_U \left[\left[\langle v_m^{(y)} \omega_m \rangle (y_1, t') \langle v_m^{(y)} \omega_m \rangle (y_2, 0) \right] \right] R[q_z] \right\}. \tag{92}$$

Appendix B: Linear Friction and Viscous Regularization of Sokhotskyi–Plemelj Formula

The Sokhotskyi–Plemelj formula adapted to our notations is

$$\int_0^\infty e^{-ity} \, \mathrm{d}t = \lim_{\alpha \to 0^+} \frac{-i}{y - i\alpha} = \pi \delta(y) - iPV\left(\frac{1}{y}\right). \tag{93}$$

In this Appendix, we consider different regularizations of the above equation that correspond to a viscous regularization or to the combination of the regularizations given by viscosity and by linear friction. We thus deduce few useful technical results.

In the following, we have to deal with the large y behavior of oscillatory integrals of the form

$$\int_0^\infty e^{-iyt} f(t) \, \mathrm{d}t. \tag{94}$$

In this respect, we will use the well known result that

$$\mathfrak{R}\left[\int_0^\infty e^{-iyt} f(t) \, \mathrm{d}t \right] = \sum_{n=1}^N \frac{f^{(2n-1)}(0)}{i^{2n} y^{2n}} + \mathrm{O}\left(\frac{1}{y^{2N+2}} \right) \tag{95}$$

and

$$\mathfrak{I}\left[\int_0^\infty e^{-iyt} f(t) \, \mathrm{d}t \right] = -\sum_{n=0}^N \frac{f^{(2n)}(0)}{i^{2n} y^{2n+1}} + \mathrm{O}\left(\frac{1}{y^{2N+3}} \right). \tag{96}$$

The above formula holds for any function f which is $(2N+1)$ times differentiable, such that $\lim_{t \to \infty}[f^{(n)}(t)] = 0$ for any $n \le 2N$ and such that $\int_0^\infty \mathrm{d}t e^{iyt} f^{(n)}(t)$ is finite for any $n \le 2N+1$. They can be found by successive part integrations.

Viscous Regularization   We consider the regularization of Plemelj formula that correspond to the effect of viscosity. We consider

$$H_\lambda(y) = \frac{1}{\lambda} \int_0^\infty e^{-i\frac{y}{\lambda}u - \lambda^2(k^2+l^2)u + \lambda l u^2 - \frac{1}{3}u^3} \, \mathrm{d}u. \tag{97}$$

We note that the real part of H_λ is an even function and the imaginary part of H_λ is an odd function.

We can also compute the integral of H_λ using the fact that

$$\int_{-\infty}^{\infty} H_\lambda(y)\mathrm{d}y = \frac{1}{2\lambda} \int_{-\infty}^{\infty} \mathrm{d}y' \int_{-\infty}^{\infty} e^{-i\frac{y'}{\lambda}t - \lambda^2(k^2+l^2)t + \lambda l t^2 - \frac{1}{3}t^3} \, \mathrm{d}t, \tag{98}$$

which can be proved by integrating Eq. (97) and with the two changes of variable $y' = -y$ and $t = -u$. Using in the last expression the integral representation of the Dirac delta, we obtain the desired result

$$\int_{-\infty}^{+\infty} H_\lambda(y) \, \mathrm{d}y = \pi. \tag{99}$$

We want now to prove that $H_\lambda(y) \underset{\lambda \to 0}{=} G_\lambda(y) + \mathcal{O}(\lambda)$ for every value of y. Let us consider $\tilde{H}_\lambda(Y) = H_\lambda(\lambda Y)$ and study the large Y asymptotic behavior of $\tilde{H}_\lambda$. Supposing small viscosity, $\lambda l \ll 1$ and $\lambda^2(k^2 + l^2) \ll 1$, and by applying Eqs. (95) and (96) to $\tilde{H}_\lambda$, we obtain for the real part

$$\Re\big[\tilde{H}_\lambda(Y)\big] \underset{1 \ll Y \ll \frac{1}{\lambda}\sqrt{\frac{2}{k^2+l^2}}}{\sim} -\frac{2}{\lambda Y^4} \qquad \text{and} \qquad \Re\big[\tilde{H}_\lambda(Y)\big] \underset{Y \gg \frac{1}{\lambda}\sqrt{\frac{2}{k^2+l^2}}}{\sim} \frac{\lambda(k^2+l^2)}{Y^2}, \tag{100}$$

while the behavior of the imaginary part is

$$\Im\big[\tilde{H}_\lambda(Y)\big] \underset{Y \gg 1}{\sim} -\frac{1}{\lambda Y}. \tag{101}$$

Observe that the crossover in the asymptotic behavior of $\Re[\tilde{H}_\lambda(Y)]$ happens for $Y \gg 1$, so that the asymptotic expansion performed is justified.

By performing the change of variables $Y = y/\lambda$, we thus obtain the asymptotic behavior of H_λ

$$\Re\big[H_\lambda(y)\big] \underset{\lambda \ll y \ll \sqrt{\frac{2}{k^2+l^2}}}{\sim} -\frac{2\lambda^3}{y^4} \qquad \text{and} \qquad \Re\big[H_\lambda(y)\big] \underset{y \gg \sqrt{\frac{2}{k^2+l^2}}}{\sim} \frac{\lambda^3(k^2+l^2)}{y^2} \tag{102}$$

for the real part, and

$$\Im\big[H_\lambda(y)\big] \underset{y \gg \lambda}{\sim} -\frac{1}{y} \tag{103}$$

for the imaginary part.

A very similar reasoning applied to the function G_λ defined in Eq. (67) leads us to the conclusion that

$$\Re\big[G_\lambda(y)\big] \underset{y \gg \lambda}{\sim} -\frac{2\lambda^3}{y^4}, \tag{104}$$

while the leading order behavior of the imaginary part remains the one found for H_λ. We can thus conclude that $H_\lambda(y) \underset{\lambda \to 0}{=} G_\lambda(y) + \mathcal{O}(\lambda)$ for every value of y.

Combination of a Viscous and a Linear Friction Regularization   We can now also discuss the regularization of Plemelj formula that corresponds to the combined effect of viscosity and linear friction, by a simple generalization of the argument given above. We start observing that the stationary vorticity-vorticity correlations is

$$g^\infty(x_1 - x_2, y_1 - y_2) = \varepsilon(k^2 + l^2)e^{ik(x_1-x_2)}e^{il(y_1-y_2)}$$
$$\times \int_0^\infty e^{-iks(y_1-y_2)t}e^{-2\nu(k^2+l^2)t+2\nu sklt^2-\frac{2}{3}\nu s^2k^2t^3}e^{-2\alpha t}\,dt + \text{C.C.};$$
(105)

we have thus to study the following oscillating integral (obtained from the one contained in the above formula by a change of variable)

$$H_{\lambda_\nu,\lambda_\alpha}(y) = \frac{1}{\lambda_\nu}\int_0^\infty e^{-i\frac{y}{\lambda_\nu}t_1 - [\lambda_\nu^2(k^2+l^2)+\frac{\lambda_\alpha}{\lambda_\nu}]t_1+\lambda_\nu lt_1^2 - \frac{1}{3}t_1^3}\,dt_1,$$
(106)

where $\lambda_\nu = (\frac{2\nu}{ks})^{1/3}$ is the length scale associated with the viscous regularization and $\lambda_\alpha = \frac{2\alpha}{ks}$ is the length scale associated with the Rayleigh friction. The discussion in the following restricts to the real part of $H_{\lambda_\nu,\lambda_\alpha}$ because the decay of the imaginary part is the same for all the regularizations.

Introducing the function $\tilde{H}_{\lambda_\nu,\lambda_\alpha}(Y) = H_{\lambda_\nu,\lambda_\alpha}(\lambda_\nu Y)$ and performing a reasoning very similar to the one of the previous section, we deduce that the real part of $\tilde{H}_{\lambda_\nu,\lambda_\alpha}(Y)$ is given by

$$\Re[\tilde{H}_{\lambda_\nu,\lambda_\alpha}(Y)] = \frac{1}{\lambda_\nu}\left[\frac{\lambda_\nu^2(k^2+l^2)+\frac{\lambda_\alpha}{\lambda_\nu}}{Y^2}\right.$$
$$\left.- \frac{(\lambda_\nu^2(k^2+l^2)+\frac{\lambda_\alpha}{\lambda_\nu})^3 + 6\lambda_\nu^3 l(k^2+l^2)+6\lambda_\alpha l+2}{Y^4} + O\left(\frac{1}{Y^6}\right)\right].$$
(107)

We thus see that three different regimes are possible in the limit of small Rayleigh friction ($\alpha/s \ll 1$) and small viscosity ($\lambda_\nu l \ll 1$): (i) the viscosity is negligible compared to the Rayleigh friction: $\lambda_\alpha/\lambda_\nu \gg 1$; (ii) the Rayleigh friction is negligible compared to the viscosity $\lambda_\alpha/\lambda_\nu \ll \lambda_\nu^2(k^2+l^2) \ll 1$ and (iii) and intermediate case $1 \gg \lambda_\alpha/\lambda_\nu \gg \lambda_\nu^2(k^2+l^2)$.

Case (i): $\lambda_\alpha/\lambda_\nu \gg 1$. We see that Eq. (107) reduces to

$$\Re[\tilde{H}_{\lambda_\nu,\lambda_\alpha}(Y)] \underset{\frac{\lambda_\alpha}{\lambda_\nu}\gg 1}{\sim} \frac{1}{\lambda_\nu}\frac{\lambda_\alpha}{\lambda_\nu}\left[\frac{1}{Y^2} - \left(\frac{\lambda_\alpha}{\lambda_\nu}\right)^2\frac{1}{Y^4} + O\left(\frac{1}{Y^6}\right)\right],$$
(108)

and thus

$$\Re[H_{\lambda_\nu,\lambda_\alpha}(y)] \underset{\lambda_\nu \ll y \ll \lambda_\alpha}{\sim} -\lambda_\alpha^3\frac{1}{y^4} \quad\text{and}\quad \Re[H_{\lambda_\nu,\lambda_\alpha}(y)] \underset{y \gg \lambda_\alpha}{\sim} \lambda_\alpha\frac{1}{y^2}.$$
(109)

We thus obtain that, even if the viscosity is negligible with respect to the Rayleigh friction, the behavior $1/y^2$ is modified by the presence of a small viscosity in $1/y^4$ for a small enough y, corresponding to $\lambda_\nu \ll y \ll \lambda_\alpha$.

Case (ii): $\lambda_\alpha/\lambda_\nu \ll \lambda_\nu^2(k^2+l^2) \ll 1$. We see that Eq. (107) reduces to the case we have already discussed in the previous section where the Rayleigh friction was not present, see Eq. (102). The only difference with respect to that case is that the crossover point between the $1/y^4$ and $1/y^2$ behavior is slightly shifted by the presence of a small Rayleigh friction.

Case (iii): $\lambda_\nu^2(k^2 + l^2) \ll \lambda_\alpha/\lambda_\nu \ll 1$. With a similar reasoning as before, we obtain

$$\Re\big[H_{\lambda_\nu,\lambda_\alpha}(y)\big] \underset{\lambda_\nu \ll y \ll \sqrt{\frac{2\lambda_\nu^3}{\lambda_\alpha}}}{\sim} -\frac{2\lambda_\nu^3}{y^4} \quad \text{and} \quad \Re\big[H_{\lambda_\nu,\lambda_\alpha}(y)\big] \underset{y \gg \sqrt{\frac{2\lambda_\nu^3}{\lambda_\alpha}}}{\sim} \lambda_\alpha \frac{1}{y^2}. \tag{110}$$

We thus see in this case the appearance of the new length scale $\sqrt{\frac{2\lambda_\nu^3}{\lambda_\alpha}}$, corresponding to the crossover between the $1/y^4$ decay given by the regularization due to the viscosity and the $1/y^2$ decay given by the Rayleigh friction.

Appendix C: Convergence of Reynolds Stress Divergence on Stationary Points

We prove in this Appendix that the divergence of the Reynolds stress $-\mathbb{E}_U[\langle v_m^{(y)}\omega_m\rangle(y)]$ converges even when evaluated at points $y = y_c$, where y_c are stationary for the zonal flow $U: U'(y_c) = 0$. Even more generically, we prove that $\mathbb{E}_U[\langle\omega_m(\mathbf{r}_1, t)v_m^{(y)}(\mathbf{r}_2, t)\rangle]$ converges, no matter for which values of $\mathbf{r}_1$ and $\mathbf{r}_2$.

In this case, to the best of our knowledge, no theoretical results are available in literature. However, strong numerical evidences are reported in [7] and permit to infer relations analogous to Eqs. (70), (71), (72) and (73) for $y = y_c$. The main differences in this case are: (i) the algebraic decay of the velocity and of the stream-function depends on the parity of the initial datum; (ii) the exponent of the algebraic decay for $y = y_c$ is different.

For initial conditions even with respect to $y \to -y$, the numerical results in [7] can be summarized by replacing Eqs. (70), (71), (72) and (73) by

$$\big|\tilde{\omega}_k(y_c, t)\big| \underset{t\to\infty}{\sim} \frac{1}{t} \tag{111}$$

$$\big|\tilde{v}_k^{(x)}(y_c, t)\big| \underset{t\to\infty}{\sim} \frac{1}{t} \tag{112}$$

$$\big|\tilde{v}_k^{(y)}(y_c, t)\big| \underset{t\to\infty}{\sim} \frac{1}{t^2}. \tag{113}$$

Moreover, the asymptotic profile $\tilde{\omega}_k^\infty(y)$ appearing in Eqs. (70), (71), (72) and (73) is equivalent to $(y - y_c)^2$ in the vicinity of y_c. We observe that these results are about the asymptotic behavior of the modulus of vorticity and velocity and that the oscillating behavior of these quantities is not known. This fact does not allow to obtain estimates for the asymptotic behavior of derivatives of the vorticity and of the velocity.

For initial conditions odd with respect to $y \to -y$, the numerical results in [7] can be summarized by replacing Eqs. (70), (71), (72) and (73) by

$$\big|\tilde{\omega}_k(y_c, t)\big| \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}} \tag{114}$$

$$\big|\tilde{v}_k^{(x)}(y_c, t)\big| \underset{t\to\infty}{\sim} \frac{1}{t^{3/2}} \tag{115}$$

$$\big|\tilde{v}_k^{(y)}(y_c, t)\big| \underset{t\to\infty}{\sim} \frac{1}{t^{3/2}}. \tag{116}$$

Again, the asymptotic profile $\tilde{\omega}_k^\infty(y)$ appearing in Eqs. (70), (71), (72) and (73) vanishes as $(y - y_c)^2$ in the vicinity of y_c.

Using these results, we can prove that the divergence of Reynolds stress at $y = y_c$ converges. It is useful here to consider $\tilde{g}_{kl}$ defined, as in Sect. 5.3, by $g_{kl}(x_1, x_2, y_1, y_2, t) = e^{ik(x_1-x_2)}\tilde{g}_{kl}(y_1, y_2, t) + \text{C.C.}$ with

$$\frac{\partial \tilde{g}_{kl}}{\partial t} + L_{U,k}^{\alpha(1)}\tilde{g}_{kl} + L_{U,-k}^{\alpha(2)}\tilde{g}_{kl} = e^{il(y_1-y_2)}. \tag{117}$$

Moreover, because the Orr mechanism works differently for initial conditions with different parity, it is useful to decompose $\tilde{g}_{kl}$ as follow

$$\tilde{g}_{kl} = \tilde{g}_{kl,cc} - i\tilde{g}_{kl,cs} + i\tilde{g}_{kl,sc} + \tilde{g}_{kl,ss} \tag{118}$$

where $g_{kl,cc}$ solves the Lyapunov equation

$$\frac{\partial \tilde{g}_{kl,cc}}{\partial t} + L_{U,k}^{0(1)}\tilde{g}_{kl,cc} + L_{U,-k}^{0(2)}\tilde{g}_{kl,cc} = \cos(ly_1)\cos(ly_2), \tag{119}$$

and $\tilde{g}_{kl,cs}, \tilde{g}_{kl,sc}, \tilde{g}_{kl,ss}$ solve the same Lyapunov equation with the right hand side replaced, respectively, by $\cos(ly_1)\sin(ly_2)$, $\sin(ly_1)\cos(ly_2)$ and $\sin(ly_1)\sin(ly_2)$. The computation needed to show the convergence of the divergence of the Reynolds stress obtained as linear transforms of the four functions $\tilde{g}_{kl,cc}, \tilde{g}_{kl,cs}, \tilde{g}_{kl,sc}$ and $\tilde{g}_{kl,ss}$ are very similar. We thus report in detail only the one corresponding to $\tilde{g}_{kl,cc}$.

With the notation $\tilde{g}_{kl,cc} = \Delta_k^{(2)} h_{kl,cc}$ and using Eq. (51), we can write

$$h_{kl,cc}^\infty(y_1, y_2) = \int_0^\infty dt_1 \, \tilde{\omega}_k(y_1, t_1)\tilde{v}_{-k}^{(y)}(y_2, t_1), \tag{120}$$

where $\tilde{\omega}_k$ is the solution to the deterministic evolution $\partial_t + L_{U,k}^0$ and $\tilde{\omega}_k(y_1, 0) = \cos(ly_1)$ and $\tilde{v}_{-k}^{(y)}(y_2, 0) = \cos(ly_2)$. The convergence of the this integral has already been established in the case $y_1 \neq y_c$ and $y_2 \neq y_c$ in Sect. 5.2.3.

For any value of y_1 and y_2, we see from Eqs. (70), (72), (111) and (113) that the absolute value of the integrand of Eq. (120) decays to zero at least as fast as $1/t^2$. The convergence of $h_{kl,cc}$ is thus ensured for all the values of y_1 and y_2.

We leave to the reader the very similar computation needed to show that also the Reynolds stresses obtained from $\tilde{g}_{kl,cs}, \tilde{g}_{kl,sc}$ and $\tilde{g}_{kl,ss}$ converge. The slowest decay of the integrand of expressions like Eq. (120) which is encountered is $1/t^{3/2}$, still sufficient to ensure the convergence of h_{kl}^∞.

We conclude stressing that the results presented in this appendix rely on conclusions drawn from numerical results reported in [7].

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